Universiteit Leiden
Wiskunde en Natuurwetenschappen

Counting Curves.

Vak:

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Docent:

Naam:

Studierichting:

Collegekaartnummer:

Fix a field $k = k^{alg}$.

1) Curves

Let $f \in k[x,y]$. Define $Z(f) \subseteq k^2$ by

$$Z(f) = \{(a,b) \in k^2 \mid f(a,b) = 0\}.$$

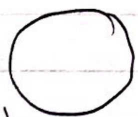
$f = 0 \rightsquigarrow Z(f) = k^2$
f unit $\Rightarrow$ $Z(f)$ empty

A subset of k^2 of this form, for $f \neq 0$, unit, is ~~an Zaffine~~ an <u>(affine) plane curve</u>.

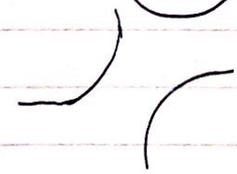
E.g. $f = x$, then $Z(f) = \{(0,b) : b \in k\}$.

eg if $k = \mathbb{C}$, then can draw $Z(f) \cap \mathbb{R}^2$, eg:

$f = x^2 + y^2 - 1$:



$f = xy - 1$



$f = x^2 + y^2 + 1$

$\underline{Q}$: When is $Z(f) = Z(g)$?

~~eg. $f = g^2$, then $Z(f$~~

eg. $g = \lambda f$, $\lambda \in k^\times$. So ask if $(f) = (g)$, noting $k[x,y]$ a domain w. units $k^\times$.

or $g = f^2$

or $f = f_1 f_2$, $g = f_1^2 f_2$

<u>Thm</u> [Nullstellensatz]: $Z(f) = Z(g) \iff \sqrt{(f)} = \sqrt{(g)}$. ($\iff$ same set of irred. factors up to $k^\times$.)

(<u>Def</u>: (f) = ideal of $k[x,y]$ gen by f,

$\sqrt{I} = \{ a \in k[x,y] : \exists n > 0 \text{ with } a^n \in I \}$.)

<u>Nb</u> <u>False</u> if $k \neq \bar{k}$, eg. if $k = \mathbb{R}$ then $Z(x^2+y^2+1) = Z(x^2+y^2+2)$

~~<u>Def</u> We say a curve $C \subseteq k^2$ is <u>irreducible</u> if~~

~~$\exists f$ irred s.t. $C = Z(f)$.~~

~~<u>eg</u> Not $C = Z(xy)$. If $C = Z(g)$ then $\sqrt{(g)} = \sqrt{(xy)}$, so~~

~~<u>Prop</u> $Z(f)$ irred $\iff$ f is a power of an irred. poly.~~

~~<u>Pf</u> $\Leftarrow$: Say $f = g^n$, g irred. Then $Z(f) = Z(g)$.~~

~~$\Rightarrow$: Say $Z(f)$ irred $\Rightarrow$ $\exists g$ irred w. $Z(f) = Z(g)$ $\overset{NSZ}{\Longrightarrow}$ $\exists g$ irred w. $\sqrt{(f)} = \sqrt{(g)}$.~~

~~Fix such a g, then $\sqrt{(g)} = (g)$ (ex).~~

~~Then $\sqrt{(f)} = (g)$, so~~

Irreducibility

eg. $Z(xy-1)$ irred.

$Z(xy)$ not irred? Pf?

Def. A curve ~~...~~ $C \subseteq k^2$ is *irred* if $\exists$ g irred s.t. $C = Z(g)$

Prop ~~...~~ $Z(f)$ irred $\Leftrightarrow$ f is a power of an irreducible.

Pf $\Leftarrow$: Say $f = g^n$, g irred, then $Z(f) = Z(g)$.

$\Rightarrow$: Say $Z(f)$ irred, & let g ~~...~~ irred s.t. $Z(f) = Z(g)$.

Then (NSZ): $\sqrt{(f)} = \sqrt{(g)}$.

Ex: $\sqrt{(g)} = (g)$.

Then $\sqrt{(f)} = (g)$, so $g^n = cf$ & $f = dg$ some $n \geq 1$, $c, d \in k[x,y]$

Then $g^n = cdg$ so $cd = g^{n-1}$ so $d = g^m u$ some $u \in k^*$, $m \geq 1$.

Then $f = ug^{m+1}$, so $f = (vg)^{m+1}$ where $v^{m+1} = u$ (uses $k = \bar{k}$)

$\square$

Eg ~~...~~ $Z((y^2 - x)^3)$ is irred.

$Z(x^2 y)$ not irred.

Smoothness

Def Let $C = Z(f)$ be a curve

{ Assume $(f) = \sqrt{(f)}$, ie f a product of distinct irred.

· Let $p \in$ ~~Z(f)~~ C. We say C is smooth at p if at least one of $\frac{\partial f}{\partial x}(p)$, $\frac{\partial f}{\partial y}(p)$ is non-zero.

· C is smooth if it is smooth at p $\forall p \in C$.

eg. $f = x^2+y^2-1$. Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$.

So if ~~[illegible]~~ $2 \in k^\times$ then p not smooth $\Rightarrow x(p) = y(p) = 0$,

but $(0,0) \notin Z(f)$. So $Z(f)$ smooth.

eg $f = xy$. Then $\frac{\partial f}{\partial x} = y$, ~~[illegible]~~ $\frac{\partial f}{\partial y} = x$, so ~~$p = (0,0)$~~ C is not smooth at $p = (0,0)$

eg $f = x$, then $\frac{\partial f}{\partial x} = 1$, so smooth.

eg $f = x^2$, then $\frac{\partial f}{\partial x} = 2x = 0$, $\frac{\partial f}{\partial y} = 0$, ?! Now write in last line of def'n

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Genus

V. important. Lots of $\equiv$ def's. All a bit hard to formulate.

Heuristic Let $k = \mathbb{C}$. Let C be smooth irred curve. ~~Then~~

~~[illegible]~~ Give C the topology from $\mathbb{C}^2$.

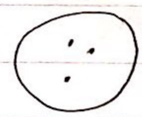
Then $\exists!\ (g, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ s.t. C is homeomorphic

to a sphere with - g handles attached

- n pts deleted.

The genus is g.

e.g. $y^2 = x^3 + 1 \rightsquigarrow$  $g = n = 1$.

Warning! homeo matters; show that  homotopic to 

$g = 0 \quad n = 3$ $\qquad$ $g = n = 1$.

Def. A irred curve $C \subseteq k^2$ has genus 0 (aka 'rational') if & only if $\exists\ a, b \in k[t]$, & $S \subseteq k$, s.t.

$$\varphi : k \setminus S \longrightarrow k^2$$
$$t \longmapsto (a(t), b(t))$$

~~[illegible]~~ factors via $C \hookrightarrow k^2$ & is non-constant ~~[illegible]~~

(can replace by surjective, or bij outside C_{sing}).

If $k = \mathbb{C}$ & C smooth, then this is $\equiv$ to the above def, & $\# S = n - 1$.

eg $C = Z(y-x^2)$. Then $\varphi: k \longrightarrow k^2$, $t \longmapsto (t, t^2)$, image of φ is exactly C.

'Rational' is a v. important property. Say again $k = \mathbb{C}$, ~~then~~ C smooth.

Look at ~~#~~ $C(\mathbb{Q})$:= pts w. rat. coords.

Rational ($g=0$): $C(\mathbb{Q}) \neq \phi \iff C(\mathbb{Q})$ infinite.

$g = 1$: $C(\mathbb{Q})$ can be any size $0 \le \# C(\mathbb{Q}) \le \# \mathbb{N}$.

$g \geq 2$: ~~#~~ $C(\mathbb{Q})$ finite.

Counting rat. curves of deg d.

Def. Let C a curve. degree of C is min. deg. of an f s.t. $Z(f) = C$.
(eg. $\deg(Z(y-x^2)) = 2$)
$\deg(\text{line}) = 1$.

~~How many~~
~~N_d := # ration~~
~~How many lines through 2 pts? 1 (if distinct!)~~
~~— — deg curves~~

Fix $d \geq 1$. Let S be a set of $3d-1$ pts in $\mathbb{C}^2$.

$N_d(S)$ = # rat. curves of deg d containing all pts in S.

$N_d = \inf_S N_d(S)$, achieved on S in a dense open of $(\mathbb{C}^2)^{3d-1}$

eg N_1 = # lines through 2 pts $= 1$
N_2 = # deg 2 curves through 5 pts $= 1$.
N_3 = # singular cubics through 8 pts $= 12$

$N_1 = 1$

$N_2 = 1$

$N_3 = 12$ (Steiner, 1848)

$N_4 = 620$ (Zeuthen, 1873)

$N_5 = 87304$ (Ran, Vainsencher, ~1990)

Thm (Kontsevich)

$$N_d = \sum_{\substack{d_A + d_B = d \\ d_A, d_B \geq 1}} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1} \right).$$

Earnt $\frac{\text{Fieldsmedal}}{4}$.