Well-Posedness of Initial Value Problems for Functional Differential and Algebraic Equations of Mixed Type

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Abstract

We study the well-posedness of initial value problems for scalar functional algebraic and differential functional equations of mixed type. We provide a practical way to determine whether such problems admit unique solutions that grow at a specified rate. In particular, we exploit the fact that the answer to such questions is encoded in an integer $n^\sharp$. We show how this number can be tracked as a problem is transformed to a reference problem for which a Wiener-Hopf splitting can be computed. Once such a splitting is available, results due to Mallet-Paret and Verduyn-Lunel can be used to compute $n^\sharp$. We illustrate our techniques by analytically studying the well-posedness of two macro-economic overlapping generations models for which Wiener-Hopf splittings are not readily available.

Key words: functional differential equations, advanced and retarded arguments, overlapping generations models, initial value problems, indeterminacy.

1 Introduction

In this paper we consider a class of initial value problems that includes the prototypes

$$\begin{array}{rcll} ax'(\xi) & = & x(\xi) + \int_{-1}^{1} x(\xi+\sigma)d\sigma & \text{for all } \xi \geq 0, \\ x(\tau) & = & \psi(\tau) & \text{for all } -1 \leq \tau \leq 0, \end{array} \qquad (1.1)$$

in which a is allowed to be any real number, including zero. We wish to determine whether such systems will admit bounded solutions for any initial condition $\psi \in C([-1,0],\mathbb{R})$ and whether such solutions are unique. If $a \neq 0$, the first line of (1.1) is called a functional differential equation of mixed type (MFDE), while if $a = 0$, we use the term functional algebraic equation of mixed type (MFAE). The word 'mixed' refers to the fact that the nonlocal term in (1.1) involves shifts in the argument of x that are both positive and negative.

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Notice that (1.1) with $a \neq 0$ differs from traditional initial value problems, in the sense that the initial condition ψ does not provide sufficient information to calculate $x'(0)$. The system (1.1) hence cannot be interpreted as an evolution equation, requiring us to consider the behaviour of x on the entire interval $[0, \infty)$ all at once.

In this sense, it seems reasonable to argue that the initial condition ψ should be taken from $C([-1, 1], \mathbb{R})$, which after all is a natural state space for (1.1) as it does allow $x'(0)$ to be computed. However, the resulting problem is highly ill-posed [13, 23]. For example, there is no guarantee that $\psi'(0)$ exists or agrees with the value for $x'(0)$ computed from ψ. One might be tempted to incorporate such a requirement into the state space, but this is just one of a sequence of increasingly intricate incompatibilities that needs to be addressed. As we will see below, the choice to take $\psi \in C([-1, 0], \mathbb{R})$ is natural both from a mathematical and an applied perspective.

Macro-Economic Modelling

The primary motivation for this paper comes from the area of macro-economic research. To set the stage, let us consider an isolated economy that starts at $t = 0$. We write $k(t)$ for the production capacity at time t, which is a measure for the amount of goods and services that can be produced. At any point in time, this capacity must be divided between investments $u(t)$ and consumption $c(t)$. The former leads to an increase in the production capacity, while the latter satisfies the immediate needs of the population. The goal is to optimize the welfare of the population, which is assumed to depend only on the consumption $c(t)$. This can be formulated as the optimal control problem

$$\max \int_0^\infty e^{-\rho t} W\big(c(t)\big) dt, \tag{1.2}$$

in which the discount factor ρ reflects how future welfare is rated relative to present welfare, while W measures the welfare associated to consumption.

Typically, time must pass before an investment actually increases the production capacity. Kydland and Prescott [18] showed that is vital to include such time-lags in any realistic model, which turns (1.2) into a delayed optimal control problem [1, 2, 24]. Hughes proved that the resulting Euler-Lagrange optimality conditions are in fact MFDEs [14]. Once hence encounters problems of the form (1.1) if one wishes to impose initial conditions at the start-up time $t = 0$.

The model (1.2) neglects the fact that populations typically consist of many individuals that have competing interests. Since their introduction by Samuelson [25], overlapping generations models have been used extensively to take this into account. Such models assume that the population can be split into agegroups, that each make separate economic decisions based upon their expectations for the future. To calculate the resulting macro-economic behaviour, one typically assumes that the decisions that are made actually create economic conditions that are compatible with the anticipations on which these decisions were based [6, 8, 11, 26].

As an illustrating example, let us discuss the work of d'Albis et al. [5], which models a population of individuals that all live for a fixed time that we scale to unity. We write $c(s, t) \geq 0$ for the consumption at time t of an individual born at time s and similarly $a(s, t)$ for the assets that such an individual owns. Everybody earns an age-independent income $w(t)$ and receives interest on their assets at the rate $r(t)$, which leads to the budget constraint

$$\frac{\partial a(s, t)}{\partial t} = r(t)a(s, t) + w(t) - c(s, t). \tag{1.3}$$

In addition, everybody is born with zero assets and may not die in debt, i.e., $a(s, s+1) \geq a(s, s) = 0$. Subject to these constraints, everybody acts to maximize his or her total life-time welfare

$$W(s) = \int_s^{s+1} \ln\big(c(s, \tau)\big) d\tau. \tag{1.4}$$

Solving the above optimization problem shows that for any $s \geq 0$ and $t \in [s, s+1]$, the optimal consumption is given by

$$c_*(s,t) = \int_s^{s+1} w(\sigma) \exp[\int_\sigma^t r(\tau) d\tau] d\sigma. \tag{1.5}$$

Writing $a_*(s,t)$ for the corresponding optimal asset allocation, we note that the total amount of capital $k(t)$ that is available in the economy is given by the sum of the assets of everybody that is alive at time t, i.e.,

$$k(t) = \int_{t-1}^t a_*(\sigma, t) d\sigma. \tag{1.6}$$

The economy features a unique commodity which can be used for both consumption and investments. We will assume that the rate Q at which this commodity can be produced at time t depends on the available amount of capital $k(t)$ and labour $l(t)$ via

$$Q\big(t, k(t), l(t)\big) = Ak(t)^\alpha \big(e(t)l(t)\big)^{1-\alpha}, \tag{1.7}$$

for some constants $A > 0$ and $0 < \alpha < 1$. The factor $e(t)$ is included to correct for the increase in labour efficiency over time. At every time t, the interest rate $r(t)$ can be interpreted as the price for capital, while the wages $w(t)$ can be seen as the price for labour. These prices can be determined by partial differentiation of Q with respect to $k(t)$ and $l(t)$, yielding

$$\begin{array}{rcl} r(t) & = & \alpha A k(t)^{\alpha-1}\big(e(t)l(t)\big)^{1-\alpha}, \\ w(t) & = & (1-\alpha)Ak(t)^\alpha e(t)^{1-\alpha} l(t)^{-\alpha}. \end{array} \tag{1.8}$$

Choosing $e(t) = k(t)$ and restricting ourselves to a fixed population size $l(t) = 1$, these expressions reduce to

$$\begin{array}{rcl} r(t) & = & \alpha A, \\ w(t) & = & (1-\alpha)Ak(t). \end{array} \tag{1.9}$$

In combination with (1.6), these identities can be used [5, Eq. A.7-A.9] to describe the dynamical behaviour of the capital market by the MFDE

$$k'(t) = Ak(t) - (1-\alpha)A\Big[\int_{t-1}^t k(\sigma)(\sigma+1-t)e^{\alpha A(t-\sigma)} d\sigma + \int_t^{t+1} k(\sigma)(t+1-\sigma)e^{\alpha A(t-\sigma)} d\sigma\Big]. \tag{1.10}$$

This equation can be turned into an autonomous MFDE by considering the new variable $e^{-\alpha At}k(t)$.

To incorporate the fact that populations do not extend arbitrarily far into the past, we will assume that (1.10) describes the dynamics of $k(t)$ for all $t \geq 0$. From the form of (1.10), it is clear that we will need to supply initial values $k(\sigma)$ for $-1 \leq \sigma \leq 0$ before our model can be used to calculate $k(t)$ for $t > 0$. It is not clear however, whether such an initial condition always leads to a unique bounded solution k of (1.10).

Indeterminacy

Venditti and his coworkers [20] coined the term indeterminacy to describe the situation where several different sequences of self-fulfilling expectations may exist simultaneously. The topic of indeterminacy has attracted significant economic interest, since it may provide some insight into the mechanism by which countries that have similar economic structures and initial conditions sometimes undergo a completely different economic development.

Many authors have considered the issue of indeterminacy in a two generation model, in which the population is divided into an old and a young group [9–11, 27]. After a fixed amount of time, the young group becomes old, the old group dies and a new young group is born. The resulting economic models can be written as discrete dynamical systems on $\mathbb{R}^n$ for which $m \le n$ initial conditions can be freely chosen. The details depend heavily on parameters such as the number of different commodities that can be exchanged and the role that pension and labour plays. Restricting oneself to trajectories that converge to an equilibrium, the degree of indeterminacy can be readily computed by subtracting the number of initial conditions m from the dimension of the stable manifold around the equilibrium under consideration, assuming that suitable non-degeneracy conditions are satisfied.

The distinguishing feature of the model (1.10) is that births and deaths occur in continuous time, rather than at discrete time intervals. First used by Cass and Yaari [3], even the simplest of such continuum models admit economic features that can only be observed in discrete models by incorporating relatively complex interactions. The price that needs to be paid is that dimension counting arguments no longer suffice to study the indeterminacy of (1.10), since the dimension of the space of initial conditions $C([-1, 0], \mathbb{R})$ and the dimension of the natural state space $C([-1, 1], \mathbb{R})$ are both infinite. Nevertheless, using the techniques developed in this paper, the notion of indeterminacy can be quantified and calculated for various economic models featuring a continuum of overlapping generations.

Initial Value Problems

The initial value problems for MFDEs that we consider in this paper can be written in the general form

$$\begin{array}{rcll} x'(\xi) & = & L\,\mathrm{ev}_\xi x & \text{for all } \xi \ge 0, \\ x(\tau) & = & \psi(\tau) & \text{for all } r_{\min} \le \tau \le 0. \end{array} \tag{1.11}$$

Here x is a continuous real-valued function on the interval $[r_{\min}, \infty)$ and the operator L is a bounded linear map from $C\big([r_{\min}, r_{\max}], \mathbb{R}\big)$ into $\mathbb{R}$. We will use the notation $\mathrm{ev}_\xi x \in C([r_{\min}, r_{\max}], \mathbb{R})$ to denote the state of x evaluated at ξ, defined by $[\mathrm{ev}_\xi x](\theta) = x(\xi + \theta)$ for all $r_{\min} \le \theta \le r_{\max}$. We require $r_{\min} \le 0$ and $r_{\max} \ge 0$ and take the initial condition ψ from the set $C([r_{\min}, 0], \mathbb{R})$.

The algebraic problems that we study can be written as

$$\begin{array}{rcll} 0 & = & M\,\mathrm{ev}_\xi x & \text{for all } \xi \ge 0, \\ x(\tau) & = & \psi(\tau) & \text{for all } r_{\min} \le \tau \le 0, \end{array} \tag{1.12}$$

in which M is a special type of linear map from $C\big([r_{\min}, r_{\max}], \mathbb{R}\big)$ into $\mathbb{R}$. In particular, we will require that a number of formal differentiations reduces the MFAE (1.12) to the MFDE (1.11). For example, differentiating our prototype system (1.1) with $a = 0$ yields $x'(\xi) = x(\xi - 1) - x(\xi + 1)$, which can be written as (1.11) with $L\phi = \phi(-1) - \phi(1)$.

We will consider the initial value problems (1.11) and (1.12) on exponentially weighted spaces. In particular, let us choose an exponential weight $\eta \in \mathbb{R}$ and consider the function space

$$BC_\eta^\oplus = \{x \in C\big([r_{\min}, \infty), \mathbb{R}\big) \mid \|x\|_\eta := \sup_{\xi \ge r_{\min}} e^{-\eta\xi} \, |x(\xi)| < \infty\}. \tag{1.13}$$

Our goal in this paper is to develop a feasible approach to determine whether (1.11) and (1.12) admit solutions $x \in BC_\eta^\oplus$ for every initial condition $\psi \in C\big([r_{\min}, 0], \mathbb{R}\big)$ and whether such solutions are unique.

Let us emphasize here that well-posedness results for the linear systems (1.11) and (1.12) also play an important role in nonlinear settings. Consider for example the nonlinear initial value problem

$$\begin{array}{rcll} x'(\xi) & = & x(\xi + 1) + x(\xi - 1) + x(\xi)^2 & \text{for all } \xi \ge 0, \\ x(\tau) & = & \psi(\tau) & \text{for all } r_{\min} \le \tau \le 0. \end{array} \tag{1.14}$$

Let us write $\mathfrak{Q}$ for the set of bounded solutions to the linearized equation $x'(\xi) = x(\xi+1) + x(\xi-1)$ posed on $\mathbb{R}_+$. In addition, write

$$Q = \mathrm{ev}_0\big(\mathfrak{Q}\big), \qquad Q_\epsilon = \{\phi \in Q \mid \|\phi\| \le \epsilon\}. \tag{1.15}$$

Using techniques developed in [16, 17], one may show that there exists a function

$$u^* : Q_\epsilon \to C([-1,1],\mathbb{R}), \qquad u^*(\phi) = \phi + O(\phi)^2, \tag{1.16}$$

such that $u^*(Q_\epsilon)$ defines a local stable manifold for the equilibrium $x = 0$. We hence see that the nonlinear initial value problem (1.14) admits solutions that decay to zero for all sufficiently small initial data $\psi \in C([-1,0],\mathbb{R})$ if and only if the linear problem (1.11) with $L\phi = \phi(1) + \phi(-1)$ is well-posed with respect to $BC_0^{\oplus}$.

Characteristic Equations

In the special case that $r_{\max} = 0$, the problem (1.11) reduces to an initial value problem for a retarded functional differential equation (RFDE). Such systems have been studied extensively during the last three decades, resulting in a rich and diverse literature on the subject. Using the theory described in [12], the well-posedness of (1.11) can be read off directly from the characteristic function $\Delta_L : \mathbb{C} \to \mathbb{C}$, that can be written as

$$\Delta_L(z) = z - L\, e^{z\cdot}. \tag{1.17}$$

Indeed, consider any $\eta \in \mathbb{R}$ with the property that the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z \ge \eta$. It then follows from [12, Theorem 7.6.1] that any $\phi \in C([r_{\min}, 0],\mathbb{R})$ can be extended to a solution $x \in BC_\eta^{\oplus}$. If this property fails, one can determine the codimension of the set of initial conditions that can be extended by studying the number and multiplicity of the roots of $\Delta_L(z) = 0$ that have $\mathrm{Re}\, z \ge \eta$.

Such a direct criterion no longer exists when $r_{\max} > 0$. The investigation is complicated by the fact that the characteristic equation $\Delta_L(z) = 0$ will in general have an infinite number of roots on both sides of the imaginary axis. For example, for the prototype system (1.1) with $a = 1$ we have

$$z\Delta_L(z) = z\big[z - 1 - \int_{-1}^{1} e^{z\sigma} d\sigma\big] = z^2 - z - e^z + e^{-z}. \tag{1.18}$$

This transcendental equation can no longer be bounded by a polynomial on the half plane $\mathrm{Re}\, z \ge 0$, as is always possible for an RFDE. The reason that we may nevertheless expect to obtain well-posedness results for (1.11) and (1.12) is that the space $C([r_{\min}, 0],\mathbb{R})$ containing the initial conditions now differs from the natural state space $C([r_{\min}, r_{\max}],\mathbb{R})$.

The key result that allows the well-posedness of (1.11) to be analyzed was obtained by Mallet-Paret and Verduyn Lunel in [22]. In particular, under a non-degeneracy condition that roughly states that the interval $[r_{\min}, r_{\max}]$ cannot be decreased, the authors show that for every $\alpha \in \mathbb{R}$ there exists a Wiener-Hopf factorization

$$(z - \alpha)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z), \tag{1.19}$$

in which Δ_{L_-} and Δ_{L_+} are the characteristic functions associated to a retarded respectively advanced functional differential equation, i.e. $\Delta_{L_\pm}(z) = z - L_\pm \exp(z\cdot)$ for some pair of operators $L_- \in \mathcal{L}(C([r_{\min}, 0],\mathbb{C}),\mathbb{C})$ and $L_+ \in \mathcal{L}(C([0, r_{\max}],\mathbb{C}),\mathbb{C})$. For any $\eta \ne \alpha \in \mathbb{R}$ for which $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$, one may compute an integer $n_L^\sharp(\eta)$ by counting the number of roots of the equations $\Delta_{L_\pm}(z) = 0$ that lie on the 'wrong' side of the line $\mathrm{Re}\, z = \eta$. It turns out that this integer $n_L^\sharp(\eta)$ is independent of the specific factorization (1.19). In addition, all information concerning the well-posedness of (1.11) with respect to $BC_\eta^{\oplus}$ can be determined from this invariant.

In practice however, it is often intractable to actually find a factorization of the form (1.19) for the symbol Δ_L. In this paper, we show how $n^{\sharp}_L(\eta)$ can still be computed in such situations by constructing a homotopy from a suitable reference system that can actually be factorized. This computation requires one to count the number of roots of the characteristic equation $\Delta_L(z) = 0$ that cross the line $\operatorname{Re} z = \eta$ as the MFDE is transformed from the reference system to the system under consideration. We will give examples in which this number can be computed analytically, but remark that this counting can very easily be performed numerically.

As can be expected, the well-posedness of the MFAE (1.12) depends heavily on properties of the related MFDE (1.11), since any solution to the first line of (1.12) will automatically satisfy (1.11). The converse however is not true and care has to be taken to isolate the superfluous solutions to (1.11). We will address this issue by using spectral projections and Laplace transform techniques.

Our main results are stated in §2 and proved in §4-§5. In §3.1 we discuss the well-posedness of the overlapping generations model (1.10) and in §3.2 we consider an additional overlapping generations model that leads to an algebraic initial value problem of the form (1.12).

2 Main Results

In this section we state our main results, which will be proved in §4-§5. We first discuss systems that are governed by a differential equation and subsequently show how these results can be used to study the class of algebraic problems that we are interested in.

2.1 Initial Value Problems for MFDEs

To set the stage, let us consider the autonomous linear homogeneous MFDE

$$x'(\xi) = L\,\mathrm{ev}_\xi x, \tag{2.1}$$

in which L is a bounded linear operator from $C([r_{\min}, r_{\max}], \mathbb{C}^n)$ into $\mathbb{C}^n$. We require $r_{\min} \le 0$ and $r_{\max} \ge 0$ and recall the notation $[\mathrm{ev}_\xi x](\theta) = x(\xi + \theta)$ for $r_{\min} \le \theta \le r_{\max}$. Later on we will restrict ourselves to the scalar case $n = 1$, but for now we allow $n \ge 1$. We write

$$\Delta_L(z) = z - L\, e^{z\cdot} I \tag{2.2}$$

for the characteristic matrix that is associated to (2.1), in which I is the $n \times n$ identity matrix.

Since we wish to consider (2.1) on the half-lines $\mathbb{R}_\pm$, let us introduce the exponentially weighted function spaces

$$\begin{array}{lcll} BC^{\ominus}_\eta & := & \{x \in C\big((-\infty, r_{\max}], \mathbb{C}^n\big) & \mid \ \sup_{\xi \le r_{\max}} e^{-\eta\xi}\,|x(\xi)| < \infty\}, \\ BC^{\oplus}_\eta & := & \{x \in C\big([r_{\min}, \infty), \mathbb{C}^n\big) & \mid \ \sup_{\xi \ge r_{\min}} e^{-\eta\xi}\,|x(\xi)| < \infty\} \end{array} \tag{2.3}$$

and write $\|x\|_\eta$ for the corresponding norms. We can now introduce the following solution sets for (2.1),

$$\begin{array}{lcl} \mathfrak{P}_L(\eta) & = & \big\{v \in BC^{\ominus}_\eta \mid v'(\xi) = L\,\mathrm{ev}_\xi v \text{ for all } \xi \le 0\big\}, \\ \mathfrak{Q}_L(\eta) & = & \big\{v \in BC^{\oplus}_\eta \mid v'(\xi) = L\,\mathrm{ev}_\xi v \text{ for all } \xi \ge 0\big\}. \end{array} \tag{2.4}$$

As in [22], it is convenient to introduce the spaces

$$\begin{array}{lcl} P_L(\eta) & = & \big\{\phi \in C([r_{\min}, r_{\max}], \mathbb{C}^n) \mid \phi = \mathrm{ev}_0 v \text{ for some } v \in \mathfrak{P}_L(\eta)\big\}, \\ Q_L(\eta) & = & \big\{\phi \in C([r_{\min}, r_{\max}], \mathbb{C}^n) \mid \phi = \mathrm{ev}_0 v \text{ for some } v \in \mathfrak{Q}_L(\eta)\big\}, \end{array} \tag{2.5}$$

which describe the initial segments of the solution sets $\mathfrak{P}_L$ and $\mathfrak{Q}_L$ in the natural state space $C([r_{\min}, r_{\max}])$. Let us also introduce the associated restriction operators

$$\begin{array}{ll} \pi^+_{P_L(\eta)} : P_L(\eta) \to C\big([0, r_{\max}], \mathbb{C}^n\big) & \phi \mapsto \phi_{|[0,r_{\max}]}, \\ \pi^-_{Q_L(\eta)} : Q_L(\eta) \to C\big([r_{\min}, 0], \mathbb{C}^n\big) & \phi \mapsto \phi_{|[r_{\min},0]}. \end{array} \tag{2.6}$$

The well-posedness properties of the initial value problem (1.11) that we wish to understand are entirely encoded in the family of restriction operators $\pi^-_{Q_L(\eta)}$. Indeed, if $\mathrm{Range}\big(\pi^-_{Q_L(\eta)}\big) = C\big([r_{\min}, 0], \mathbb{C}\big)$ for some $\eta \in \mathbb{R}$, then for any $\phi \in C\big([r_{\min}, 0], \mathbb{C}\big)$ the initial value problem (1.11) has a solution $x \in BC^{\oplus}_{\eta}$. If $\mathrm{Ker}\big(\pi^-_{Q_L(\eta)}\big) = \{0\}$, then such solutions are unique. We will say that (1.11) is well-posed with respect to the space $BC^{\oplus}_{\eta}$ if and only if $\pi^-_{Q_L(\eta)}$ is an isomorphism from $Q_L(\eta)$ onto $C([r_{\min}, 0], \mathbb{C})$.

The following proposition shows that for appropriate values of η, the state space $C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is decomposed by $P_L(\eta)$ and $Q_L(\eta)$. In addition, the restriction operators (2.6) are Fredholm, which means that their kernels are finite dimensional, while their ranges are closed and of finite codimension. We recall that the index of a Fredholm operator F is determined by the formula

$$\mathrm{ind}(F) = \dim \mathrm{Ker}(F) - \mathrm{codim}\,\mathrm{Range}(F). \tag{2.7}$$

We remark that these results can be easily obtained by applying exponential shifts to the theory developed in [22, §3].

Proposition 2.1 (see [22, §3]). *Consider the linear system (2.1) and choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the spaces $P_L(\eta)$ and $Q_L(\eta)$ are closed and satisfy*

$$C([r_{\min}, r_{\max}], \mathbb{C}^n) = P_L(\eta) \oplus Q_L(\eta). \tag{2.8}$$

In addition, there exist constants $K > 0$ and $\alpha > 0$ such that

$$\|\mathrm{ev}_\xi v\| \le K e^{(\eta+\alpha)\xi} \,\|\mathrm{ev}_0 v\| \tag{2.9}$$

for any $v \in \mathfrak{P}_L(\eta)$ and $\xi \le 0$, while also

$$\|\mathrm{ev}_\xi w\| \le K e^{(\eta-\alpha)\xi} \,\|\mathrm{ev}_0 w\| \tag{2.10}$$

for any $w \in \mathfrak{Q}_L(\eta)$ and $\xi \ge 0$. Finally, the operators $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$ defined in (2.6) are Fredholm, with

$$\mathrm{ind}(\pi^+_{P_L(\eta)}) + \mathrm{ind}(\pi^-_{Q_L(\eta)}) = -n. \tag{2.11}$$

To obtain more detailed information on the restriction operators $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$, we need to impose the following additional restriction on the linear operator L.

(HL) There exist quantities $s_\pm \ge 0$ and non-singular matrices $J_\pm$ such that the following asymptotic expansions hold,

$$\begin{array}{lcll} \Delta_L(z) & = & z^{-s_+} e^{z r_{\max}} (J_+ + o(1)) & \text{as } z \to \infty, \\ \Delta_L(z) & = & z^{-s_-} e^{z r_{\min}} (J_- + o(1)) & \text{as } z \to -\infty. \end{array} \tag{2.12}$$

We remark that (HL) is significantly weaker than the atomicity condition used in [22, Eq. (2.3)], which requires $s_\pm = 0$ to hold in (HL). Such a condition is violated by the economic models studied in this paper. Nevertheless, the techniques developed in [22, §5] can still be used to obtain the following result, which lies at the basis for a further understanding of $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$.

Proposition 2.2 (see [22, Thm 5.2]). *Consider the linear system (2.1) and suppose that (HL) is satisfied. Then for any monic polynomial p of degree n, there exist linear operators*

$$L_- \in \mathcal{L}(C([r_{\min}, 0], \mathbb{C}^n), \mathbb{C}^n), \qquad L_+ \in \mathcal{L}(C([0, r_{\max}], \mathbb{C}^n), \mathbb{C}^n), \tag{2.13}$$

with associated characteristic matrices

$$\Delta_{L_\pm}(z) = zI - L_\pm e^{z\cdot} I \tag{2.14}$$

for which the splitting

$$p(z) \det \Delta_L(z) = \det \Delta_{L_-}(z) \det \Delta_{L_+}(z) \tag{2.15}$$

holds.

Proof. It suffices to show that the proof of [22, Thm. 5.2] still holds for the weaker condition (HL). The atomicity condition [22, Eq. (2.3)] is only used to verify the conditions associated with a Phragmén-Lindelöf theorem [19, Thm. I.21] that asserts that entire functions that grow at most exponentially on $\mathbb{C}$ and polynomially on the real and imaginary axes, are in fact polynomials. Allowing $s_\pm > 0$ in (2.12) does not destroy these required growth estimates. □

The splitting (2.15) is referred to as a Wiener-Hopf factorization for the symbol Δ_L and we will call any such triplet (p, L_-, L_+) a Wiener-Hopf triplet for L. In general, such triplets need not be unique. Indeed, in [22] a mechanism is given by which pairs of roots of the characteristic equations $\det \Delta_{L_\pm}(z) = 0$ may be interchanged. Nevertheless, it turns out to be possible to extract a quantity that does not depend on the chosen splitting (2.15). To this end, let us consider any Wiener-Hopf triplet (p, L_-, L_+) for L and pick an $\eta \in \mathbb{R}$ for which the equation $p(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. We now introduce the quantity

$$n^\sharp_L(\eta) = n^+_{L_+}(\eta) - n^-_{L-}(\eta) + n^0_p(\eta) \tag{2.16}$$

that is defined by

$$\begin{array}{rcl} n^-_{L_-}(\eta) & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L_-}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \\ n^+_{L_+}(\eta) & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L_+}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}, \\ n^0_p(\eta) & = & \#\{z \in \mathbb{C} \mid p(z) = 0 \text{ and } \operatorname{Re} z > \eta\}. \end{array} \tag{2.17}$$

This quantity $n^\sharp_L(\eta)$ is invariant in the following sense.

Proposition 2.3 (see [22, Thm. 5.2]). *Consider the linear system (2.1) and suppose that (HL) is satisfied. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the quantity $n^\sharp_L(\eta)$ is invariant across all Wiener-Hopf triplets (p, L_-, L_+) for L that have $p(\eta + i\nu) \neq 0$ for all $\nu \in \mathbb{R}$.*

Proof. The remarks made in the proof of Proposition 2.2 also apply here, ensuring that the proof of [22, Thm. 5.2] remains valid. □

In the special case that (2.1) is scalar, the quantities $n^\sharp_L(\eta)$ can be used to characterize the kernels and ranges of the Fredholm operators $\pi^-_{Q_L(\eta)}$ and $\pi^+_{P_L(\eta)}$. This dimension restriction is related to the fact that the splitting (2.15) only features the determinant of Δ_L.

Proposition 2.4 (see [22, Thms. 6.1-2]). *Consider a scalar version of the linear system (2.1) and suppose that (HL) is satisfied. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the following identities hold,*

$$\begin{array}{rclrcl} \dim \operatorname{Ker} \pi^+_{P_L(\eta)} & = & \max\{-n^\sharp_L(\eta), 0\}, & \operatorname{codim} \operatorname{Range} \pi^+_{P_L(\eta)} & = & \max\{n^\sharp_L(\eta), 0\}, \\ \dim \operatorname{Ker} \pi^-_{Q_L(\eta)} & = & \max\{n^\sharp_L(\eta) - 1, 0\}, & \operatorname{codim} \operatorname{Range} \pi^-_{Q_L(\eta)} & = & \max\{1 - n^\sharp_L(\eta), 0\}. \end{array} \tag{2.18}$$

Proof. To see that the proofs of [22, Thms. 6.1-2] still work with the weaker condition (HL), we note that the stronger atomicity condition is only used once in a setting that is not related to the application of a Phragmén-Lindelöf theorem. This occurs in the proof of [22, Lem. 5.9], where it is needed to establish the non-degeneracy of the Hale inner product for delay equations. Careful inspection however shows that [22, Lem. 5.9] is only needed in the special case $m = 0$. It therefore suffices to show that the sets of generalized eigenfunctions associated to $L_\pm$ are complete. This can be done by noting that the factorization (2.15) ensures that the asymptotic growth rates (2.12) are shared by $\Delta_{L_\pm}$ and subsequently applying [12, Cor. 7.8.1]. □

In principle, we now have sufficient information to answer the well-posedness question for the scalar initial value problem (1.11). Indeed, for any η for which $\Delta_L(z) = 0$ has no roots with $\operatorname{Re} z = \eta$, the problem (1.11) is well-posed with respect to the space $BC^{\oplus}_{\eta}$ if and only if $n^{\sharp}_{L}(\eta) = 1$. However, as discussed in the introduction, it is often intractable to find Wiener-Hopf triplets for a prescribed operator L. This often prevents us from computing $n^{\sharp}_{L}(\eta)$ directly from (2.16).

Our first main result addresses this difficulty and allows $n^{\sharp}_{L}(\eta)$ to be calculated in settings where a Wiener-Hopf triplet is not readily available for the system (2.1) under consideration. The only requirement is that a Wiener-Hopf triplet is available for some reference system that can be continuously transformed into the original system without violating (HL). Please note however that the exponents $s_\pm$ appearing in this condition (HL) need not remain constant during this transformation.

Theorem 2.5 (see §4). *Consider a continuous path*

$$\Gamma : [0, 1] \to \mathcal{L}\big(C([r_{\min}, r_{\max}], \mathbb{C}^n), \mathbb{C}^n\big) \tag{2.19}$$

and suppose that the operators $\Gamma(\mu)$ *satisfy (HL) for all* $0 \leq \mu \leq 1$*. Fix any* $\eta \in \mathbb{R}$ *and suppose that the characteristic equation* $\det \Delta_{\Gamma(\mu)}(z) = 0$ *admits roots with* $\operatorname{Re} z = \eta$ *for only finitely many values of* $\mu \in [0, 1]$ *and that* $\mu \in (0, 1)$ *for all such* μ*. Then we have the identity*

$$n^{\sharp}_{\Gamma(1)}(\eta) - n^{\sharp}_{\Gamma(0)}(\eta) = -\operatorname{cross}(\Gamma, \eta), \tag{2.20}$$

in which the crossing number $\operatorname{cross}(\Gamma, \eta)$ *denotes the net number of roots of the characteristic equation* $\det \Delta_{\Gamma(\mu)}(z) = 0$*, counted with multiplicity, that cross the line* $\operatorname{Re} z = \eta$ *from left to right as* μ *increases from* 0 *to* 1*.*

We note that the formula [22, Eq. (6.7)] can be seen as a special case of this theorem, that applies only to operators $L : C([r_{\min}, r_{\max}], \mathbb{C}) \to \mathbb{C}$ that can be written as

$$L\phi = \sum_{j=0}^{N} A_j \phi(r_j) \tag{2.21}$$

for some integer N, constants $A_j \in \mathbb{C}$ and shifts $r_{\min} \leq r_j \leq r_{\max}$. This formula was obtained by embedding $\Gamma(0)$ and $\Gamma(1)$ into a non-autonomous MFDE

$$x'(\xi) = L(\xi) \operatorname{ev}_\xi x \tag{2.22}$$

that has $L(-\infty) = \Gamma(0)$ and $L(\infty) = \Gamma(1)$ and subsequently invoking a spectral flow result [21, Thm. C]. This latter result requires (2.21) to hold, while the restriction to scalar equations comes from the fact that the identities (2.18) are used.

Our result is obtained using more direct techniques that also work when $n > 1$ and do not suffer from the point-shift restriction (2.21). We remark that many examples, including the economic models studied in this paper, violate this restriction. Of course, we have to admit that the ability to calculate the invariant $n^{\sharp}_{L}(\eta)$ if $n > 1$ is of limited value at present, since no analogue of Proposition 2.4 is currently available. In future work we plan to remedy this situation. In particular, we are hopeful that in situations where the characteristic equation $\det \Delta_L(z) = 0$ does not admit high-multiplicity eigenvalues, information on $\pi^{-}_{Q_L(\eta)}$ and $\pi^{+}_{P_L(\eta)}$ can still be obtained from $n^{\sharp}_{L}(\eta)$.

2.2 Initial Value Problems for MFAEs

We will now turn our attention to algebraic equations of the form

$$0 = M \operatorname{ev}_\xi x, \tag{2.23}$$

in which M is a bounded linear operator from $C([r_{\min}, r_{\max}], \mathbb{C}^n)$ into $\mathbb{C}^n$ that can be closely related to a differential system of the form (2.1). In order to clarify this relationship, we introduce the characteristic matrix

$$\delta_M(z) = -M e^{z \cdot} I \tag{2.24}$$

that is associated to (2.23). The restriction on M that we need in this paper can now be captured by the following condition on the characteristic matrices.

(HM) There exist an integer $\ell \geq 1$ and an operator $L \in \mathcal{L}\big(C([r_{\min}, r_{\max}], \mathbb{C}^n), \mathbb{C}^n\big)$ such that

$$\beta(z - \alpha)^\ell \delta_M(z) = \Delta_L(z) \tag{2.25}$$

for some $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$.

This condition is related to the fact that we require any solution to the MFAE (2.23) to also satisfy the MFDE (2.1). The reader may wish to keep in mind the example MFAE

$$x(\xi) = \int_{-1}^{1} x(\xi + \sigma) d\sigma, \tag{2.26}$$

for which we have

$$\begin{array}{lcl} M\phi & = & -\phi(0) + \int_{-1}^{1} \phi(\sigma) d\sigma, \\ \delta_M(z) & = & 1 - \int_{-1}^{1} e^{z\sigma} d\sigma = \frac{1}{z}[z + e^{-z} - e^{z}], \end{array} \tag{2.27}$$

which implies that (HM) is satisfied with $\ell = 1$, $\alpha = 0$, $\beta = 1$ and

$$L\phi = \phi(1) - \phi(-1). \tag{2.28}$$

Indeed, a single differentiation of (2.26) easily yields that $x'(\xi) = L \operatorname{ev}_\xi x$ for this choice of L.

We will be interested in the solution spaces

$$\begin{array}{lcl} \mathfrak{p}_M(\eta) & = & \big\{ v \in BC_\eta^\ominus \mid 0 = M \operatorname{ev}_\xi v \text{ for all } \xi \leq 0 \big\}, \\ \mathfrak{q}_M(\eta) & = & \big\{ v \in BC_\eta^\oplus \mid 0 = M \operatorname{ev}_\xi v \text{ for all } \xi \geq 0 \big\}. \end{array} \tag{2.29}$$

Our second main result relates these spaces $\mathfrak{p}_M(\eta)$ and $\mathfrak{q}_M(\eta)$ to their counterparts $\mathfrak{P}_L(\eta)$ and $\mathfrak{Q}_L(\eta)$ that were defined for the differential equation (2.1). The result exploits the explicit constructions in the proof of [15, Prp. 4.2], which show that if (2.25) holds for a single $\alpha \in \mathbb{C}$, it will in fact hold[1] for any $\alpha \in \mathbb{C}$. In particular, the operator L' appearing in the result below can be computed explicitly from the operator L appearing in (HM).

Theorem 2.6 (see §5). *Consider the algebraic equation (2.23) and suppose that (HM) is satisfied. Choose any $\eta \in \mathbb{R}$ for which the characteristic equation $\det \delta_M(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then there exists a bounded linear operator $L' : C([r_{\min}, r_{\max}], \mathbb{C}^n) \to \mathbb{C}^n$ such that*

$$\beta(z - \eta)^\ell \delta_M(z) = \Delta_{L'}(z), \tag{2.30}$$

in which ℓ and β are the constants appearing in (HM). In addition, for every sufficiently small $\epsilon > 0$ we have

$$\mathfrak{p}_M(\eta) = \mathfrak{P}_{L'}(\eta + \epsilon), \qquad \mathfrak{q}_M(\eta) = \mathfrak{Q}_{L'}(\eta - \epsilon). \tag{2.31}$$

With this result in hand, the theory developed above to describe the spaces $\mathfrak{P}_L$ and $\mathfrak{Q}_L$ associated to the MFDE (2.1) can also be utilized for the algebraic system (2.23).

[1] After suitably modifying L, but keeping ℓ and β fixed.

3 Examples

In this section, we consider two example initial value problems that arise from economic overlapping generations models. The first example will be an MFDE problem of the form (1.11), the second example will be an MFAE problem of the form (1.12). For both examples, we will find suitable reference systems that admit an explicit Wiener-Hopf factorization. It will be possible to analytically track the roots of the characteristic equations that cross the imaginary axis, as we move from the reference systems to the original systems.

3.1 Well-posedness of an MFDE

As our first example, we consider the MFDE (1.10), which describes the dynamics of the capital market for the overlapping generations model discussed in the introduction. For ease of notation, we introduce the parameter $\beta = (1-\alpha)A$ and consider the new variable $x(\xi) = k(\xi)e^{-\alpha A \xi}$. In terms of this new variable, (1.10) reduces to the linear autonomous MFDE

$$x'(\xi) = L\,\mathrm{ev}_\xi x := \beta x(\xi) - \beta \int_{-1}^{0} (1+\sigma)x(\xi+\sigma)d\sigma - \beta \int_{0}^{1} (1-\sigma)x(\xi+\sigma)d\sigma. \tag{3.1}$$

We will impose the following condition on the model parameters, which basically states that in the economy under consideration, the reward for labour is high relative to the return rate on capital.

(HP) We have $0 < \alpha A < 1$ and $\beta > 1$.

The characteristic matrix Δ_L associated to (3.1) can be written as

$$\Delta_L(z) = z - \beta + \beta \int_{-1}^{0} \int_{\sigma}^{\sigma+1} e^{z\tau} d\tau d\sigma. \tag{3.2}$$

Lemma 3.1. *Assume that (HP) holds. Then the characteristic equation $\Delta_L(z) = 0$ admits precisely two real roots, namely $z = z_- < 0$ and $z = 0$. These two roots are the only solutions to $\Delta_L(z) = 0$ in the vertical strip $z_- \le \mathrm{Re}\, z \le 0$.*

Proof. It is easy to verify the limits $\lim_{p\to\pm\infty} \Delta_L(p) = \infty$ and check that $\Delta_L(0) = 0$ and $\Delta_L'(0) = 1$. Using

$$\Delta_L''(z) = \beta \int_{-1}^{0} \int_{\sigma}^{\sigma+1} \tau^2 e^{z\tau} d\tau d\sigma \tag{3.3}$$

we see that $\Delta_L''(p) > 0$ for $p \in \mathbb{R}$, which completes the proof of the statements concerning the real roots of the characteristic equation.

Suppose now that $\Delta_L(p + iq) = 0$ for some $q \in \mathbb{R}$ and $z_- \le p \le 0$. Isolating the real part of this equation, we find

$$\beta \int_{-1}^{0} \int_{\sigma}^{\sigma+1} e^{p\tau} \cos(q\tau) d\tau d\sigma = \beta - p. \tag{3.4}$$

However, properties of the cosine together with the observation $\Delta_L(p) \le 0$ yield the inequalities

$$\beta \int_{-1}^{0} \int_{\sigma}^{\sigma+1} e^{p\tau} \cos(q\tau) d\tau d\sigma \le \beta \int_{-1}^{0} \int_{\sigma}^{\sigma+1} e^{p\tau} d\tau d\sigma \le \beta - p. \tag{3.5}$$

Observing that equality can hold only when $p \in \{z_-, 0\}$ and $q = 0$ completes the proof. □

Since a Wiener-Hopf triplet is not readily available for L, let us incorporate the system (3.1) into the family of MFDEs

$$\begin{array}{rcl} x'(\xi) = \Gamma(\mu)\mathrm{ev}_\xi & := & \beta x(\xi) - \beta\int_{-1}^{0}(1+\sigma)x(\xi+\sigma)d\sigma - \beta\int_{0}^{1}(1-\sigma)x(\xi+\sigma)d\sigma \\ & & -(1-\mu)\big[\beta x(\xi+1) - \frac{1}{\beta}x(\xi-1)\big] \\ & = & L\,\mathrm{ev}_\xi x - (1-\mu)\big[\beta x(\xi+1) - \frac{1}{\beta}x(\xi-1)\big], \end{array} \tag{3.6}$$

parametrized by $\mu \in [0,1]$. We easily find

$$\Delta_{\Gamma(\mu)}(z) = \Delta_L(z) + (1-\mu)\big[\beta e^z - \frac{1}{\beta}e^{-z}\big]. \tag{3.7}$$

Notice that $\Gamma(1) = L$, while $\Gamma(0)$ admits the Wiener-Hopf factorization

$$z\Delta_{\Gamma(0)}(z) = \Delta_{L_-}(z)\Delta_{L_+}(z), \tag{3.8}$$

in which the characteristic matrices

$$\begin{array}{rcl} \Delta_{L_-}(z) & = & z + \int_{-1}^{0} e^{z\sigma}d\sigma - \frac{1}{\beta}e^{-z}, \\ \Delta_{L_+}(z) & = & z - \beta + \beta e^z, \end{array} \tag{3.9}$$

correspond, respectively, to the delayed equation

$$w'(\xi) = L_-\,\mathrm{ev}_\xi w := -\int_{-1}^{0} w(\xi+\sigma)d\sigma + \frac{1}{\beta}w(\xi-1) \tag{3.10}$$

and the advanced equation

$$v'(\xi) = L_+\,\mathrm{ev}_\xi v := \beta v(\xi) - \beta v(\xi+1). \tag{3.11}$$

We now set out to compute $n^\sharp_{\Gamma(0)}(\eta)$ for sufficiently small $|\eta|$. We will need to use the following two results.

Lemma 3.2. *Assume that (HP) holds and that $\Delta_{L_+}(z) = 0$ for some $z \in \mathbb{C}$. Then either $z = 0$ or $z \notin \mathbb{R}$ with $\mathrm{Re}\, z > 0$.*

Proof. It is easy to check that $\Delta_{L_+}(0) = 0$. Since $\Delta'_{L_+}(p) = 1 + \beta e^p > 0$ for all $p \in \mathbb{R}$, this is the only real root. Now suppose that $\Delta_{L_+}(p + iq) = 0$ for some pair $p \le 0$ and $q \in \mathbb{R}$. Isolating the real part of the characteristic equation, we find

$$\beta e^p \cos q = \beta - p. \tag{3.12}$$

On the other hand, using $\Delta_{L_+}(p) \le 0$, we find

$$\beta e^p \cos q \le \beta e^p \le \beta - p, \tag{3.13}$$

with equality only when $p = 0$ and $q \in 2\pi\mathbb{N}$. Noticing that $\mathrm{Im}\,\Delta_{L_+}(2\pi i\ell) = 2\pi\ell$ for any integer ℓ completes the proof. □

Lemma 3.3. *Assume that (HP) holds and that $\Delta_{L_-}(z) = 0$ for some $z \in \mathbb{C}$. Then we have $\mathrm{Re}\, z < 0$.*

Proof. Notice first that $\Delta_{L_-}(0) = 1 - \frac{1}{\beta} > 0$. Second, observe that

$$\Delta'_{L_-}(z) = 1 + \frac{1}{\beta}e^{-z} + \int_{-1}^{0} \sigma e^{z\sigma}d\sigma. \tag{3.14}$$

An easy computation shows that for $p \geq 0$ we have

$$-\frac{1}{2} \leq \int_{-1}^{0} \sigma e^{p\sigma} d\sigma < 0, \tag{3.15}$$

which means that $\Delta'_{L_-}(p) > 0$ and also $\Delta_{L_-}(p) > 0$ for all $p \geq 0$. In addition, for any $q \neq 0$ we have

$$q^{-1} \operatorname{Im} \Delta_{L_-}(p + iq) = 1 + \tfrac{1}{\beta} e^{-p} \tfrac{\sin q}{q} + \int_{-1}^{0} e^{p\sigma} \tfrac{\sin q\sigma}{q} d\sigma. \tag{3.16}$$

For the remainder of the proof, let us fix $p \geq 0$. This allows us to obtain the estimate

$$\begin{array}{rcl} q^{-1} \operatorname{Im} \Delta_{L_-}(p + iq) & \geq & 1 + \tfrac{1}{\beta} e^{-p} \tfrac{\sin q}{q} - \left| \int_{-1}^{0} e^{p\sigma} \tfrac{\sin q\sigma}{q} d\sigma \right| \\ & \geq & 1 + \tfrac{1}{\beta} e^{-p} \tfrac{\sin q}{q} - \left| \int_{-1}^{0} \sigma e^{p\sigma} d\sigma \right| \\ & \geq & \tfrac{1}{2} + \tfrac{1}{\beta} e^{-p} \tfrac{\sin q}{q}. \end{array} \tag{3.17}$$

For $0 < q < \pi$ we have $\sin q > 0$ and hence also $q^{-1} \operatorname{Im} \Delta_{L_-}(p + iq) > 0$. For $q \geq \pi$ we find

$$q^{-1} \Delta_{L_-}(p + iq) \geq \frac{1}{2} - \frac{1}{\pi\beta} e^{-p} > \frac{1}{2} - \frac{1}{\pi} > 0, \tag{3.18}$$

which in view of the symmetry $q \mapsto -q$ completes the proof. □

The formula (2.16) can now be used to conclude that $n^{\sharp}_{\Gamma(0)}(\eta) = 1$ for all sufficiently small $|\eta|$. Before Theorem 2.5 can be applied to calculate $n^{\sharp}_{L}(\eta)$, we will need to check whether $\Delta_{\Gamma(\mu)}(z) = 0$ admits roots in the vicinity of the imaginary axis as the parameter μ is varied.

Lemma 3.4. *Assume that (HP) holds. For every $\mu \in [0, 1)$, the characteristic equation $\Delta_{\Gamma(\mu)}(z) = 0$ has no roots with $\operatorname{Re} z = 0$.*

Proof. For convenience, let us write $\overline{\mu} = 1 - \mu$. Note that $\Delta_{\Gamma(\mu)}(0) = \overline{\mu}(\beta - \beta^{-1}) > 0$ for all $\overline{\mu} > 0$. It therefore suffices to show that $\Delta_{\Gamma(\mu)}(iq) \neq 0$ for all $q > 0$ and $\mu \in [0, 1)$. Assuming the contrary, we obtain the system

$$\begin{array}{rcl} \beta q^2 & = & 2\beta - 2\beta \cos q + \overline{\mu} q^2 (\beta - \beta^{-1}) \cos q, \\ q & = & -\overline{\mu}(\beta + \beta^{-1}) \sin q. \end{array} \tag{3.19}$$

In view of $\beta > 1$ and $q > 0$, the second line implies that we must have $\sin q < 0$ and hence $q > \pi$. Using $\sin q = -\sqrt{1 - \cos^2 q}$, this implies

$$\overline{\mu}^2 \beta^{-2} (\beta^2 + 1)^2 [1 - \cos^2 q] > \pi^2. \tag{3.20}$$

Furthermore, substituting the second line of (3.19) into the first line and using the fact that $\sin q \neq 0$, we find the second order equation

$$\overline{\mu}^3 (\beta^2 - 1)(\beta^2 + 1)^2 \cos^2 q + (\beta^2 + 1)^2 \overline{\mu}^2 (\overline{\mu}(\beta^2 - 1) - \beta^2) \cos q + 2\beta^4 - \beta^2 \overline{\mu}^2 (\beta^2 + 1)^2 = 0. \tag{3.21}$$

This implies that

$$\begin{array}{rcl} 1 - \cos^2 q & = & 1 + \frac{2\beta^4 - \beta^2 \overline{\mu}^2 (\beta^2+1)^2}{\overline{\mu}^3 (\beta^2-1)(\beta^2+1)^2} + \frac{\overline{\mu}(\beta^2-1) - \beta^2}{\overline{\mu}(\beta^2-1)} \cos q \\ & = & \frac{(\overline{\mu}^3 (\beta^2-1) - \overline{\mu}^2 \beta^2)(\beta^2+1)^2 + 2\beta^4}{\overline{\mu}^3 (\beta^2-1)(\beta^2+1)^2} + \frac{\overline{\mu}(\beta^2-1) - \beta^2}{\overline{\mu}(\beta^2-1)} \cos q \\ & = & \frac{\overline{\mu}(\beta^2-1) - \beta^2}{\overline{\mu}(\beta^2-1)} (1 + \cos q) + \frac{2\beta^4}{\overline{\mu}^3 (\beta^2-1)(\beta^2+1)^2} \end{array} \tag{3.22}$$

and leads to the inequality

$$\frac{\overline{\mu}(\overline{\mu}(\beta^2 - 1) - \beta^2)(\beta^2 + 1)^2}{\beta^2 (\beta^2 - 1)} (1 + \cos q) + \frac{2\beta^2}{\overline{\mu}(\beta^2 - 1)} > \pi^2. \tag{3.23}$$

Solving the quadratic equation (3.21) yields

$$\begin{array}{rcl} 1+\cos q & = & \frac{\beta^2+\overline{\mu}(\beta^2-1)}{2\overline{\mu}(\beta^2-1)}\Big[1\pm\sqrt{1-8\frac{\beta^4}{(\beta^2+1)^2(\beta^2+\overline{\mu}(\beta^2-1))^2}\frac{\beta^2-1}{\overline{\mu}}}\,\Big] \\ & > & \frac{2\beta^4}{\overline{\mu}^2(\beta^2+1)^2(\beta^2+\overline{\mu}(\beta^2-1))}. \end{array} \tag{3.24}$$

Here we have used the inequality

$$1+\sqrt{1-x}\geq 1-\sqrt{1-x}>\frac{1}{2}x, \tag{3.25}$$

which holds for any $0<x\leq 1$. Since $\overline{\mu}(\beta^2-1)<\beta^2$, the inequality (3.23) now leads to

$$\pi^2<\frac{2\beta^2}{\overline{\mu}(\beta^2-1)}\Big[\frac{\overline{\mu}(\beta^2-1)-\beta^2}{\beta^2+\overline{\mu}(\beta^2-1)}+1\Big]=\frac{4\beta^2}{\beta^2+\overline{\mu}(\beta^2-1)}<4, \tag{3.26}$$

which clearly is a contradiction. □

Corollary 3.5. *Assume that (HP) holds. Then for any sufficiently small $\epsilon>0$, we have the identities*

$$n_L^{\sharp}(-\epsilon)=0, \qquad n_L^{\sharp}(+\epsilon)=1. \tag{3.27}$$

Proof. Lemma 3.4 guarantees that we only have to consider the trajectory of the simple root $z=0$ of the characteristic equation $\Delta_{\Gamma(\mu)}(z)=0$ at $\mu=1$ as this parameter is varied. Writing this root as $z_*(\mu)$, we may use the implicit function theorem to compute

$$\frac{dz_*}{d\mu}\,|_{\mu=1}=-[\Delta'_{\Gamma(1)}(0)]^{-1}[\frac{d}{d\mu}\Delta_{\Gamma(\mu)}(0)]_{|\mu=1}=\beta-\beta^{-1}>0. \tag{3.28}$$

Thus as μ increases to one, the root $z_*(\mu)$ crosses the line $\operatorname{Re} z=-\epsilon$ from left to right for all sufficiently small $\epsilon>0$, but it does not cross the line $\operatorname{Re} z=+\epsilon$. In the terminology of Theorem 2.5, this means that $\operatorname{cross}(\Gamma,-\epsilon)=1$ and $\operatorname{cross}(\Gamma,+\epsilon)=0$, which concludes the proof. □

We conclude from Proposition 2.4 that the initial value problem (1.11) with L as in (3.1) is well-posed with respect to the space $BC^{\oplus}_{+\epsilon}$ for every small $\epsilon>0$. Notice that the equation $\Delta_L(z)=0$ admits only the simple root $z=0$ on the imaginary axis, which contributes a constant eigenfunction. This allows us to use Lemma 5.4 to strengthen our result slightly and state that (1.11) is also well-posed with respect to $BC^{\oplus}_0$.

3.2 Well-posedness of an MFAE

Our second example features the algebraic equation

$$0=M\,\mathrm{ev}_\xi\,x:=-A(\rho)x(\xi)+\int_{-1}^{0}x(\xi+\sigma)(1+\sigma)d\sigma+\int_{0}^{1}x(\xi+\sigma)(1-\sigma)d\sigma \tag{3.29}$$

with $\rho>0$, in which the constant $A(\rho)$ is given by

$$A(\rho)=\int_0^1 e^{-\rho\sigma}d\sigma\int_0^1 e^{\rho\sigma}d\sigma=2\rho^{-2}(\cosh\rho-1)>1. \tag{3.30}$$

This equation is encountered [4, 8] when one studies an overlapping generations model that is similar to the one described in the introduction, but now with a discounted welfare function

$$W(s)=\int_s^{s+1}e^{-\rho\tau}\ln c(s,\tau)d\tau, \tag{3.31}$$

fixed wages $w(t) = 1$ and a population that grows at the rate $e^{\rho t}$. Nonlinear versions of this model are discussed in [8], but we restrict ourselves to the linear case here. The function p appearing in (3.29) is related to the interest rate by means of

$$x(\xi) = \exp[-\int_0^\xi r(\tau)d\tau]. \tag{3.32}$$

The characteristic equation associated to (3.29) is given by

$$\delta_M(z) = A(\rho) - \int_{-1}^0 (1+\sigma)e^{z\sigma}d\sigma - \int_0^1 (1-\sigma)e^{z\sigma}d\sigma. \tag{3.33}$$

Lemma 3.6. *There exists $\eta_* > 0$ such that the characteristic equation $\delta_M(z) = 0$ has precisely two real roots $z = \pm\eta_*$ and no other complex roots in the strip $-\eta_* \leq \mathrm{Re}\, z \leq \eta_*$.*

Proof. Notice first that $\delta_M(-z) = \delta_M(z)$ and $\delta_M(0) = A(\rho) - 1 > 0$. For $z \neq 0$, we will use the representation

$$\delta_M(z) = A(\rho) - z^{-2}\big[e^z + e^{-z} - 2\big]. \tag{3.34}$$

Differentiation yields

$$-z^3 e^{-z}\delta_M'(z) = \wp(z) := z - 2 + 4e^{-z} - (2+z)e^{-2z}. \tag{3.35}$$

Since $\wp(0) = \wp'(0) = 0$ and

$$\wp''(p) = 4e^{-2p}(e^p - p - 1) > 0 \tag{3.36}$$

for all $p > 0$, Taylor's formula implies that $\delta_M'(p) < 0$ for $p > 0$, which establishes that $\delta_M(z) = 0$ has precisely two real roots $z = \pm\eta_*$, for some $\eta_* > 0$.

Let us now write $\widetilde{\delta}_M(z) = z^2\delta_M(z)$ and compute

$$\begin{array}{rcl} \mathrm{Re}\,\widetilde{\delta}_M(p+iq) & = & A(\rho)(p^2-q^2) + 2 - (e^p + e^{-p})\cos q, \\ \mathrm{Im}\,\widetilde{\delta}_M(p+iq) & = & 2A(\rho)pq - (e^p - e^{-p})\sin q. \end{array} \tag{3.37}$$

It is not hard to verify that $\widetilde{\delta}_M(iq) \neq 0$ for all $q \neq 0$. Let us assume therefore that $\widetilde{\delta}_M(p+iq) = 0$ for some $0 < p \leq \eta_*$. The second line of (3.37) can be used to isolate an expression for q. Substituting this into the q^2 term in the first line of (3.37) and using $\sin^2 q + \cos^2 q = 1$, we see that

$$(e^p - e^{-p})^2\cos^2 q - 4A(\rho)p^2(e^p + e^{-p})\cos q + \big[4p^4A(\rho)^2 + 8A(\rho)p^2 - (e^p - e^{-p})^2\big] = 0, \tag{3.38}$$

which can be solved to yield the solutions $\cos q = c_\pm^*$, with

$$c_\pm^* = \pm 1 + 2p^2A(\rho)\frac{(e^{\frac{1}{2}p} \mp e^{-\frac{1}{2}p})^2}{(e^p - e^{-p})^2}. \tag{3.39}$$

Since $c_+^* > 1$, we need only consider c_-^*. Our assumption on p implies $\widetilde{\delta}_M(p) \geq 0$, which means

$$p^2A(\rho) \geq (e^{\frac{1}{2}p} - e^{-\frac{1}{2}p})^2. \tag{3.40}$$

We thus find $c_-^* \geq 1$ with equality if and only if $p = \eta_*$, in which case the second line of (3.37) immediately yields $q = 0$. □

An easy integration by parts yields

$$z\delta_M(z) = A(\rho)\Delta_L(z) = A(\rho)z - \int_0^{-1} e^{z\sigma} d\sigma - \int_0^1 e^{z\sigma} d\sigma, \tag{3.41}$$

which shows that (3.30) is closely related to the MFDE

$$x'(\xi) = L\,\mathrm{ev}_\xi x := A(\rho)^{-1} \int_0^{-1} x(\xi + \sigma) d\sigma + A(\rho)^{-1} \int_0^1 x(\xi + \sigma) d\sigma. \tag{3.42}$$

In view of Theorem 2.6, we now set out to compute $n_L^\sharp(-\epsilon)$ for sufficiently small $\epsilon > 0$. Noting the symmetry $\delta_M(-z) = \delta_M(z)$, let us consider the retarded differential equation

$$v'(\xi) = L_-\mathrm{ev}_\xi v := iA(\rho)^{-\frac{1}{2}}[v(\xi - 1) - v(\xi)], \tag{3.43}$$

together with the advanced differential equation

$$v'(\xi) = L_+\mathrm{ev}_\xi v := iA(\rho)^{-\frac{1}{2}}[v(\xi) - v(\xi + 1)]. \tag{3.44}$$

The associated characteristic functions satisfy

$$\begin{array}{rcl} \Delta_{L_-}(z) & = & z - iA(\rho)^{-\frac{1}{2}}(e^{-z} - 1), \\ \Delta_{L_+}(z) & = & z + iA(\rho)^{-\frac{1}{2}}(e^{z} - 1) \end{array} \tag{3.45}$$

and obviously $\Delta_{L_-}(-z) = -\Delta_{L_+}(z)$. A simple computation yields

$$\begin{array}{rcl} \Delta_{L_-}(z)\Delta_{L_+}(z) & = & z^2 - izA(\rho)^{-\frac{1}{2}}\big(e^{-z} - e^{z}\big) + A(\rho)^{-1}\big[2 - e^{z} - e^{-z}\big] \\ & = & z^2 - izA(\rho)^{-\frac{1}{2}}\big(e^{-z} - e^{z}\big) - zA(\rho)^{-1}[\int_0^{-1} e^{z\sigma} d\sigma + \int_0^1 e^{z\sigma} d\sigma] \\ & = & z\Big[\Delta_L(z) - iA(\rho)^{-\frac{1}{2}}\big(e^{-z} - e^{z}\big)\Big]. \end{array} \tag{3.46}$$

Let us therefore embed (3.42) into the family of MFDEs

$$\begin{array}{rcl} p'(\xi) = \Gamma(\mu)\mathrm{ev}_\xi p & := & A(\rho)^{-1} \int_0^{-1} p(\xi + \sigma) d\sigma + A(\rho)^{-1} \int_0^1 p(\xi + \sigma) d\sigma \\ & & + i(1 - \mu)A(\rho)^{-\frac{1}{2}} p(\xi - 1) - i(1 - \mu)A(\rho)^{-\frac{1}{2}} p(\xi + 1). \end{array} \tag{3.47}$$

Notice that $\Gamma(1) = L$, while

$$\Delta_{\Gamma(\mu)} = \Delta_L(z) - i(1 - \mu)A(\rho)^{-\frac{1}{2}}\big(e^{-z} - e^{z}\big) \tag{3.48}$$

and $\Gamma(0)$ admits the Wiener-Hopf factorization

$$z\Delta_{\Gamma(0)} = \Delta_{L_-}(z)\Delta_{L_+}(z). \tag{3.49}$$

Lemma 3.7. *Suppose that $\Delta_{L_-}(z) = 0$ for some $z \in \mathbb{C}$. Then either $z = 0$ or $\mathrm{Re}\, z < 0$. The root at $z = 0$ is a simple root, i.e., $\Delta'_{L_-}(0) \neq 0$.*

Proof. The identities $\Delta_{L_-}(0) = 0$ and $\Delta'_{L_-}(0) = 1 + iA(\rho)^{-\frac{1}{2}} \neq 0$ can be verified directly. Observe furthermore that

$$\begin{array}{rcl} \mathrm{Re}\, \Delta_{L_-}(p + iq) & = & p - A(\rho)^{-\frac{1}{2}} e^{-p} \sin q, \\ \mathrm{Im}\, \Delta_{L_-}(p + iq) & = & q + A(\rho)^{-\frac{1}{2}}\big[1 - e^{-p}\cos q\big]. \end{array} \tag{3.50}$$

Looking for solutions to $\Delta_{L_-}(z) = 0$, we may use the identity $\sin^2 q + \cos^2 q = 1$ to find

$$e^{2p}\big(A(\rho)q^2 + 2A(\rho)^{\frac{1}{2}} q + A(\rho)p^2 + 1 - e^{-2p}\big) = 0, \tag{3.51}$$

which can be solved to yield

$$q = q_\pm(p) = -A(\rho)^{-\frac{1}{2}}\big[1 \pm \sqrt{e^{-2p} - A(\rho)p^2}\,\big]. \tag{3.52}$$

Let us now suppose that $p \geq 0$ and that $q_\pm(p) \in \mathbb{R}$. Recalling that $A(\rho) > 1$, we may estimate

$$0 \geq q_\pm(p) = -A(\rho)^{-\frac{1}{2}}\left|1 \pm \sqrt{e^{-2p} - A(\rho)p^2}\right| \geq -2A(\rho)^{-\frac{1}{2}} > -2 > -\pi, \tag{3.53}$$

which in view of the requirement

$$\sin q = pe^p A(\rho)^{\frac{1}{2}} \geq 0 \tag{3.54}$$

implies that $q = 0$ and hence also $p = 0$. □

In view of the symmetry $\Delta_{L_-}(z) = -\Delta_{L_+}(-z)$, we may now conclude that

$$n^\sharp_{\Gamma(0)}(-\epsilon) = n^+_{L_+}(-\epsilon) - n^-_{L_-}(-\epsilon) + n^0_z(-\epsilon) = 0 - 1 + 1 = 0 \tag{3.55}$$

for any sufficiently small $\epsilon > 0$. The transition from $\Gamma(0)$ to $\Gamma(1)$ is studied in the following result.

Lemma 3.8. *Besides the simple root at $z = 0$, the characteristic equation $\Delta_{\Gamma(\mu)}(z) = 0$ has no roots on the imaginary axis for any $\mu \in [0, 1]$.*

Proof. The statement concerning the simple root at $z = 0$ can be verified directly. Let us therefore suppose that $\Delta_{\Gamma(\mu)}(iq) = 0$ for some $\mu \in [0, 1]$ and $q \in \mathbb{R} \setminus \{0\}$, i.e.,

$$A(\rho)\Delta_{\Gamma(\mu)}(iq) = -2(1-\mu)A(\rho)^{\frac{1}{2}} \sin q + iq^{-1}\big[q^2 A(\rho) - 2 + 2\cos q\big] = 0. \tag{3.56}$$

Using a Taylor expansion, we find that for any $q > 0$ there exists $0 < q' < q$ such that

$$q^2 A(\rho) - 2 + 2\cos q = \frac{1}{2}q^2[2A(\rho) - 2\cos q'] > 0. \tag{3.57}$$

A similar argument works for $q < 0$. □

Using Theorem 2.5 we hence conclude that $n^\sharp_L(-\epsilon) = 0$ for all sufficiently small $\epsilon > 0$. In view of Theorem 2.6, this means that the initial value problem (1.12) with M as in (3.29) is not well-posed with respect to the space $BC^\oplus_0$.

To repair this, let us recall the constant η_* that appears in Lemma 3.6. We consider any $\eta > \eta_*$ that is sufficiently close to η_* to ensure that $\delta_M(z) = 0$ only has the simple root $z = \eta_*$ in the strip $0 \leq \operatorname{Re} z \leq \eta$. Writing L' for the operator $L'\phi = L\phi - \eta A(\rho)^{-1}M\phi$, we find

$$(z - \eta)\delta_M(z) = A(\rho)\Delta_L(z) - \eta\delta_M(z) = A(\rho)\Delta_{L'}(z). \tag{3.58}$$

Notice that there is a bijective correspondence between the roots of the equation $\Delta_{L'}(z) = 0$ and those of $\Delta_L(z) = 0$. The simple root at $z = 0$ of the latter equation is moved to $z = \eta$, but the rest of the roots remain fixed. An application of Theorem 2.5 hence yields

$$n^\sharp_{L'}(\eta - \epsilon) = 1 + n^\sharp_L(-\epsilon) = 1 \tag{3.59}$$

for all sufficiently small $\epsilon > 0$. We hence see that the initial value problem (1.12) with M as in (3.29) is well-posed with respect to the space $BC^\oplus_\eta$. Since $z = \eta_*$ is a simple root of $\Delta_{L'}(z) = 0$, we can argue as for the previous example that this well-posedness also holds with respect to the space $BC^\oplus_{\eta_*}$.

4 Continuity of $n^\sharp$

In this section we prove Theorem 2.5. We will proceed much along the lines of [22, §5] and show how the Wiener-Hopf splitting (2.15) can be obtained in a fashion that is robust under small perturbations of L. Let us start by considering the equation

$$x'(\xi) = L \operatorname{ev}_\xi x \tag{4.1}$$

for some bounded linear operator $L : C([r_{\min}, r_{\max}], \mathbb{C}^n) \to \mathbb{C}^n$. Our first result implies that once $n^\sharp_L(\eta)$ is known for a specific value of η, one only needs to study the characteristic equation $\det \Delta_L(z) = 0$ to obtain $n^\sharp_L(\eta)$ for all other appropriate values of η. In particular, one does not need to have a Wiener-Hopf triplet for L.

Lemma 4.1. *Consider the system (4.1) and suppose that (HL) is satisfied. Pick any two real numbers $\eta_1 < \eta_2$ and suppose that the characteristic equation $\det \Delta_L(z) = 0$ has m roots in the vertical strip $\eta_1 \le \operatorname{Re} z \le \eta_2$, in which each root is counted according to its multiplicity. Suppose furthermore that each of these roots has $\eta_1 < \operatorname{Re} z < \eta_2$. Then we have the identity*

$$n^\sharp_L(\eta_2) = n^\sharp_L(\eta_1) + m. \tag{4.2}$$

Proof. Choose a monic polynomial p that has degree n and has $p(z) \neq 0$ for all $\operatorname{Re} z \ge \eta_1$. Proposition 2.2 guarantees that there exist linear operators $L_- \in \mathcal{L}(C([r_{\min}, 0], \mathbb{C}^n), \mathbb{C}^n)$ and $L_+ \in \mathcal{L}(C([0, r_{\max}], \mathbb{C}^n), \mathbb{C}^n)$ such that

$$p(z) \det \Delta_L(z) = \det \Delta_{L_-}(z) \det \Delta_{L_+}(z). \tag{4.3}$$

Using $n^0_p(\eta_1) = n^0_p(\eta_2) = 0$, this allows us to compute

$$n^\sharp_L(\eta_i) = n^+_{L_+}(\eta_i) - n^-_{L_-}(\eta_i) \tag{4.4}$$

for $i = 1, 2$. Let us write m_+ for the number of roots of the characteristic equation $\det \Delta_{L_+}(z) = 0$ that have $\eta_1 < \operatorname{Re} z < \eta_2$ and m_- for the analogous quantity associated to the equation $\det \Delta_{L_-} = 0$. As usual, each root should be counted according to its multiplicity. In view of (4.3), we must have $m = m_+ + m_-$. It is easy to see that $n^+_{L_+}(\eta_2) = n^+_{L_+}(\eta_1) + m_+$ and $n^-_{L_-}(\eta_2) = n^-_{L_-}(\eta_1) - m_-$. The identity (4.2) now follows immediately from (4.4). □

We now move on to study parameter-dependent versions of (4.1). To set the stage, let us pick any $\mu_0 \in \mathbb{R}$ and consider a C^0-smooth map

$$L : U \to \mathcal{L}\big(C([r_{\min}, r_{\max}], \mathbb{C}^n), \mathbb{C}^n\big), \tag{4.5}$$

in which U is an open interval containing μ_0. We will assume that $L(\mu)$ satisfies (HL) for every $\mu \in U$.

The next result states that for sufficiently negative η, solutions in $\mathfrak{Q}_{L(\mu)}(\eta)$ automatically satisfy a retarded differential equation that depends continuously on the parameter μ. Of course an analogous result holds for the space $\mathfrak{P}_L(\eta)$ if η is sufficiently large. We remark that we will use the notation

$$\pi^- : C\big([r_{\min}, r_{\max}], \mathbb{C}^n\big) \to C\big([r_{\min}, 0], \mathbb{C}^n\big), \qquad \phi \mapsto \phi_{|[r_{\min}, 0]}. \tag{4.6}$$

Lemma 4.2. *Pick any sufficiently negative $\eta \in \mathbb{R}$. Then there exists an open set $U' \subset U$ with $\mu_0 \in U'$, together with a C^0-smooth map*

$$L_- : U' \to \mathcal{L}\big(C([r_{\min}, 0], \mathbb{C}^n), \mathbb{C}^n\big) \tag{4.7}$$

such that the differential equation

$$v'(\xi) = L_-(\mu)\pi^- \operatorname{ev}_\xi v, \qquad \xi \ge 0, \tag{4.8}$$

holds for any $\mu \in U'$ and $v \in \mathfrak{Q}_{L(\mu)}(\eta)$.

Proof. For convenience, let us use the shorthand $L_0 = L(\mu_0)$. As established in [22, Lem. 5.5], we can ensure that the map

$$\pi^-_{Q_{L_0}(\eta)} : Q_{L_0}(\eta) \to C([r_{\min}, 0], \mathbb{C}^n) \tag{4.9}$$

is injective by choosing η to be sufficiently close to $-\infty$. Without loss of generality, we may assume that $\det \Delta_{L_0}(z) = 0$ has no roots with $\operatorname{Re} z = \eta$. Using techniques very similar to those developed in [16, §5], we can construct a C^0-smooth operator

$$u^* : U' \to \mathcal{L}\big(Q_{L_0}(\eta), C([r_{\min}, r_{\max}], \mathbb{C}^n)\big), \tag{4.10}$$

with $u^*(\mu_0) = I$, in such a way that $Q_{L(\mu)}(\eta) = u^*(\mu)\big(Q_{L_0}(\eta)\big)$.

Write $R = \operatorname{Range}\big(\pi^-_{Q_{L_0}(\eta)}\big) \subset C([r_{\min}, 0], \mathbb{C}^n)$ and observe that the Fredholm properties in Proposition 2.1 imply that this space is closed and of finite codimension. This allows us to fix a finite dimensional complement $R_\perp$ such that $R \oplus R_\perp = C([r_{\min}, 0], \mathbb{C}^n)$. We will write π_R and $\pi_{R_\perp}$ for the accompanying projections.

We now introduce, for any $\mu \in U'$, the linear map $\Psi(\mu) : R \oplus R_\perp \to R \oplus R_\perp$ that acts as

$$\Psi(\mu)\big(\psi, \psi_\perp\big) = \big(\pi_R \pi^- u^*(\mu)[\pi^-_{Q_{L_0}(\eta)}]^{-1}\psi, \psi_\perp + \pi_{R_\perp}\pi^- u^*(\mu)[\pi^-_{Q_{L_0}(\eta)}]^{-1}\psi\big). \tag{4.11}$$

Note that Ψ depends C^0-smoothly on μ when viewed as a map from $U' \to \mathcal{L}\big(C([r_{\min}, 0], \mathbb{C}^n)\big)$, with $\Psi(\mu_0) = I$. This means that $\Psi(\mu)$ is invertible for all $\mu \in U'$, possibly after decreasing the size of U'.

As in [22, §5], we define $L_-(\mu_0)$ by writing $L_-(\mu_0)\phi = L_0[\pi^-_{Q_{L_0}(\eta)}]^{-1}\phi$ for $\phi \in R$ and arbitrarily extending $L_-(\mu_0)$ to a bounded linear map on $C([r_{\min}, 0], \mathbb{C}^n)$. We are now in a position to define

$$L_-(\mu)\phi = L_-(\mu_0)\pi_{R_\perp}[\Psi(\mu)]^{-1}\phi + L(\mu)u^*(\mu)[\pi^-_{Q_{L_0}(\eta)}]^{-1}\pi_R[\Psi(\mu)]^{-1}\phi. \tag{4.12}$$

Recall that for any $\varphi \in Q_{L(\mu)}(\eta)$, there exists $\rho \in Q_{L_0}(\eta)$ such that $\varphi = u^*(\mu)\rho$. Writing $\psi = \pi^-\rho \in R$, we obviously have $\rho = [\pi^-_{Q_{L_0}(\eta)}]^{-1}\psi$ and hence $\varphi = u^*(\mu)[\pi^-_{Q_{L_0}(\eta)}]^{-1}\psi$. This means $\pi^-\varphi = \Psi(\psi, 0)$ and hence $L_-(\mu)\pi^-\varphi = L(\mu)\varphi$, as desired. □

We are now ready to study the characteristic equations

$$\begin{array}{rcl} \Delta_{L(\mu)}(z) & = & zI - L(\mu)e^{z\cdot}I, \\ \Delta_{L_\pm(\mu)}(z) & = & zI - L_\pm(\mu)e^{z\cdot}I, \end{array} \tag{4.13}$$

in which the operators $L_\pm(\mu)$ are those that are defined by Lemma 4.2 and its analogue for $\mathfrak{P}_L(\eta)$. As a consequence of this result, the functions $(z, \mu) \mapsto \Delta_{L(\mu)}(z)$ and $(z, \mu) \mapsto \Delta'_{L(\mu)}(z)$ are continuous, as are $(z, \mu) \mapsto \Delta_{L_\pm(\mu)}(z)$ and $(z, \mu) \mapsto \Delta'_{L_\pm(\mu)}(z)$. Notice in addition that

$$\Delta_{L(\mu)}(z) = zI + O(1), \qquad \operatorname{Im} z \to \pm\infty, \tag{4.14}$$

uniformly for z in vertical strips of the complex plane and μ in compact subsets of U'. Such estimates also hold for the characteristic matrices $\Delta_{L_\pm(\mu)}$.

Let us pick η_- sufficiently close to $-\infty$ and η_+ sufficiently close to $+\infty$ in such a way that $\eta_- < \eta_+$ holds, that $\pi^-_{Q_{L(\mu_0)}(\eta_-)}$ and $\pi^+_{P_{L(\mu_0)}(\eta_+)}$ are both injective and that $\det \Delta_{L_\pm(\mu_0)}(z) \neq 0$ and $\det \Delta_{L(\mu_0)}(z) \neq 0$ for all z with $\operatorname{Re} z \in \{\eta_-, \eta_+\}$. This choice enables us to define the sets

$$\begin{array}{rcl} \Sigma_\mu & = & \{z \in \mathbb{C} \mid \det \Delta_{L(\mu)}(z) = 0 \text{ and } \eta_- \leq \operatorname{Re} z \leq \eta_+\}, \\ \Sigma^\pm_\mu & = & \{z \in \mathbb{C} \mid \det \Delta_{L_\pm(\mu)}(z) = 0 \text{ and } \eta_- \leq \operatorname{Re} z \leq \eta_+\}, \end{array} \tag{4.15}$$

for $\mu \in U'$, in which each root is included according to its multiplicity, together with the associated polynomials

$$\begin{array}{rcl} \wp_\mu(z) & = & \prod_{\lambda \in \Sigma_\mu} (z - \lambda), \\ \wp_\mu^{\pm}(z) & = & \prod_{\lambda \in \Sigma_\mu^{\pm}} (z - \lambda). \end{array} \tag{4.16}$$

We also write

$$\varrho_\mu(z) = [\det \Delta_{L(\mu)}(z)]^{-1} \det \Delta_{L_+(\mu)}(z) \det \Delta_{L_-(\mu)}(z) \frac{\wp_\mu(z)}{\wp_\mu^+(z)\wp_\mu^-(z)}. \tag{4.17}$$

Lemma 4.3. *There exists an open subset $U' \subset U$, with $\mu_0 \in U'$, such that the elements of Σ_μ and $\Sigma_\mu^{\pm}$ depend continuously on $\mu \in U'$, with*

$$\#\Sigma_\mu = \#\Sigma_{\mu_0}, \qquad \#\Sigma_\mu^{\pm} = \#\Sigma_{\mu_0}^{\pm}. \tag{4.18}$$

In addition, for every $\mu \in U'$, the function ϱ_μ is a polynomial of degree

$$\deg \varrho_\mu = n + \#\Sigma_{\mu_0} - \#\Sigma_{\mu_0}^+ - \#\Sigma_{\mu_0}^-. \tag{4.19}$$

The roots of this polynomial vary continuously with μ.

Proof. The estimate (4.14) ensures that the elements in the sets $\Sigma_\mu^{\pm}$ and Σ_μ can be a-priori bounded. The identities (4.18) now follow immediately from the argument principle.

To see that ϱ_μ is an entire function, it suffices to check that this function has no poles λ with $\operatorname{Re} \lambda < \eta_-$ or $\operatorname{Re} \lambda > \eta_+$. Supposing to the contrary that such a pole does exist, we have that $z = \lambda$ is a root of order $\ell \geq 1$ for the characteristic equation $\det \Delta_{L(\mu)}(\lambda) = 0$. Without loss of generality, we will assume that $\operatorname{Re} \lambda < \eta_-$. Let us now consider any polynomial p for which the function $x(\xi) = e^{\lambda \xi} p(\xi)$ satisfies $x \in \mathfrak{Q}_{L(\mu)}(\eta_-)$. Lemma 4.2 implies that x also satisfies the delay equation $x'(\xi) = L_-(\mu)\pi^- \mathrm{ev}_\xi x$, which implies that $z = \lambda$ is a root of the characteristic equation $\det \Delta_{L_-(\mu)}(z) = 0$ of order ℓ or greater. This yields a contradiction.

Proceeding similarly as in the proof of [22, Thm 5.1], a theorem of Phragmén-Lindelöf type ensures that for each $\mu \in U'$, the function

$$z \mapsto r_\mu(z) := z^{-n} \varrho_\mu(z) \frac{\wp_\mu^+(z)\wp_\mu^-(z)}{\wp_\mu(z)} \tag{4.20}$$

is a rational function with $r_\mu(\infty) = 1$. Combining (4.20) with (4.18) now shows that ϱ_μ must be a polynomial of the degree specified by (4.19). An additional application of the argument principle shows that the roots of this polynomial depend continuously on μ. □

Notice that the identity (4.17) resembles the Wiener-Hopf factorization (2.15). Using the root-swapping techniques developed in [22, §5], the superfluous polynomial factors in (4.17) can be systematically eliminated. We describe this process in the proof of the next result, which essentially tells us how $n^\sharp_{L(\mu)}(\eta)$ can be determined directly from (4.17). The continuity of the elements of Σ_μ and $\Sigma_\mu^{\pm}$ can subsequently be used to show that $n^\sharp_{L(\mu)}(\eta)$ is invariant under small changes of μ, as long as the line $\operatorname{Re} z = \eta$ avoids the eigenvalues associated to $L(\mu)$ and $L_\pm(\mu)$.

Lemma 4.4. *Consider any $\eta \in \mathbb{R}$ for which the characteristic equations $\det \Delta_{L(\mu_0)}(z) = 0$ and $\det \Delta_{L_\pm(\mu_0)}(z) = 0$ have no roots with $\operatorname{Re} z = \eta$. Then there exists an open set $U' \subset U$, with $\mu_0 \in U$, such that*

$$n^\sharp_{L(\mu)}(\eta) = n^\sharp_{L(\mu_0)}(\eta) \tag{4.21}$$

for all $\mu \in U'$.

Proof. Pick any monic polynomial p of degree n such that $p(z) = 0$ admits no roots with $\operatorname{Re} z \geq \eta$. For the moment, we fix a $\mu \in U$ that is sufficiently close to μ_0. Our goal is to define, for some integer $\ell_* > 1$, a sequence of monic polynomials q^{ℓ}_{in} and q^{ℓ}_{out} together with a sequence of operators $L^{\ell}_{-} \in \mathcal{L}(C([r_{\min}, 0], \mathbb{C}^n), \mathbb{C}^n)$ and $L^{\ell}_{+} \in \mathcal{L}(C([0, r_{\max}], \mathbb{C}^n), \mathbb{C}^n)$, that are indexed by $1 \leq \ell \leq \ell_*$ and satisfy the following properties.

(i) For every $1 \leq \ell < \ell_*$ we have

$$\deg q^{\ell+1}_{\mathrm{in}} = \deg q^{\ell+1}_{\mathrm{out}} < \deg p^{\ell}_{\mathrm{in}} = \deg p^{\ell}_{\mathrm{out}}. \tag{4.22}$$

(ii) For every $1 \leq \ell \leq \ell_*$, the equations $q^{\ell}_{\mathrm{in}}(z) = 0$ and $q^{\ell}_{\mathrm{out}}(z) = 0$ do not admit roots with $\operatorname{Re} z = \eta$.

(iii) We have $\deg q^{\ell_*}_{\mathrm{in}} = \deg q^{\ell_*}_{\mathrm{out}} = 0$.

(iv) For every $1 \leq \ell \leq \ell_*$, the following factorization holds,

$$p(z) \det \Delta_{L(\mu)}(z) = \frac{\det \Delta_{L^{\ell}_{-}}(z) \det \Delta_{L^{\ell}_{+}}(z) q^{\ell}_{\mathrm{in}}(z)}{q^{\ell}_{\mathrm{out}}(z)}. \tag{4.23}$$

Notice that once we have found such a sequence, items (iii) and (iv) imply that the set $(p, L^{\ell_*}_{-}, L^{\ell_*}_{+})$ is a Wiener-Hopf triplet for L, which will allow us to compute $n^{\sharp}_{L(\mu)}(\eta)$.

Let us introduce the quantities

$$\begin{array}{rcl} \widetilde{n}^{\ell}_{+} & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L^{\ell}_{+}}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}, \\ \widetilde{n}^{\ell}_{-} & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L^{\ell}_{-}}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \end{array} \tag{4.24}$$

together with

$$\begin{array}{rcl} \widetilde{m}^{\ell}_{+,\mathrm{in}} & = & \#\{z \in \mathbb{C} \mid q^{\ell}_{\mathrm{in}}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \\ \widetilde{m}^{\ell}_{-,\mathrm{in}} & = & \#\{z \in \mathbb{C} \mid q^{\ell}_{\mathrm{in}}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}, \\ \widetilde{m}^{\ell}_{+,\mathrm{out}} & = & \#\{z \in \mathbb{C} \mid q^{\ell}_{\mathrm{out}}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \\ \widetilde{m}^{\ell}_{-,\mathrm{out}} & = & \#\{z \in \mathbb{C} \mid q^{\ell}_{\mathrm{out}}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}. \end{array} \tag{4.25}$$

We claim that we can define the sequences mentioned above in such a way that the following identity holds for all $1 \leq \ell \leq \ell_*$,

$$n^{\sharp}_{L(\mu)}(\eta) = \widetilde{n}^{\ell}_{+} - \widetilde{n}^{\ell}_{-} - \frac{1}{2}\big[\widetilde{m}^{\ell}_{+,\mathrm{in}} - \widetilde{m}^{\ell}_{-,\mathrm{in}} + \widetilde{m}^{\ell}_{-,\mathrm{out}} - \widetilde{m}^{\ell}_{+,\mathrm{out}}\big]. \tag{4.26}$$

The definition of $n^{\sharp}_{L(\mu)}(\eta)$ given in (2.16) implies that (4.26) certainly holds for $\ell = \ell_*$, so we will only need to prove that the right hand side of (4.26) is invariant.

To establish our claims, we start by remarking that (iv) is satisfied for $\ell = 1$ if we write $L^{1}_{\pm} = L_{\pm}(\mu)$ and

$$\begin{array}{rcl} q^{1}_{\mathrm{in}}(z) & = & \wp_{\mu}(z) p(z), \\ q^{1}_{\mathrm{out}}(z) & = & \wp^{+}_{\mu}(z) \wp^{-}_{\mu}(z) \varrho_{\mu}(z). \end{array} \tag{4.27}$$

Lemma 4.3 implies that q^{1}_{in} and q^{1}_{out} have the same degree and that (ii) holds for $\ell = 1$.

We now iteratively define $q^{\ell+1}_{\mathrm{in}}$, $q^{\ell+1}_{\mathrm{out}}$ and $L^{\ell+1}_{\pm}$ by arbitrarily choosing a root $z = \lambda_{\mathrm{out}}$ of the equation $q^{\ell}_{\mathrm{out}}(z) = 0$ and writing

$$q^{\ell+1}_{\mathrm{out}}(z) = q^{\ell}_{\mathrm{out}}(z)/(z - \lambda_{\mathrm{out}}). \tag{4.28}$$

Since the left hand side of (4.23) is analytic in z, at least one of the following three recipes can be followed.

(A) Suppose that $q^{\ell}_{\rm in}(\lambda_{\rm out}) = 0$. Write $q^{\ell+1}_{\rm in}(z) = q^{\ell}_{\rm in}(z)/(z - \lambda_{\rm out})$ and keep $L^{\ell+1}_{\pm} = L^{\ell}_{\pm}$ fixed. If $\lambda > \eta$, both $\widetilde{m}_{+,\rm in}$ and $\widetilde{m}_{+,\rm out}$ will decrease by one, which ensures that the right hand side of (4.26) does not change. A similar argument works if $\lambda < \eta$.

(B) Suppose that $\det \Delta^{\ell}_{L^{\ell}_{-}}(\lambda_{\rm out}) = 0$. In view of (i), there exists $\lambda_{\rm in} \in \mathbb{C}$ for which $q^{\ell}_{\rm in}(\lambda_{\rm in}) = 0$. We may now use [22, Lem. 5.8] to construct $L^{\ell+1}_{-}$ in such a way that

$$\det \Delta_{L^{\ell+1}_{-}}(z) = \frac{z - \lambda_{\rm in}}{z - \lambda_{\rm out}} \det \Delta_{L^{\ell}_{-}}(z). \tag{4.29}$$

Furthermore, we keep $L^{\ell+1}_{+} = L^{\ell}_{+}$ fixed and write

$$q^{\ell+1}_{\rm in}(z) = q^{\ell}_{\rm in}(z)/(z - \lambda_{\rm in}). \tag{4.30}$$

If $\lambda_{\rm out}$ and $\lambda_{\rm in}$ lie on the same side of η, none of the quantities in (4.24) and (4.25) change. If $\lambda_{\rm in} < \eta < \lambda_{\rm out}$, then $\widetilde{n}_{-}$ will decrease by one. However, both $\widetilde{m}_{-,\rm in}$ and $\widetilde{m}_{+,\rm out}$ will decrease by one, ensuring that the right hand side of (4.26) remains invariant. The remaining case $\lambda_{\rm out} < \eta < \lambda_{\rm in}$ can be treated similarly.

(C) Suppose that $\det \Delta^{\ell}_{L^{\ell}_{+}}(\lambda) = 0$. One can proceed similarly as in (B), now applying [22, Lem. 5.8] to construct $L^{\ell+1}_{+}$.

To complete the proof, it now suffices to observe that our choice (4.27) allows Lemma 4.3 to be invoked. This allows us to establish that the quantities $\widetilde{n}^1_{\pm}$, $\widetilde{m}^1_{\pm,\rm in}$ and $\widetilde{m}^1_{\pm,\rm out}$ will not depend on $\mu \in U'$ as long as U' is chosen to be sufficiently small. □

Proof of Theorem 2.5. For every $\mu \in [0,1]$, one may choose a suitable $\eta_{\mu} \in \mathbb{R}$ and use Lemma 4.4 to find an open neighbourhood $U'_{\mu} \subset [0,1]$, with $\mu \in U'$, for which the identity

$$n^{\sharp}_{\Gamma(\mu')}(\eta_{\mu}) = n^{\sharp}_{\Gamma(\mu)}(\eta_{\mu}) \tag{4.31}$$

holds for all $\mu' \in U'_{\mu}$. The intervals $U'_{\mu} \subset [0,1]$ clearly form an open covering of $[0,1]$, allowing us to extract a finite set $\mu_1 < \mu_2 < \ldots < \mu_N$ with the property that $[0,1] = U'_{\mu_1} \cup \ldots \cup U'_{\mu_N}$. In view of Lemma 4.1, we will assume that $\mu_1 = 0$ and $\mu_N = 1$, with $\eta_{\mu_1} = \eta_{\mu_N} = \eta$. Since the interval $[0,1]$ is connected, we may choose $\mu_{j+\frac{1}{2}}$ for $j = 1, \ldots, N-1$ that satisfy $\mu_{j+\frac{1}{2}} \in U'_{\mu_j} \cap U'_{\mu_{j+1}}$. Using Lemma 4.1 we may compute

$$\begin{array}{lcl} n^{\sharp}_{\Gamma(1)}(\eta) - n^{\sharp}_{\Gamma(0)}(\eta) & = & \sum_{j=1}^{N-1} \#\{z \in \mathbb{C} \mid \det \Delta_{\Gamma(\mu_{j+\frac{1}{2}})}(z) = 0 \text{ and } \eta_{\mu_j} < \operatorname{Re} z < \eta_{\mu_{j+1}}\} \\ & & - \sum_{j=1}^{N-1} \#\{z \in \mathbb{C} \mid \det \Delta_{\Gamma(\mu_{j+\frac{1}{2}})}(z) = 0 \text{ and } \eta_{\mu_{j+1}} < \operatorname{Re} z < \eta_{\mu_j}\}, \end{array} \tag{4.32}$$

in which each root is counted according to its multiplicity. The formula (2.20) can now be easily verified. □

5 Functional algebraic equations of mixed type

In this section, we set out to prove Theorem 2.6. To this end, we will study the algebraic equation

$$0 = M \operatorname{ev}_{\xi} x \tag{5.1}$$

for some bounded linear operator $M : C([r_{\min}, r_{\max}], \mathbb{C}^n) \to \mathbb{C}^n$. Let us first introduce the exponentially weighted space

$$BC^{+}_{\eta} \quad := \quad \{x \in C\big([0,\infty), \mathbb{C}^n\big) \quad | \quad \sup_{\xi \geq 0} e^{-\eta \xi} \, |x(\xi)| < \infty\}. \tag{5.2}$$

In addition, let us recall the space $\text{NBV}([r_{\min}, r_{\max}], \mathbb{C}^{n\times n})$ that contains all $\mathbb{C}^{n\times n}$-valued functions μ that are defined on the interval $[r_{\min}, r_{\max}]$, are right-continuous on the interval $(r_{\min}, r_{\max})$, have bounded total variation and have $\mu(r_{\min}) = 0$.

As a consequence of the Riesz representation theorem, there exists a unique

$$\mu \in \text{NBV}([r_{\min}, r_{\max}], \mathbb{C}^{n\times n}) \tag{5.3}$$

for which the representation

$$M\phi = \int_{r_{\min}}^{r_{\max}} d\mu(\sigma)\phi(\sigma) \tag{5.4}$$

holds for all $\phi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$.

Throughout this section, we will assume that for some integer $\ell \geq 1$, the function μ appearing in (5.4) can be embedded into a sequence

$$\mu_i \in \text{NBV}([r_{\min}, r_{\max}], \mathbb{C}^{n\times n}), \qquad i = 1, \dots, \ell, \tag{5.5}$$

that has $\mu_\ell = \mu$ and satisfies the following properties.

(hμ1) For any integer $1 \leq i \leq \ell - 1$ and $\sigma \in [r_{\min}, r_{\max}]$, we have

$$\mu_i(\sigma) = -D\mu_{i+1}(\sigma). \tag{5.6}$$

(hμ2) For any integer $1 \leq i \leq \ell - 1$, we have $\mu_i(r_{\max}) = 0$.

(hμ3) There exists $\zeta \in \text{NBV}([r_{\min}, r_{\max}], \mathbb{C}^{n\times n})$ for which

$$\mu_1(\sigma) \quad = \quad -H(\sigma) + \textstyle\int_{r_{\min}}^{\sigma} \zeta(\tau)d\tau, \tag{5.7}$$

with $H(\sigma) = 1$ for all $\sigma \geq 0$ and $H(\sigma) = 0$ for all $\sigma < 0$.

We remark that [15, Prop. 3.1] shows that, up to a multiplicative constant, the linear operator M satisfies these criteria if and only if M satisfies the condition (HM) appearing in §2.

Using the function ζ appearing in (hμ3), we introduce the function $\mu_* \in \text{NBV}([r_{\min}, r_{\max}], \mathbb{C}^{n\times n})$ that is given by

$$\mu_*(\sigma) = -\zeta(\sigma) + \zeta(r_{\max})H(\sigma - r_{\max}) \tag{5.8}$$

and consider the associated MFDE

$$x'(\xi) = L\,\text{ev}_\xi x := \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)x(\xi + \sigma) = \zeta(r_{\max})x(\xi + r_{\max}) - \int_{r_{\min}}^{r_{\max}} d\zeta(\sigma)x(\xi + \sigma). \tag{5.9}$$

Introducing the characteristic matrices

$$\delta_i(z) = -\int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma)e^{z\sigma} \tag{5.10}$$

together with

$$\Delta_L(z) = z - \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)e^{z\sigma} = z - e^{r_{\max}z}\zeta(r_{\max}) + \int_{r_{\min}}^{r_{\max}} d\zeta(\sigma)e^{z\sigma}, \tag{5.11}$$

we can clarify the relationship between the different measures that we have introduced.

Lemma 5.1. *For any integer $1 \le i \le \ell$, we have the identity*

$$z^i \delta_i(z) = \Delta_L(z). \tag{5.12}$$

Proof. Using (Hμ3), we see that

$$\delta_1(z) = 1 - \int_{r_{\min}}^{r_{\max}} \zeta(\sigma) e^{z\sigma} d\sigma. \tag{5.13}$$

Integrating by parts, we compute

$$\int_{r_{\min}}^{r_{\max}} e^{z\sigma} d\zeta(\sigma) = \zeta(r_{\max}) e^{r_{\max} z} - z \int_{r_{\min}}^{r_{\max}} e^{z\sigma} \zeta(\sigma) d\sigma, \tag{5.14}$$

which in combination with (5.11) establishes the claim for $i = 1$. If $1 < i \le \ell$, a further integration by parts using (hμ1) and (hμ2) yields

$$\begin{array}{lcl} \delta_{i-1}(z) & = & -\int_{r_{\min}}^{r_{\max}} d\mu_{i-1}(\sigma) e^{z\sigma} = z \int_{r_{\min}}^{r_{\max}} \mu_{i-1}(\sigma) e^{z\sigma} d\sigma = -z \int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma) e^{z\sigma} \\ & = & z \delta_i(z), \end{array} \tag{5.15}$$

which completes the proof. □

Lemma 5.2. *Consider any $\phi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$. We have the identity*

$$\begin{array}{lcl} z \int_{r_{\min}}^{r_{\max}} d\mu_1(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau & = & \phi(0) + \int_{r_{\min}}^{r_{\max}} d\mu_1(\sigma) \phi(\sigma) \\ & & \quad + \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau. \end{array} \tag{5.16}$$

In addition, for any integer $1 < i \le \ell$ we have

$$\begin{array}{lcl} z \int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau & = & \int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma) \phi(\sigma) \\ & & \quad + \int_{r_{\min}}^{r_{\max}} d\mu_{i-1}(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau. \end{array} \tag{5.17}$$

Proof. Setting out to establish (5.16), we observe that

$$\int_{r_{\min}}^{r_{\max}} d\mu_1(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau = \int_{r_{\min}}^{r_{\max}} \zeta(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau d\sigma. \tag{5.18}$$

An integration by parts shows that

$$\begin{array}{lcl} \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau & = & -\int_{r_{\min}}^{r_{\max}} \zeta(\sigma) \phi(\sigma) d\sigma \\ & & + z \int_{r_{\min}}^{r_{\max}} \zeta(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau d\sigma. \end{array} \tag{5.19}$$

Noticing that

$$\int_{r_{\min}}^{r_{\max}} d\mu_1(\sigma) \phi(\sigma) = \int_{r_{\min}}^{r_{\max}} \zeta(\sigma) \phi(\sigma) d\sigma - \phi(0) \tag{5.20}$$

completes the proof of (5.16).

For $1 < i \le \ell$, we may use the boundary condition $\mu_{i-1}(r_{\max}) = 0$ to compute

$$\begin{array}{lcl} \int_{r_{\min}}^{r_{\max}} d\mu_{i-1}(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau & = & -z \int_{r_{\min}}^{r_{\max}} \mu_{i-1}(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau d\sigma \\ & & \quad + \int_{r_{\min}}^{r_{\max}} \mu_{i-1}(\sigma) \phi(\sigma) d\sigma \\ & = & z \int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \phi(\tau) d\tau \\ & & \quad - \int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma) \phi(\sigma), \end{array} \tag{5.21}$$

which establishes (5.17). □

We remark that repeated application of Lemma 5.2 yields the identity

$$\begin{array}{rcl} \phi(0) + \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau}\phi(\tau)d\tau & = & z^\ell \int_{r_{\min}}^{r_{\max}} d\mu_\ell(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau}\phi(\tau)d\tau \\ & & - \sum_{i=0}^{\ell-1} z^i \int_{r_{\min}}^{r_{\max}} d\mu_{i+1}(\sigma)\phi(\sigma). \end{array} \tag{5.22}$$

This identity can be used to study the relation between the algebraic equation (5.1) and the differential equation (5.9).

Lemma 5.3. *Consider any $\eta \in \mathbb{R}$ and a function $x \in BC_\eta^\oplus$. Then x solves the algebraic equation (5.1) for all $\xi \geq 0$ if and only if x solves the differential equation (5.9) for $\xi \geq 0$ and in addition satisfies the identities*

$$\int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma)x(\sigma) = 0 \tag{5.23}$$

for all integers $1 \leq i \leq \ell$.

Proof. Let us consider any $x \in BC_\eta^\oplus$ and define the function $v \in BC_\eta^+$ via

$$v(\xi) = -\int_{r_{\min}}^{r_{\max}} d\mu_\ell(\sigma)x(\xi + \sigma). \tag{5.24}$$

For any z with $\operatorname{Re} z > \eta$, the Laplace transform $\widetilde{v}(z)$ is well-defined and given by

$$\begin{array}{rcl} \widetilde{v}(z) & = & \int_0^\infty e^{-z\xi} v(\xi) d\xi = -\int_{r_{\min}}^{r_{\max}} \int_0^\infty d\mu_\ell(\sigma)x(\xi+\sigma)d\xi \\ & = & -\int_{r_{\min}}^{r_{\max}} d\mu_\ell(\sigma)e^{z\sigma}\big(\widetilde{x}(z) + \int_\sigma^0 e^{-z\tau}x(\tau)d\tau\big) \\ & = & \delta_\ell(z)\widetilde{x}(z) - \int_{r_{\min}}^{r_{\max}} d\mu_\ell(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau, \end{array} \tag{5.25}$$

in which we have used Fubini's theorem to change the order of integration. Similarly, if $x \in BC_\eta^\oplus$ and $x' \in BC_\eta^+$, then we may write

$$w(\xi) = x'(\xi) - \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)x(\xi + \sigma) \tag{5.26}$$

and compute the Laplace transform $\widetilde{w}(z)$ for any z with $\operatorname{Re} z > \eta$. A similar computation as above and an application of (5.22) yields

$$\begin{array}{rcl} \widetilde{w}(z) & = & \Delta_L(z)\widetilde{x}(z) - x(0) - \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau \\ & = & \Delta_L(z)\widetilde{x}(z) - z^\ell \int_{r_{\min}}^{r_{\max}} d\mu_\ell(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau + \sum_{i=0}^{\ell-1} z^i \int_{r_{\min}}^{r_{\max}} d\mu_{i+1}(\sigma)x(\sigma). \end{array} \tag{5.27}$$

Now, suppose that $x \in BC_\eta^\oplus$ satisfies the algebraic equation (5.1). The identities (5.23) can be easily verified by differentiating (5.1) and subsequently using integration by parts together with the boundary condition (hμ3). Using [15, Prop 4.2(iii)], we may conclude that $x' \in BC_\eta^+$. This means that the Laplace transform $\widetilde{w}(z)$ is well-defined for $\operatorname{Re} z > \eta$. Comparing (5.25) and (5.27), noting that $\widetilde{v}(z) = 0$ and using (5.23), we see that also $\widetilde{w}(z) = 0$, which implies that x satisfies the differential equation (5.9). The converse statement can be easily established by inspection of (5.23), (5.25) and (5.27). ☐

In order to establish Theorem 2.6, we will need to improve our understanding of the criteria (5.23). To do this, we will use the spectral projection $\Pi_{\mathrm{sp}} \in \mathcal{L}\big(C([r_{\min}, r_{\max}], \mathbb{C}^n)\big)$ that is associated to the root $z = 0$ of the characteristic equation $\det \Delta_L(z) = 0$. We recall from [17, §4] that this spectral projection is given by

$$[\Pi_{\mathrm{sp}}\phi](\theta) = \operatorname{Res}_{z=0} e^{z\theta}\Delta_L(z)^{-1}\big[\phi(0) + \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}\phi(\tau)d\tau\big]. \tag{5.28}$$

This projection can be used to characterize the difference between the two spaces $Q_L(\pm\epsilon)$.

Lemma 5.4. *Suppose that the characteristic equation* $\det \Delta_L(z) = 0$ *admits no roots on the imaginary axis besides* $z = 0$*. Then for any sufficiently small* $\epsilon > 0$*, we have the characterization*

$$Q_L(-\epsilon) = \{\phi \in Q_L(\epsilon) \mid \Pi_{\mathrm{sp}}\phi = 0\}, \tag{5.29}$$

together with the direct sum decomposition

$$Q_L(\epsilon) = Q_L(-\epsilon) \oplus \mathrm{Range}(\Pi_{\mathrm{sp}}). \tag{5.30}$$

Proof. Since $Q_L(-\epsilon)$ is closed and $\mathrm{Range}(\Pi_{\mathrm{sp}})$ is a finite dimensional subspace of $Q_L(\epsilon)$ that intersects trivially with $Q_L(-\epsilon)$, it suffices to show that (5.29) holds. Let us therefore consider any $x \in \mathfrak{Q}_L(\epsilon)$. Using (5.27) and applying the inverse Laplace transform, we find that x satisfies

$$x(\xi) \quad = \quad \tfrac{1}{2\pi i}\int_{2\epsilon - i\infty}^{2\epsilon + i\infty} e^{z\xi}\Delta_L(z)^{-1}\Big[x(0) + \int_{r_{\min}}^{r_{\max}} d\mu_*(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}x(\tau)d\tau\Big]. \tag{5.31}$$

Let us suppose that $\Pi_{\mathrm{sp}}\mathrm{ev}_0 x = 0$. Comparing (5.31) with (5.28), we see that the residue at zero vanishes, allowing the integration contour in (5.31) to be shifted to the line $-2\epsilon + i\mathbb{R}$. Arguing similarly as in the proof of [7, Lem. I.5.3.], we may now conclude that x decays exponentially, which implies $x \in \mathfrak{Q}_L(-\epsilon)$.

On the other hand, suppose that $\phi \in Q_L(-\epsilon)$ satisfies $\Pi_{\mathrm{sp}}\phi = \psi \neq 0$. Since ψ is a polynomial, we see that $\phi - \psi \in Q_L(\epsilon) \setminus Q_L(-\epsilon)$ and by construction $\Pi_{\mathrm{sp}}(\phi - \psi) = 0$. This contradicts our conclusion above. □

Comparing the characterization (5.29) with the identity (2.31) that we wish to establish, we see that it now suffices to relate the spectral projection Π_{sp} to the integral criteria (5.23). This is clarified in the following result.

Lemma 5.5. *Suppose that* $\det \delta_M(0) \neq 0$*. Then any* $\phi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ *satisfies* $\Pi_{\mathrm{sp}}\phi = 0$ *if and only if*

$$\int_{r_{\min}}^{r_{\max}} d\mu_i(\sigma)\phi(\sigma) = 0 \tag{5.32}$$

holds for all integers $1 \leq i \leq \ell$*.*

Proof. Since $\Delta_L(z) = z^\ell \delta_M(z)$, we find that $\Delta_L(z)^{-1}$ can be written as

$$\Delta_*(z)^{-1} = z^{-\ell}(A_0 + A_1 z + \ldots + A_{\ell-1}z^{\ell-1}) + O(1) \tag{5.33}$$

as $z \to 0$, with $\det A_0 \neq 0$. Inspecting the representation (5.28) for the spectral projection Π_{sp} and applying the identity (5.22), we find

$$\begin{array}{rcl} -[\Pi_{\mathrm{sp}}\phi](\theta) & = & \mathrm{Res}_{z=0}\, z^{-\ell}[\sum_{j=0}^{\ell-1}\frac{1}{j!}z^j\theta^j][\sum_{k=0}^{\ell-1}A_k z^k][\sum_{m=0}^{\ell-1} z^m \int_{r_{\min}}^{r_{\max}} d\mu_{m+1}(\sigma)\phi(\sigma)] \\ & = & \sum_{j=0}^{\ell-1} b_j \theta^j \end{array} \tag{5.34}$$

for some set $\{b_0, \ldots, b_{\ell-1}\} \subset \mathbb{C}^n$. Matching powers shows that for any integer $0 \leq i \leq \ell - 1$, we have

$$b_i = \sum_{k=0}^{\ell-1-i} A_k \int_{r_{\min}}^{r_{\max}} d\mu_{\ell-i-k}(\sigma)\phi(\sigma). \tag{5.35}$$

In view of the fact that A_0 is invertible, the condition $b_0 = \ldots = b_{\ell-1} = 0$ is equivalent to the requirement that (5.32) holds for all integers $1 \leq i \leq \ell$, which completes the proof. □

Proof of Theorem 2.6. For $\eta = 0$, the statement follows by combining Lemma's 5.3, 5.4 and 5.5. The case $\eta \neq 0$ can be treated by applying exponential shifts to the system (2.23). □

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