Lin's Method and Homoclinic Bifurcations for Functional Differential Equations of Mixed Type

H. J. Hupkes [a,*], S. M. Verduyn Lunel [a]

[a] *Mathematical Insitute of the University of Leiden*

Abstract

We extend Lin's method for use in the setting of parameter-dependent nonlinear functional differential equations of mixed type (MFDEs). We show that the presence of M-homoclinic and M-periodic solutions that bifurcate from a prescribed homoclinic connection can be detected by studying a finite dimensional bifurcation equation. As an application, we describe the codimension two orbit-flip bifurcation in the setting of MFDEs.

Key words: mixed type functional differential equation, homoclinic bifurcation, Lin's method, orbit-flip bifurcation, finite dimensional reduction, exponential dichotomies, advanced and retarded arguments.

1 Introduction

The main purpose of this paper is to provide a framework that facilitates the detection of solutions to a parameter-dependent nonlinear functional differential equation of mixed type

$$x'(\xi) = G(x_\xi, \mu), \tag{1.1}$$

that bifurcate from a prescribed homoclinic or heteroclinic connection. Here x is a continuous $\mathbb{C}^n$-valued function and for any $\xi \in \mathbb{R}$ the state $x_\xi \in C([r_{\min}, r_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $r_{\min} \le 0$ and $r_{\max} \ge 0$, hence the nonlinearity G may depend on advanced and retarded arguments simultaneously. The parameter μ is taken from an open subset of $\mathbb{R}^p$, for some integer $p \ge 1$.

The fact that travelling wave solutions to lattice differential equations are described by functional differential equations of mixed type (MFDEs) forms one of the primary motivations for this paper.

*Current address corresponding author:

H.J. Hupkes
Division of Applied Mathematics
Brown University
182 George Street
Providence, RI 02912
tel. 1-401-863-7422

Email address: hjhupkes@dam.brown.edu (H. J. Hupkes).

As exhibited in detail in [8], lattice differential equations have many modelling applications in a wide range of scientific disciplines. As a consequence they are attracting a considerable amount of interest, both from an applied as well as a theoretical perspective. One of the driving forces in these investigations is the desire to apply the powerful tools that are currently available for ODEs to the infinite dimensional setting of (1.1). The constructions in [10, 11] concerning finite dimensional center manifolds, which describe the behaviour of solutions to (1.1) in the vicinity of equilibria and periodic solutions, should be seen in this light.

In the present work we continue this approach, by studying solutions to (1.1) that remain orbitally close to a prescribed homoclinic or heteroclinic solution q that solves (1.1) at $\mu = \mu_0$. We will be particularly interested in the construction of M-homoclinic and M-periodic orbits, which loosely speaking wind around the principal orbit q exactly M times, before converging to an equilibrium or repeating their pattern. More precisely, we will fix a Poincaré section that intersects the trajectory of q at q_0 in a transverse fashion and study solutions that pass through this section M times. We will show that for μ sufficiently close to μ_0, one may construct solutions that satisfy these winding properties, up to M possible discontinuities that occur exactly at the Poincaré section. Moreover, our construction will force these jumps to be contained in some finite dimensional subset of this section. This crucial reduction allows us to search for M-homoclinic and M-periodic orbits by studying the roots of M finite dimensional bifurcation equations, that effectively measure the size of the jumps.

This construction is known as Lin's method and was originally developed by Lin [13] in order to study systems that depend upon a single parameter. Sandstede generalized the method in such a way that bifurcations with higher codimensions could also be incorporated [16]. Our approach here should be seen as a subsequent generalization of this latter framework to the infinite dimensional context of (1.1). In addition, we will show that the bifurcation equations that describe the size of the jumps have a similar asymptotic form as those derived for the ODE version of (1.1). This provides a bridge that will allow classical bifurcation results obtained for ODEs to be directly lifted to the mixed type functional differential equation (1.1).

We mention here that very recently Lin's method was used to study homoclinic solutions to a reversible lattice differential equation, in the neighbourhood of a prescribed symmetric homoclinic connection [3]. The approach in [3] however cannot be used to detect bifurcating periodic solutions.[1] In addition, the choice to use the Hilbert space $\mathbb{C}^n \times L^2([r_{\min}, r_{\max}], \mathbb{C}^n)$ as a state space for (1.1) causes the nonlinearity to have a domain and therefore requires the use of a proper functional-analytic setup. We prefer to avoid such complications and therefore choose to work with the traditional state space $C([r_{\min}, r_{\max}], \mathbb{C}^n)$. This will enable us to stay very close to the finite dimensional framework developed in [13, 16] and should considerably ease the application of our results.

Historically, the primary motivation for the work by Lin and Sandstede mentioned above, was the classification of the bifurcations that homoclinic solutions to generic ODEs with one or two parameters may undergo. In a sequence of papers, Shilnikov [17, 18, 19, 20, 21, 22] presented an alternative for generic ODE versions of (1.1) with $p = 1$. In particular, the ODE either admits precisely one branch of large-period periodic solution that bifurcates from the homoclinic orbit q for $\mu > \mu_0$ or $\mu < \mu_0$, or else admits symbolic dynamics for all μ sufficiently close to μ_0. The existence of the unique periodic orbit was generalized to semilinear parabolic PDEs and delay equations by Chow and Deng [1] using semigroup techniques. Sandstede lifted the result concerning the presence of symbolic dynamics to parabolic PDEs that have a sectorial linear part [16].

According to Yanagida [23], the generic non-resonant bifurcations of codimension two that a hyperbolic homoclinic solution to an ODE may undergo, are the inclination-flip and the orbit-flip bifurcations. The former of these has been analyzed by several authors [7, 12] using Lyapunov-Schmidt techniques, that unfortunately break down when studying the orbit-flip bifurcation. However, employing the adaptation of Lin's method discussed above, Sandstede obtained a general description of this bifurcation for ODEs in [16]. In Section 2 we will use our bridge to lift this result and characterize the orbit-flip bifurcation for (1.1).

[1] This restriction was lifted in a sequel [4] that appeared simultaneously with the present paper.

The first obstacle that needs to be overcome in any bifurcation analysis involving MFDEs, is that the linearized problems one encounters are ill-posed and therefore do not generate a semiflow. It is known that exponential dichotomies form a very powerful tool when dealing with ill-posed problems, since they split the state space into separate parts that do admit a semiflow. The existence of such exponential splittings for parameter-independent homogeneous linear MFDEs, was established independently and simultaneously by Verduyn Lunel and Mallet-Paret [15] on the one hand and Härterich and coworkers [6] on the other, using very different methods.

A second obstacle is that there is no immediate way to write down a variation-of-constants formula that solves inhomogeneous MFDEs. This is caused by the fact that the inhomogeneity will simply be a $\mathbb{C}^n$-valued function, while the projections associated to the exponential dichotomies act on the state space $C([r_{\min}, r_{\max}], \mathbb{C}^n)$. Such a difficulty was also encountered in the study of retarded differential equations, i.e. (1.1) with $r_{\max} = 0$. It was resolved by the development of so-called sun-star techniques [2], which allow both the system under consideration and its relevant spectral projections to be lifted to the appropriate extended state space $\mathbb{C}^n \times L^\infty([r_{\min}, 0], \mathbb{C}^n)$. Unfortunately, these constructions are based on a semigroup approach and therefore break down when $r_{\min} < 0 < r_{\max}$, due to the ill-posedness mentioned above. In view of this fact, a third obstacle arises when one wishes to study systems that depend on a parameter, since robustness theorems for exponential dichotomies are generally established using a variation-of-constants formula.

In previous work [9, 10, 11], the absence of a variation-of-constants formula was circumvented by utilizing variants of the Greens function that was constructed by Mallet-Paret for autonomous MFDEs [14]. Continuing in this spirit, we will use the Fredholm theory developed in [14] for nonautonomous MFDEs to construct inverses for inhomogeneous MFDEs on half-lines. By carefully combining these inverses with the exponential splittings developed in [15], we are able to construct exponential dichotomies for parameter-dependent MFDEs without using a variation-of-constants formula. In addition, this setup will allow us to obtain precise estimates on the speed at which the projections associated to these dichotomies approach the limiting spectral projections at $\pm\infty$. We will also be able to isolate the portion of the state space that corresponds to a specific eigenvalue of one of these spectral projections. These results can be found in Sections 3 to 5 and provide the machinery that we require to construct the bridge between ODEs and MFDEs.

In Section 2 we state our main results, which describe Lin's method in the setting of MFDEs and give an explicit expression for the leading order terms in the bifurcation equations. In addition, we characterize the orbit-flip bifurcation for MFDEs. In Section 6 we construct the candidate M-homoclinic and M-periodic orbits, that satisfy (1.1) up to M jumps. Our approach in that section broadly follows the presentation in [16], but we avoid the smooth coordinate changes that are used there, since these are often problematic in an infinite dimensional setting. Instead, these coordinate changes are only applied after the problem has been reduced to a finite dimensional one. Finally, in Sections 7 and 8 we obtain estimates on the size of the error that is made if one only considers the leading order terms when measuring the size of the M jumps.

2 Main Results

Consider for some integer $N \geq 0$ the general nonlinear functional differential equation of mixed type

$$x'(\xi) = G(x(\xi + r_0), \ldots, x(\xi + r_N), \mu) = G(x_\xi, \mu), \tag{2.1}$$

in which x should be seen as a mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some $n \geq 1$. The shifts $r_j \in \mathbb{R}$ may have either sign and we will assume that they are ordered as $r_0 < \ldots < r_N$, with $r_0 \leq 0$ and $r_N \geq 0$. Introducing $r_{\min} = r_0$ and $r_{\max} = r_N$, we write $X = C([r_{\min}, r_{\max}], \mathbb{C}^n)$ for the state space associated to (2.1). The state of a function x at $\xi \in \mathbb{R}$ will be denoted by $x_\xi \in X$ or alternatively $\mathrm{ev}_\xi x \in X$ and is defined by $x_\xi(\theta) = x(\xi + \theta)$ for $r_{\min} \leq \theta \leq r_{\max}$. The parameter μ is taken from an open subset $U \subset \mathbb{R}^p$ for some integer $p \geq 1$. For convenience, we will use both of the representations for G that were introduced in (2.1) interchangeably throughout the sequel, but the details should be clear from the context.

We will need the following assumptions on the nonlinearity G. We remark that the parameter-independence of the equilibria is not a real restriction, as this can always be achieved by means of a change of variables.

(HG) The nonlinearity $G : X \times U \to \mathbb{C}^n$ is C^{k+2} smooth for some integer $k \geq 2$. In addition, it admits D distinct equilibria $q^* \in \mathbb{C}^n$, which we label as $q^*_{(1)}$ through $q^*_{(D)}$. These equilibria do not depend on the parameter μ, i.e., we have $G(q^*_{(i)}, \mu) = 0$ for all $\mu \in U$ and all integers $1 \leq i \leq D$.

It is important to understand the linearizations of (2.1) around these equilibrium solutions. To this end, we define $L^{(i)}(\mu) = D_1 G(q^*_{(i)}, \mu)$ and consider the homogeneous linear MFDE

$$x'(\xi) = L^{(i)}(\mu) x_\xi = \sum_{j=0}^{N} A_j^{(i)}(\mu) x(\xi + r_j). \tag{2.2}$$

Associated to this linear MFDE one has the characteristic matrix

$$\Delta^{(i)}(z, \mu) = zI - L^{(i)}(\mu) e^{z\cdot} = zI - \sum_{j=0}^{N} A_j^{(i)}(\mu) e^{z r_j}. \tag{2.3}$$

We will need the following assumption on the linearizations, which basically states that all equilibria are hyperbolic.

(HL) For all integers $1 \leq i \leq D$ and all $\mu \in U$, the characteristic equation $\det \Delta^{(i)}(z, \mu) = 0$ admits no roots with $\operatorname{Re} z = 0$.

Now let us assume that for $\mu = \mu_0$, equation (2.1) has a heteroclinic solution q that connects the equilibria q^*_- and q^*_+. Inserting $x(\xi) = q(\xi) + v(\xi)$ into (2.1), we find the variational MFDE

$$v'(\xi) = D_1 G(q_\xi, \mu_0) v_\xi + R(\xi, v_\xi, \mu), \tag{2.4}$$

which is no longer autonomous. Associated to the linear part of this equation we define the operator $\Lambda : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ that is given by

$$[\Lambda v](\xi) = v'(\xi) - D_1 G(q_\xi, \mu_0) v_\xi = v'(\xi) - \sum_{j=0}^{N} A_j(\xi) v(\xi + r_j), \tag{2.5}$$

with $A_j(\xi) = D_j G\big(q(\xi + r_0), \ldots, q(\xi + r_N), \mu_0\big)$. It is possible to define an operator $\Lambda^* : W^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^n)$ that can be interpreted as an adjoint for Λ under suitable conditions. This adjoint is given by

$$[\Lambda^* w](\xi) = w'(\xi) + \sum_{j=0}^{N} A_j(\xi - r_j)^* w(\xi - r_j). \tag{2.6}$$

We will write $Y = C([-r_{\max}, -r_{\min}], \mathbb{C}^n)$ for the state space associated to the adjoint (2.6) and ev^*_ξ for the associated evaluation operator, which now maps into Y. The coupling between Λ and Λ^* is provided through the Hale inner product, which is given by

$$\langle \psi, \phi \rangle_\xi = \psi(0)^* \phi(0) - \sum_{j=0}^{N} \int_0^{r_j} \psi(\theta - r_j)^* A_j(\xi + \theta - r_j) \phi(\theta) d\theta, \tag{2.7}$$

for any $\phi \in X$ and $\psi \in Y$. The following condition on the operator Λ ensures that the Hale inner product is nondegenerate, in the sense that if $\langle \psi, \phi \rangle_\xi = 0$ for all $\psi \in Y$ and some $\phi \in X$, then $\phi = 0$. A proof for this fact can be found in [15].

(HB) The matrices $A_0(\xi)$ and $A_N(\xi)$ are nonsingular for every $\xi \in \mathbb{R}$.

Let $\mathcal{I} \subset \mathbb{R}$ be an interval. To state our results, we use the following family of Banach spaces, parametrized by $\eta \in \mathbb{R}$,

$$BC_\eta(\mathcal{I}, \mathbb{C}^n) \quad = \quad \Big\{ x \in C(\mathcal{I}, \mathbb{C}^n) \mid \|x\|_\eta := \sup_{\xi \in \mathcal{I}} e^{-\eta \xi} |x(\xi)| < \infty \Big\}. \tag{2.8}$$

We also need to consider the finite dimensional kernels

$$\begin{array}{lcl} \mathcal{K} & = & \big\{ b \in BC_0(\mathbb{R}, \mathbb{C}^n) \mid \Lambda b = 0 \big\}, \\ \mathcal{K}^* & = & \big\{ d \in BC_0(\mathbb{R}, \mathbb{C}^n) \mid \Lambda^* d = 0 \big\}. \end{array} \tag{2.9}$$

Let us write $X_0 = \{\phi \in X \mid \phi = b_0 \text{ for some } b \in \mathcal{K}\}$ and choose $\widehat{X}$ in such a way that $X = \widehat{X} \oplus X_0$. In addition, we write $Y_0 = \{\psi \in Y \mid \psi = d_0 \text{ for some } d \in \mathcal{K}^*\}$ and define the space

$$\widehat{X}_\perp = \big\{ \phi \in \widehat{X} \mid \langle \psi, \phi \rangle_0 = 0 \text{ for all } \psi \in Y_0 \big\}. \tag{2.10}$$

We note that $\widehat{X}_\perp \subset \widehat{X}$ is closed and of finite codimension, which allows us to fix a finite dimensional complement Γ and write $X = X_0 \oplus \widehat{X}_\perp \oplus \Gamma$.

Proposition 2.1. *Consider the nonlinear equation (2.1) and suppose that (HG), (HL) and (HB) are satisfied. There exists a small neighbourhood $U' \subset U$, with $\mu_0 \in U'$, a small constant $\epsilon > 0$ and two C^{k+1}-smooth maps $u^- : U' \to BC_{+\epsilon}\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ and $u^+ : U' \to BC_{-\epsilon}\big([r_{\min}, \infty), \mathbb{C}^n\big)$, such that the following properties are satisfied.*

(i) For any $\mu \in U'$, the function $x(\xi) = q(\xi) + u^-(\mu)(\xi)$ satisfies the nonlinear equation (2.1) for all $\xi \leq 0$. In addition, the function $x(\xi) = q(\xi) + u^+(\mu)(\xi)$ satisfies (2.1) for all $\xi \geq 0$.

(ii) For all $\mu \in U'$, we have the identities

$$\begin{array}{lcl} \mathrm{ev}_0 u^-(\mu) & \in & \widehat{X}_\perp \oplus \Gamma, \\ \mathrm{ev}_0 u^+(\mu) & \in & \widehat{X}_\perp \oplus \Gamma. \end{array} \tag{2.11}$$

(iii) For all $\mu \in U'$, we have $\xi^\infty(\mu) := \mathrm{ev}_0 u^-(\mu) - \mathrm{ev}_0 u^+(\mu) \in \Gamma$.

(iv) For any $d \in \mathcal{K}^*$, we have the Melnikov identity

$$D_\mu [\langle \mathrm{ev}_0^* d, \xi^\infty(\mu) \rangle_0]_{|\mu = \mu_0} = \int_{-\infty}^{\infty} d(\xi')^* D_2 G(q_{\xi'}, \mu_0) d\xi'. \tag{2.12}$$

These maps are locally unique, in the sense that there exists $\delta > 0$ such that any pair $(\widetilde{u}^+, \widetilde{u}^-)$ that satisfies (i) through (iii) for some $\mu \in U'$ and also has $\widetilde{u}^+ \in BC_0\big([r_{\min}, \infty), \mathbb{C}^n\big)$, $\widetilde{u}^- \in BC_0\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ and $\|\widetilde{u}^\pm\|_0 < \delta$, must satisfy $\widetilde{u}^+ = u^+(\mu)$ and $\widetilde{u}^- = u^-(\mu)$.

We remark that the condition (HB) ensures that the Hale inner product is nondegenerate, which means that the inner product appearing in (2.12) is a valid way of measuring the gap between the local stable and unstable manifolds of (2.1). If one is merely interested in studying heteroclinic orbits that bifurcate from a prescribed heteroclinic connection, then Proposition 2.1 already reduces this problem to a finite dimensional one. Indeed, item (iii) implies that one has to search for the roots of a C^{k+1}-smooth function defined on Γ.

For the purpose of this paper however, let us consider a family of heteroclinic connections $\{q_j\}_{j \in \mathcal{J}}$, in which $\mathcal{J} \subset \mathbb{Z}$ is a possibly infinite set of subsequent integers. We emphasize here that these connections need not be distinct, thus any heteroclinic connection can appear in the family an arbitrary number of times. We write $\mathcal{J}^* \subset \mathbb{Z} + \frac{1}{2}$ for the set of half-integers $\mathcal{J}^* = \{j \pm \frac{1}{2}\}_{j \in \mathcal{J}}$, that will be related to the boundary conditions that tie the connections together. In particular, we will assume that the family $\{q_j\}_{j\in\mathcal{J}}$ connects the equilibria $\{q^*_\ell\}_{\ell\in\mathcal{J}^*}$, i.e.,

$$\lim_{\xi\to\pm\infty} q_j(\xi) = q^*_{j\pm\frac{1}{2}}. \tag{2.13}$$

Our aim is to construct solutions x to (2.1) that subsequently intersect the Poincaré sections $\mathrm{ev}_0 q_j + \widehat{X}^{(j)}_\perp + \Gamma^{(j)}$ close to $\mathrm{ev}_0 q_j$ at prescribed times T_j. To this end, we look for solutions to (2.1) that can be written as

$$\begin{array}{rclr} x(T_j+\xi) &=& q_j(\xi) + u_j^-(\mu)(\xi) + v_j^-(\mu)(\xi), & \omega_j^- + r_{\min} \le \xi \le r_{\max}, \\ x(T_j+\xi) &=& q_j(\xi) + u_j^+(\mu)(\xi) + v_j^+(\mu)(\xi), & r_{\min} \le \xi \le \omega_j^+ + r_{\max}, \end{array} \tag{2.14}$$

in which we will take $\omega_j^+ = -\omega_{j+1}^- = \omega_{j+\frac{1}{2}}$, for some family $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ that has $T_{j+1} - T_j = 2\omega_{j+\frac{1}{2}}$, wherever this is defined. If $\mathcal{J}$ is finite, i.e., $\mathcal{J} = \{1,\ldots,M\}$, then we can supply boundary conditions by requiring either $\lim_{\xi\to-\infty} x(\xi) = q^*_{\frac{1}{2}}$ and $\lim_{\xi\to\infty} x(\xi) = q^*_{M+\frac{1}{2}}$ if we are looking for a heteroclinic connection or $\mathrm{ev}_{\omega_1^-} x = \mathrm{ev}_{\omega_M^+} x$ if we are interested in periodic orbits.

The main result of this paper shows that if the prescribed crossing times T_j are sufficiently far apart, the search for solutions x of the form (2.14) is equivalent to the search for roots of a smooth function defined on the collection of finite dimensional spaces $\{\Gamma^{(j)}\}_{j\in\mathcal{J}}$.

Theorem 2.2. *Consider the nonlinear equation (2.1) and suppose that (HG), (HL) and (HB) are satisfied. Furthermore, consider a family of heteroclinic connections $\{q_j\}_{j\in\mathcal{J}}$ that satisfies (2.13). There exists an $\Omega > 0$ and an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, such that for any family $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ that has $\omega_\ell \ge \Omega$ for all $\ell \in \mathcal{J}^*$, there exist two families of functions $v_j^- : U' \to C\big([\omega_j^- + r_{\min}, r_{\max}], \mathbb{C}^n\big)$ and $v_j^+ : U' \to C\big([r_{\min}, \omega_j^+ + r_{\max}], \mathbb{C}^n\big)$, defined for $j \in \mathcal{J}$, that satisfy the following properties.*

(i) For any $\mu \in U'$ and $j \in \mathcal{J}$, the function $x(\xi) = q_j(\xi) + u_j^-(\mu)(\xi) + v_j^-(\mu)(\xi)$ satisfies the nonlinear equation (2.1) for all $\omega_j^- \le \xi \le 0$. In addition, the function $x(\xi) = q_j(\xi) + u_j^+(\mu)(\xi) + v_j^+(\xi)$ satisfies (2.1) for all $0 \le \xi \le \omega_j^+$.

(ii) For any $\mu \in U'$ and any $j \in \mathcal{J}$, we have $\mathrm{ev}_0 v_j^-(\mu) \in \widehat{X}^{(j)}_\perp \oplus \Gamma^{(j)}$ and similarly $\mathrm{ev}_0 v_j^+(\mu) \in \widehat{X}^{(j)}_\perp \oplus \Gamma^{(j)}$.

(iii) For any $\mu \in U'$ and $j \in \mathcal{J}$, the following boundary conditions are satisfied,

$$\mathrm{ev}_{\omega_{j+1}^-} v_{j+1}^-(\mu) - \mathrm{ev}_{\omega_j^+} v_j^+(\mu) = \mathrm{ev}_{\omega_j^+}[q_j + u_j^+(\mu)] - \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu)]. \tag{2.15}$$

If the family $\mathcal{J}$ is finite with M elements and $-\infty < \omega_1^- = -\omega_M^+$, then v_{M+1}^- should be read as v_1^-. If however $\omega_1^- = -\infty$ and $\omega_M^+ = \infty$, then (2.15) holds for all $1 \le j < M$ and one has the additional limits

$$\lim_{\xi\to-\infty} v_1^-(\mu)(\xi) \;=\; 0, \qquad \lim_{\xi\to\infty} v_M^+(\mu)(\xi) \;=\; 0. \tag{2.16}$$

(iv) For any $\mu \in U'$ and any $j \in \mathcal{J}$, we have $\xi_j(\mu) \in \Gamma^{(j)}$, in which $\xi_j(\mu)$ denotes the gap $\mathrm{ev}_0[v_j^-(\mu) - v_j^+(\mu)]$.

The two families $\{v_j^\pm\}_{j\in\mathcal{J}}$ are locally unique in a sense similar to the one described in Proposition 2.1. In addition, these functions $v_j^\pm$ depend C^k-smoothly on μ, while the gaps ξ_j depend C^k-smoothly on the pair $(\mu, \{\omega_\ell\}_{\ell\in\mathcal{J}^*})$. Finally, for any $d \in \mathcal{K}^*$ and $j \in \mathcal{J}$, we can estimate $\xi_j(\mu)$ according to

$$\begin{array}{rcl} \langle \mathrm{ev}_0^* d, \xi_j(\mu)\rangle_0 &=& \langle \mathrm{ev}^*_{\omega_j^+} d, \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u_{j+1}^-(\mu) - q^*_{j+\frac{1}{2}}]\rangle_{\omega_j^+} \\ && -\, \langle \mathrm{ev}^*_{\omega_j^-} d, \mathrm{ev}_{\omega_{j-1}^+}[q_{j-1} + u_{j-1}^+(\mu) - q^*_{j-\frac{1}{2}}]\rangle_{\omega_j^-} + \mathcal{R}_j. \end{array} \tag{2.17}$$

The error term $\mathcal{R}_j$ enjoys the following estimate, for some positive constants C_1 and C_2,

$$\mathcal{R}_j \quad \leq \quad \|\mathrm{ev}_0^* d\| \, [C_1 \, |\mu - \mu_0| \, e^{-2\alpha\omega} + C_2 e^{-3\alpha\omega}]. \tag{2.18}$$

Here we have introduced $\omega = \min_{\ell \in \mathcal{J}^*}\{\omega_\ell\}$, while $\alpha > 0$ is sufficiently small to ensure that the characteristic equations $\det \Delta^{(i)} z = 0$ have no roots with $|\mathrm{Re}\, z| \leq \alpha$ for all $1 \leq i \leq D$.

We note here that sharper estimates for the remainder terms $\mathcal{R}_j$ can be found in Sections 7 and 8, where we also provide estimates on the derivatives of $\mathcal{R}_j$ with respect to μ and the family $\{\omega_\ell\}_{\ell \in \mathcal{J}^*}$. In combination with these estimates, Theorem 2.2 allows bifurcation problems for the infinite dimensional system (2.1) to be treated on a similar footing as bifurcation problems for ODEs.

The orbit-flip bifurcation

To illustrate the application range of Theorem 2.2, we lift a result obtained by Sandstede [16] that describes the homoclinic orbit-flip bifurcation for ODEs. We proceed by stating the assumptions on the system (2.1) that we will need.

(OF1) The nonlinearity G is C^{k+2}-smooth with $k \geq 2$. The parameter space U is two dimensional and contains the origin, i.e., $0 \in U \subset \mathbb{R}^2$. The nonlinear differential equation (2.1) has an equilibrium at $x = 0$ for all $\mu \in U$.

(OF2) There exists a homoclinic solution q to (2.1) at $\mu = 0$ that satisfies $\lim_{\xi \to \pm\infty} q(\xi) = 0$. The kernel $\mathcal{K} = \mathcal{K}(\Lambda) \subset BC_0(\mathbb{R}, \mathbb{C}^n)$ associated to the linearization (2.5) of the nonlinear equation (2.1) around this orbit q, is one dimensional and satisfies

$$\mathcal{K} = \mathrm{span}\big\{q'\big\}. \tag{2.19}$$

We remark here that the Fredholm theory developed by Mallet-Paret [14], which is recalled here in Section 3, now implies that the kernel $\mathcal{K}^* = \mathcal{K}(\Lambda^*)$ associated to the adjoint of the linearization (2.5) is also one dimensional. In particular, for some $d \in BC_0(\mathbb{R}, \mathbb{C}^n)$ we may write

$$\mathcal{K}^* = \mathrm{span}\big\{d\big\}. \tag{2.20}$$

(OF3) There exist constants $\eta_-^f < 0$ and $\eta_+^f > 0$, such that for every $\mu \in U$, the characteristic equation $\det \Delta(z, \mu) = 0$ associated to the equilibrium of (2.1) at $x = 0$ has precisely two eigenvalues $z = \lambda_\pm(\mu)$ in the strip $\eta_-^f \leq \mathrm{Re}\, z \leq \eta_+^f$. These eigenvalues are simple roots of the characteristic equation and there exist constants $\eta_\pm^s$ such that the following inequalities are satisfied for all $\mu \in U$,

$$\eta_-^f < \lambda_-(\mu) < \eta_-^s < 0 < \eta_+^s < \lambda_+(\mu) < \eta_+^f. \tag{2.21}$$

Throughout the sequel we will often use the shorthands $\lambda_\pm = \lambda_\pm(0)$. The spectral splitting in (OF3) ensures that we can decompose the state spaces X and Y as

$$X = \mathcal{M}_c(\mu) \oplus \mathcal{M}_{\lambda_-(\mu)} \oplus \mathcal{M}_{\lambda_+(\mu)}, \qquad Y = \mathcal{M}_c^* \oplus \mathcal{M}_{-\lambda_-}^* \oplus \mathcal{M}_{-\lambda_+}^*, \tag{2.22}$$

in which $\mathcal{M}_{\lambda_\pm(\mu)}$ are the one dimensional eigenspaces associated to the eigenvalues $\lambda_\pm(\mu)$ and $\mathcal{M}_c(\mu)$ is a closed complement, while the starred spaces are defined similarly. The spectral projections $\Pi_{\mathcal{M}_{\lambda_\pm(\mu)}}$ and $\Pi_{\mathcal{M}^*_{-\lambda_\pm}}$ onto these eigenspaces can be written in terms of the Hale inner product [5]. More precisely, there exist $\psi_\pm(\mu) \in Y$ and $\phi_\pm(\mu) \in X$ such that

$$\Pi_{\mathcal{M}_{\lambda_\pm(\mu)}} \phi = \langle \psi_\pm(\mu), \phi \rangle_{\infty,\mu} \phi_\pm(\mu), \qquad \Pi_{\mathcal{M}^*_{-\lambda_\pm}} \psi = \langle \psi, \phi_\pm \rangle_{\infty,\mu} \psi_\pm, \tag{2.23}$$

again with the shorthands $\psi_\pm = \psi_\pm(0)$ and $\phi_\pm = \phi_\pm(0)$. Let us now consider the functions $u^\pm(\mu)$ introduced in Proposition 2.1, together with the jump $\xi^\infty(\mu)$. We also need to introduce the functions $\Phi_\pm : U' \to \mathbb{R}$ given by

$$\Phi_\pm(\mu) \quad = \quad \lim_{\xi\to\pm\infty} e^{-\lambda_\mp(\mu)\xi}\langle \psi_\mp(\mu), \mathrm{ev}_\xi\big(q + u^\pm(\mu)\big)\rangle_{\pm\infty,\mu}. \tag{2.24}$$

In a similar fashion we define the scalars

$$\Phi^*_\pm \quad = \quad \lim_{\xi\to\pm\infty} e^{\lambda_\pm\xi}\langle \mathrm{ev}^*_\xi d, \phi_\pm\rangle_{\pm\infty,0}. \tag{2.25}$$

Using arguments very similar to those given in [14, Section 7], one may show that both $\Phi_\pm$ depend C^k-smoothly on μ.

(OF4) We have the identities $\Phi_+(0) = 0$, $\Phi_-(0) \neq 0$ and $\Phi^*_\pm \neq 0$. In particular, q approaches its limit in forward time at an exponential rate faster than η^f_-, but behaves generically as $\xi \to -\infty$, while d behaves generically at both $\pm\infty$.

(OF5) The Melnikov integral $\int_{-\infty}^{\infty} d(\xi')^* D_2G(q_{\xi'}, 0)d\xi' \in \mathbb{R}^2$ and the derivative $[D\Phi_+](0) \in \mathbb{R}^2$ are linearly independent.

This condition allows us to redefine the coordinates on the parameter space U to ensure that

$$\begin{array}{rcl} \mu_1 & = & \Phi_+(\mu_1, \mu_2), \\ \mu_2 & = & \langle d, \xi^\infty(\mu_1, \mu_2)\rangle_0. \end{array} \tag{2.26}$$

In the event that $\lambda_+ > -\eta^f_-$ we need to strengthen the condition (OF3) and give a more detailed description of the negative part of the spectrum associated to the limiting equation.

(OF6) There exist constants $\eta^{ff}_- < \eta^f_- < 0$ and $\eta^f_+ > 0$ such that for all $\mu \in U$, the characteristic equation $\det \Delta(z, \mu) = 0$ associated to the equilibrium of (2.1) at $x = 0$ has precisely three eigenvalues $z = \lambda_\pm(\mu)$ and $z = \lambda^f_-(\mu)$ in the strip $\eta^{ff}_- \le \mathrm{Re}\, z \le \eta^f_+$. These eigenvalues are simple roots of the characteristic equation and there exist constants $\eta^s_\pm$ such that the following inequalities are satisfied for all $\mu \in U$,

$$\eta^{ff}_- < \lambda^f_-(\mu) < \eta^f_- < \lambda_-(\mu) < \eta^s_- < 0 < \eta^s_+ < \lambda_+(\mu) < \eta^f_+. \tag{2.27}$$

Writing Φ^f_+ and Φ^{*f}_- for the quantities associated to this eigenvalue λ^f_- that are analogous to those defined for $\lambda_\pm$ in (2.24) and (2.25), we have $\Phi^f_+(0) \neq 0$ and $\Phi^{*f}_- \neq 0$.

After all these preparations, we are almost ready to apply Theorem 2.2 and describe the orbit-flip bifurcation for functional differential equations of mixed type. It merely remains to define the type of solutions to (2.1) in which we are interested. To this end, consider any pair of positive constants (δ, Ω), where δ should be seen as small and Ω as large. Let us consider a solution x to (2.1) that satisfies the limits $\lim_{\xi\to\pm\infty} x(\xi) = 0$. Suppose that there exist exactly M distinct values $\{\xi_j\}_{j=1}^M$ for which $\mathrm{ev}_{\xi_j} x \in \mathrm{ev}_0 q + \widehat{X}_\perp + \Gamma$, with $\|\mathrm{ev}_{\xi_j} x - \mathrm{ev}_0 q\| < \delta$. Suppose furthermore that x remains δ-close to q, in the sense that there exists a nondecreasing function $j_* : \mathbb{R} \to \{1, \ldots, M\}$ with $j_*(\xi_j) = j$ such that $\|\mathrm{ev}_\xi x - \mathrm{ev}_{\xi-\xi_{j_*(\xi)}} q\| < \delta$ for all $\xi \in \mathbb{R}$. Finally, suppose that for any pair $1 \le j_1, j_2 \le M$, we have $|\xi_{j_1} - \xi_{j_2}| > \Omega$. Then we will refer to x as a (δ, Ω, M)-homoclinic solution. Similarly, let us consider a periodic solution x to (2.1) with minimal period ω. If x also satisfies the conditions above, where the values ξ_j should now be interpreted modulo ω, then we will call x a (δ, Ω, M)-periodic solution.

Theorem 2.3. *Consider the nonlinear equation (2.1) and assume that the conditions (OF1) through (OF5) and (HB) are satisfied, with $\lambda_+ \neq -\lambda_-$. In the event that $\lambda_+ \ge -\eta^f_-$, assume furthermore that (OF6) is satisfied and that $\lambda_+ \neq -\lambda^f_-$. Then upon fixing $\delta > 0$ sufficiently small and $\Omega > 0$ sufficiently large, one of the following three alternatives must hold.*

(A) *(Homoclinic Continuation) We have $\lambda_+ < -\lambda_-$. For all sufficiently small pairs (μ_1, μ_2), with $\mu_2 > 0$, equation (2.1) admits precisely one $(\delta, \Omega, 1)$-periodic solution. For all sufficiently small $|\mu_1|$, there exists precisely one $(\delta, \Omega, 1)$-homoclinic solution to (2.1) with $\mu_2 = 0$. For all integers $M \geq 2$, there are no (δ, Ω, M)-periodic and (δ, Ω, M)-homoclinic solutions to (2.1).*

(B) *(Homoclinic Doubling) We have $-\lambda_- < \lambda_+ < -\eta_-^f$. Excluding the line $\mu_2 = 0$, there are two curves that extend from the origin in parameter space on which codimension one bifurcations occur. More precisely, there is a branch of $(\delta, \Omega, 2)$-homoclinic solutions that passes through the origin and a curve emanating from the origin at which a period-doubling bifurcation takes place, turning $(\delta, \Omega, 1)$-periodic solutions into $(\delta, \Omega, 2)$-periodic solutions.*

(C) *(Homoclinic Cascade) We have $-\lambda_-^f < \lambda_+$. For every $M \geq 1$ there is a branch of (δ, ω, M)-homoclinic solutions to (2.1) that emerges from the origin in parameter space. In addition, branches of codimension-one period-fold and period-doubling bifurcations emerge from the origin and there is an open wedge in parameter space in which (2.1) admits symbolic dynamics.*

We refer to [16] for a more graphic description of these three bifurcation scenarios.

3 Preliminaries

In this section we develop some preliminary results for the linear inhomogeneous system

$$x'(\xi) = L(\xi)x_\xi + f(\xi) = \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j) + f(\xi), \tag{3.1}$$

in which we take $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ and $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$. We will assume throughout this section that the complex $n \times n$ matrix valued functions A_j are continuous and that the shifts r_j are ordered according to $r_0 < \ldots < r_N$, again with $r_0 \leq 0$ and $r_N \geq 0$. Our main goal here is to develop a method to solve (3.1) on half-lines in weighted function spaces, which will allow us to construct exponential dichotomies for linear MFDEs in the sequel. To prepare for this, we state the Fredholm properties for (3.1) that were obtained by Mallet-Paret [14] and show how (3.1) transforms under exponential shifts.

The system (3.1) is said to be asymptotically hyperbolic if the limits $A_j^\pm = \lim_{\xi \to \pm\infty} A_j(\xi)$ exist for all integers $0 \leq j \leq N$, while the characteristic equations $\det \Delta^\pm(z) = 0$ associated to these limiting equations do not have any roots on the imaginary axis. Here we have defined

$$\Delta^\pm(z) = zI - \sum_{j=0}^{N} A_j^\pm e^{z r_j}. \tag{3.2}$$

We recall the linear operator $\Lambda : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ associated to (3.1) that is given by

$$[\Lambda x](\xi) = x'(\xi) - \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j), \tag{3.3}$$

together with the formal adjoint $\Lambda^* : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$[\Lambda^* y](\xi) = y'(\xi) + \sum_{j=0}^{N} A_j(\xi - r_j)^* y(\xi - r_j). \tag{3.4}$$

The following important result, that describes the relation between the Fredholm operators Λ and Λ^*, is due to Mallet-Paret and can be found in [14].

Theorem 3.1. *Assume that (3.1) is asymptotically hyperbolic. Then both Λ and Λ^* are Fredholm operators, with Fredholm indices given by*

$$\mathrm{ind}(\Lambda) = -\mathrm{ind}(\Lambda^*) = \dim \mathcal{K}(\Lambda) - \dim \mathcal{K}(\Lambda^*). \tag{3.5}$$

Every element in the kernels $\mathcal{K}(\Lambda)$ and $\mathcal{K}(\Lambda^*)$ decays exponentially as $\xi \to \pm\infty$, while the relation between Λ and Λ^* is given by the following identities,

$$\begin{array}{lcl} \mathcal{R}(\Lambda) & = & \Big\{ h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} d(\xi')^* h(\xi') d\xi' = 0 \textit{ for every } d \in \mathcal{K}(\Lambda^*) \Big\}, \\ \mathcal{R}(\Lambda^*) & = & \Big\{ h \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} b(\xi')^* h(\xi') d\xi' = 0 \textit{ for every } b \in \mathcal{K}(\Lambda) \Big\}. \end{array} \tag{3.6}$$

In the special case that the functions $A_j(\xi)$ do not depend on ξ, the operator Λ is invertible and there exists a Greens function $G : \mathbb{R} \to \mathbb{C}^{n \times n}$ such that

$$[\Lambda^{-1} f](\xi) = \int_{-\infty}^{\infty} G(\xi - \xi') f(\xi') d\xi'. \tag{3.7}$$

The Fourier transform of the function G is given by $\widehat{G}(\eta) = \Delta^{-1}(i\eta)$, which implies that G decays exponentially at both $\pm\infty$.

For our purposes in this paper, we will need to study the action of Λ on function spaces with exponentially weighted norms. We therefore introduce the notation $e_\nu f = e^{\nu \cdot} f(\cdot)$ for any $\nu \in \mathbb{R}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n)$. In addition, we introduce the family of exponentially weighted spaces

$$\begin{array}{lcl} L^\infty_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in L^\infty(\mathbb{R}, \mathbb{C}^n) \right\}, \\ W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) & = & \left\{ x \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^n) \mid e_{-\eta} x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \right\}, \end{array} \tag{3.8}$$

with norms given by $\|x\|_{L^\infty_\eta} = \|e_{-\eta} x\|_{L^\infty}$ and similarly $\|x\|_{W^{1,\infty}_\eta} = \|e_{-\eta} x\|_{W^{1,\infty}}$.

To study how Λ behaves under the action of e_η, let us define the shifted operator $\Lambda_\eta : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ that acts as

$$[\Lambda_\eta x](\xi) = x'(\xi) - \eta x(\xi) - \sum_{j=0}^{N} A_j(\xi) e^{-\eta r_j} x(\xi + r_j). \tag{3.9}$$

In addition, we write $\Delta^\pm_\eta$ for the characteristic equations associated to the shifted operator Λ_η. It is not hard to check that

$$\begin{array}{lcl} \Lambda e_\eta x & = & e_\eta \Lambda_{-\eta} x, \\ \Delta^\pm_\eta(z) & = & \Delta^\pm(z - \eta). \end{array} \tag{3.10}$$

Using the definition of the adjoint Λ^* in (3.4), one may also easily conclude that we have the identity

$$(\Lambda_\eta)^* = (\Lambda^*)_{-\eta}. \tag{3.11}$$

In this fashion we can define the Fredholm operator $\Lambda_{(\eta)} : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ by means of

$$\Lambda_{(\eta)} = e_\eta \circ \Lambda_{-\eta} \circ e_{-\eta}. \tag{3.12}$$

In a similar fashion we define $\Lambda^*_{(\eta)} : W^{1,\infty}_\eta(\mathbb{R}, \mathbb{C}^n) \to L^\infty_\eta(\mathbb{R}, \mathbb{C}^n)$ by

$$\Lambda^*_{(\eta)} = e_\eta \circ (\Lambda^*)_{-\eta} \circ e_{-\eta}. \tag{3.13}$$

The next proposition provides the appropriate generalization of Theorem 3.1.

Proposition 3.2. *Assume that (3.1) is asymptotically autonomous and in addition that the characteristic equations* $\det \Delta^{\pm}(z) = 0$ *have no roots with* $\operatorname{Re} z = \eta$*. Then both* $\Lambda_{(\eta)} : W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ *and* $\Lambda^*_{(-\eta)} : W^{1,\infty}_{-\eta}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}_{-\eta}(\mathbb{R}, \mathbb{C}^n)$ *are Fredholm operators, with*

$$\operatorname{ind}(\Lambda_{(\eta)}) = -\operatorname{ind}(\Lambda^*_{(-\eta)}) = \dim \mathcal{K}(\Lambda_{(\eta)}) - \dim \mathcal{K}(\Lambda^*_{(-\eta)}). \tag{3.14}$$

For every element b *in* $\mathcal{K}(\Lambda_{(\eta)})$*, the function* $e_{-\eta}b$ *decays exponentially at both* $\pm\infty$*, while for any* d *in* $\mathcal{K}(\Lambda^*_{(-\eta)})$ *we have that* $e_{\eta}d$ *decays exponentially at both* $\pm\infty$*. The relation between* $\Lambda_{(\eta)}$ *and* $\Lambda^*_{(-\eta)}$ *is given by the following identities,*

$$\begin{array}{rcl} \mathcal{R}(\Lambda_{(\eta)}) & = & \left\{ h \in L^{\infty}(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} d(\xi')^* h(\xi') = 0 \text{ for every } d \in \mathcal{K}(\Lambda^*_{(-\eta)}) \right\}, \\ \mathcal{R}(\Lambda^*_{(-\eta)}) & = & \left\{ h \in L^{\infty}(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} b(\xi')^* h(\xi') = 0 \text{ for every } b \in \mathcal{K}(\Lambda_{(\eta)}) \right\}. \end{array} \tag{3.15}$$

Proof. The result follows using Theorem 3.1 and the identities

$$\begin{array}{rcl} \mathcal{K}(\Lambda_{(\eta)}) & = & e_{\eta}\mathcal{K}(\Lambda_{-\eta}), \\ \mathcal{K}(\Lambda^*_{(-\eta)}) & = & e_{-\eta}\mathcal{K}((\Lambda^*)_{\eta}) = e_{-\eta}\mathcal{K}((\Lambda_{-\eta})^*), \\ \mathcal{R}(\Lambda_{(\eta)}) & = & e_{\eta}\mathcal{R}(\Lambda_{-\eta}), \\ \mathcal{R}(\Lambda^*_{(-\eta)}) & = & e_{-\eta}\mathcal{R}((\Lambda^*)_{\eta}) = e_{-\eta}\mathcal{R}((\Lambda_{-\eta})^*), \end{array} \tag{3.16}$$

together with the identities $\Delta^{\pm}_{-\eta}(z) = \Delta^{\pm}(z + \eta)$. □

We now introduce parameter dependence into our main linear equation (3.1). In particular, we study the system

$$x'(\xi) = L(\xi, \mu)x_{\xi} + f(\xi) = \sum_{j=0}^{N} A_j(\xi, \mu)x(\xi + r_j) + f(\xi), \tag{3.17}$$

in which the parameter μ is taken from an open set $U \subset \mathbb{R}^p$ for some $p \geq 1$. We write $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ for the parameter-dependent version of (3.3). Throughout the remainder of this section, we will assume that Λ depends C^k-smoothly on the parameter $\mu \in U$ for some integer $k \geq 0$.

We set out here to define a solution operator for (3.17) on half-lines that also depends smoothly on the parameter μ, in the neighbourhood of some fixed parameter $\mu_0 \in U$. To this end, let us introduce the shorthands $\mathcal{K} = \mathcal{K}\big(\Lambda(\mu_0)\big)$ and $\mathcal{R} = \mathcal{R}\big(\Lambda(\mu_0)\big)$. Consider two arbitrary complements $\mathcal{K}^{\perp}$ for $\mathcal{K}$ and $\mathcal{R}^{\perp}$ for $\mathcal{R}$, which allow us to write

$$W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \;=\; \mathcal{K} \oplus \mathcal{K}^{\perp}, \qquad L^{\infty}(\mathbb{R}, \mathbb{C}^n) \;=\; \mathcal{R} \oplus \mathcal{R}^{\perp}. \tag{3.18}$$

The projections associated to this splitting of $L^{\infty}(\mathbb{R}, \mathbb{C}^n)$ will be denoted by $\pi_{\mathcal{R}}$ and $\pi_{\mathcal{R}^{\perp}}$. Note that for μ sufficiently close to μ_0, we have that $\pi_{\mathcal{R}}\Lambda(\mu) : \mathcal{K}^{\perp} \to \mathcal{R}$ is invertible, with a C^k-smooth inverse $\mu \mapsto [\pi_{\mathcal{R}}\Lambda(\mu)]^{-1} \in \mathcal{L}(\mathcal{R}, \mathcal{K}^{\perp})$. Upon choosing a sufficiently small neighbourhood $U' \subset U$, with $\mu_0 \in U'$, we can hence define a C^k-smooth function $h : U' \to \mathcal{L}(\mathcal{K}, \mathcal{K}^{\perp})$ via

$$h(\mu)(b) = -[\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}\Lambda(\mu)b. \tag{3.19}$$

Observe first that we have $h(\mu_0) = 0$ by construction. In addition, this definition ensures that for $\mu \in U'$ the infinite dimensional problem to find $x \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)$ that solves $\Lambda(\mu)x = f$, is equivalent to the search for a solution $b \in \mathcal{K}$ of

$$\pi_{\mathcal{R}^{\perp}}[\Lambda(\mu)](b + h(\mu)b) = \pi_{\mathcal{R}^{\perp}}f - \pi_{\mathcal{R}^{\perp}}\Lambda(\mu)[\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}f. \tag{3.20}$$

This can be seen by substituting

$$x = [\pi_{\mathcal{R}}\Lambda(\mu)]^{-1}\pi_{\mathcal{R}}f + b + h(\mu)b. \tag{3.21}$$

These considerations allow us to define a quasi-inverse for Λ that solves (3.17) in the sense of the following result.

Proposition 3.3. *Consider the parameter-dependent inhomogeneous system (3.17) and fix a parameter $\mu_0 \in U$ for which (3.17) is asymptotically hyperbolic. Then there exists an open subset $U' \subset U$, with $\mu_0 \in U'$, together with a C^k-smooth function*

$$\mathcal{C} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), \mathcal{R}^\perp\big) \tag{3.22}$$

and a C^k-smooth quasi-inverse

$$\Lambda^{\mathrm{qinv}} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), W^{1,\infty}(\mathbb{R}, \mathbb{C}^n)\big), \tag{3.23}$$

such that the following properties hold.

(i) For all $\mu \in U'$ we have

$$\dim \mathcal{K}\big(\Lambda(\mu)\big) \leq \dim \mathcal{K}\big(\Lambda(\mu_0)\big). \tag{3.24}$$

(ii) For all $\mu \in U'$ and all $f \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ we have the identity

$$\Lambda(\mu)\Lambda^{\mathrm{qinv}}(\mu) f = f + \mathcal{C}(\mu) f. \tag{3.25}$$

In addition, the restriction of $\mathcal{C}(\mu_0)$ to the set $\mathcal{R}$ vanishes identically.

Proof. Item (i) can be confirmed by noting that

$$\dim \mathcal{K}(\Lambda(\mu)) = \dim \mathcal{K} - \operatorname{rank} \pi_{\mathcal{R}^\perp} \Lambda(\mu)[I + h(\mu)] \leq \dim \mathcal{K}. \tag{3.26}$$

To establish item (ii), we choose $\mathcal{C}$ and Λ^{qinv} according to

$$\begin{array}{lcl} \Lambda^{\mathrm{qinv}}(\mu) f & = & [\pi_{\mathcal{R}} \Lambda(\mu)]^{-1} \pi_{\mathcal{R}} f, \\ \mathcal{C}(\mu) f & = & -\pi_{\mathcal{R}^\perp} f + \pi_{\mathcal{R}^\perp} \Lambda(\mu) [\pi_{\mathcal{R}} \Lambda(\mu)]^{-1} \pi_{\mathcal{R}} f. \end{array} \tag{3.27}$$

A simple calculation is now sufficient to conclude the proof. □

In order to define a solution operator for (3.17) on half-lines, we will need to utilize the freedom we still have to choose the complements $\mathcal{K}^\perp$ and $\mathcal{R}^\perp$ in a special fashion. To do this, we will need to assume that condition (HB) holds, i.e., we demand that both $\det A_0(\xi, \mu_0)$ and $\det A_N(\xi, \mu_0)$ are non-zero for all $\xi \in \mathbb{R}$.

Lemma 3.4. *Consider the parameter-dependent linear system (3.17) and suppose that condition (HB) holds for this system at $\mu = \mu_0$, for some $\mu_0 \in U$. Write $n_d = \dim \mathcal{K}\big(\Lambda^*(\mu_0)\big)$ and choose a basis $\{d^i\}_{i=1}^{n_d}$ for $\mathcal{K}\big(\Lambda^*(\mu_0)\big)$. For any $\xi \in \mathbb{R}$ there exists a set of functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ such that for any pair of integers $1 \leq i, j \leq n_d$ we have*

$$\int_{-r_{\max}}^{-r_{\min}} d^i(\xi + \theta)^* \psi^j(\theta) d\theta = \delta_{ij}. \tag{3.28}$$

Proof. The explicit representation in (3.4) shows that the condition (HB) also holds for the linear system associated to the adjoint $\Lambda^*(\mu_0)$. This implies that any $d \in \mathcal{K}^*$ that has $\mathrm{ev}_\xi^* d = 0$, must satisfy $d = 0$. The set of elements $\{\mathrm{ev}_\xi^* d^i\}_{i=d}^{n_d} \subset Y$ is thus linearly independent. In particular, this means that the $n_d \times n_d$ matrix Z with entries $Z_{ij} = \big(\mathrm{ev}_\xi^* d^i, \mathrm{ev}_\xi^* d^j\big)$ is invertible, where $(,)$ denotes the integral inner product

$$(\psi, \phi) = \int_{-r_{\max}}^{-r_{\min}} \psi(\theta)^* \phi(\theta) d\theta. \tag{3.29}$$

For any integer $1 \le j \le n_d$ we now choose

$$\psi^j = \sum_{k=1}^{n_d} \mathrm{ev}_\xi^* d^k Z_{kj}^{-1}. \tag{3.30}$$

A simple calculation shows that indeed

$$(\mathrm{ev}_\xi^* d^i, \psi^j) = \sum_{k=1}^{n_d} (\mathrm{ev}_\xi^* d^i, \mathrm{ev}_\xi^* d^k) Z_{kj}^{-1} = \sum_{k=1}^{n_d} Z_{ik} Z_{kj}^{-1} = \delta_{ij}. \tag{3.31}$$

□

We will use Lemma 3.4 to explicitly construct a representation for $\pi_{\mathcal{R}}$ and $\pi_{\mathcal{R}^\perp}$. Indeed, let us write $r = r_{\max} - r_{\min}$ and fix an arbitrary $\xi_0 \le -4r$. In addition, for any integer $1 \le i \le n_d$ we let $g^i \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ denote the function that has $\mathrm{ev}_{\xi_0}^* g^i = \psi^i$, while $g^i(\xi') = 0$ for all $\xi' < \xi_0 - r_{\max}$ and $\xi' > \xi_0 - r_{\min}$. Here the functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ arise from an application of Lemma 3.4 with $\xi = \xi_0$. Since the set $\{g^i\}_{i=1}^{n_d}$ is linearly independent, we can now explicitly define the projection

$$\pi_{\mathcal{R}^\perp} f \quad = \quad \textstyle\sum_{i=1}^{n_d} [\int_{-\infty}^{\infty} d^i(\xi')^* f(\xi') d\xi'] g^i. \tag{3.32}$$

This enables us to define an inverse for $\Lambda(\mu)$ on the positive half-line. Indeed, consider the operator $\Lambda_+^{-1}(\mu) : L^\infty\big([0,\infty), \mathbb{C}^n\big) \to W^{1,\infty}\big([r_{\min}, \infty), \mathbb{C}^n\big)$ given by

$$\Lambda_+^{-1}(\mu) f = \Lambda^{\mathrm{qinv}}(\mu) E f, \tag{3.33}$$

in which $[Ef](\xi) = 0$ for all $\xi < 0$ and $[Ef](\xi) = f(\xi)$ for all $\xi \ge 0$. Since $g^i(\xi) = 0$ for all $\xi \ge 0$ and all integers $1 \le i \le n_d$, an application of (3.25) immediately implies that for all $\xi \ge 0$ we have

$$[\Lambda(\mu) \Lambda_+^{-1}(\mu) f](\xi) = f(\xi). \tag{3.34}$$

In a similar fashion an inverse $\Lambda_-^{-1}(\mu) : L^\infty\big((-\infty, 0], \mathbb{C}^n\big) \to W^{1,\infty}\big((-\infty, r_{\max}], \mathbb{C}^n\big)$ can be constructed for the negative half-line. Both these inverses depend C^k-smoothly on the parameter $\mu \in U'$.

4 Exponential Dichotomies

In this section we study exponential splittings for the homogeneous counterpart of the linear system (3.1), which we will write as

$$x'(\xi) = L(\xi) x_\xi = \sum_{j=0}^{N} A_j(\xi) x(\xi + r_j). \tag{4.1}$$

Throughout this entire section we will assume that the functions A_j are continuous. In addition, we will assume that (4.1) is asymptotically hyperbolic and that the condition (HB) holds.

We start by stating the main theorem which we set out to prove in this section. We remark that a similar result was previously obtained in a Hilbert space setting [6]. In addition, an exponential splitting in a Banach space setting can already be found in [15]. The construction developed there is summarized in Proposition 4.2 and provides exponential dichotomies that are defined on the full line. However, in order to use Lin's method we will also need to consider dichotomies that are defined on half-lines. In addition, the results in [15] do not allow us to control the limiting behaviour of the projections associated to the dichotomies as precisely as is needed here.

The results in this section provide these necessary extensions. The splittings obtained in this manner will allow the freedom that remains when solving the inhomogeneous system (3.1) to be controlled in a detailed fashion. In particular, they will facilitate the construction of stable and unstable manifolds for the nonlinear system (2.1) in Section 6.

Theorem 4.1. *Consider the linear system (4.1). There exist constants $K > 0$, $\alpha_S > 0$ and $\alpha_Q > 0$, such that for every $\xi \geq 0$ there is a splitting*

$$X = Q(\xi) \oplus S(\xi), \tag{4.2}$$

such that each $\phi \in Q(\xi)$ can be extended to a solution $E\phi \in C\big([\xi + r_{\min}, \infty), \mathbb{C}^n\big)$ of the homogeneous equation (4.1) on the interval $[\xi, \infty)$, while each $\psi \in S(\xi)$ can be extended to a function $E\psi \in C\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big)$ that satisfies the homogeneous equation (4.1) on the interval $[0, \xi]$. In addition, we have the exponential estimates

$$\begin{array}{lclll} |[E\phi](\xi')| & \leq & K e^{-\alpha_Q |\xi' - \xi|} \, \|\phi\| & \text{for every } \phi \in Q(\xi) & \text{and} \quad \xi' \geq \xi, \\ |[E\psi](\xi')| & \leq & K e^{-\alpha_S |\xi' - \xi|} \, \|\psi\| & \text{for every } \psi \in S(\xi) & \text{and} \quad 0 \leq \xi' \leq \xi. \end{array} \tag{4.3}$$

These spaces are invariant, in the sense that for any $0 \leq \xi' \leq \xi$ and any $\psi \in S(\xi)$, we have $\mathrm{ev}_{\xi'} E\psi \in S(\xi')$, together with a similar identity for $\phi \in Q(\xi)$. Finally, the projections $\Pi_{Q(\xi)}$ and $\Pi_{S(\xi)}$ depend continuously on $\xi \geq 0$ and there exists a constant C such that $\left\|\Pi_{Q(\xi)}\right\| \leq C$ and $\left\|\Pi_{S(\xi)}\right\| \leq C$ for all $\xi \geq 0$.

Throughout this section, we will follow the notation employed in [15]. For any $\xi \in \mathbb{R}$, we will consider the space $\mathcal{P}(\xi)$ that consists of all bounded solutions to (4.1) on the interval $(-\infty, \xi]$, together with the space $\mathcal{Q}(\xi)$ that consist of all bounded solutions to (4.1) on $[\xi, \infty)$. Notice that any bounded function b that satisfies (4.1) on the entire line will have both $b \in \mathcal{P}(\xi)$ and $b \in \mathcal{Q}(\xi)$. It is therefore convenient to introduce normalized spaces $\widehat{\mathcal{P}}(\xi)$ and $\widehat{\mathcal{Q}}(\xi)$ which do not contain such kernel elements. To be more precise, let us recall the operators Λ and Λ^* defined in (2.5) and (2.6) that are associated to (4.1), together with their kernels $\mathcal{K} = \mathcal{K}(\Lambda)$ and $\mathcal{K}^* = \mathcal{K}(\Lambda^*)$. In addition, let us write

$$\begin{array}{lcl} \mathcal{P}(\xi) & = & \big\{ x \in BC_0\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big) \mid x'(\xi') = L(\xi') x_{\xi'} \text{ for all } \xi' \in (-\infty, \xi] \big\}, \\ \mathcal{Q}(\xi) & = & \big\{ x \in BC_0\big([\xi + r_{\min}, \infty), \mathbb{C}^n\big) \mid x'(\xi') = L(\xi') x_{\xi'} \text{ for all } \xi' \in [\xi, \infty) \big\}, \\ \widehat{\mathcal{P}}(\xi) & = & \big\{ x \in \mathcal{P}(\xi) \mid \int_{-\infty}^{\min(\xi + r_{\max}, 0)} b(\xi')^* x(\xi') d\xi' = 0 \text{ for all } b \in \mathcal{K} \big\}, \\ \widehat{\mathcal{Q}}(\xi) & = & \big\{ x \in \mathcal{Q}(\xi) \mid \int_{+\infty}^{\max(\xi + r_{\min}, 0)} b(\xi')^* x(\xi') d\xi' = 0 \text{ for all } b \in \mathcal{K} \big\}. \end{array} \tag{4.4}$$

As in [15], we also introduce the following spaces, that describe the initial conditions associated to the spaces above and the kernels $\mathcal{K}$ and $\mathcal{K}^*$.

$$\begin{array}{lcl} P(\xi) & = & \big\{ \phi \in X \mid \phi = x_\xi \text{ for some } x \in \mathcal{P}(\xi) \big\}, \\ Q(\xi) & = & \big\{ \phi \in X \mid \phi = x_\xi \text{ for some } x \in \mathcal{Q}(\xi) \big\}, \\ \widehat{P}(\xi) & = & \big\{ \phi \in X \mid \phi = x_\xi \text{ for some } x \in \widehat{\mathcal{P}}(\xi) \big\}, \\ \widehat{Q}(\xi) & = & \big\{ \phi \in X \mid \phi = x_\xi \text{ for some } x \in \widehat{\mathcal{Q}}(\xi) \big\}, \\ B(\xi) & = & \big\{ \phi \in X \mid \phi = b_\xi \text{ for some } b \in \mathcal{K} \big\}, \\ B^*(\xi) & = & \big\{ \phi \in Y \mid \phi = d_\xi \text{ for some } d \in \mathcal{K}^* \big\}. \end{array} \tag{4.5}$$

The reader familiar with [15] will notice that the definitions of $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ have been adapted slightly to accomodate for the half-line setting of Theorem 4.1. In particular, the upper bounds of the defining integrals are now constant for $\xi \geq 0$ respectively $\xi \leq 0$. This choice ensures that $\widehat{P}(\xi)$ is invariant on the positive half-line and $\widehat{Q}(\xi)$ is invariant on the negative half-line, but does not affect any of the results in [15].

The following result was obtained in [15] and shows that $P(\xi)$ and $Q(\xi)$ together span X up to a finite dimensional complement, that can be described explicitly in terms of the Hale inner product.

Proposition 4.2. *Consider the homogeneous linear system (4.1). For any $\xi \in \mathbb{R}$, let $Z(\xi) \subset X$ be the closed subspace of finite codimension that is given by*

$$Z(\xi) = \big\{ \phi \in X \mid \langle \psi, \phi \rangle_\xi = 0 \text{ for every } \psi \in B^*(\xi) \big\}. \tag{4.6}$$

Then we have the direct sum decomposition

$$Z(\xi) = \widehat{P}(\xi) \oplus \widehat{Q}(\xi) \oplus B(\xi). \tag{4.7}$$

Our main contribution in this section is to provide an explicit complement for $Z(\xi)$ that will allow us to enlarge the space $\widehat{P}(\xi)$ and obtain a set $S(\xi)$ that satisfies the properties in Theorem 4.1. To do this, we will employ a very useful property of the Hale inner product. In particular, fix an interval $[\xi_-, \xi_+]$ and consider an arbitrary function $z \in W^{1,1}_{\mathrm{loc}}([\xi_- - r_{\max}, \xi_+ - r_{\min}])$ together with an arbitrary function $x \in W^{1,1}_{\mathrm{loc}}[\xi_- + r_{\min}, \xi_+ + r_{\max}]$. Then for every $\xi \in [\xi_-, \xi_+]$, we can perform the computation

$$\begin{array}{rcl} D_\xi \langle \mathrm{ev}^*_\xi z, \mathrm{ev}_\xi x \rangle_\xi & = & D_\xi \big[z(\xi)^* x(\xi) - \sum_{j=0}^N \int_\xi^{\xi + r_j} z(\theta - r_j)^* A_j(\theta - r_j) x(\theta) d\theta \big] \\ & = & z'(\xi)^* x(\xi) + z(\xi)^* x'(\xi) - \sum_{j=0}^N z(\xi)^* A_j(\xi) x(\xi + r_j) \\ & & + z(\xi - r_j)^* A_j(\xi - r_j) x(\xi) \\ & = & z(\xi)^* [\Lambda x](\xi) + [\Lambda^* z](\xi)^* x(\xi). \end{array} \tag{4.8}$$

Lemma 4.3. *Consider the homogeneous linear system (4.1). Let $\{d^i\}_{i=1}^{n_d}$ be a basis for the kernel $\mathcal{K}^*$ and recall the constant $r = r_{\max} - r_{\min}$. Then for every $\xi \geq 0$ and every integer $1 \leq i \leq n_d$, there exists a function $y^i_{(\xi)} \in C\big((-\infty, \xi + r_{\max}], \mathbb{C}^n\big)$ that satisfies the following properties.*

(i) For every $\xi \geq 0$ and every integer $1 \leq i \leq n_d$, we have $[\Lambda y^i_{(\xi)}](\xi') = 0$ for all $\xi' \geq -3r$ and all $\xi' \leq -5r$.

(ii) For any pair $0 \leq \xi' \leq \xi$ and any pair of integers $1 \leq i, j \leq n_d$, we have the identity

$$\langle \mathrm{ev}^*_{\xi'} d^i, \mathrm{ev}_{\xi'} y^j_{(\xi)} \rangle_{\xi'} = \delta_{ij}. \tag{4.9}$$

(iii) Fix an integer $1 \leq i \leq n_d$ and a constant $0 \leq \xi'$. Then the function $\xi \mapsto \mathrm{ev}_{\xi'} y^i_{(\xi)}$ depends continuously on ξ, for $\xi' \leq \xi$.

(iv) Consider any triple $0 \leq \xi' \leq \xi_1 \leq \xi_2$. Then for any integer $1 \leq i \leq n_d$ we have

$$\mathrm{ev}_{\xi'} [y^i_{(\xi_1)} - y^i_{(\xi_2)}] \in \widehat{P}(\xi'). \tag{4.10}$$

(v) For every $\xi \geq 0$ and every integer $1 \leq i \leq n_d$, we have the integral condition

$$\int_{-\infty}^0 b(\xi')^* y_{(\xi)}(\xi') d\xi' = 0, \tag{4.11}$$

which holds for all $b \in \mathcal{K}(\Lambda)$.

Proof. Fix $\xi_0 = -4r$ and consider the functions $\{\psi^i\}_{i=1}^{n_d} \subset Y$ that were constructed in Lemma 3.4 for $\xi = \xi_0$. As in Section 3, define the functions $g^i \in L^\infty(\mathbb{R}, \mathbb{C}^n)$ that have $\mathrm{ev}^*_{\xi_0} g^i = \psi^i$, while $g^i = 0$ elsewhere. For the remainder of this proof, fix an integer $1 \leq i \leq n_d$. Consider a sequence $\xi_k = k \to \infty$ and define $y_{(k)} = \Lambda^{-1}_{(k)} g^i$, where the inverse $\Lambda^{-1}_{(k)}$ should be interpreted as the analogue of Λ_-^{-1} for the half-line $(-\infty, \xi_k]$. Note that by adding an appropriate element in $\mathcal{K}$ to $y_{(k)}$ we can ensure that the integral condition (4.11) is satisfied. For any integer $1 \leq j \leq n_d$ we can use (4.8) together with the exponential decay of d^j at $-\infty$ to compute

$$\langle \mathrm{ev}^*_\xi d^j, \mathrm{ev}_\xi y_{(k)} \rangle_\xi = \int_{-\infty}^\xi d^j(\xi')^* [\Lambda y_{(k)}](\xi') d\xi' = (\mathrm{ev}^*_{\xi_0} d^j, \psi^i) = \delta_{ij}. \tag{4.12}$$

Choose a continuous function $\chi : [0, \infty) \to [0, 1]$ such that χ is zero near even integers and one near odd integers. Write

$$y_{(\xi)} = \chi(2\xi) y_{(\lceil \xi \rceil)} + [1 - \chi(2\xi)] y_{(\lceil \xi + \frac{1}{2} \rceil)}, \tag{4.13}$$

in which $\lceil \xi \rceil$ denotes the smallest integer that is larger or equal to ξ. With this definition it is easy to see that the properties (i) through (v) all hold. □

The functions defined in Lemma 4.3 are sufficient to construct the space $S(\xi)$ appearing in Theorem 4.1. Indeed, we will use the spaces

$$\begin{array}{rcl} \mathcal{S}(\xi) & = & \widehat{\mathcal{P}}(\xi) \oplus \mathrm{span}\{y^i_{(\xi)}\}_{i=1}^{n_d}, \\ S(\xi) & = & \widehat{P}(\xi) \oplus \mathrm{span}\{\mathrm{ev}_\xi y^i_{(\xi)}\}_{i=1}^{n_d}. \end{array} \tag{4.14}$$

The following result should be seen as the appropriate generalization of Theorem 4.2 in [15] and shows that functions in $\mathcal{S}$ automatically decay exponentially.

Proposition 4.4. *Consider the homogeneous linear system (4.1). Let the sets $\mathcal{S}(\xi) \subset X$ for $\xi \geq 0$ be defined as in (4.14). Then there exist constants $K > 0$ and $\alpha_S > 0$ such that for all $\xi \geq 0$ and all $\xi' \leq \xi$, we have*

$$|x(\xi')| \leq K e^{-\alpha_S(\xi - \xi')} \|x_\xi\|, \tag{4.15}$$

for every $x \in \mathcal{S}(\xi)$.

Proof. As in [15] it suffices to prove the following two statements.

(i) There exists $\sigma > -r_{\max}$ such that for all $\xi \geq 0$ and all $y \in \mathcal{S}(\xi)$, we have

$$|y(\xi')| \leq \frac{1}{2} \sup_{s < \xi + r_{\max}} |y(s)| \ \text{ for all } \xi' \leq \xi - \sigma. \tag{4.16}$$

(ii) There exists $K > 0$ such that for all $\xi \geq 0$ and all $y \in \mathcal{S}(\xi)$, we have

$$|y(\xi')| \leq K \|\mathrm{ev}_\xi y\| \ \text{ for all } \xi' \leq \xi + r_{\max}. \tag{4.17}$$

Assuming that (i) fails, we have sequences $\sigma^j \to \infty$, $\xi^j \geq 0$ and $y^j \in \mathcal{S}(\xi^j)$ such that

$$\left|y^j(-\sigma^j + \xi^j)\right| \geq \tfrac{1}{2}, \qquad \sup_{s < \xi^j + r_{\max}} \left|y^j(s)\right| = 1. \tag{4.18}$$

Suppose first that $-\sigma^j + \xi^j$ is unbounded, i.e., $-\sigma^j + \xi^j \to \pm\infty$ after passing to a subsequence. Writing $z^j(\xi') = y^j(\xi' - \sigma^j + \xi^j)$, an application of Ascoli's theorem yields a convergent subsequence $z^j \to z$. Notice that $z(0) \geq \frac{1}{2}$, which means that z is a nontrivial bounded solution on $\mathbb{R}$ of one of the limiting equations at $\pm\infty$. This situation is however precluded by the hyperbolicity of these limiting equations.

Now suppose that, possibly after passing to a subsequence, we have $-\sigma^j + \xi^j \to \beta^0$. Using the fact that $[\Lambda y^j](\xi') = 0$ for $\xi \geq r_{\min}$, together with the limit $\xi^j \to \infty$, we may apply Ascoli-Arzela to conclude that $y^j \to y_*$ uniformly on compact subsets of $[r_{\min}, \infty)$. Since we also have $[\Lambda y_*](\xi') = 0$ for all $\xi' \geq 0$, we conclude that $\mathrm{ev}_0 y_* \in Q(0)$. However, this immediately implies that for any $\psi \in B^*(0)$ we have $\langle \psi, \mathrm{ev}_0 y_* \rangle_0 = 0$. In view of the identity

$$\Lambda y^j = \sum_{i=1}^{n_d} g^i \langle \mathrm{ev}_0^* d^i, \mathrm{ev}_0 y^j \rangle_0, \tag{4.19}$$

this however implies that $\Lambda y^j \to 0$ uniformly on every compact subset of $\mathbb{R}$. This allows us to apply Ascoli-Arzela on the entire line, by means of which we obtain the convergence $y^j \to y_*$, which is again uniform on compacta. In addition, we have $\Lambda y_* = 0$, which now means that $y_* \in \mathcal{K}$. However, this is precluded by the integral condition (4.11).

Let us now suppose that (ii) fails, which implies that for some sequence $K^j \to \infty$, $\xi^j \geq 0$ and $y^j \in \mathcal{S}(\xi^j)$, we have

$$\sup_{s<\xi^j + r_{\max}} \left|y^j(s)\right| = K^j \left\|\mathrm{ev}_{\xi^j} y^j\right\| = 1. \tag{4.20}$$

In view of (i), this means that there exists a sequence $\sigma^j \in [-r_{\min}, \sigma]$ such that $\left|y^j(-\sigma^j + \xi^j)\right| = 1$.

Suppose that ξ^j is unbounded. We find $y^j(\xi' + \xi^j) \to z(\xi')$ where $z : (-\infty, r_{\max}] \to \mathbb{C}^n$ is a bounded solution of the limiting equation at $+\infty$. Since the sequence σ^j is bounded, z does not vanish identically. Since $\left\|\mathrm{ev}_{\xi^j} y^j\right\| = 1/K^j \to 0$, we have $\|z_0\| = 0$ and hence z can be extended to a bounded nontrivial solution of the limiting equation at $+\infty$ on the entire line. Again, this is precluded by the hyperbolicity of this limiting equation.

Now assume that, possibly after passing to a subsequence, we have $\xi^j \to \xi^* \geq 0$. Since $\mathrm{ev}_{\xi^j} y^j \to 0$, we can use Ascoli-Arzela to find the convergence $y^j \to y_*$, which is now uniform on the interval $[-r + r_{\min}, \xi^* + r_{\max}]$. In addition, we have $[\Lambda y_*](\xi') = 0$ for all $\xi' \in [-r, \xi^*]$. If $\xi^* \geq \sigma$, this fact is precluded by the non-degeneracy condition (HB), since we also have $\mathrm{ev}_{\xi^*} y_* = 0$. In the case where $\xi^* < \sigma$, we can again use (4.19) to obtain the convergence $y^j \to y_*$, which this time is uniform on compact subsets of $(-\infty, \xi^* + r_{\max}]$. As before, the condition (HB) now leads to a contradiction. □

Notice that we have now obtained a splitting

$$X = S(\xi) \oplus Q(\xi) \tag{4.21}$$

that satisfies nearly all of the properties stated in Theorem 4.1. It remains only to consider the statements concerning the projections $\Pi_{S(\xi)}$ and $\Pi_{Q(\xi)}$. We will address these issues in the remainder of this section by establishing the continuity of these projections and studying the limiting behaviour as $\xi \to \infty$. In Section 5 we will show how these estimates can be improved if one has detailed information concerning the rate at which $L(\xi)$ approaches its limits as $\xi \to \pm\infty$. For the moment however, let us recall the splitting

$$X = P(\infty) \oplus Q(\infty) \tag{4.22}$$

associated to the autonomous limit of (4.1) at $+\infty$, which was established in [15].

Lemma 4.5. *Consider the linear homogeneous system (4.1). The following limit holds with respect to the norm on $\mathcal{L}(S(\xi), X)$,*

$$\left\|[I - \Pi_{P(\infty)}]_{|S(\xi)}\right\| \to 0 \quad as \quad \xi \to \infty. \tag{4.23}$$

In addition, for the norm on $\mathcal{L}(Q(\xi), X)$ we have the similar limit

$$\left\|[I - \Pi_{Q(\infty)}]_{|Q(\xi)}\right\| \to 0 \quad as \quad \xi \to \infty. \tag{4.24}$$

Proof. The second limit was established in [15], so we restrict ourselves to the first limit here. Choose an arbitrarily small $\epsilon > 0$ and fix $C > 0$ sufficiently large to ensure that for all $\xi \in \mathbb{R}$, the inequality

$$\sum_{j=0}^{N} |A_j(\xi) e^{\alpha_S r_j}| \leq C \tag{4.25}$$

holds. Recalling the constants K and α_S from Proposition 4.4, pick $\xi_0 > 0$ sufficiently large to ensure that $4(1 + C)^2 K \exp(-\alpha_S \xi_0) < \frac{\epsilon}{2}$ and also

$$\sum_{j=0}^{N} \left|A_j(\xi') - A_j^+\right| < \frac{\epsilon}{2K} \tag{4.26}$$

for all $\xi' \geq \xi_0$. Fix any $\xi \geq 2\xi_0 + r_{\max}$. Consider an arbitrary $y \in \mathcal{S}(\xi)$ and write $\phi = \mathrm{ev}_\xi y \in S(\xi)$. Notice first that $\phi_{|[r_{\min},0]} \in C^1([r_{\min}, 0], \mathbb{C}^n)$. We can hence approximate ϕ with a sequence of C^1-smooth functions ϕ^k that have $\phi^k(\theta) = \phi(\theta)$ for all $\theta \in [-1, 0]$. Let us extend these functions to C^1-smooth functions y^k on the line, with $\mathrm{ev}_\xi y^k = \phi^k$ but also $y^k(\xi') = y(\xi')$ for all $0 \leq \xi' \leq \xi$. Notice that we may construct the functions y^k in such a way to ensure that the following estimate holds for all $\xi' \leq 0$,

$$\left|Dy^k(\xi')\right| + \left|y^k(\xi')\right| \leq 2[\left|Dy^k(0)\right| + \left|y^k(0)\right|]. \tag{4.27}$$

In particular, this means that for all $\xi' \leq \xi$ we have the bound

$$\left|Dy^k(\xi')\right| + \left|y^k(\xi')\right| \leq 2K(1 + C)[e^{-\alpha_S \xi} + e^{-\alpha_S(\xi - \xi')}] \left\|\phi^k\right\|. \tag{4.28}$$

Now, for any C^1-smooth function y we have the representation

$$\Pi_{Q(\infty)} \mathrm{ev}_\xi y = \mathrm{ev}_\xi \Lambda_\infty^{-1}[I - H_\xi]\Lambda_\infty y, \tag{4.29}$$

in which we have introduced the notation $[\Lambda_\infty x](\xi') = x'(\xi') - \sum_{j=0}^N A_j^+ x(\xi' + r_j)$, together with the Heaviside function H_ξ that satisfies $H_\xi(\xi') = I$ if $\xi' \geq \xi$ and zero otherwise. Observing that

$$[\Lambda_\infty y^k](\xi') = [\Lambda y^k](\xi') + \sum_{j=0}^N [A_j(\xi') - A_j^+]\mathrm{ev}_{\xi'} y^k, \tag{4.30}$$

we may compute

$$\begin{array}{rcl} \left\|[I - H_\xi]\Lambda_\infty y^k\right\|_{L^\infty(\mathbb{R},\mathbb{C}^n)} & \leq & \sup_{\xi' \leq \xi_0} \left|Dy^k(\xi')\right| + C\left\|\mathrm{ev}_{\xi'} y^k\right\| + \sup_{\xi_0 \leq \xi' \leq \xi} \frac{\epsilon}{2K}\left\|\mathrm{ev}_{\xi'} y^k\right\| \\ & \leq & 4K(1+C)^2 e^{-\alpha_S \xi_0}\left\|\phi^k\right\| + \frac{\epsilon}{2}\left\|\phi^k\right\| \\ & \leq & \epsilon\left\|\phi^k\right\|. \end{array} \tag{4.31}$$

This however means that for some constant $C' > 0$ we have

$$\left\|\Pi_{Q(\infty)}\phi^k\right\| \leq \epsilon C'\left\|\phi^k\right\|, \tag{4.32}$$

which concludes the proof due to the continuity of $\Pi_{Q(\infty)}$. □

Lemma 4.6. *Consider the system (4.1) and suppose that (HB) is satisfied. For $\xi \geq 0$, write $\Gamma(\xi) = \mathrm{span}\{\mathrm{ev}_\xi y^i_{(\xi)}\}_{i=1}^{n_d}$ and consider the splitting*

$$X = \widehat{P}(\xi) \oplus \Gamma(\xi) \oplus Q(\xi) \tag{4.33}$$

with the corresponding projection operators $\Pi_{\widehat{P}(\xi)}$, $\Pi_{\Gamma(\xi)}$ and $\Pi_{Q(\xi)}$. Then we have the following limits for any $\xi_0 \geq 0$,

$$\begin{array}{rclr} \left\|[I - \Pi_{\widehat{P}(\xi_0)}]_{|\widehat{P}(\xi)}\right\| & \to & 0 & \text{as } \xi \to \xi_0, \\ \left\|[I - \Pi_{Q(\xi_0)}]_{|Q(\xi)}\right\| & \to & 0 & \text{as } \xi \to \xi_0, \\ \left\|[I - \Pi_{\Gamma(\xi_0)}]_{|\Gamma(\xi)}\right\| & \to & 0 & \text{as } \xi \to \xi_0. \end{array} \tag{4.34}$$

Proof. The statements concerning $\widehat{P}(\xi)$ and $Q(\xi)$ were established in [15]. The limit involving $\Gamma(\xi)$ follows easily using the finite dimensionality of $\Gamma(\xi)$ and item (iii) in Lemma 4.3. □

Lemma 4.7. *Consider an arbitrary $\xi_0 \geq 0$. The projections $\Pi_{Q(\xi)}$ can be uniformly bounded for all $\xi \geq \xi_0$.*

Proof. Assuming the statement is false, let us consider a sequence ξ^j and $\phi^j \in X$ that has $\xi^j \geq \xi_0$ and $\|\phi^j\| = 1$ for all integers $j \geq 1$, while $\|\Pi_{Q(\xi^j)}\phi^j\| \to \infty$ as $j \to \infty$. Let us first assume that ξ^j is bounded, which after passing to a subsequence implies that $\xi^j \to \xi_*$ for some $\xi_* \geq \xi_0$. Let us write $\sigma^j = \Pi_{\Gamma(\xi^j)}\phi^j$, $p^j = \Pi_{\widehat{P}(\xi^j)}\phi^j$ and $q^j = \Pi_{Q(\xi^j)}\phi^j$. Defining $\kappa_j = \|\sigma^j\| + \|p^j\| + \|q^j\|$, let us also introduce the bounded sequence $\widetilde{\sigma}^j = \kappa_j^{-1}\sigma^j$ and similarly defined sequences $\widetilde{p}^j$ and $\widetilde{q}^j$. In addition, we introduce $\widetilde{\sigma}^j_* = \Pi_{\Gamma(\xi_*)}\widetilde{\sigma}^j$ and similarly $\widetilde{p}^j_* = \Pi_{\widehat{P}(\xi_*)}\widetilde{p}^j$ and $\widetilde{q}^j_* = \Pi_{Q(\xi_*)}\widetilde{q}^j$. Using Lemma 4.6 we obtain the following limits as $j \to \infty$,

$$\begin{array}{lcl} \widetilde{\sigma}^j + \widetilde{p}^j + \widetilde{q}^j & \to & 0, \\ \widetilde{\sigma}^j - \widetilde{\sigma}^j_* & \to & 0, \\ \widetilde{p}^j - \widetilde{p}^j_* & \to & 0, \\ \widetilde{q}^j - \widetilde{q}^j_* & \to & 0. \end{array} \tag{4.35}$$

Since $\Gamma(\xi_*)$ is finite dimensional, we can pass to a subsequence and obtain $\widetilde{\sigma}^j_* \to \sigma_*$ This implies the following limit as $j \to \infty$,

$$\sigma_* + \widetilde{p}^j_* + \widetilde{q}^j_* \to 0. \tag{4.36}$$

We now introduce the truncation operators $\pi^+ : X \to C\big([0, r_{\max}], \mathbb{C}^n\big)$ and $\pi^- : X \to C\big([r_{\min}, 0], \mathbb{C}^n\big)$. Using the exponential estimates on $\mathcal{Q}(\xi_*)$ and $\widehat{\mathcal{P}}(\xi_*)$, it is not hard to see that the restriction of π^+ to $Q(\xi_*)$ is compact, as is the restriction of π^- to $\widehat{P}(\xi_*)$. After passing to a subsequence, we thus find that $\pi^+\widetilde{q}^j_*$ and hence also $\pi^+\widetilde{p}^j_*$ converge uniformly on $[0, r_{\max}]$. Invoking a similar argument involving π^- we conclude that as $j \to \infty$, we must have $\widetilde{p}^j_* \to p_*$ and $\widetilde{q}^j_* \to q_*$ for some $p_* \in \widehat{P}(\xi_*)$ and $q_* \in Q(\xi_*)$. In view of (4.36), this leads to a contradiction, since $\|\sigma_*\| + \|p_*\| + \|q_*\| = 1$.

It remains to consider the case that $\xi^j \to \infty$. However, using the splitting $X = S(\xi) \oplus Q(\xi)$ and the limits in Lemma 4.5, we can obtain a contradiction in the same fashion as above. ☐

Corollary 4.8. *Consider the linear homogeneous system (4.1) and recall the splittings*

$$X = S(\xi) \oplus Q(\xi), \tag{4.37}$$

that hold for $\xi \geq 0$. The projections $\Pi_{S(\xi)}$ and $\Pi_{Q(\xi)}$ depend continuously on $\xi \in \mathbb{R}$. In addition, we have the limits

$$\lim_{\xi \to \infty} \|\Pi_{Q(\xi)} - \Pi_{Q(\infty)}\| = 0, \qquad \lim_{\xi \to \infty} \|\Pi_{S(\xi)} - \Pi_{P(\infty)}\| = 0. \tag{4.38}$$

Proof. The limit for $\Pi_{Q(\xi)}$ as $\xi \to \infty$ can be seen by writing

$$\Pi_{Q(\xi)} - \Pi_{Q(\infty)} = [I - \Pi_{Q(\infty)}]\Pi_{Q(\xi)} - [I - \Pi_{P(\infty)}]\Pi_{S(\xi)} \tag{4.39}$$

and using the limits in Lemma 4.5, together with the uniform bounds for $\Pi_{Q(\xi)}$ and $\Pi_{S(\xi)}$ that follow from Lemma 4.7. The other statements follow analogously. ☐

Proof of Theorem 4.1. The spaces $S(\xi)$ can be defined as in 4.14, while the spaces $Q(\xi)$ can be defined as in (4.5). The decay rates in (4.3) for $S(\xi)$ follow from Proposition 4.4, while Theorem 4.2 in [15] provides the rates for $Q(\xi)$. The continuity of the projections $\Pi_{S(\xi)}$ and $\Pi_{Q(\xi)}$ follow from Corollary 4.8 and the boundedness of these projections follows from Lemma 4.7. ☐

5 Parameter Dependent Exponential Dichotomies

In this section we show how homogeneous linear systems of the form

$$x'(\xi) = L(\xi, \mu)x_\xi = \sum_{j=0}^{N} A_j(\xi, \mu)x(\xi + r_j), \tag{5.1}$$

which depend on a parameter $\mu \in U$, can be incorporated into the framework developed in the previous section. Throughout this section, we will assume that the linear operators $\Lambda(\mu) : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$ associated to (5.1) by means of (3.3) depend C^k-smoothly on the parameter μ for some integer $k \geq 0$. In addition, we will assume that (HB) holds for some parameter $\mu_0 \in U$, that (5.1) is asymptotically hyperbolic for $\mu \in U$ and that the limiting operators $L^\pm$ depend C^k-smoothly on μ.

The first part of this section is devoted to the proof of Theorem 5.1, which shows that the exponential splittings in Section 4 can be constructed in such a way, that the relevant spaces and projections depend smoothly on the parameter μ. The price we have to pay is that we lose the invariance of $S(\xi, \mu)$, but for our purposes this will be irrelevant.

In the second part of this section we study the limiting behaviour of the projection operators $\Pi_{Q(\xi,\mu)}$ and $\Pi_{S(\xi,\mu)}$. Theorem 5.4 improves upon the estimates in the previous section and relates the convergence $\Pi_{Q(\xi,\mu)} \to \Pi_{Q(\infty,\mu)}$ to the rate at which $L(\xi, \mu)$ approaches its limit as $\xi \to \infty$. Together, Theorems 5.1 and 5.4 should be seen as the analogue of Lemma 1.1 in [16].

Theorem 5.1. *Consider the linear homogeneous system (5.1). There exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, such that for all $\mu \in U'$ and all $\xi \geq 0$ we have the splitting*

$$X = Q(\xi, \mu) \oplus S(\xi, \mu). \tag{5.2}$$

In addition, there exist constants $K > 0$, $\alpha_S > 0$ and $\alpha_Q > 0$, such that each $\phi \in Q(\xi, \mu)$ can be extended to a solution $E\phi$ of the homogeneous equation (5.1) on $[\xi, \infty)$, while each $\psi \in S(\xi, \mu)$ can be extended to a function $E\psi$ that is defined on the interval $[r_{\min}, \xi + r_{\max}]$ and satisfies the homogeneous equation (5.1) on $[0, \xi]$. The maps $\mu \mapsto \Pi_{Q(\xi,\mu)}$ and $\mu \mapsto \Pi_{S(\xi,\mu)}$ are C^k-smooth and all derivatives can be bounded independently of $\xi \geq 0$. Moreover, we have the following exponential estimates for all integers $0 \leq \ell \leq k$,

$$\begin{array}{lcll} \left\| D^\ell_\mu \mathrm{ev}_{\xi'} E \Pi_{Q(\xi,\mu)} \right\| & \leq & K e^{-\alpha_Q |\xi' - \xi|} & \text{for every } \xi' \geq \xi, \\ \left\| D^\ell_\mu \mathrm{ev}_{\xi'} E \Pi_{S(\xi,\mu)} \right\| & \leq & K e^{-\alpha_S |\xi' - \xi|} & \text{for every } 0 \leq \xi' \leq \xi. \end{array} \tag{5.3}$$

Our approach towards establishing Theorem 5.1 will be to construct the parameter-dependent spaces $Q(\xi, \mu)$ and $S(\xi, \mu)$ separately, using the implicit function theorem to represent these spaces as graphs over $Q(\xi, \mu_0)$ and $S(\xi, \mu_0)$. The exponential estimates will follow essentially from those established in the previous section for (5.1) with $\mu = \mu_0$.

Lemma 5.2. *Consider the exponential splitting $X = Q(\xi) \oplus S(\xi)$ for $\xi \geq 0$, as defined in Theorem 4.1 for the system (5.1) with $\mu = \mu_0$. Then there exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, together with a family of C^k-smooth functions $u^*_{Q(\xi)} : U' \to \mathcal{L}(Q(\xi), X)$, parametrized by $\xi \geq 0$, such that for all $\mu \in U'$ we have $\mathcal{R}(u^*_{Q(\xi)}(\mu)) = Q(\xi, \mu)$, with $\Pi_{Q(\xi)} u^*_{Q(\xi)}(\mu) = I$ and $[I - \Pi_{Q(\xi)}] u^*_{Q(\xi)}(\mu) \to 0$ as $\mu \to \mu_0$, uniformly for $\xi \geq 0$. In addition, there exist constants $K > 0$ and $\alpha_Q > 0$ such that for all $\mu \in U'$, all pairs $\xi' \geq \xi \geq 0$ and all integers $0 \leq \ell \leq k$, we have*

$$\left\| D^\ell_\mu \mathrm{ev}_{\xi'} E u^*_{Q(\xi)}(\mu) \right\|_{\mathcal{L}(Q(\xi), X)} \leq K e^{-\alpha_Q |\xi' - \xi|}. \tag{5.4}$$

Proof. We recall the C^k-smooth operator

$$\mathcal{C} : U' \to \mathcal{L}\big(L^\infty(\mathbb{R}, \mathbb{C}^n), \mathcal{R}(\Lambda(\mu_0))^\perp\big) \tag{5.5}$$

defined in Proposition 3.3 and we choose a basis for $\mathcal{R}(\Lambda(\mu_0))^\perp$ in such a way that the support of each basis function is contained in $[-4r - r_{\max}, -4r - r_{\min}] \subset (-\infty, 0)$. We also recall the constants $K > 0$ and $\alpha_Q > 0$ obtained by an application of Theorem 4.1 to the system (5.1) at $\mu = \mu_0$.

For any $\xi \geq 0$, let us consider the map $\mathcal{G} : U \to \mathcal{L}\big(BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$ that is given by

$$\mathcal{G}(\mu)u = \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u - E\Pi_{Q(\xi)} \mathrm{ev}_\xi \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u. \tag{5.6}$$

Here we have introduced the notation $[L(\mu)u](\xi) = L(\xi, \mu)u_\xi$. We first note that $\mathcal{G}$ is well-defined, since the extension operator E indeed maps $Q(\xi)$ into $BC_{-\alpha_Q}\big([r_{\min} + \xi, \infty), \mathbb{C}^n\big)$ due to the exponential estimates in Theorem 4.1. To be more precise, note that for some constant $K_1 > 0$, the $\mathcal{L}\big(Q(\xi), BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$-norm of this extension is given by

$$\|E\| \leq K_1 e^{\alpha_Q \xi}. \tag{5.7}$$

Notice also that for some constant $C_1 > 0$ the $\mathcal{L}\big(BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n), X\big)$-norm of the evaluation operator $\mathrm{ev}_{\xi'}$ is bounded by

$$\|\mathrm{ev}_{\xi'}\| \leq C_1 e^{-\alpha_Q \xi'}. \tag{5.8}$$

The C^k-smoothness of $\mu \mapsto L(\mu)$ now implies that $\mathcal{G}$ is C^k-smooth as a map from U into $\mathcal{L}\big(BC_{-\alpha_Q}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$. By taking μ sufficiently close to μ_0 we can achieve the following bounds, simultaneously for all $\xi \geq 0$ and every integer $1 \leq \ell \leq k$,

$$\begin{array}{lcl} \|\mathcal{G}(\mu)\| & \leq & \frac{1}{2}, \\ \big\|D^\ell_\mu \mathcal{G}(\mu)\big\| & \leq & C_2, \end{array} \tag{5.9}$$

in which we have introduced a constant $C_2 > 0$. The first estimate in (5.9) implies that for all μ sufficiently close to μ_0 and all $\xi \geq 0$, we can define the linear maps

$$v^*_{Q(\xi)}(\mu) : Q(\xi) \to BC_{-\alpha_Q}\big([r_{\min} + \xi, \infty), \mathbb{C}^n\big), \ \phi \mapsto [1 - \mathcal{G}(\mu)]^{-1} E\phi, \tag{5.10}$$

together with $u^*_{Q(\xi)}(\mu) = \mathrm{ev}_\xi v^*_{Q(\xi)}(\mu)$. The exponential estimates (5.4) follow directly from this representation of $u^*_{Q(\xi)}(\mu)$, together with (5.7), (5.8) and (5.9). In addition, it is immediately clear from our choice of $\mathcal{G}$ that $\Pi_{Q(\xi)} u^*_{Q(\xi)}(\mu) = I$. The remainder term can be bounded using the identity

$$[I - \Pi_{Q(\xi)}] u^*_{Q(\xi)}(\mu) = \mathrm{ev}_\xi \big[[I - \mathcal{G}(\mu)]^{-1} - I\big] E, \tag{5.11}$$

which approaches 0 as $\mu \to \mu_0$. Again, this limit can be obtained simultaneously for all $\xi \geq 0$ by using (5.7) and (5.8).

We now set out to prove that $\mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big) = \mathcal{Q}(\xi, \mu)$. Suppose therefore that $u = v^*_{Q(\xi)}(\mu)\phi$ for some $\phi \in Q(\xi)$. Notice that u necessarily satisfies the following identity for all $\xi' \geq \xi$,

$$\begin{array}{rcl} [\Lambda(\mu)u](\xi') & = & [L(\xi', \mu_0) - L(\xi', \mu)]\mathrm{ev}_{\xi'} E\phi + [L(\xi', \mu) - L(\xi', \mu_0)]\mathrm{ev}_{\xi'} u \\ & & \quad + [L(\xi', \mu_0) - L(\xi', \mu)]\mathrm{ev}_{\xi'} \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u \\ & & \quad - [L(\xi', \mu_0) - L(\xi', \mu)]\mathrm{ev}_{\xi'} E \Pi_{Q(\xi)} \mathrm{ev}_\xi \Lambda^{\mathrm{qinv}}_{(-\alpha_Q)}(\mu_0)[L(\mu) - L(\mu_0)]u \\ & = & [L(\xi', \mu_0) - L(\xi', \mu)]\mathrm{ev}_{\xi'} u + [L(\xi', \mu) - L(\xi', \mu_0)]\mathrm{ev}_{\xi'} u = 0. \end{array} \tag{5.12}$$

This means that $v^*_{Q(\xi)}(\mu)$ indeed maps into $\mathcal{Q}(\xi, \mu)$.

It remains to show that $\mathcal{Q}(\xi, \mu) \subset \mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big)$. Supposing this is not the case, pick $q^1_\mu \in \mathcal{Q}(\xi, \mu)$ with $q^1_\mu \notin \mathcal{R}\big(v^*_{Q(\xi)}(\mu)\big)$ and write $\phi = \Pi_{Q(\xi)} \mathrm{ev}_\xi q^1_\mu$ and $q^2_\mu = v^*_{Q(\xi)}(\mu)\phi$. Writing $q_\mu = q^1_\mu - q^2_\mu$, we have $q_\mu \in \mathcal{Q}(\xi, \mu)$ with $\Pi_{Q(\xi)} \mathrm{ev}_\xi q_\mu = 0$. Noticing that $[L(\mu) - L(\mu_0)]q_\mu = \Lambda(\mu_0) q_\mu$, we find that for some $q_{\mu_0} \in \mathcal{Q}(\xi)$ we must have

$$\begin{array}{rcl} \mathcal{G}(\mu) q_\mu & = & q_\mu + q_{\mu_0} - E\Pi_{Q(\xi)} \mathrm{ev}_\xi [q_\mu + q_{\mu_0}] \\ & = & q_\mu + q_{\mu_0} - q_{\mu_0} = q_\mu \end{array} \tag{5.13}$$

and hence $q_\mu \in \mathcal{K}\big(I - \mathcal{G}(\mu)\big) = \{0\}$, which concludes the proof. □

In the next proposition, a similar approach is used to construct $S(\xi, \mu)$. Notice however that this construction will be treated as a definition, as there is no canonical way to define $S(\xi, \mu)$ as was possible for $Q(\xi, \mu)$.

Lemma 5.3. *Consider the exponential splitting $X = Q(\xi) \oplus S(\xi)$ for $\xi \geq 0$ as defined in Theorem 4.1 for the system (5.1) with $\mu = \mu_0$. Then there exists an open neighbourhood $U' \subset U$, with $\mu_0 \in U'$, together with a family of C^k-smooth functions $u^*_{S(\xi)} : U \to \mathcal{L}(S(\xi, \mu_0), X)$, parametrized by $\xi \geq 0$, such that for all $\mu \in U'$ we have $\Pi_{S(\xi)} u^*_{S(\xi)}(\mu) = I$ and $[I - \Pi_{S(\xi)}] u^*_{S(\xi)}(\mu) \to 0$ as $\mu \to \mu_0$, uniformly for $\xi \geq 0$. In addition, there exist constants $K > 0$ and $\alpha_S > 0$, such that for all $\mu \in U'$, all pairs $0 \leq \xi' \leq \xi$ and all integers $0 \leq \ell \leq k$, we have*

$$\left\| D^\ell_\mu \mathrm{ev}_{\xi'} E u^*_{S(\xi)}(\mu) \right\|_{\mathcal{L}(S(\xi), X)} \leq K e^{-\alpha_S |\xi' - \xi|}. \tag{5.14}$$

Finally, for all $\mu \in U'$ and all $\xi \geq 0$, the range $\mathcal{R}\big(u^*_{S(\xi)}(\mu)\big) \subset X$ is closed.

Proof. We can proceed in the same fashion as in the proof of Lemma 5.2, although we here need to use the function space $BC_{\alpha_S}\big([r_{\min}, \xi + r_{\max}], \mathbb{C}^n\big)$. To see that $\mathcal{R}\big(u^*_{S(\xi)}(\mu)\big)$ is closed, consider a sequence $\phi^j \in S(\xi)$, write $\psi^j = u^*_{S(\xi)}(\mu)\phi^j$ and assume that $\psi^j \to \psi_*$. Since $\Pi_{S(\xi)}\psi^j = \phi^j$, we also have $\phi^j \to \Pi_{S(\xi)}\psi_* := \phi_*$. Since $u^*_{S(\xi)}(\mu)$ is bounded, we have $u^*_{S(\xi)}(\mu)[\phi^j - \phi_*] \to 0$ and hence $\psi_* = u^*_{S(\xi)}(\mu)\phi_*$. □

Proof of Theorem 5.1. We first establish the splitting $X = Q(\xi, \mu) \oplus S(\xi, \mu)$. To this end, consider the family of maps $U^*_\xi : U' \to \mathcal{L}\big(Q(\xi) \oplus S(\xi)\big)$ defined by

$$U^*_\xi(\mu)(\phi, \psi) = \big(\Pi_{Q(\xi)}\big[u^*_{Q(\xi)}(\mu)\phi + u^*_{S(\xi)}(\mu)\psi\big], \Pi_{S(\xi)}\big[u^*_{Q(\xi)}(\mu)\phi + u^*_{S(\xi)}(\mu)\psi\big]\big). \tag{5.15}$$

Since $\Pi_{Q(\xi)} u^*_{S(\xi)} \to 0$ as $\mu \to \mu_0$ and similarly $\Pi_{S(\xi)} u^*_{Q(\xi)} \to 0$, uniformly for $\xi \geq 0$, we find that by choosing the neighbourhood U' small enough, we can ensure that $U^*_\xi(\mu)$ is invertible for all $\mu \in U'$ and all $\xi \geq 0$, with a bound on the inverse and the first k derivatives of this inverse with respect to μ that is uniform for $\mu \in U'$ and $\xi \geq 0$. This allows us to define the projections

$$\begin{array}{rcl} \Pi_{S(\xi,\mu)} & = & u^*_{S(\xi)}(\mu)\Pi_{S(\xi)}[U^*_\xi(\mu)]^{-1}, \\ \Pi_{Q(\xi,\mu)} & = & u^*_{Q(\xi)}(\mu)\Pi_{Q(\xi)}[U^*_\xi(\mu)]^{-1}. \end{array} \tag{5.16}$$

It is easy to see that indeed $\Pi^2_{Q(\xi,\mu)} = \Pi_{Q(\xi,\mu)}$ and similarly $\Pi^2_{S(\xi,\mu)} = \Pi_{S(\xi,\mu)}$. Also $\Pi_{Q(\xi,\mu)} + \Pi_{S(\xi,\mu)} = I$. These functions $\mu \mapsto \Pi_{Q(\xi,\mu)}$ and $\mu \mapsto \Pi_{S(\xi,\mu)}$ are C^k-smooth as functions $U' \to \mathcal{L}(X)$, which follows from the C^k-smoothness of $u^*_{Q(\xi)}$, $u^*_{S(\xi)}$ and U^*_ξ. In addition, since we have estimates on the first k derivatives of these functions with respect to μ that are uniform for $\mu \in U'$ and $\xi \geq 0$, the same holds for the derivatives of the projections. The exponential estimates (5.3) now follow from (5.4) and (5.14). □

Throughout the remainder of this section we will consider the limiting behaviour of the projections $\Pi_{S(\xi,\mu)}$ and $\Pi_{Q(\xi,\mu)}$ as $\xi \to \infty$. The next result describes the speed at which these projections approach their limiting values $\Pi_{P(\infty,\mu)}$ and $\Pi_{Q(\infty,\mu)}$.

Theorem 5.4. *Consider the linear system (5.1) and suppose that for some $\alpha_- < 0$, the characteristic equations $\det \Delta^+(z, \mu) = 0$ have no roots in the strip $\alpha_- \leq \operatorname{Re} z \leq 0$ for $\mu \in U$, where Δ^+ is the characteristic matrix associated to the limiting system at $+\infty$. Suppose furthermore that for some $\alpha^f_- \leq \alpha_-$, all $\xi \in \mathbb{R}$ and some constant $C > 0$ we have the bound*

$$\big\| L(\xi, \mu) - L^+(\mu) \big\|_{\mathcal{L}(X, \mathbb{C}^n)} \leq C\big[\, |\mu - \mu_0|\, e^{\alpha_- \xi} + e^{\alpha^f_- \xi}\big], \tag{5.17}$$

for $\mu \in U$, in which $L^+(\mu)$ denotes the linear operator (3.1) associated to the limiting system at $+\infty$. Then there exists a constant $K > 0$ and an open neighbourhood $U' \subset U$ with $\mu_0 \in U'$, such that the following bound holds for all $\xi \geq 0$ and all $\mu \in U'$,

$$\big\| \Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)} \big\| \leq K\big[\, |\mu - \mu_0|\, e^{\alpha_- \xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha^f_- \xi}\big]. \tag{5.18}$$

In addition, suppose that for some $C > 0$ and all integers $0 \le \ell \le k$, we have

$$\left\| D^\ell_\mu [L(\xi,\mu) - L^+(\mu)] \right\|_{\mathcal{L}(X,\mathbb{C}^n)} \le C \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi} + e^{\alpha^f_- \xi} \big], \tag{5.19}$$

for all $\mu \in U$. Then there exists a constant $K > 0$, an open neighbourhood $U' \subset U$ with $\mu_0 \in U'$ such that for all $\xi \ge 0$, all integers $0 \le \ell \le k$ and all $\mu \in U'$, we have the bound

$$\left\| D^\ell_\mu [\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)}] \right\| \le K \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha^f_- \xi} \big]. \tag{5.20}$$

Our approach towards proving these bounds will be to provide sharper versions of the results previously established in Lemma 4.5. The next two Lemma's will show that the quantity $\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)}$ indeed behaves as prescribed above, using an explicit Greens function representation of $\Pi_{Q(\infty,\mu)}$ and $\Pi_{P(\infty,\mu)}$. The calculations in these Lemma's can also be used to study the derivatives $D^\ell_\mu [\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)}]$, after an appropriate reformulation.

Lemma 5.5. *Consider the setting of Theorem 5.4 and suppose that (5.17) holds. Then there exists a constant K_1 such that*

$$\left\| [I - \Pi_{P(\infty,\mu)}]_{|S(\xi,\mu)} \right\| \le K_1 \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi} + e^{\alpha^f_- \xi} + e^{(\alpha_- - \alpha_S)\xi} \big], \tag{5.21}$$

for all $\mu \in U'$ and $\xi \ge 0$.

Proof. Consider a $\phi \in S(\xi,\mu)$. We recall the sequences ϕ^j and y^j of C^1-smooth functions that were introduced in the proof of Lemma 4.5, with $\phi^j \to \phi$ as $j \to \infty$ and $\mathrm{ev}_\xi y^j = \phi^j$. We will give a detailed estimate of the quantity $\mathrm{ev}_\xi z^j$ defined by

$$z^j = \Lambda^{-1}_\infty(\mu)[I - H_\xi]\Lambda_\infty(\mu) y^j. \tag{5.22}$$

To this end, we recall the Greens functions G from Theorem 3.1 that satisfy $G(\xi,\mu) \le K_2 e^{\alpha_- \xi}$ for $\xi \ge 0$ and allow us to write

$$[\Lambda^{-1}_\infty(\mu) f](\xi) = \int_{-\infty}^{\infty} G(\xi - s,\mu) f(s) ds. \tag{5.23}$$

Using this representation, we introduce the shorthands $z = z^j$ and $y = y^j$ and calculate

$$\begin{array}{lcl} |z(\xi)| & = & \left| \int_{-\infty}^0 G(\xi - s,\mu)[\Lambda_\infty(\mu) y](s) + \int_0^\xi G(\xi - s)[\Lambda_\infty(\mu) y](s) ds \right| \\ & \le & K_3 e^{-\alpha_S \xi} \|\phi\| \int_{-\infty}^0 e^{\alpha_-(\xi - s)} ds + \int_0^\xi e^{\alpha_-(\xi - s)} \|L(\xi,\mu) - L^+(\mu)\| \, \|\mathrm{ev}_s y\| \, ds \\ & \le & K_4 e^{-\alpha_S \xi} e^{\alpha_- \xi} \|\phi\| \\ & & \quad + K_5 \|\phi\| \int_0^\xi e^{\alpha_-(\xi - s)} [|\mu - \mu_0| \, e^{\alpha_- s} + e^{\alpha^f_- s}] e^{-\alpha_S(\xi - s)} ds \\ & \le & K_4 e^{(\alpha_- - \alpha_S)\xi} \|\phi\| \\ & & \quad + K_6 \|\phi\| \, e^{(\alpha_- - \alpha_S)\xi} \big[\, |\mu - \mu_0| \, [e^{\alpha_S \xi} - 1] + [e^{(\alpha_S + \alpha^f_- - \alpha_-)\xi} - 1] \big]. \end{array} \tag{5.24}$$

Similar estimates for $z(\xi + \theta)$, with $r_{\min} \le \theta \le r_{\max}$, complete the proof. □

Lemma 5.6. *Consider the setting of Theorem 5.4 and suppose that (5.17) holds. Then there exists a constant $K_1 > 0$ such that*

$$\left\| [I - \Pi_{Q(\infty,\mu)}]_{|Q(\xi,\mu)} \right\| \le K_1 \big[\, |\mu - \mu_0| \, e^{\alpha_- \xi} + e^{\alpha^f_- \xi} \big] \tag{5.25}$$

for all $\mu \in U'$ and all $\xi \ge 0$.

Proof. Consider a similar setup as in the proof of Lemma 5.5, now with $y^j \in \mathcal{Q}(\xi, \mu)$. This time, we need to estimate the quantity $\mathrm{ev}_\xi z^j$, with

$$z^j = \Lambda_\infty^{-1}(\mu) H_\xi \Lambda_\infty(\mu) y^j. \tag{5.26}$$

For all $\xi' \geq \xi$ we compute

$$[\Lambda_\infty(\mu) y^k](\xi') = [\Lambda(\mu) y^j](\xi') + [L(\xi', \mu) - L^+(\mu)] y^j_{\xi'} = [L(\xi', \mu) - L^+(\mu)] y^j_{\xi'}, \tag{5.27}$$

since $y^j \in \mathcal{Q}(\xi, \mu)$. The estimate is now immediate. □

Proof of Theorem 5.4. We will consider the identity

$$\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)} = \Pi_{P(\infty,\mu)} \Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu)} \Pi_{S(\xi,\mu)} \tag{5.28}$$

in more detail. In particular, let us write

$$\Pi_{Q(\infty,\mu)} \Pi_{S(\xi,\mu)} = \Lambda_\infty^{-1}(\mu) F(\xi, \mu) \Pi_{S(\xi)} [U_\xi^*(\mu)]^{-1}, \tag{5.29}$$

in which $F(\xi, \mu) : S(\xi, \mu_0) \to L^\infty\big(\mathbb{R}, \mathbb{C}^n\big)$ is defined by

$$[F(\xi, \mu)\phi](\xi') = [L(\xi', \mu) - L^+(\mu)] \mathrm{ev}_{\xi'} v^*_{S(\xi)}(\mu)\phi \tag{5.30}$$

for all $0 \leq \xi' \leq \xi$, while $[F(\xi, \mu)\phi](\xi') = 0$ for $\xi' \geq \xi$ and

$$[F(\xi, \mu)\phi](\xi') = [\Lambda_\infty(\mu) y(\xi, \mu)\phi](\xi') \tag{5.31}$$

for $\xi' \leq 0$. Here the quantity $y(\xi, \mu)\phi$ denotes a C^1-smooth function that is defined on the interval $(-\infty, r_{\max}]$ and has $\mathrm{ev}_0 y(\xi, \mu)\phi = \mathrm{ev}_0 v^*_{S(\xi)}(\mu)\phi$. In addition, Lemma 5.3 implies that this function can be chosen in such a way that it depends C^k-smoothly on μ and also satisfies the following estimate, for all $\xi' \leq r_{\max}$ and all integers $0 \leq \ell \leq k$,

$$\left\| D^\ell_\mu \mathrm{ev}_{\xi'} y(\xi, \mu)\phi \right\| + \left\| D^\ell_\mu \mathrm{ev}_{\xi'} y'(\xi, \mu)\phi \right\| \leq K_1 e^{-\alpha_S \xi} \left\| \phi \right\|. \tag{5.32}$$

The representation (5.29) together with the estimates (5.14) and (5.32) now allow us to mimic the calculation in (5.24) to obtain the estimate

$$\left\| D^\ell_\mu [\Pi_{Q(\infty,\mu)} \Pi_{S(\xi,\mu)}] \right\| \leq K_2 \big[\left|\mu - \mu_0\right| e^{\alpha_- \xi} + e^{(\alpha_- - \alpha_S)\xi} + e^{\alpha_-^f \xi} \big], \tag{5.33}$$

which holds for all integers $0 \leq \ell \leq k$ and all $\xi \geq 0$. An easier computation for the derivatives of the quantity $\Pi_{P(\infty,\mu)} \Pi_{Q(\xi,\mu)}$ completes the proof. □

To conclude this section, we show how we can isolate the part of $S(\xi, \mu)$ that decays at the rate of the leading positive eigenvalue of the characteristic matrix Δ^+. To this end, consider any $\nu > 0$ such that the characteristic equation $\det \Delta^+(z, \mu_0) = 0$ has no roots with $\mathrm{Re}\, z = \nu$. This allows us to perform the spectral decomposition

$$X = P_\nu(\infty, \mu_0) \oplus Q(\infty, \mu_0) \oplus \Gamma_{0,\nu}, \tag{5.34}$$

in which $\Gamma_{0,\nu}$ is the finite dimensional generalized eigenspace associated to the roots of $\det \Delta^+(z, \mu_0) = 0$ that have $0 < \mathrm{Re}\, z < \nu$. By nature of the spectral projection, we have the identity $\Pi_{\Gamma_{0,\nu}} + \Pi_{P_\nu(\infty,\mu_0)} = \Pi_{P(\infty,\mu_0)}$.

Let us now introduce the operator $U_\xi \in \mathcal{L}\big(P_\nu(\infty, \mu_0) \oplus \Gamma_{0,\nu} \oplus Q(\infty, \mu_0)\big)$, that is defined by

$$(\psi, \gamma, \phi) \quad \mapsto \quad [\Pi_{P_\nu(\infty,\mu_0)} \oplus \Pi_{\Gamma_{0,\nu}} \oplus \Pi_{Q(\infty,\mu_0)}][\Pi_{S_\nu(\xi,\mu)}\psi + \Pi_{S(\xi,\mu)}\gamma + \Pi_{Q(\xi,\mu)}\phi]. \tag{5.35}$$

Here we have introduced the space $S_\nu(\xi,\mu)$, that should be seen as the analogue of $S(\xi,\mu)$ after application of an exponential shift $e_{-\nu}$ to the system (5.1). We claim that U_ξ is close to the identity for ξ large enough and μ sufficiently close to μ_0. To see this, we compute

$$\begin{array}{rcl} \Pi_{P_\nu(\infty,\mu_0)} U_\xi(\psi,\gamma,\phi) & = & \psi + \Pi_{P_\nu(\infty,\mu_0)}[\Pi_{S_\nu(\xi,\mu)} - \Pi_{P_\nu(\infty,\mu_0)}]\psi + \Pi_{P_\nu(\infty,\mu_0)}[\Pi_{S(\xi,\mu)} - \Pi_{P(\infty,\mu_0)}]\gamma \\ & & + \Pi_{P_\nu(\infty,\mu_0)}[\Pi_{Q(\xi,\mu)} - \Pi_{Q(\infty,\mu_0)}]\phi. \end{array} \tag{5.36}$$

Similar estimates for the other projections complete the proof of the claim. This allows us to obtain the following splitting, for all sufficiently large ξ,

$$X = S^f(\xi,\mu) \oplus S^s(\xi,\mu) \oplus Q(\xi,\mu), \tag{5.37}$$

in which we have $\Pi_{S^f(\xi,\mu)} + \Pi_{S^s(\xi,\mu)} - \Pi_{P(\infty,\mu_0)} \to 0$ as $\xi \to \infty$ and $\mu \to \mu_0$. In addition, we have the identities

$$\begin{array}{rcl} S^f(\xi,\mu) & = & \Pi_{S_\nu(\xi,\mu)}\big(P_\nu(\infty,\mu_0)\big), \\ S^s(\xi,\mu) & = & \Pi_{S(\xi,\mu)}\big(\Gamma_{0,\nu}\big). \end{array} \tag{5.38}$$

6 Lin's Method for MFDEs

Now that the necessary machinery for linear systems has been developed, we are ready to consider the nonlinear functional differential equation of mixed type,

$$x'(\xi) = G(x_\xi,\mu) \tag{6.1}$$

and study bifurcations from heteroclinic connections. Our approach in this section was strongly inspired by the presentation in [16], but the notation here will differ somewhat. This is primarily due to the fact that we have to adapt the framework developed by Sandstede to an infinite dimensional setting and need to avoid the use of a variation-of-constants formula.

To set the stage, let q be a heteroclinic solution to (6.1) at some parameter $\mu = \mu_0$, that connects the two equilibria $q_\pm \in \mathbb{C}^n$. We set out to find solutions to (6.1) that remain close to q, for parameters that have $\mu \approx \mu_0$. We therefore write $x = q + u$ and find the variational equation

$$\begin{array}{rcl} u'(\xi) & = & G(q_\xi + u_\xi,\mu) - q'(\xi) \\ & = & G(q_\xi + u_\xi,\mu) - G(q_\xi,\mu_0) \\ & = & [G(q_\xi + u_\xi,\mu) - G(q_\xi,\mu_0) - D_1G(q_\xi,\mu_0)u_\xi - D_2G(q_\xi,\mu_0)(\mu-\mu_0)] \\ & & \quad + D_1G(q_\xi,\mu_0)u_\xi + D_2G(q_\xi,\mu_0)(\mu-\mu_0) \\ & = & \mathcal{N}(\xi,u_\xi,\mu) + D_1G(q_\xi,\mu_0)u_\xi + D_2G(q_\xi,\mu_0)(\mu-\mu_0), \end{array} \tag{6.2}$$

in which the nonlinearity $\mathcal{N}$ is given explicitly by

$$\mathcal{N}(\xi,\phi,\mu) = G(q_\xi + \phi,\mu) - G(q_\xi,\mu_0) - D_1G(q_\xi,\mu_0)\phi - D_2G(q_\xi,\mu_0)(\mu-\mu_0). \tag{6.3}$$

Throughout this entire section we will assume that the conditions (HG), (HL) and (HB) are satisfied. We therefore obtain the bound $\mathcal{N}(\xi,\phi,\mu) = O\big((|\mu-\mu_0| + \|\phi\|)^2\big)$ as $\mu \to \mu_0$ and $\phi \to 0$. This estimate holds uniformly for all $\xi \in \mathbb{R}$, due to the fact that the heteroclinic connection q can be uniformly bounded.

We write Λ for the operator (2.5) associated to the linear part of (6.2), i.e., for $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^n)$ we have

$$[\Lambda u](\xi) = u'(\xi) - D_1G(q_\xi,\mu_0)u_\xi. \tag{6.4}$$

Throughout the sequel, we use the following splitting of the state space X, that is associated to the linearization (6.4),

$$X = \widehat{P}(0) \oplus \widehat{Q}(0) \oplus B(0) \oplus \Gamma(0). \tag{6.5}$$

We pick two constants $\alpha_- < 0 < \alpha_+$ in such a way that the characteristic equations $\det \Delta^{\pm}(z) = 0$ associated to (6.4) have no roots in the strip $\alpha_- \leq \mathrm{Re}\, z \leq \alpha_+$. To ease the notation throughout this section, we now introduce the shorthands

$$\begin{array}{rclrcl} BC^+_{\alpha_-} & = & BC_{\alpha_-}\big([0,\infty),\mathbb{C}^n\big), & BC^-_{\alpha_+} & = & BC_{\alpha_+}\big((-\infty,0],\mathbb{C}^n\big), \\ BC^{\oplus}_{\alpha_-} & = & BC_{\alpha_-}\big([r_{\min},\infty),\mathbb{C}^n\big), & BC^{\ominus}_{\alpha_+} & = & BC_{\alpha_+}\big((-\infty,r_{\max}],\mathbb{C}^n\big). \end{array} \tag{6.6}$$

We recall the inverses for Λ on half-lines that were constructed in (3.33). In particular, we will use the appropriately defined inverses $\Lambda_+^{-1} = \Lambda^{\mathrm{qinv}}_{(\alpha_-)}(\mu_0)$ to ensure that for any $f \in BC^+_{\alpha_-}$ we can find $x \in BC^{\oplus}_{\alpha_-}$ with $\Lambda x = f$ on $[0,\infty)$, with the analogous properties for $\Lambda_-^{-1} = \Lambda^{\mathrm{qinv}}_{(\alpha_+)}(\mu_0)$.

Lemma 6.1. *Consider the linearization (6.4). For every pair of functions (g^-,g^+) that has $g^- \in BC^-_{\alpha_+}$ and $g^+ \in BC^+_{\alpha_-}$, there exists a unique pair $(u^-,u^+) = L_1(g^-,g^+)$, with $u^- \in BC^{\ominus}_{\alpha_+}$ and $u^+ \in BC^{\oplus}_{\alpha_-}$, such that the following properties hold.*

(i) We have the identities

$$\begin{array}{rclll} [\Lambda u^-](\xi') & = & g^-(\xi') & \text{for all} & \xi' \leq 0, \\ [\Lambda u^+](\xi') & = & g^+(\xi') & \text{for all} & \xi' \geq 0. \end{array} \tag{6.7}$$

(ii) We have $\mathrm{ev}_0 u^- \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$ and similarly $\mathrm{ev}_0 u^+ \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$.

(iii) We have $\mathrm{ev}_0[u^- - u^+] \in \Gamma(0)$, with

$$\langle \mathrm{ev}_0^* d, \mathrm{ev}_0[u^- - u^+]\rangle_0 = \int_{-\infty}^0 d(\xi')^* g^-(\xi') d\xi' + \int_0^\infty d(\xi')^* g^+(\xi') d\xi', \tag{6.8}$$

for any $d \in \mathcal{K}(\Lambda^*)$.

The linear map L_1 is bounded as a map from $BC^-_{\alpha_+} \times BC^+_{\alpha_-}$ into $BC^{\ominus}_{\alpha_+} \times BC^{\oplus}_{\alpha_-}$.

Proof. One may easily check that the choice

$$\begin{array}{rcl} u^- & = & \Lambda_-^{-1} g^- - E\Pi_{B(0)} \mathrm{ev}_0 \Lambda_-^{-1} g^- + E\Pi_{\widehat{P}(0)}[\Lambda_+^{-1} g^+ - \Lambda_-^{-1} g^-], \\ u^+ & = & \Lambda_+^{-1} g^+ - E\Pi_{B(0)} \mathrm{ev}_0 \Lambda_+^{-1} g^+ + E\Pi_{\widehat{Q}(0)}[\Lambda_-^{-1} g^- - \Lambda_+^{-1} g^+], \end{array} \tag{6.9}$$

ensures that all the required properties hold, using the identity (4.8) to verify (iii). □

Proof of Proposition 2.1. In order to find the functions $u^-(\mu)$ and $u^+(\mu)$ that satisfy the properties stated in Proposition 2.1, it suffices to solve the nonlinear fixed point problem

$$(u^-,u^+) = L_1\big(\mathcal{N}(u^-,\mu) + D_2G(q,\mu_0)(\mu-\mu_0), \mathcal{N}(u^+,\mu) + D_2G(q,\mu_0)(\mu-\mu_0)\big). \tag{6.10}$$

Here the maps $\mathcal{N}$ and D_2G should be viewed as substitution operators, i.e., for any $\xi' \geq 0$ we have $\mathcal{N}(u^+,\mu)(\xi') = \mathcal{N}(\xi', \mathrm{ev}_{\xi'} u^+, \mu)$, together with similar identities for $D_2G(q,\mu_0)$ and $\mathcal{N}(u^-,\mu)$. By construction we have that $(0,0)$ is a solution to this problem at $\mu = \mu_0$. The definition of $\mathcal{N}$ in (6.3) ensures that, by taking μ sufficiently close to μ_0 and by restricting u^+ to a small ball in $BC^{\oplus}_{\alpha_-}$, we may achieve

$$\big\|[D_2\mathcal{N}](\xi, \mathrm{ev}_\xi u^+, \mu)\big\| \leq C[\big\|\mathrm{ev}_\xi u^+\big\| + |\mu - \mu_0|] \tag{6.11}$$

for all $\xi \geq 0$. Now consider the ball $B_\delta(0) \subset BC^{\ominus}_{\alpha_+} \times BC^{\oplus}_{\alpha_-}$ around the pair $(0,0)$ that has radius $\delta > 0$. Choosing δ sufficiently small, (6.11) implies that the right hand side of (6.10) is a contraction on $B_\delta(0)$. In addition, choosing a sufficiently small neighbourhood $U' \subset U$ ensures that the right hand side of (6.10) maps $B_\delta(0)$ into itself. Together with the implicit function theorem, these observations show that for each $\mu \in U'$, equation (6.10) has a unique solution in $B_\delta(0)$, that depends C^{k+1}-smoothly on μ. We lose one order of smoothness here, due to the fact that the substitution operator $\mathcal{N}$ is only C^{k+1}-smooth [16]. □

We now proceed towards establishing Theorem 2.2. In order to meet the boundary conditions in item (iii), we will need to insert $x^{\pm} = q(\xi) + u^{\pm}(\mu)(\xi) + v^{\pm}(\xi)$ into the nonlinear equation (6.1). We find that $v^{\pm}$ must solve the equations

$$\begin{array}{rcll} [D_\xi v^-](\xi) & = & \mathcal{M}^-(\xi, \mathrm{ev}_\xi v^-, \mu) + D_1 G(q_\xi + \mathrm{ev}_\xi u^-(\mu), \mu)\mathrm{ev}_\xi v^-, & \xi \leq 0, \\ [D_\xi v^+](\xi) & = & \mathcal{M}^+(\xi, \mathrm{ev}_\xi v^+, \mu) + D_1 G(q_\xi + \mathrm{ev}_\xi u^+(\mu), \mu)\mathrm{ev}_\xi v^+, & \xi \geq 0, \end{array} \tag{6.12}$$

in which the nonlinearities $\mathcal{M}^{\pm}$ are given by

$$\mathcal{M}^{\pm}(\xi, \phi, \mu) = G\big(q_\xi + \mathrm{ev}_\xi u^{\pm}(\mu) + \phi, \mu\big) - D_1 G\big(q_\xi + \mathrm{ev}_\xi u^{\pm}(\mu), \mu\big)\phi - G(q_\xi + \mathrm{ev}_\xi u^{\pm}, \mu). \tag{6.13}$$

Let us write $\Lambda(\mu)$ for the operator (2.5) associated to the inhomogeneous linearization

$$v'(\xi) = D_1 G\big(q_\xi + \widetilde{\mathrm{ev}}_\xi u(\mu), \mu\big)v_\xi + f(\xi), \tag{6.14}$$

in which we have $\widetilde{\mathrm{ev}}_\xi u(\mu) = \mathrm{ev}_\xi u^+(\mu)$ for $\xi \geq 0$ and $\widetilde{\mathrm{ev}}_\xi u(\mu) = \mathrm{ev}_\xi u^-(\mu)$ for $\xi \leq 0$. We remark here that the matrix-valued functions $A_j(\xi, \mu)$ associated to (6.14) that were introduced in (3.17) are no longer continuous at $\xi = 0$ for $\mu \neq \mu_0$, but this will not matter for our purposes here. For convenience, we introduce the following shorthands for $\omega_+ > 0$ and $\omega_- < 0$,

$$\begin{array}{rclrcl} C^+_{(\omega^+)} & = & C\big([0, \omega^+], \mathbb{C}^n\big), & C^-_{(\omega^-)} & = & C\big([\omega^-, 0], \mathbb{C}^n\big), \\ C^\oplus_{(\omega^+)} & = & C\big([r_{\min}, \omega^+ + r_{\max}], \mathbb{C}^n\big), & C^\ominus_{(\omega^-)} & = & C\big([\omega^- + r_{\min}, r_{\max}], \mathbb{C}^n\big). \end{array} \tag{6.15}$$

We also recall the splitting $X = Q(\xi, \mu) \oplus S(\xi, \mu)$ that holds for all $\xi \geq 0$. Similarly, for all $\xi \leq 0$ we will use the splitting $X = P(\xi, \mu) \oplus R(\xi, \mu)$. Here we have introduced the spaces $R(\xi, \mu)$, that should be seen as the natural counterparts of $S(\xi, \mu)$ on the negative half-line.

Lemma 6.2. *Consider the parameter-dependent inhomogeneous linear system (6.14). Then there exists a neighbourhood $U' \subset U$, with $\mu_0 \in U'$ and a constant $\Omega > 0$, such that for every $\mu \in U'$, every pair $\omega^- < -\Omega < \Omega < \omega^+$, every pair $(g^-, g^+) \in C^-_{(\omega^-)} \times C^+_{(\omega^+)}$ and every pair $(\phi^-, \phi^+) \in Q(-\infty) \times P(\infty)$, there exists a unique pair $(v^-, v^+) \in C^\ominus_{(\omega^-)} \times C^\oplus_{(\omega^+)}$ that satisfies the following properties.*

(i) The functions $v^{\pm}$ satisfy the linear system

$$\begin{array}{lll} [\Lambda(\mu)v^-](\xi') = g^-(\xi') & \textit{for all} & \omega^- \leq \xi' \leq 0, \\ [\Lambda(\mu)v^+](\xi') = g^+(\xi') & \textit{for all} & 0 \leq \xi' \leq \omega^+. \end{array} \tag{6.16}$$

(ii) We have $\mathrm{ev}_0 v^-(\mu) \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$ and similarly $\mathrm{ev}_0 v^+(\mu) \in \widehat{P}(0) \oplus \widehat{Q}(0) \oplus \Gamma(0)$.

(iii) The gap between v^- and v^+ at zero satisfies $\mathrm{ev}_0[v^-(\mu) - v^+(\mu)] \in \Gamma(0)$.

(iv) The functions $v^{\pm}$ satisfy the boundary conditions

$$\begin{array}{rcl} \Pi_{Q(-\infty)}\mathrm{ev}_{\omega^-} v^- & = & \phi^-, \\ \Pi_{P(\infty)}\mathrm{ev}_{\omega^+} v^+ & = & \phi^+, \end{array} \tag{6.17}$$

in which we have introduced the shorthands $\Pi_{Q(-\infty)} = \Pi_{Q(-\infty, \mu_0)}$ and $\Pi_{P(\infty)} = \Pi_{P(\infty, \mu_0)}$.

This pair (v^-, v^+) will be denoted by

$$(v^-, v^+) = L_3\big(g^-, g^+, \phi^-, \phi^+, \mu, \omega^-, \omega^+\big), \tag{6.18}$$

in which L_3 is a linear operator with respect to the first four variables that depends C^{k+1}-smoothly on μ, with a norm that can be bounded independently of $\omega^{\pm}$.

In addition, consider any $d \in \mathcal{K}^*$ and write $d^+ = Eu^*_{Q^*(0)}\mathrm{ev}^*_0 d$ and $d^- = Eu^*_{P^*(0)}\mathrm{ev}^*_0 d$. Then the following identity holds for the gap at zero,

$$\begin{array}{lcl} \langle \mathrm{ev}^*_0 d^-, \mathrm{ev}_0 v^- \rangle_{0,\mu} & = & \langle \mathrm{ev}^*_0 d^+, \mathrm{ev}_0 v^+ \rangle_{0,\mu} \\ & & + \langle \mathrm{ev}^*_{\omega^-} d^-, \mathrm{ev}_{\omega^-} v^- \rangle_{\omega^-,\mu} - \langle \mathrm{ev}^*_{\omega^+} d^+, \mathrm{ev}_{\omega^+} v^+ \rangle_{\omega^+,\mu} \\ & & + \int_{\omega^-}^0 d^-(\xi')^* g^-(\xi') d\xi' + \int_0^{\omega^+} d^+(\xi')^* g^+(\xi') d\xi'. \end{array} \tag{6.19}$$

Proof. We first define the functions $w^+ = \Lambda_+^{-1}(\mu) g^+$ and $w^- = \Lambda_-^{-1}(\mu) g^-$. In order to satisfy the conditions (ii) through (iv), we now set out to find $\psi^{B+} \in B(0)$, $\psi^{B-} \in B(0)$, $\psi^{\widehat{Q}} \in \widehat{Q}(0)$, $\psi^{\widehat{P}} \in \widehat{P}(0)$, $\psi^S \in P(\infty)$ and $\psi^R \in Q(-\infty)$ that satisfy the linear system

$$\begin{array}{lcl} -\Pi_{B(0)} w_0^+ & = & \psi^{B+} + \Pi_{B(0)} \mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)} \psi^S, \\ -\Pi_{B(0)} w_0^- & = & \psi^{B-} + \Pi_{B(0)} \mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)} \psi^R, \\ -\Pi_{\widehat{Q}(0)} [w_0^- - w_0^+] & = & -\psi^{\widehat{Q}} + \Pi_{\widehat{Q}(0)} u^*_{P(0)}(\mu) [\psi^{\widehat{P}} + \psi^{B-}] \\ & & \quad + \Pi_{\widehat{Q}(0)} \big[\mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)} \psi^R - \mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)} \psi^S \big], \\ -\Pi_{\widehat{P}(0)} [w_0^- - w_0^+] & = & \psi^{\widehat{P}} - \Pi_{\widehat{P}(0)} u^*_{Q(0)}(\mu) [\psi^{\widehat{Q}} + \psi^{B+}] \\ & & \quad + \Pi_{\widehat{P}(0)} \big[\mathrm{ev}_0 E \Pi_{R(\omega^-,\mu)} \psi^R - \mathrm{ev}_0 E \Pi_{S(\omega^+,\mu)} \psi^S \big], \\ \Pi_{P(\infty)} [\phi^+ - w^+_{\omega^+}] & = & \psi^S + \Pi_{P(\infty)} \mathrm{ev}_{\omega^+} E u^*_{Q(0)}(\mu) [\psi^{\widehat{Q}} + \psi^{B+}] \\ & & \quad + \Pi_{P(\infty)} [\Pi_{S(\omega^+,\mu)} - \Pi_{P(\infty)}] \psi^S, \\ \Pi_{Q(-\infty)} [\phi^- - w^-_{\omega^-}] & = & \psi^R + \Pi_{Q(-\infty)} \mathrm{ev}_{\omega^-} E u^*_{P(0)}(\mu) [\psi^{\widehat{P}} + \psi^{B-}] \\ & & \quad + \Pi_{Q(-\infty)} [\Pi_{R(\omega^-,\mu)} - \Pi_{Q(-\infty)}] \psi^R. \end{array} \tag{6.20}$$

Then upon writing $\psi^Q = \psi^{\widehat{Q}} + \psi^{B+}$ and $\psi^P = \psi^{\widehat{P}} + \psi^{B-}$ and defining

$$\begin{array}{lcl} v^+ & = & w^+ + E u^*_{Q(0)}(\mu) \psi^Q + E \Pi_{S(\omega^+,\mu)} \psi^S, \\ v^- & = & w^- + E u^*_{P(0)}(\mu) \psi^P + E \Pi_{R(\omega^-,\mu)} \psi^R, \end{array} \tag{6.21}$$

we see that the properties (i) through (iv) are satisfied. The exponential estimates in Theorem 5.1, together with the results established in Lemma 5.2 and Theorem 5.4, ensure that by choosing a sufficiently small neighbourhood $U' \subset U$, with $\mu_0 \in U'$ and a sufficiently large constant $\Omega > 0$, the system (6.20) can always be solved. Moreover, the inverse of the linear operator associated to (6.20) depends C^{k+1}-smoothly on μ.

To verify the identity (6.19), it suffices to observe that for any continuous function d that satisfies $\Lambda^*(\mu) d = 0$ on the interval $[0, \xi]$, we have

$$\langle \mathrm{ev}^*_0 d, \mathrm{ev}_0 x \rangle_{0,\mu} \; = \; \langle \mathrm{ev}^*_\xi d, \mathrm{ev}_\xi x \rangle_{\xi,\mu} + \int_\xi^0 d(\xi')^* [\Lambda(\mu) x](\xi') d\xi'. \tag{6.22}$$

To see the uniqueness of the pair (v^-, v^+) that has now been constructed, consider any continuous function $y \in C^{\oplus}_{(\omega^+)}$ that has $\Lambda(\mu) y = 0$ on $[0, \omega^+]$. Writing $z = E \Pi_{S(\omega^+,\mu)} \mathrm{ev}_{\omega^+} y$, we find that $\Lambda(\mu)(y - z) = 0$ on $[0, \omega^+]$, while $\mathrm{ev}_{\omega^+}[y - z] \in Q(\omega^+, \mu)$. This implies that $\mathrm{ev}_0[y - z] \in Q(0, \mu)$, which in turn means $y \in \mathcal{Q}(0, \mu) + \mathcal{S}(\omega^+, \mu)$, with some abuse of notation. It is thus sufficient to show that

$$\begin{array}{lcl} \mathcal{S}(\omega^+, \mu) & = & \Pi_{S(\omega^+,\mu)} \big(P(\infty) \big), \\ \mathcal{R}(\omega^-, \mu) & = & \Pi_{R(\omega^-,\mu)} \big(Q(-\infty) \big), \end{array} \tag{6.23}$$

but these identities follow directly from the discussion at the end of Section 5. ☐

We are now ready to consider a family of heteroclinic connections $\{q_j\}_{j \in \mathcal{J}}$ that connect the equilibria $\{q^*_\ell\}_{\ell \in \mathcal{J}^*}$, i.e.,

$$\lim_{\xi \to \pm\infty} q_j(\xi) = q^*_{j \pm \frac{1}{2}}. \tag{6.24}$$

For any $j \in \mathcal{J}$, let us write $\Lambda^{(j)}(\mu)$ for the linear operator (6.14) that is associated to the heteroclinic connection q_j and $\widehat{P}^{(j)}(0)$, $\widehat{Q}^{(j)}(0)$ and $\Gamma^{(j)}(0)$ for the spaces appearing in (6.5).

Lemma 6.3. *Consider the nonlinear equation (6.1) and a family of heteroclinic connections* $\{q_j\}_{j\in\mathcal{J}}$ *that satisfy (6.24). Then there exists a neighbourhood* $U' \subset U$*, with* $\mu_0 \in U'$ *and a constant* $\Omega > 0$*, such that for every* $\mu \in U'$*, every family* $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ *that has* $\omega_\ell > \Omega$ *for all* $\ell \in \mathcal{J}^*$*, every family* $\{g_j^-, g_j^+\}_{j\in\mathcal{J}}$ *with* $(g_j^-, g_j^+) \in C^-_{(\omega_j^-)} \times C^+_{(\omega_j^+)}$*, and every family* $\{\Phi_\ell\}_{\ell\in\mathcal{J}^*} \in X^{\mathcal{J}^*}$*, there is a unique family* $\{v_j^-, v_j^+\}_{j\in\mathcal{J}}$ *with* $(v_j^-, v_j^+) \in C^{\ominus}_{(\omega_j^-)} \times C^{\oplus}_{(\omega_j^+)}$*, that satisfies the following properties.*

(i) For every $j \in \mathcal{J}$*, the pair* (v_j^-, v_j^+) *solves the linear system*

$$\begin{array}{lll} [\Lambda^{(j)}(\mu)v_j^-](\xi') = g_j^-(\xi') & \text{for all} & \omega_j^- \le \xi' \le 0, \\ [\Lambda^{(j)}(\mu)v_j^+](\xi') = g_j^+(\xi') & \text{for all} & 0 \le \xi' \le \omega_j^+. \end{array} \tag{6.25}$$

(ii) For every $j \in \mathcal{J}$*, we have* $\mathrm{ev}_0 v_j^- \in \widehat{P}^{(j)}(0) \oplus \widehat{Q}^{(j)}(0) \oplus \Gamma^{(j)}(0)$ *and similarly* $\mathrm{ev}_0 v_j^+ \in \widehat{P}^{(j)}(0) \oplus \widehat{Q}^{(j)}(0) \oplus \Gamma^{(j)}(0)$*.*

(iii) For every $j \in \mathcal{J}$*, the gap between* $v_j^\pm$ *at zero satisfies* $\mathrm{ev}_0[v_j^- - v_j^+] \in \Gamma^{(j)}(0)$*.*

(iv) For every $\ell \in \mathcal{J}^*$*, we have the boundary condition*

$$\mathrm{ev}_{\omega^+_{\ell-\frac{1}{2}}} v^+_{\ell-\frac{1}{2}} - \mathrm{ev}_{\omega^-_{\ell+\frac{1}{2}}} v^-_{\ell+\frac{1}{2}} = \Phi_\ell, \tag{6.26}$$

which should be interpreted in the sense of item (iii) in Theorem 2.2.

This family $\{v_j^-, v_j^+\}$ *will be denoted by*

$$\{v_j^-, v_j^+\} = L_4\big(\{g_j^-, g_j^+\}, \{\Phi_\ell\}, \mu, \{\omega_\ell\}\big), \tag{6.27}$$

in which L_4 *is a linear operator with respect to the first two variables that depends* C^{k+1}*-smoothly on* μ*, with a norm that can be bounded independently of the family* $\{\omega_\ell\}$*.*

Proof. It suffices to choose a family $\{\phi_j^-, \phi_j^+\}_{j\in\mathcal{J}}$, with $\phi_j^- \in Q^{(j)}(-\infty)$ and $\phi_j^+ \in P^{(j)}(\infty)$, such that the family of solutions defined by $(v_j^-, v_j^+) = L^3\big(g_j^-, g_j^+, \phi_j^-, \phi_j^+, \mu, \omega_j^-, \omega_j^+\big)$ satisfies the following boundary condition for every $\ell \in \mathcal{J}^*$,

$$\begin{array}{lll} \Pi^{(\ell-\frac{1}{2})}_{P(\infty)}\big[\Phi_\ell + \mathrm{ev}_{\omega^-_{\ell+\frac{1}{2}}} L_3^-(g^-_{\ell+\frac{1}{2}}, g^+_{\ell+\frac{1}{2}}, 0, 0)\big] & = & \phi^+_{\ell-\frac{1}{2}} + K^+_{\ell-\frac{1}{2}}\big(\phi^-_{\ell+\frac{1}{2}}, \phi^+_{\ell+\frac{1}{2}}\big), \\ \Pi^{(\ell+\frac{1}{2})}_{Q(-\infty)}\big[\mathrm{ev}_{\omega^+_{\ell-\frac{1}{2}}} L_3^+(g^-_{\ell-\frac{1}{2}}, g^+_{\ell-\frac{1}{2}}, 0, 0) - \Phi_\ell\big] & = & \phi^-_{\ell+\frac{1}{2}} + K^-_{\ell+\frac{1}{2}}\big(\phi^-_{\ell-\frac{1}{2}}, \phi^+_{\ell-\frac{1}{2}}\big). \end{array} \tag{6.28}$$

Here we have introduced the obvious shorthand $L_3 = (L_3^-, L_3^+)$ and dropped the dependence of L_3 on μ and $\omega^\pm$. For any $j \in \mathcal{J}$ we can inspect (6.20) and obtain the bounds

$$\begin{array}{lll} \|K_j^+\| & \le & K_1 e^{\alpha_P \omega^-_{j+1}} + K_2 \left\| \Pi^{(j+1)}_{R(\omega^-_{j+1}, \mu)} - \Pi^{(j+1)}_{Q(-\infty)} \right\|, \\ \|K_j^-\| & \le & K_1 e^{-\alpha_Q \omega^+_{j-1}} + K_2 \left\| \Pi^{(j-1)}_{S(\omega^+_{j-1}, \mu)} - \Pi^{(j-1)}_{P(\infty)} \right\|, \end{array} \tag{6.29}$$

which ensures that the right hand side of (6.28) is close to the identity, for sufficiently large $\Omega > 0$ and a sufficiently small neighbourhood $U' \subset U$. □

With this result we are ready to establish the existence of the family $\{v_j^-, v_j^+\}_{j\in\mathcal{J}}$ that appears in Theorem 2.2. We will defer the proof of the estimates (2.18) to the next section.

Proof of Theorem 2.2. In order to find the family $\{v_j^-, v_j^+\}$, we will first fix the family $\{\omega_\ell\}$ and solve the fixed point problem

$$\{v_j^-, v_j^+\} = L_4\big(\{\mathcal{M}^-(v_j^-, \mu), \mathcal{M}^+(v_j^+, \mu)\}, \{\Phi_\ell\}, \mu, \{\omega_\ell\}\big). \tag{6.30}$$

First note that for some $C > 0$ we can make the estimate

$$\big\|D_2\mathcal{M}^+(\xi, \phi, \mu)\big\| \le C \|\phi\|, \tag{6.31}$$

uniformly for $\xi \ge 0$ and $\mu \in U'$. This allows us to proceed as in the proof of Proposition 2.1 to obtain families $v_j^\pm\big(\{\Phi_\ell\}, \mu, \{\omega_\ell\}\big)$ that solve (6.30), for small values of $\{\Phi_\ell\}$ and μ sufficiently close to μ_0. Moreover, these families depend C^k-smoothly on $\{\Phi_\ell\}$ and μ, where we have again lost an order of smoothness due to our use of the substitution operators $\mathcal{M}^\pm$. Upon choosing a sufficiently large constant $\Omega > 0$ and subsequently using (2.15) to pick the appropriate (small) values for $\Phi_\ell = \Phi_\ell(\mu)$, the fixed point of (6.30) will satisfy the properties (i) through (iv) in Theorem 2.2. Since $\Phi_\ell(\mu)$ depends C^k-smoothly on μ, the fixed point of (6.30) will share this property.

It remains to consider the smoothness of the jumps with respect to the family $\{\omega_\ell\}$. Let us therefore fix a sufficiently large $\overline{\Omega} > \Omega$ and reconsider the setting of Lemma 6.2. Instead of looking for a pair $(v^-, v^+) \in C^\ominus_{(\omega^-)} \times C^\oplus_{(\omega^+)}$ that satisfies the properties (i) through (iv), we will look for a pair $(v^-, v^+) \in C^\ominus_{(-\overline{\Omega})} \times C^\oplus_{(\overline{\Omega})}$ that satisfies these properties, still with the original quantities $\omega^\pm$ that have $|\omega^\pm| < \overline{\Omega}$. In order to solve this modified problem, let us adapt the action of the extension operator E on the space $S(\omega^+, \mu)$, to ensure that $E\psi \in C^\oplus_{\overline{\Omega}}$ for $\psi \in S(\omega^+, \mu)$, with a similar modification for the space $R(\omega^-, \mu)$. The exact details are irrelevant, as long as we still have $\Lambda(\mu)E\psi = 0$ on the interval $[0, \omega^+]$. After this modification, it again suffices to solve the linear system (6.20). To see that $\Pi_{S(\omega^+,\mu)}$ depends smoothly on ω^+, we note that for any ω_* we can redefine the space $\mathcal{S}(\omega_*, \mu)$ so that it contains solutions to (5.1) on the slightly larger half-line $[-1, \omega_*]$. We can then obtain solutions to (5.1) on the interval $[0, \omega^+]$ with $\omega^+ = \omega_* + \Delta\omega$, by solving (5.1) with $A_j(\xi, \mu, \Delta\omega) = A_j(\xi + \Delta\omega, \mu)$ and shifting the resulting function to the right by $\Delta\omega$. This observation allows us to treat the parameter ω^+ on the same footing as μ. We emphasize that these modifications do not affect the pair (v^-, v^+) when viewed as functions in $C^\oplus_{(\omega^-)} \times C^\ominus_{(\omega^+)}$, due to the uniqueness result in Lemma 6.2. Applying similar modifications to Lemma 6.3 and the construction above now completes the proof, using the estimates for $\mathcal{R}_j$ that are obtained in the next section. □

7 The remainder term

Our goal in this section is to obtain estimates on the size of the remainder term $\mathcal{R}_j$ that features in (2.17). To set the stage, assume that for some $j \in \mathcal{J}$ we have $d \in \mathcal{K}\big((\Lambda^{(j)})^*\big)$ with $\|\mathrm{ev}_0^* d\| = 1$. We also recall the functions $d^+(\mu) \in \mathcal{Q}^*(0, \mu)$ and $d^-(\mu) \in \mathcal{P}^*(0, \mu)$ that are defined by

$$\begin{array}{lcl} d^+(\mu) & = & Eu^*_{Q^*(0)}(\mu)\mathrm{ev}_0^* d, \\ d^-(\mu) & = & Eu^*_{P^*(0)}(\mu)\mathrm{ev}_0^* d. \end{array} \tag{7.1}$$

In this section, we will study the slightly modified remainder term $\widetilde{\mathcal{R}}_j$, that is given by

$$\begin{array}{lcl} \widetilde{\mathcal{R}}_j & = & \langle \mathrm{ev}_0^* d^-, \mathrm{ev}_0 v^- \rangle_{0,\mu} - \langle \mathrm{ev}_0^* d^+, \mathrm{ev}_0 v^+ \rangle_{0,\mu} \\ & & \quad - \langle \mathrm{ev}^*_{\omega_j^+} d^+, \mathrm{ev}_{\omega_{j+1}^-}[q_{j+1} + u^-_{j+1}(\mu) - q^*_{j+\frac{1}{2}}] \rangle_{\omega_j^+,\mu} \\ & & \quad + \langle \mathrm{ev}^*_{\omega_j^-} d^-, \mathrm{ev}_{\omega_{j-1}^+}[q_{j-1} + u^+_{j-1}(\mu) - q^*_{j-\frac{1}{2}}] \rangle_{\omega_j^-,\mu}. \end{array} \tag{7.2}$$

In the terminology of Theorem 2.2, we see that the difference satisfies $\Big|\widetilde{\mathcal{R}}_j - \mathcal{R}_j\Big| = O(|\mu - \mu_0|\, e^{-2\alpha\omega})$, but our final estimate on $\widetilde{\mathcal{R}}_j$ will satisfy the improved bound

$$\widetilde{\mathcal{R}}_j \le \|\mathrm{ev}_0^* d\| \,[C_1 |\mu - \mu_0|^2 e^{-2\alpha\omega} + C_2 e^{-3\alpha\omega}]. \tag{7.3}$$

To simplify our arguments, we introduce the following quantities that are associated to the boundary conditions in (2.15),

$$\begin{array}{rcl} \Theta_j^{\pm} & = & \mathrm{ev}_{\omega_j^{\pm}}[q_j + u_j^{\pm}(\mu) - q^*_{j\pm\frac{1}{2}}], \\ \Phi_j^{+} & = & \Pi^{(j)}_{P(\infty)}\big(\mathrm{ev}_{\omega_j^{+}}[q_j + u_j^{+}(\mu)] - \mathrm{ev}_{\omega_{j+1}^{-}}[q_{j+1} + u_{j+1}^{-}(\mu)]\big), \\ \Phi_j^{-} & = & \Pi^{(j)}_{Q(-\infty)}\big(\mathrm{ev}_{\omega_j^{-}}[q_j + u_j^{-}(\mu)] - \mathrm{ev}_{\omega_{j-1}^{+}}[q_{j-1} + u_{j-1}^{+}(\mu)]\big). \end{array} \tag{7.4}$$

We also introduce the supremum norms $\|\Theta\| = \sup_{j\in\mathcal{J}} \left\|\Theta_j^{\pm}\right\|$ and similarly $\|\Phi\| = \sup_{j\in\mathcal{J}} \left\|\Phi_j^{\pm}\right\|$. In addition, we introduce the terms

$$\begin{array}{rcl} r_j^{+} & = & \left\|\Pi_{P(\infty)}[\Pi_{S(\omega_j^{+},\mu)} - \Pi_{P(\infty,\mu)}]\Pi_{P(\infty)}\right\| + \left\|D_{\omega_j^{+}}\Pi_{P(\infty)}[\Pi_{S(\omega_j^{+},\mu)} - \Pi_{P(\infty,\mu)}]\Pi_{P(\infty)}\right\| \\ & & + \left\|\Pi_{P(\infty)}[\Pi_{P(\infty,\mu)} - \Pi_{P(\infty,\mu_0)}]\Pi_{P(\infty)}\right\| \\ r_j^{-} & = & \left\|\Pi_{Q(-\infty)}[\Pi_{R(\omega_j^{-},\mu)} - \Pi_{Q(-\infty,\mu)}]\Pi_{Q(-\infty)}\right\| + \left\|D_{\omega_j^{-}}\Pi_{Q(-\infty)}[\Pi_{R(\omega_j^{-},\mu)} - \Pi_{Q(-\infty,\mu)}]\Pi_{Q(-\infty)}\right\| \\ & & + \left\|\Pi_{Q(-\infty)}[\Pi_{Q(-\infty,\mu)} - \Pi_{Q(-\infty,\mu_0)}]\Pi_{Q(-\infty)}\right\|. \end{array} \tag{7.5}$$

We wish to remark at this point that

$$\left\|\Pi_{P(\infty)}[\Pi_{P(\infty,\mu)} - \Pi_{P(\infty,\mu_0)}]\Pi_{P(\infty)}\right\| = O(|\mu - \mu_0|^2), \tag{7.6}$$

with a similar identity holding for the projections on the Q-spaces. To see this, we mimic the constructions of $S(\xi,\mu)$ and $Q(\xi,\mu)$ in Section 5 and write $P(\infty,\mu)$ and $Q(\infty,\mu)$ as graphs over $P(\infty)$ and $Q(\infty)$, using functions $u^*_{P(\infty)}$ and $u^*_{Q(\infty)}$ that should be seen as the analogues of $u^*_{S(\xi)}$ and $u^*_{Q(\xi)}$. We also introduce $U^*_\infty(\mu) : X \to X$ via $U^*_\infty(\mu) = u^*_{P(\infty)}(\mu)\Pi_{P(\infty)} + u^*_{Q(\infty)}(\mu)\Pi_{Q(\infty)}$, which is invertible for μ close to μ_0. This allows us to write $\Pi_{P(\infty,\mu)} = u^*_{P(\infty)}(\mu)\Pi_{P(\infty)}[U^*_\infty(\mu)]^{-1}$ and hence

$$\begin{array}{rcl} D_\mu \Pi_{P(\infty,\mu)} & = & [D_\mu u^*_{P(\infty)}](\mu)\Pi_{P(\infty)}[U^*_\infty(\mu)]^{-1} \\ & & -u^*_{P(\infty)}(\mu)\Pi_{P(\infty)}[U^*_\infty(\mu)]^{-1}[D_\mu u^*_{P(\infty)}](\mu)\Pi_{P(\infty)}[U^*_\infty(\mu)]^{-1} \\ & & -u^*_{P(\infty)}(\mu)\Pi_{P(\infty)}[U^*_\infty(\mu)]^{-1}[D_\mu u^*_{Q(\infty)}](\mu)\Pi_{Q(\infty)}[U^*_\infty(\mu)]^{-1}. \end{array} \tag{7.7}$$

Wedging this expression between $\Pi_{P(\infty)}$ and evaluating this derivative at $\mu = \mu_0$ yields

$$\begin{array}{rcl} [\Pi_{P(\infty)} D_\mu \Pi_{P(\infty,\mu)} \Pi_{P(\infty)}]_{|\mu=\mu_0} & = & \Pi_{P(\infty)}[D_\mu u^*_{P(\infty)}](\mu_0)\Pi_{P(\infty)} \\ & & -\Pi_{P(\infty)}[D_\mu u^*_{P(\infty)}](\mu_0)\Pi_{P(\infty)} \\ & & -\Pi_{P(\infty)}[D_\mu u^*_{Q(\infty)}](\mu_0)\Pi_{Q(\infty)}\Pi_{P(\infty)} \\ & = & 0. \end{array} \tag{7.8}$$

Our main focus will be to study the rate at which the error terms $\widetilde{R}_j$ decay as the quantities $\{\omega_\ell\}_{\ell\in\mathcal{J}^*}$ tend towards infinity. In order to eliminate the need to keep track of constants, we introduce the notation

$$a\big(\mu, \{\omega_\ell\}\big) \le_* b\big(\mu, \{\omega_\ell\}\big) \tag{7.9}$$

to indicate that there exists a constant $C > 0$ such that for all $\mu \in U'$ and families $\{\omega_\ell\}$ that have $\omega_\ell > \Omega$ for all $\ell \in \mathcal{J}^*$, we have the inequality

$$a\big(\mu, \{\omega_\ell\}\big) \le C b\big(\mu, \{\omega_\ell\}\big). \tag{7.10}$$

As a final preparation, we will assume that for every $j \in \mathcal{J}$ we have obtained the splittings

$$\begin{array}{rcl} X & = & S^s(\omega_j^{+},\mu) \oplus S^f(\omega_j^{+},\mu) \oplus Q(\omega_j^{+},\mu), \\ X & = & R^s(\omega_j^{-},\mu) \oplus R^f(\omega_j^{-},\mu) \oplus P(\omega_j^{-},\mu), \end{array} \tag{7.11}$$

as introduced at the end of Section 5. We write α_S^f and α_R^f for the exponential rates associated to the fast spaces $S^f(\omega^+,\mu)$ and $R^f(\omega^-,\mu)$. In view of this more detailed splitting, we modify the definition of $v^\pm$ in (6.21) to make it read

$$\begin{array}{rcl} v^+ & = & w^+ + Eu^*_{Q(0)}(\mu)\phi^Q + E\Pi_{S^f(\omega^+,\mu)}\psi^S + E\Pi_{S^s(\omega^+,\mu)}\psi^S, \\ v^- & = & w^- + Eu^*_{P(0)}(\mu)\phi^P + E\Pi_{R^f(\omega^-,\mu)}\psi^R + E\Pi_{R^s(\omega^-,\mu)}\psi^R. \end{array} \tag{7.12}$$

Our first goal will be to fix a $j \in \mathcal{J}$, consider small boundary values ϕ^+ and ϕ^- and get estimates on the solution of the nonlinear fixed point problem

$$(v^-, v^+) = L_3\big(\mathcal{M}^-(v^-,\mu), \mathcal{M}^+(v^+,\mu), \phi^-, \phi^+, \mu, \omega^-, \omega^+\big), \tag{7.13}$$

in terms of ϕ^+ and ϕ^-. We proceed by introducing the notation $\mathbf{w}^- = (w^-, \psi^R, \psi^P)$, $\mathbf{w}^+ = (w^+, \psi^Q, \psi^S)$ and $\mathbf{w} = (w^-, w^+, \psi^R, \psi^P, \psi^Q, \psi^S) \in \mathbf{W}$, where $\mathbf{W}$ denotes the space

$$\mathbf{W} = C^\ominus_{(\omega^-)} \times C^\oplus_{(\omega^+)} \times Q(-\infty) \times P(0) \times Q(0) \times P(\infty). \tag{7.14}$$

The problem (7.13) can now be written as

$$\mathbf{w} = [I-K]^{-1}\Big([J_0B_0 + J_1B_1 + J_2B_2]\big(\mathcal{M}^-(\mathbf{w}^-), \mathcal{M}^+(\mathbf{w}^+)\big) + J_3(\phi^-,\phi^+)\Big), \tag{7.15}$$

in which the operators B_0, B_1 and B_2 act as

$$\begin{array}{rcl} B_0(g^-,g^+) & = & (\Lambda_-^{-1}(\mu)g^-, \Lambda_+^{-1}(\mu)g^+), \\ B_1(g^-,g^+) & = & \mathrm{ev}_0 B_0(g^-,g^+), \\ B_2(g^-,g^+) & = & \big(\mathrm{ev}_{\omega^-}, \mathrm{ev}_{\omega^+}\big)B_0(g^-,g^+), \end{array} \tag{7.16}$$

while the precise form of the operators $K \in \mathcal{L}(\mathbf{W})$ and J_i can be found by inspection of (6.20). Note that for any $\mathbf{b} \in \mathbf{W}$ we have the bound

$$\big\|[I-K]^{-1}\mathbf{b} - [I+K]\mathbf{b}\big\| \leq [I - \|K\|]^{-1}\|K\|\,\|K\mathbf{b}\|. \tag{7.17}$$

Now consider the first order estimate $\mathbf{w}_0 = [I-K]^{-1}J_3(\phi^-,\phi^+)$. Upon introducing the quantities

$$\begin{array}{rcl} T_0^+ & = & e^{-\alpha_S\omega^+}\big\|\Pi_{S^s(\omega^+,\mu)}\phi^+\big\| + e^{-\alpha_S^f\omega^+}\|\phi^+\|, \\ T_0^- & = & e^{\alpha_R\omega^-}\big\|\Pi_{R^s(\omega^-,\mu)}\phi^-\big\| + e^{\alpha_R^f\omega^-}\|\phi^-\|, \\ T_1^+ & = & r^+e^{-\alpha_S\omega^+}\|\phi^+\|, \\ T_1^- & = & r^-e^{\alpha_R\omega^-}\|\phi^-\|, \end{array} \tag{7.18}$$

together with $T_0 = T_0^- + T_0^+$ and $T_1 = T_1^- + T_1^+$, we find $\mathbf{w}_0 = (0, 0, \psi^R, \psi^P, \psi^Q, \psi^S)$, with

$$\begin{array}{rcl} \big\|\psi^R - \phi^-\big\| & \leq_* & r^-\|\phi^-\| + e^{\alpha_P\omega^-}[T_0 + T_1^+], \\ \big\|\psi^P\big\| & \leq_* & T_0 + T_1, \\ \big\|\psi^Q\big\| & \leq_* & T_0 + T_1, \\ \big\|\psi^S - \phi^+\big\| & \leq_* & r^+\|\phi^+\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-]. \end{array} \tag{7.19}$$

In order to see that these are in fact all the terms, we note that we can use a separate norm on $\mathbf{W}$ for each of the components. In particular, the operator K remains bounded, independently of $\omega^+ \geq \Omega$ and $\omega^- \leq -\Omega$, after the scalings $\widetilde{\psi}^S \sim e^{-\alpha_S\omega^+}\psi^S$ and $\widetilde{\psi}^R \sim e^{\alpha_R\omega^-}\psi^R$, which allows us to get the estimates on ψ^P and ψ^Q. To obtain the estimate on ψ^S, one can use the scalings $\widetilde{\psi}^Q \sim e^{-\alpha_Q\omega^+}\psi^Q$, $\widetilde{\psi}^P \sim e^{-\alpha_Q\omega^+}\psi^P$ and $\widetilde{\psi}^R = e^{\alpha_R\omega^-}e^{-\alpha_Q\omega^+}\psi^R$.

We now include the higher order terms using the expansion

$$\mathbf{w} \quad = \quad \mathbf{w}_0 + [I-K]^{-1}[J_0B_0 + J_1B_1 + J_2B_2]\big(\mathcal{M}^-(\mathbf{w}_0^-), \mathcal{M}^+(\mathbf{w}_0^+)\big) + (V^-, V^+), \tag{7.20}$$

in which $\|V^{\pm}\|_0 \leq_* \|\phi\|^3$, with $\phi = (\phi^+, \phi^-)$. We thus find that the fixed point $\mathbf{w} = (w^-, w^+, \psi^R, \psi^P, \psi^Q, \psi^S)$ of (7.15) can be bounded by

$$
\begin{array}{lll}
\|\widehat{w}^-\|_0 & \leq_* & e^{-2\alpha_S \omega^+} \|\phi\|^2 , \\
\|\widetilde{w}^-\|_{-\alpha_R} & \leq_* & e^{\alpha_R \omega^-} \|\phi\|^2 , \\
\|\widehat{w}^+\|_0 & \leq_* & e^{2\alpha_R \omega^-} \|\phi\|^2 , \\
\|\widetilde{w}^+\|_{\alpha_S} & \leq_* & e^{-\alpha_S \omega^+} \|\phi\|^2 , \\
\left\|\psi^R - \phi^-\right\| & \leq_* & r^- \|\phi^-\| + e^{\alpha_P \omega^-}[T_0 + T_1^+] + \|\phi\|^2 , \\
\left\|\psi^P\right\| & \leq_* & T_0 + T_1 + e^{-\alpha_S \omega^+} \|\phi\|^2 + e^{\alpha_R \omega^-} \|\phi\|^2 , \\
\left\|\psi^Q\right\| & \leq_* & T_0 + T_1 + e^{-\alpha_S \omega^+} \|\phi\|^2 + e^{\alpha_R \omega^-} \|\phi\|^2 , \\
\left\|\psi^S - \phi^+\right\| & \leq_* & r^+ \|\phi^+\| + e^{-\alpha_Q \omega^+}[T_0 + T_1^-] + \|\phi\|^2 ,
\end{array}
\tag{7.21}
$$

in which we have split $w^{\pm} = \widehat{w}^{\pm} + \widetilde{w}^{\pm}$. Adding higher order terms does not change these estimates.

We are now ready to move on to the full system. We will use (6.28) to find the family $\{\phi_j^-, \phi_j^+\}$ in terms of the boundary conditions $\{\Phi_j^-\}$ and $\{\Phi_j^+\}$. To this end, we reformulate (6.28) as follows,

$$
\begin{array}{rcl}
\phi_j^+ & = & \Phi_j^+ + \Pi_{P(\infty)}\big[\mathrm{ev}_{\omega_{j+1}^-} w_{j+1}^- + \mathrm{ev}_{\omega_{j+1}^-} u_{P(0)}^* \psi_{j+1}^P\big] \\
& & \quad + \Pi_{P(\infty)}[\Pi_{R^s(\omega_{j+1}^-,\mu)} + \Pi_{R^f(\omega_{j+1}^-,\mu)} - \Pi_{Q(-\infty)}]\psi_{j+1}^R , \\
\phi_j^- & = & \Phi_j^- + \Pi_{Q(-\infty)}\big[\mathrm{ev}_{\omega_{j-1}^+} w_{j-1}^+ + \mathrm{ev}_{\omega_{j-1}^+} u_{Q(0)}^* \psi_{j-1}^Q\big] \\
& & \quad + \Pi_{Q(-\infty)}[\Pi_{S^s(\omega_{j-1}^+,\mu)} + \Pi_{S^f(\omega_{j-1}^+,\mu)} - \Pi_{P(\infty)}]\psi_{j-1}^S .
\end{array}
\tag{7.22}
$$

We first set out to find the lowest order terms, i.e., we compute

$$
\{\phi_j^{(1)-}, \phi_j^{(1)+}\} = [D\{\phi_j^-, \phi_j^+\}](0)\big(\{\Phi_j^-\}, \{\Phi_j^+\}\big) = [I - \mathbf{K}]^{-1}(\{\Phi_j^-\}, \{\Phi_j^+\}), \tag{7.23}
$$

for some linear operator $\mathbf{K}$. We can use the estimate (7.21) to bound the components of $\mathbf{K}$ by

$$
\begin{array}{rcl}
\left|K_j^+(\{c^-\},\{c^+\})\right| & \leq_* & r_{j+1}^- \left\|c_{j+1}^-\right\| + r_{j+1}^- e^{\alpha_P \omega_{j+1}^-} e^{-\alpha_S \omega_{j+1}^+} \left\|c_{j+1}^+\right\| \\
& & + e^{\alpha_P \omega_{j+1}^-} e^{\alpha_R \omega_{j+1}^-} \left\|\Pi_{R^s(\omega_{j+1}^-,\mu)} c_{j+1}^-\right\| \\
& & + e^{\alpha_P \omega_{j+1}^-} e^{-\alpha_S \omega_{j+1}^+} \left\|\Pi_{S^s(\omega_{j+1}^+,\mu)} c_{j+1}^+\right\| \\
& & + e^{\alpha_P \omega_{j+1}^-}[e^{\alpha_R^f \omega_{j+1}^-} \left\|c_{j+1}^-\right\| + e^{-\alpha_S^f \omega_{j+1}^+} \left\|c_{j+1}^+\right\|] \\
& & + e^{\alpha_P \omega_{j+1}^-} e^{-\alpha_S \omega_{j+1}^+} r_{j+1}^+ \left\|c_{j+1}^+\right\| , \\
\left|K_j^-(\{c^-\},\{c^+\})\right| & \leq_* & r_{j-1}^+ \left\|c_{j-1}^+\right\| + r_{j-1}^+ e^{-\alpha_Q \omega_{j-1}^+} e^{\alpha_R \omega_{j-1}^-} \left\|c_{j-1}^-\right\| \\
& & + e^{-\alpha_Q \omega_{j-1}^+} e^{\alpha_R \omega_{j-1}^-} \left\|\Pi_{R^s(\omega_{j-1}^-,\mu)} c_{j-1}^-\right\| \\
& & + e^{-\alpha_Q \omega_{j-1}^+} e^{-\alpha_S \omega_{j-1}^+} \left\|\Pi_{S^s(\omega_{j+1}^+,\mu)} c_{j-1}^+\right\| \\
& & + e^{-\alpha_Q \omega_{j-1}^+}[e^{\alpha_R^f \omega_{j-1}^-} \left\|c_{j-1}^-\right\| + e^{-\alpha_S^f \omega_{j-1}^+} \left\|c_{j-1}^+\right\|] \\
& & + e^{-\alpha_Q \omega_{j-1}^+} e^{\alpha_R \omega_{j-1}^-} r_{j-1}^- \left\|c_{j-1}^-\right\| .
\end{array}
\tag{7.24}
$$

Let us now introduce the scaling factors

$$
\begin{array}{rcl}
\widetilde{\phi}_j^+ & \sim & e^{-\alpha_P \omega_{j+1}^-} \phi_j^+ , \\
\widetilde{\phi}_j^- & \sim & e^{\alpha_Q \omega_{j-1}^+} \phi_j^- .
\end{array}
\tag{7.25}
$$

In terms of these scaled variables, the operator K can still be bounded independently of the family $\{\omega_\ell\}_{\ell \in \mathcal{J}^*}$, as long as $\omega_\ell > \Omega$ for all $\ell \in \mathcal{J}^*$. We wish to invoke the general estimate (7.17) using these scaled variables. Let us therefore split up $\mathbf{K}(\{\Phi^-\}, \{\Phi^+\}) = \{a^-, a^+\} + \{b^-, b^+\}$, in which

$$
\begin{array}{rcl}
\left\|b_j^-\right\| & \leq_* & r_{j-1}^+ \left\|\Phi_{j-1}^+\right\| , \\
\left\|b_j^+\right\| & \leq_* & r_{j+1}^- \left\|\Phi_{j+1}^-\right\| ,
\end{array}
\tag{7.26}
$$

while the family $\{a^-, a^+\}$ can be estimated using the scaled norm according to

$$\left\|\{a^-, a^+\}\right\|_{\mathrm{sc}} \leq_* S_0, \tag{7.27}$$

where we have introduced the quantity

$$\begin{array}{rcl} S_0 & = & \sup_{j\in\mathcal{J}} \Big\{ e^{\alpha_R \omega_j^-} \left\|\Pi_{R^s(\omega_j^-,\mu)} \Phi_j^-\right\| + e^{-\alpha_S \omega_j^+} \left\|\Pi_{S^s(\omega_j^+,\mu)} \Phi_j^+\right\| \\ & & + e^{\alpha_R^f \omega_j^-} \left\|\Phi_j^-\right\| + e^{-\alpha_S^f \omega_j^+} \left\|\Phi_j^+\right\| \\ & & + e^{\alpha_R \omega_j^-} [r_j^- + r_j^+] \left\|\Phi_j^-\right\| + e^{-\alpha_S \omega_j^+} [r_j^+ + r_j^-] \left\|\Phi_j^+\right\| \\ & & + e^{\alpha_R \omega_j^-} r_{j-1}^+ \left\|\Phi_{j-1}^+\right\| + e^{-\alpha_S \omega_j^+} r_{j+1}^- \left\|\Phi_{j+1}^-\right\| \Big\}. \end{array} \tag{7.28}$$

We now compute $\mathbf{K}(\{b^-, b^+\}) = \{e^-, e^+\} + \{f^-, f^+\}$ and obtain the bounds

$$\begin{array}{lcl} \left\|e_j^-\right\| & \leq_* & r_{j-1}^+ r_j^- \left\|\Phi_j^-\right\|, \\ \left\|e_j^+\right\| & \leq_* & r_{j+1}^- r_j^+ \left\|\Phi_j^+\right\|, \\ \left\|\{f^-, f^+\}\right\|_{\mathrm{sc}} & \leq_* & S_0. \end{array} \tag{7.29}$$

Since the family $\{e^-, e^+\}$ is now bounded componentwise by the family $\{\Phi^-, \Phi^+\}$, we may write

$$\begin{array}{lcl} \left\|\phi_j^{(1)-} - \Phi_j^-\right\| & \leq_* & r_{j-1}^+ \left\|\Phi_{j-1}^+\right\| + e^{-\alpha_Q \omega_{j-1}^+} S_0, \\ \left\|\phi_j^{(1)+} - \Phi_j^+\right\| & \leq_* & r_{j+1}^- \left\|\Phi_{j+1}^-\right\| + e^{\alpha_P \omega_{j+1}^-} S_0. \end{array} \tag{7.30}$$

Adding the second order terms, we arrive at

$$\begin{array}{lcl} \left\|\phi_j^- - \Phi_j^-\right\| & \leq_* & r_{j-1}^+ [\left\|\Phi_{j-1}^+\right\| + \|\Phi\|^2] + e^{-\alpha_Q \omega_{j-1}^+} [S_0 + \|\Phi\|^2], \\ \left\|\phi_j^+ - \Phi_j^+\right\| & \leq_* & r_{j+1}^- [\left\|\Phi_{j+1}^-\right\| + \|\Phi\|^2] + e^{\alpha_P \omega_{j+1}^-} [S_0 + \|\Phi\|^2]. \end{array} \tag{7.31}$$

We are now finally in a position to estimate the error term. To this end, we write $\widetilde{\mathcal{R}}_j = \widetilde{\mathcal{R}}_j^+ + \widetilde{\mathcal{R}}_j^-$ and represent the two parts in the following manner,

$$\begin{array}{rcl} \widetilde{\mathcal{R}}_j^- & = & \int_{\omega_j^-}^0 d^-(\xi') \mathcal{M}^-(\xi', \mathrm{ev}_{\xi'} v_j^-, \mu) d\xi' + \langle \mathrm{ev}_{\omega_j^-}^* d^-, \phi_j^- - \Phi_j^- \rangle_{\omega_j^-,\mu} \\ & & + \langle \mathrm{ev}_{\omega_j^-}^* d^-, [\Pi_{R(\omega_j^-,\mu)} - \Pi_{Q(-\infty)}] \mathrm{ev}_{\omega_j^-} v_j^- \rangle_{\omega_j^-,\mu} \\ & & - \langle \mathrm{ev}_{\omega_j^-}^* d^-, [\Pi_{R(\omega_j^-,\mu)} - \Pi_{Q(-\infty)}][\Theta_j^- - \Theta_{j-1}^+] \rangle_{\omega_j^-,\mu} \\ & & + \langle \mathrm{ev}_{\omega_j^-}^* d^-, \Pi_{R(\omega_j^-,\mu)} \Theta_j^- \rangle_{\omega_j^-,\mu}, \\ \widetilde{\mathcal{R}}_j^+ & = & \int_0^{\omega_j^+} d^+(\xi') \mathcal{M}^+(\xi', \mathrm{ev}_{\xi'} v_j^+, \mu) d\xi' - \langle \mathrm{ev}_{\omega_j^+}^* d^+, \phi_j^+ - \Phi_j^+ \rangle_{\omega_j^+,\mu} \\ & & - \langle \mathrm{ev}_{\omega_j^+}^* d^+, [\Pi_{S(\omega_j^+,\mu)} - \Pi_{P(\infty)}] \mathrm{ev}_{\omega_j^+} v_j^+ \rangle_{\omega_j^+,\mu} \\ & & + \langle \mathrm{ev}_{\omega_j^+}^* d^+, [\Pi_{S(\omega_j^+,\mu)} - \Pi_{P(\infty)}][\Theta_j^+ - \Theta_{j+1}^-] \rangle_{\omega_j^+,\mu} \\ & & - \langle \mathrm{ev}_{\omega_j^+}^* d^+, \Pi_{S(\omega_j^+,\mu)} \Theta_j^+ \rangle_{\omega_j^+,\mu}. \end{array} \tag{7.32}$$

In order to complete our estimate, observe that $\Pi_{S(\omega_j^+,\mu)} \Theta_j^+ \leq_* \|\Theta\|^2$, because the function $q_j + u_j^+(\mu)$ is contained in the stable manifold of $q_{j+\frac{1}{2}}^*$. Notice also that for some small constant $\epsilon > 0$, we may write $d^+(\xi) = O(e^{-(\alpha_S+\epsilon)\xi})$ as $\xi \to \infty$, since the characteristic equation $\det \Delta^+(z) = 0$ associated to the equilibrium $q_{j+\frac{1}{2}}^*$ has no roots in the strip $0 \leq \mathrm{Re}\, z \leq \alpha_S$. Putting everything together we obtain the following result, which completes the proof of Theorem 2.2.

Lemma 7.1. *Consider the setting of Theorem 2.2. For every $j \in \mathcal{J}$, we have the following estimate for the error term $\widetilde{\mathcal{R}}_j$ that is defined in (7.2),*

$$\begin{array}{rcl} \widetilde{\mathcal{R}}_j & \leq_* & e^{-\alpha_S \omega_+^j} [\|\Theta\|^2 + (r_{j+1}^- + r_j^+) \|\Theta\| + e^{\alpha_P \omega_{j+1}^-} S_0] \\ & & + e^{\alpha_R \omega_j^-} [\|\Theta\|^2 + (r_{j-1}^+ + r_j^-) \|\Theta\| + e^{-\alpha_Q \omega_{j-1}^+} S_0]. \end{array} \tag{7.33}$$

8 Derivative of the remainder term

The main goal of this section is to provide an estimate for the quantities $D_{\omega_\ell}\widetilde{\mathcal{R}}_j$, for $j \in \mathcal{J}$ and $\ell \in \mathcal{J}^*$. Recalling the fixed point problem (7.15), together with the solution $\mathbf{w} = \mathbf{w}(\phi^-, \phi^+, \omega^-, \omega^+) \in \mathbf{W}$, we set out to compute the derivatives $D_{\phi^\pm}\mathbf{w}$ and $D_{\omega^\pm}\mathbf{w}$. We start with the observation

$$\begin{array}{rcl} D_{\omega^\pm}\mathbf{w} & = & [I-K]^{-1}[D_{\omega^\pm}K]\mathbf{w} \\ & & +[I-K]^{-1}D_{\omega^\pm}[J_0B_0 + J_1B_1 + J_2B_2]\big(\mathcal{M}^-(\mathbf{w}^-), \mathcal{M}^+(\mathbf{w}^+)\big). \end{array} \tag{8.1}$$

Inspection of (6.20) yields the identities

$$\begin{array}{rcl} [D_{\omega^-}K_{\psi^R}]\mathbf{w} & \leq_* & r^- \left\|\psi^R\right\| + e^{\alpha_P\omega^-}\left\|\psi^P\right\|, \\ {[D_{\omega^-}K_{\psi^P}]\mathbf{w}} & \leq_* & e^{\alpha_R\omega^-}\left\|\Pi_{R^s(\omega^-,\mu)}\psi^R\right\| + e^{\alpha_R^f\omega^-}\left\|\psi^R\right\|, \\ {[D_{\omega^+}K_{\psi^P}]\mathbf{w}} & \leq_* & e^{-\alpha_S\omega^+}\left\|\Pi_{S^s(\omega^+,\mu)}\psi^S\right\| + e^{-\alpha_S^f\omega^+}\left\|\psi^S\right\|, \\ {[D_{\omega^-}K_{\phi^Q}]\mathbf{w}} & \leq_* & e^{\alpha_R\omega^-}\left\|\Pi_{R^s(\omega^-,\mu)}\psi^R\right\| + e^{\alpha_R^f\omega^-}\left\|\psi^R\right\|, \\ {[D_{\omega^+}K_{\phi^Q}]\mathbf{w}} & \leq_* & e^{-\alpha_S\omega^+}\left\|\Pi_{S^s(\omega^+,\mu)}\psi^S\right\| + e^{-\alpha_S^f\omega^+}\left\|\psi^S\right\|, \\ {[D_{\omega^+}K_{\psi^S}]\mathbf{w}} & \leq_* & r_+\left\|\psi^S\right\| + e^{-\alpha_Q\omega^+}\left\|\psi^Q\right\|. \end{array} \tag{8.2}$$

Let us write $\mathbf{Dw}_\mathbf{0}^\pm = [I-K]^{-1}[D_{\omega^\pm}K]\mathbf{w}$. Utilizing the bounds (7.21) and performing a calculation in the spirit of the previous section now yields the estimates

$$\begin{array}{rcl} (\mathbf{Dw}_\mathbf{0}^+)_{\psi^R} & \leq_* & e^{\alpha_P\omega^-}\big[T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\left\|\phi\right\|^2\big], \\ (\mathbf{Dw}_\mathbf{0}^+)_{\psi^P} & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\left\|\phi\right\|^2, \\ (\mathbf{Dw}_\mathbf{0}^+)_{\psi^Q} & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\left\|\phi\right\|^2, \\ (\mathbf{Dw}_\mathbf{0}^+)_{\psi^S} & \leq_* & r^+\left\|\phi^+\right\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-] + \left\|\phi\right\|^2. \end{array} \tag{8.3}$$

Inserting this back into (8.1), we find the following estimate for the derivative $D_{\omega^+}\mathbf{w}$, where $\mathbf{w} = (w^-, w^+, \psi^R, \psi^P, \psi^Q, \psi^S)$ is the solution of the fixed point problem (7.15),

$$\begin{array}{rcl} \|D_{\omega^+}\widehat{w}^-\|_0 & \leq_* & e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\left\|\phi\right\|^2, \\ \|D_{\omega^+}\widetilde{w}^-\|_{-\alpha_R} & \leq_* & e^{\alpha_R\omega^-}e^{\alpha_P\omega^-}e^{-\alpha_S\omega^+}\left\|\phi\right\|^2, \\ \|D_{\omega^+}\widehat{w}^+\|_0 & \leq_* & e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\left\|\phi\right\|^2, \\ \|D_{\omega^+}\widetilde{w}^+\|_{\alpha_S} & \leq_* & e^{-\alpha_S\omega^+}r^+\left\|\phi\right\|^2, \\ \left\|D_{\omega^+}\psi^R\right\| & \leq_* & e^{\alpha_P\omega^-}\big[T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-]\big] \\ & & \quad + e^{-\alpha_S\omega^+}[e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-} + e^{\alpha_P\omega^-}]\left\|\phi\right\|^2, \\ \left\|D_{\omega^+}\psi^P\right\| & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\left\|\phi\right\|^2, \\ \left\|D_{\omega^+}\psi^Q\right\| & \leq_* & T_0^+ + T_1^+ + e^{-\alpha_S\omega^+}e^{-\alpha_Q\omega^+}[T_0^- + T_1^-] + e^{-\alpha_S\omega^+}\left\|\phi\right\|^2, \\ \left\|D_{\omega^+}\psi^S\right\| & \leq_* & r^+\left\|\phi^+\right\| + e^{-\alpha_Q\omega^+}[T_0 + T_1^-] + \left\|\phi\right\|^2. \end{array} \tag{8.4}$$

Using a similar calculation, we also obtain

$$\begin{array}{rcl} \|[D_\phi\widehat{w}^-](\Delta\phi^+, \Delta\phi^-)\|_0 & \leq_* & e^{-2\alpha_S\omega^+}\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \|[D_\phi\widetilde{w}^-](\Delta\phi^+, \Delta\phi^-)\|_{-\alpha_R} & \leq_* & e^{\alpha_R\omega^-}\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \|[D_\phi\widehat{w}^+](\Delta\phi^+, \Delta\phi^-)\|_0 & \leq_* & e^{2\alpha_R\omega^-}\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \|[D_\phi\widetilde{w}^+](\Delta\phi^+, \Delta\phi^-)\|_{\alpha_S} & \leq_* & e^{-\alpha_S\omega^+}\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \left\|[D_\phi\psi^R](\Delta\phi^+, \Delta\phi^-)\right\| & \leq_* & \left\|\Delta\phi^-\right\| + e^{-\alpha_S\omega^+}e^{\alpha_P\omega^-}\left\|\Delta\phi^+\right\| + e^{-\alpha_S\omega^+}\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \left\|[D_\phi\psi^P](\Delta\phi^+, \Delta\phi^-)\right\| & \leq_* & \Delta T_0 + \Delta T_1 + [e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \left\|[D_\phi\psi^Q](\Delta\phi^+, \Delta\phi^-)\right\| & \leq_* & \Delta T_0 + \Delta T_1 + [e^{-\alpha_S\omega^+} + e^{\alpha_R\omega^-}]\left\|\phi\right\|\left\|\Delta\phi\right\|, \\ \left\|[D_\phi\psi^S](\Delta\phi^+, \Delta\phi^-)\right\| & \leq_* & \left\|\Delta\phi^+\right\| + e^{\alpha_R\omega^-}e^{-\alpha_Q\omega^+}\left\|\Delta\phi^-\right\| + e^{\alpha_R\omega^-}\left\|\phi\right\|\left\|\Delta\phi\right\|. \end{array} \tag{8.5}$$

These expressions can be used to determine the derivatives $D_{\omega_\ell}\phi_j^\pm$ for $j \in \mathcal{J}$ and $\ell \in \mathcal{J}^*$ using the boundary conditions in (7.22). Let us therefore fix some $j^* \in \mathcal{J}$ and determine the family $\{b_j^-, b_j^+\}_{j\in\mathcal{J}}$ that describes the derivatives of the family $\{\phi^-, \phi^+\}$ with respect to $\omega_{j^*}^+$ up to first order in $\|\Phi\|$, i.e., $\left\|D_{\omega_{j^*}^+}\phi_j^\pm - b_j^\pm\right\| \leq_* \|\Phi\|^2$. Careful inspection of (7.22) shows that we must solve the coupled system

$$\begin{array}{rcl} b_j^+ & = & \mathbf{B}_j^+ + L_j^+\big(b_{j+1}^-, b_{j+1}^+\big), \\ b_j^- & = & \mathbf{B}_j^- + L_j^-\big(b_{j-1}^-, b_{j-1}^+\big), \end{array} \tag{8.6}$$

in which the norms of $L_j^\pm$ share the estimates for $K_j^\pm$ given in (7.24), while the initial value $\mathbf{B}$ can be bounded as

$$\begin{array}{rcl} \left\|\mathbf{B}_j^+ - \delta_{jj^*}D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| & \leq_* & \delta_{j,j^*-1}\big[e^{\alpha_P\omega_{j^*}^-}\left\|D_{\omega_{j^*}^+}\psi_{j^*}^P\right\| + r_{j^*}^-\left\|D_{\omega_{j^*}^+}\psi_{j^*}^R\right\|\big] \\ \left\|\mathbf{B}_j^-\right\| & \leq_* & \delta_{j,j^*+1}e^{-\alpha_Q\omega_{j^*}^+}[\left\|\psi_{j^*}^Q\right\| + \left\|D_{\omega_j^+}\psi_{j^*}^Q\right\|] \\ & & \quad + \delta_{j,j^*+1}r_{j^*}^+[\left\|\psi_{j^*}^S\right\| + \left\|D_{\omega_{j^*}^+}\psi_j^S\right\|]. \end{array} \tag{8.7}$$

As in the previous section, a small number of applications of the operator family $\{L_j^-, L_j^+\}$, together with the scaling (7.25), enables us to obtain an estimate on the solution to the coupled system (8.6). We obtain

$$\begin{array}{rcl} \left\|b_j^+ - \delta_{jj^*}D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| & \leq_* & \delta_{jj^*}r_{j^*+1}^-r_{j^*}^+\left\|\phi_{j^*}^+\right\| + e^{\alpha_P\omega_{j+1}^-}\mathbf{E}, \\ \left\|b_j^-\right\| & \leq_* & \delta_{j,j^*+1}[r_{j^*}^+\left\|D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| + r_{j^*}^+\left\|\phi_{j^*}^+\right\|] + e^{-\alpha_Q\omega_{j-1}^+}\mathbf{E}, \end{array} \tag{8.8}$$

in which we have defined the quantity

$$\begin{array}{rcl} \mathbf{E} = \mathbf{E}_{(j^*)}^+ & = & T_0^{j^*} + T_1^{j^*} + r_{j^*}^+e^{\alpha_R\omega_{j^*+1}^-}\left\|\phi_{j^*}^+\right\| + r_{j^*}^-e^{-\alpha_S\omega_{j^*}^+}\left\|\phi_{j^*}^+\right\| \\ & & + e^{-\alpha_S\omega_{j^*}^+}\left\|\Pi_{S^s(\omega_{j^*}^+,\mu)}D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| + e^{-\alpha_S^f\omega_{j^*}^+}\left\|D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| \\ & & + [e^{-\alpha_S\omega_{j^*}^+} + e^{\alpha_R\omega_{j^*+1}^-}]r_{j^*}^+\left\|D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\| + r_{j^*}^-e^{-\alpha_S\omega_{j^*}^+}\left\|D_{\omega_{j^*}^+}\Phi_{j^*}^+\right\|. \end{array} \tag{8.9}$$

Of course, similar estimates can be obtained for the derivatives with respect to $\omega_{j^*+1}^-$. In order to combine these estimates, we now fix $\ell^* \in \mathcal{J}^*$ and introduce the following quantities for any $\ell \in \mathcal{J}^*$,

$$\begin{array}{rcl} |\Phi_\ell|_1 & = & \left|\Phi_{\ell-\frac{1}{2}}^+\right| + \left|\Phi_{\ell+\frac{1}{2}}^-\right| + \left|D_{\omega_{\ell^*}}\Phi_{\ell-\frac{1}{2}}^+\right| + \left|D_{\omega_{\ell^*}}\Phi_{\ell+\frac{1}{2}}^-\right|, \\ \left|\Phi_\ell^S\right|_{1,s} & = & \left|\Pi_{S^s(\omega_\ell,\mu)}\Phi_{\ell-\frac{1}{2}}^+\right| + \left|\Pi_{S^s(\omega_\ell,\mu)}D_{\omega_{\ell^*}}\Phi_{\ell-\frac{1}{2}}^+\right|, \\ \left|\Phi_\ell^R\right|_{1,s} & = & \left|\Pi_{R^s(-\omega_\ell,\mu)}\Phi_{\ell+\frac{1}{2}}^-\right| + \left|\Pi_{R^s(-\omega_\ell,\mu)}D_{\omega_{\ell^*}}\Phi_{\ell+\frac{1}{2}}^-\right|, \\ r_\ell & = & r_{\ell-\frac{1}{2}}^+ + r_{\ell+\frac{1}{2}}^-. \end{array} \tag{8.10}$$

We also introduce the quantity S_1, which should be seen as the sum of the quantities $\mathbf{E}_{(\ell^*-\frac{1}{2})}^+ + \mathbf{E}_{(\ell^*+\frac{1}{2})}^-$, after insertion of the inequalities (7.31),

$$\begin{array}{rcl} S_1 & = & e^{-\alpha_S\omega_{\ell^*}}\left|\Phi_{\ell^*}^S\right|_{1,s} + e^{-\alpha_R\omega_{\ell^*}}\left|\Phi_{\ell^*}^R\right|_{1,s} \\ & & + [e^{-\alpha_S^f\omega_{\ell^*}} + e^{-\alpha_R^f\omega_{\ell^*}}]\left|\Phi_{\ell^*}\right|_1 \\ & & + [e^{-\alpha_S\omega_{\ell^*}} + e^{-\alpha_R\omega_{\ell^*}}]r_\ell\left|\Phi_{\ell^*}\right|_1 \\ & & + e^{-\alpha_R\omega_{\ell^*-1}}\left|\Phi_{\ell^*-1}^R\right|_{1,s} + e^{-\alpha_R^f\omega_{\ell^*-1}}\left|\Phi_{\ell^*-1}\right|_1 + r_{\ell^*-1}e^{-\alpha_R\omega_{\ell^*-1}}\left|\Phi_{\ell^*-1}\right|_1 \\ & & + e^{-\alpha_S\omega_{\ell^*+1}}\left|\Phi_{\ell^*+1}^S\right|_{1,s} + e^{-\alpha_S^f\omega_{\ell^*+1}}\left|\Phi_{\ell^*+1}\right|_1 + r_{\ell^*+1}e^{-\alpha_S\omega_{\ell^*+1}}\left|\Phi_{\ell^*+1}\right|_1 \\ & & + e^{-\alpha_S\omega_{\ell^*}}e^{-\alpha_P\omega_{\ell^*}}S_0 + e^{-\alpha_R\omega_{\ell^*-1}}e^{-\alpha_Q\omega_{\ell^*-1}}S_0 \\ & & + e^{-\alpha_R\omega_{\ell^*}}e^{-\alpha_Q\omega_{\ell^*}}S_0 + e^{-\alpha_S\omega_{\ell^*+1}}e^{-\alpha_P\omega_{\ell^*+1}}S_0 \\ & & + \|\Phi\|^2. \end{array} \tag{8.11}$$

We are now ready to put everything together. Using (8.8) together with the definitions above and inserting the second order terms in the appropriate places, we obtain the estimates

$$\begin{array}{lcl} \left\| D_{\omega_{\ell^*}} [\phi^+_{\ell^*-\frac{1}{2}} - \Phi^+_{\ell^*-\frac{1}{2}}] \right\| & \leq_* & e^{-\alpha_P \omega_{\ell^*}} S_1 + r^-_{\ell^*+\frac{1}{2}} \left| \Phi_{\ell^*} \right|_1 \\ & & \quad + r^-_{\ell^*+\frac{1}{2}} [e^{-\alpha_Q \omega_{\ell^*}} S_0 + r^+_{\ell^*-\frac{1}{2}} e^{-\alpha_P \omega_{\ell^*}} S_0] + \left\| \Phi \right\|^2 , \\ \left\| D_{\omega_{\ell^*}} [\phi^-_{\ell^*+\frac{1}{2}} - \Phi^-_{\ell^*+\frac{1}{2}}] \right\| & \leq_* & e^{-\alpha_Q \omega_{\ell^*}} S_1 + r^+_{\ell^*-\frac{1}{2}} \left| \Phi_{\ell^*} \right|_1 \\ & & \quad + r^+_{\ell^*-\frac{1}{2}} [e^{-\alpha_P \omega_{\ell^*}} S_0 + r^-_{\ell^*+\frac{1}{2}} e^{-\alpha_Q \omega_{\ell^*}} S_0] + \left\| \Phi \right\|^2 , \\ \left\| D_{\omega_{\ell^*}} \phi^+_j \right\| & \leq_* & e^{\alpha_P \omega^-_{j+1}} S_1, \quad \text{for all } j \neq \ell^* - \frac{1}{2}, \\ \left\| D_{\omega_{\ell^*}} \phi^-_j \right\| & \leq_* & e^{-\alpha_Q \omega^+_{j-1}} S_1, \quad \text{for all } j \neq \ell^* + \frac{1}{2}. \end{array} \tag{8.12}$$

With these estimates in hand, we can move on and analyze (7.32) in order to obtain estimates for the quantities $D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_j$. Care has to be taken to distinguish the terms in (7.32) that depend directly on ω_{ℓ^*}, from those that only depend on this quantity through the family of boundary terms $\{\phi^-, \phi^+\}$. Using methods similar to those employed here to estimate the derivatives $\left| D_\mu \widetilde{\mathcal{R}}_j \right|$ and $\left| D_\mu D_{\omega^*_\ell} \widetilde{\mathcal{R}}_j \right|$, we obtain the following result.

Lemma 8.1. *Consider the setting of Theorem 2.2 and recall the error terms (7.2). Fix an $\ell^* \in \mathcal{J}^*$ and let $j \in \mathcal{J}$ be such that $j \neq \ell^* \pm \frac{1}{2}$. Then the following estimates hold for the error terms $\{\widetilde{\mathcal{R}}\}$,*

$$\begin{array}{lcl} \left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_{\ell^*-\frac{1}{2}} \right| & \leq_* & \left| \widetilde{\mathcal{R}}_{\ell^*-\frac{1}{2}} \right| + e^{-\alpha_S \omega_{\ell^*}} [\left\| \Theta \right\| + r_{\ell^*}] \left| \Phi_\ell \right|_1 \\ & & \quad + e^{-\alpha_S \omega_{\ell^*}} e^{-\alpha_P \omega_{\ell^*}} S_1 \\ & & \quad + e^{-\alpha_R \omega_{\ell^*-1}} e^{-\alpha_Q \omega_{\ell^*-1}} S_1, \\ \left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_{\ell^*+\frac{1}{2}} \right| & \leq_* & \left| \widetilde{\mathcal{R}}_{\ell^*+\frac{1}{2}} \right| + e^{-\alpha_R \omega_{\ell^*}} [\left\| \Theta \right\| + r_{\ell^*}] \left| \Phi_\ell \right|_1 \\ & & \quad + e^{-\alpha_R \omega_{\ell^*}} e^{-\alpha_Q \omega_{\ell^*}} S_1 \\ & & \quad + e^{-\alpha_S \omega_{\ell^*+1}} e^{-\alpha_P \omega_{\ell^*+1}} S_1, \\ \left| D_{\omega_{\ell^*}} \widetilde{\mathcal{R}}_j \right| & \leq_* & e^{-\alpha_S \omega^+_j} e^{\alpha_P \omega^-_{j+1}} S_1 + e^{\alpha_R \omega^-_j} e^{-\alpha_Q \omega^+_{j-1}} S_1. \end{array} \tag{8.13}$$

In addition, for all $j \in \mathcal{J}$ we have the estimates

$$\begin{array}{lcl} \left| D_\mu \widetilde{\mathcal{R}}_j \right| & \leq_* & \left| \mu - \mu_0 \right| e^{-2\alpha\omega} + e^{-3\alpha\omega}, \\ \left| D_\mu D_{\omega^*_\ell} \widetilde{\mathcal{R}}_j \right| & \leq_* & \left| \mu - \mu_0 \right| e^{-2\alpha\omega} + e^{-3\alpha\omega}, \end{array} \tag{8.14}$$

in which α and ω are defined as in Theorem 2.2.

We are now ready to consider the orbit-flip bifurcation for (2.1). An application of Theorem 2.2 to the setting of Theorem 2.3 yields a finite dimensional bifurcation equation, that is very similar to the one obtained in Chapter 4 of [16]. The calculations contained in that chapter carry over to our setting and can hence be used to establish Theorem 2.3.

9 Acknowledgements

The authors wish to thank an anonymous referee for valuable comments and suggestions. The first author also acknowledges support from the Netherlands Organisation for Scientific Research (NWO).

References

[1] Chow, S. N. and Deng, B. (1989), Bifurcation of a Unique Periodic Orbit from a Homoclinic Orbit in Infinite-Dimensional Systems. *Trans. Amer. Math. Soc.* **312**, 539–587.

[2] Diekmann, O., van Gils, S. A., Verduyn-Lunel, S. M., and Walther, H. O. (1995), *Delay Equations.* Springer-Verlag, New York.

[3] Georgi, M. (2008), Bifurcations from Homoclinic Orbits to Non-Hyperbolic Equilibria in Reversible Lattice Differential Equations. *Nonlinearity* **21**, 735–763.

[4] Georgi, M., Jukes, A., Lamb, J., and Webster, K., Homoclinic Period Blow-up and the Detection of Multipulse Solutions in Lattice Differential Equations. *Preprint.*

[5] Hale, J. K. and Verduyn-Lunel, S. M. (1993), *Introduction to Functional Differential Equations.* Springer–Verlag, New York.

[6] Härterich, J., Sandstede, B., and Scheel, A. (2002), Exponential Dichotomies for Linear Non-Autonomous Functional Differential Equations of Mixed Type. *Indiana Univ. Math. J.* **51**, 1081–1109.

[7] Homburg, A. J., Kokubu, H., and Krupa, M. (1994), The Cusp Horseshoe and its Bifurcations from Inclination-Flip Homoclinic Orbits. *Ergod. Theory Dyn. Syst.* **14**, 667–693.

[8] Hupkes, H. J. (2008), Invariant Manifolds and Applications for Functional Differential Equations of Mixed Type. *PhD Thesis.*

[9] Hupkes, H. J., Augeraud-Veron, E., and Verduyn-Lunel, S. M. (2008), Center Projections for Smooth Difference Equations of Mixed Type. *J. Diff. Eqn.* **244**, 803–835.

[10] Hupkes, H. J. and Verduyn-Lunel, S. M., Center Manifolds for Periodic Functional Differential Equations of Mixed Type. *J. Diff. Eq., to appear.*

[11] Hupkes, H. J. and Verduyn-Lunel, S. M. (2007), Center Manifold Theory for Functional Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **19**, 497–560.

[12] Kisaka, M., Kokubu, H., and Oka, H. (1993), Bifurcation to N-homoclinic Orbits and N-periodic orbits in Vector Fields. *J. Dyn. Diff. Eq.* **5**, 305–358.

[13] Lin, X. B. (1990), Using Melnikov's Method to Solve Shilnikov's Problems. *Proc. Roy. Soc. Edinburgh* **116**, 295–325.

[14] Mallet-Paret, J. (1999), The Fredholm Alternative for Functional Differential Equations of Mixed Type. *J. Dyn. Diff. Eq.* **11**, 1–48.

[15] Mallet-Paret, J. and Verduyn-Lunel, S. M., Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differential Equations. *J. Diff. Eq., to appear.*

[16] Sandstede, B. (1993), Verzweigungstheorie Homokliner Verdopplungen. *PhD Thesis.*

[17] Shilnikov, L. P. (1962), Some Cases of Generation of Periodic Motions in an n-dimensional Space. *Soviet Math. Dokl.* **3**, 394–397.

[18] Shilnikov, L. P. (1965), A Case of the Existence of a Countable Number of Periodic Motions. *Soviet Math. Dokl.* **6**, 163–166.

[19] Shilnikov, L. P. (1966), On the Generation of a Periodic Motion from a Trajectory which Leaves and Re-enters a Saddle-Saddle State of Equilibrium. *Soviet Math. Dokl.* **7**, 1155–1158.

[20] Shilnikov, L. P. (1968), On the Generation of a Periodic Motion from Trajectories Doubly Asymptotic to an Equilibrium State of Saddle Type. *Math. USSR Sbornik* **6**, 427–437.

[21] Shilnikov, L. P. (1969), On a New Type of Bifurcation of Multidimensional Dynamical Systems. *Soviet Math. Dokl.* **10**, 1368–1371.

[22] Shilnikov, L. P. (1970), A Contribution to a Problem of the Structure of an Extended Neighborhood of a Rough Equilibrium State of Saddle-Focus Type. *Math. USSR Sbornik* **10**, 91–102.

[23] Yanagida, E. (1987), Branching of Double Pulse Solutions from Single Solutions in Nerve Axon Equations. *J. Diff. Eq.* **66**, 243–262.