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A Néron model of the universal jacobian

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1) Classical situation: Néron model over a 1-d base.

Def: Let:
- S reg. Noeth. scheme (~~énoncé of dimension 1~~)
- $U \subseteq S$ dense open
- A/U abelian scheme.

DNE for now

A Néron model (NM) for A/S is a smooth separated ~~scheme~~ group ~~scheme~~ (alg. space) N/S, together with an isom $A \to N|_U$, s.t.

$\forall T \to S$ smooth, $\forall f: T_U \to N_U$,
$\exists !\ F: T \to N$ s.t. $F|_{T_U} = f$.

~~(unique if exists)~~ (*)

Eg: $S = \operatorname{Spec} \mathbb{C}[[t]]$, $U = S \setminus \{t=0\}$, E/U elliptic. Let $\mathcal{E}$ be min. reg. model of E over S, ~~then~~ & write $\operatorname{sm}(\mathcal{E}/S)$ for ~~the~~ locus where $\mathcal{E} \to S$ smooth. Then $\operatorname{Sm}(\mathcal{E}/S)$ is NM of E.

~~Eg: every sectio~~

Thm [Néron, 1965]: In the above setup ($\dim S = 1$), NMs always exist.

Applications: too many to list - play a key role in just about all big thms. on abelian varieties.

(*)
- unique up to unique iso if exists;
- sections of A over U extend uniquely to sections of N/S (take $T=S$)
- finite type over S if exists
- ~~badly behaved ...~~
- 'best model of degenerating family'.

2) Higher dimensional base-schemes S; examples & questions.

~~2.~~ 2.1: Example

~~eg.~~ S = small nbd. of origin $\underline{o} \in \mathbb{A}^2_{\mathbb{C}}$, coords u, v.

Define C by $y^2 = (x-1)(x-1-u)(x+1)(x+1+v) \subset \mathbb{P}_S(1,1,2)$,

& fix a section e through smooth locus.

Then C_S is an elliptic curve outside $uv=0$. ($U := S \setminus \{uv=0\}$)

What happens at $u=v=0$?

~~Given~~ Pick $\lambda \in \mathbb{C}^*$, & let L_λ be the line $u=\lambda v$ through o in S.
Restricting to L_λ, we get an elliptic curve over an open, so (by Néron's thm) it has a NM, $\Rightarrow$ a gp. scheme ~~over L_λ~~ N_λ over L_λ.
Restricting to the origin, we get a gp. scheme that should 'fill in' the degenerating elliptic curve.

Problem: this group scheme '$N_{\lambda_{\underline{o}}}$' depends continuously on λ.

- So we don't get a 'nice' model of the gp. scheme $C_{/U}$.
- Given dependence on λ, maybe should blow up origin of S?

Various approaches to the 'correct' model of the gp. scheme
- Caporaso, ~~[illegible]~~, Esteves, ~~[illegible]~~ Melo,

<u>2.2) Theory</u>

The example suggests things aren't so nice, let's now try to make this precise.

Naive def:

<u>Def</u>: Keep old NM def, cross out "$\mathbb{Q}$-Cartier of dim 1"

<u>Question 1)</u>: Do they always exist?

~~1) If not,~~

Prev. eg. suggests 'no'. Maybe better:

<u>Question 2)</u> Do NMs always exist if we allow blowups/alteration of S?

3) Motivation. Why should we care whether NMs exist? We describe a connection to arithmetic problems...

Conj 1 (UBC): Fix $g > 0$. Then $\exists$ constant $B = B(g)$ s.t. $\forall$ ab. vars $A/_{\mathbb{Q}}$ of dim g, we have $\# A(\mathbb{Q})_{tors} \leq B$.

(If $g = 1$, thm [Mazur, Kamenny, Merel])

Conjecture 2) Let $S/_{\mathbb{Q}}$ be a variety (integral sep. scheme of fin. type), & let $A/_S$ jacobian scheme. Let $d \geq 1$ integer, & let $\sigma \in A(S)$ a section of infinite order. Then

$$T(d) := \{ p \in S(\bar{\mathbb{Q}}) \mid [k(p):\mathbb{Q}] \leq d \ \& \ \sigma(p) \text{ torsion in } A_p(\bar{\mathbb{Q}}) \}$$

is not Zariski dense in S.

[If dim $S = 1$, this is a thm of S'man & Tate].

Thm [H]: Conjecture 1 is $\equiv$ to conjecture 2.

Thm [H]: Let $S/_{\mathbb{Q}}$ a variety & $A/_S$ abelian. Suppose $\exists$ a proper model $\bar{S}/_{\mathbb{Z}}$ of S such that A admits a NM over $\bar{S}$ (perhaps after altering $\bar{S}$). Then conj. 2 holds for $A/_S$.

Unfortunately:

Thm [H]: Let $E/_U$, $U \subseteq S$ be as in example. Let $f: S' \to S$ any proper surj (e.g. alteration). Then f^*E does NOT admit a NM over S'.

This is a special case of a result ~~classes~~ classifying exactly when a jacobian admits a NM, & behaviour under base change.

[Optional extra: $C/_S$ nodal. Given $s \in S$, let Γ_s = dual graph with each edge labelled by the 'direction in which the singularity survives', e.g.

Def: $C/_S$ aligned at s iff $\forall$ cycles $\gamma \in \Gamma_s$, $\forall$ labels e_1, e_2 on γ, $\exists\, n_1, n_2 > 0$ s.t.

$$e_1^{n_1} = e_2^{n_2}.$$

Thm: $C/_S$ aligned iff $\mathrm{Jac}(C/_U)$ admits a NM $/_S$.

Thm [H]: $f: S' \to S$ proper surj. Then $C/_S$ aligned iff $f^*C/_{S'}$ aligned.

4) Positive results. We would like to quantify the failure of existence of NMs. We have two approaches:

1) an algebraic version of Hain's 'height jumping' from Hodge theory
2) look at the ~~minimal basechange~~ 'largest' basechange after which a NM exists.

We focus on (2) today. We work with stacks so that we can treat the universal case. The def of NM extends w/o change to that situation.

Def: Let
- S reg. loc. noeth. sep. alg. stack;
- $U \subseteq S$ dense open;
- A/U abelian scheme;
- Let T a reg. loc. noeth. stack, & $f: T \to S$ a morphism

We say T is 'NM-admitting' if
- $f^{-1}U$ is dense in T;
- f^*A has a NM over T.

A universal NM-admitting morphism is a terminal object in the 2-cat of NM admitting morphisms.

IOW: a UNMA morphism is a morphism $\varphi: \tilde{S} \to S$ s.t.:
- $\varphi^{-1}U$ dense in $\tilde{S}$
- φ^*A has a NM on $\tilde{S}$
- ~~$\tilde{S}$ is the universal such~~

Any other $f: T \to S$ w. these properties factors 'uniquely' via φ.

Thm [H]: Fix $n, g \geq 0$ s.t. $2g-2+n>0$, & set $S = \overline{M}_{g,n}$ for the DM stack of stable curves of genus g w. n marked pts. Let $U = M_{g,n} \subseteq S = \overline{M}_{g,n}$, & let A = jacobian of universal curve over U. Then A/S admits a UNMA morphism $\varphi: \tilde{M}_{g,n} \to \overline{M}_{g,n}$

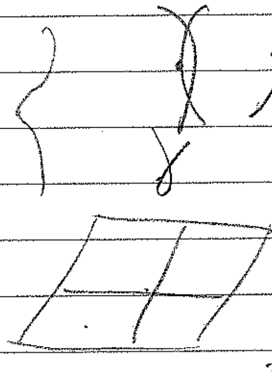
- φ is lf type
- φ is an iso. over $M_{g,n}$
- φ is q-cpct iff it's an isomorphism
- φ satisfies VCP for DVRs mapping generic pt to $M_{g,n}$.
- φ is rel. rep. by alg. spaces.

Kapranov's $\overline{P}_g$?
Mumford's moduli of enriched structures (cont. cpct?)
(limit linear series)

6) Example revisited.

To conclude, let's look at a more concrete eg of a UNMA morphism.

Let $A/_{U} \subseteq S$ be as in prev. eg (S nbhd of 0 in $\mathbb{A}^2_{\mathbb{C}}$, u, v, $U = S\setminus\{uv=$



$C : y^2 = (x-1)(x-1-u)(x+1)(x+1+v)$,
e section,
$A/_{U} = (C_U, e_U)$ elliptic curve over U.

Then $A/_{S}$ admits a UNMA $\tilde{S} \to S$, & we can give an explicit description:

- Set $S_0 = S$, $U_0 = U$, $D_0 = S \setminus U$ (reduced ~~divisor~~ closed subscheme), ~~$V_0 = S_0$~~
- Let B_0 = locus where D_0 is singular (ie. $u=v=0$), $V_0 = S_0 \setminus B_0$
- Let $S_1 \xrightarrow{f_1} S_0$ = blowup of S_0 at B_0.
- Let $D_1 = f_1^* D_0$ (w. reduced scheme structure),
  B_1 = locus where D_1 singular
  $V_1 = S_1 \setminus B_1$
  $\varphi_1 : V_0 \to V_1$ obvious open immersion
- Let $S_2 \xrightarrow{f_2} S_1$ = blowup of S_1 at B_1, repeat.

Let $\tilde{S}$ = colim of system $V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \dots$,
Then $\tilde{S} \xrightarrow{f} S$ is UNMA.
- $\tilde{S}$ integral, ~~loc noeth~~
- f loc. fin. pres.
- $\tilde{S}$ is not q-cpt.
- ~~Take~~ $\tilde{S} \to S$ sat. VCP for DVR mapping generic pt to U.