# Néron models: what and why?

- Take an elliptic curve$/\mathbb{Q}$, say $E_{\mathbb{Q}}: y^2 = x^3 + x^2 + 7$. $\subset \mathbb{P}^2$

- There is a group law on the rational points; say 3 pts sum to zero iff they are collinear (need to choose a flex pt as origin).

- The equation for $E$ makes sense$/\mathbb{Z}$ (in general, we can scale coefficients), and so we can consider the surface

$$E_{\mathbb{Z}_p} = \operatorname{Spec}\left(\frac{\mathbb{Z}_p[x,y]}{(y^2-x^3-x^2-7)}\right) \quad \left(\text{take closure in } \mathbb{P}^2_{\mathbb{Z}}\right).$$

The group law on $E_{\mathbb{Q}}$ can be written in terms of polynomial equations: if $P_i = (x_i, y_i)$, and say $P_1 + P_2 = P_3$, then generally we have

$$x_3 = \lambda^2 + 1 - x_1 - x_2, \quad \lambda = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_3 = -\lambda x_3 - \left(\frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}\right)$$

We can again view these as living in $\mathbb{Z}_p(x,y)$.

<u>Question</u>: Does this give us a group law on $E_{\mathbb{Z}_p}$?

Given all these equations over $\mathbb{Z}_p$, we can reduce mod $p$ to get a curve $E_{\mathbb{F}_p}$ and equations for addition.

<u>Simpler Question</u>: Do these equations give us a group law on $E_{\mathbb{F}_p}$?

<u>Answer</u>: It depends!

- If $E_p$ is non-singular, then <u>yes</u>.

- If $E_p$ is singular, then the surface $E_{\mathbb{Z}_p}$ may not be regular; in this case, blow up to resolve singularities on $E_{\mathbb{Z}_p}$.
  $\rightsquigarrow$ a (minimal) regular model $\overline{E_{\mathbb{Z}_p}}$.

[See end of notes for comments on smoothness, regularity, ...].

<u>Then</u> we get a gp. law on the <u>smooth</u> pts of $E_{\mathbb{F}_p}$.

<u>Note</u> Smooth pts of $E_{\mathbb{F}_p}$ lift to pts of $E_{\mathbb{Z}_p}$ <u>AND</u> vice versa <u>if</u> we have a <u>regular</u> surface.

~~Note If $E_{\mathbb{Z}_p}$ and minimal regular~~

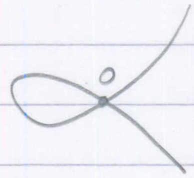
- The group law on $E_{/\mathbb{Q}}$ can be described in terms of isomorphism classes of invertible sheaves on $E/_{\mathbb{Q}}$.

- Everything we said above about 'reducing equations for the group law mod p' can be phrased in terms of isomorphism classes of invertible sheaves on $E_{\mathbb{F}_p}$ (& the same holds for $E_{\mathbb{Z}_p}$).

- However, the fact that equations <u>exist</u> is important in proving representability of the Picard functor (see later talks...).

- Why is this useful in practice? Because we can deal with higher genus curves (∃ many other answers!)

Example 1: $E_{\mathbb{Q}}: y^2 = x^3 + x^2 + p$, $p \neq 2$. $E_{\mathbb{F}_p}: y^2 = x^3 + x^2$. This is singular @ 0; we can draw



We can construct a morphism from $\mathbb{P}^1_{\mathbb{F}_p}$ to $E_{\mathbb{F}_p}$ which is birational as follows:

$$\mathbb{P}^1_{\mathbb{F}_p} \longrightarrow E_{\mathbb{F}_p}$$
$$t \longmapsto (t^2-1,\ t^3-t)$$

[~~ex~~ CHECK]

The two pts $t = 1$ & $t = -1$ on $\mathbb{P}^1$ are both sent to the sing. pt, elsewhere the map has a regular inverse.

~~Fact (see later talk)~~

Fact/ex: $E_{\mathbb{Z}}: y^2 = x^3 + x^2 + p$ is regular, ~~...~~ (~~as~~ it is a surface).

The eq'ns ~~...~~ for addition give $t_3 = t_1 t_2$ ~~...~~
($t_i \mapsto (x_i, y_i)$). [check - may have sign or something missing!]

Magic: Once everything is correctly defined, we can consider arbitrary genus; eg. $y^2 = x^9 + p$, $(g = 4)$.

[Say: ~~at~~ generic / smooth fibres comp. irreducible].

Easy calculation: The special fibre of the Néron model of the jacobian is $\cong$

$$\mathbb{G}_m \times \Gamma, \quad \Gamma = \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(unipotent)

In general, you get an extension of the jacobian of the normalisation of the special fibre (see a later talk) by a group like this.