Lin's Method and Homoclinic Bifurcations for Functional Differential Equations of Mixed Type

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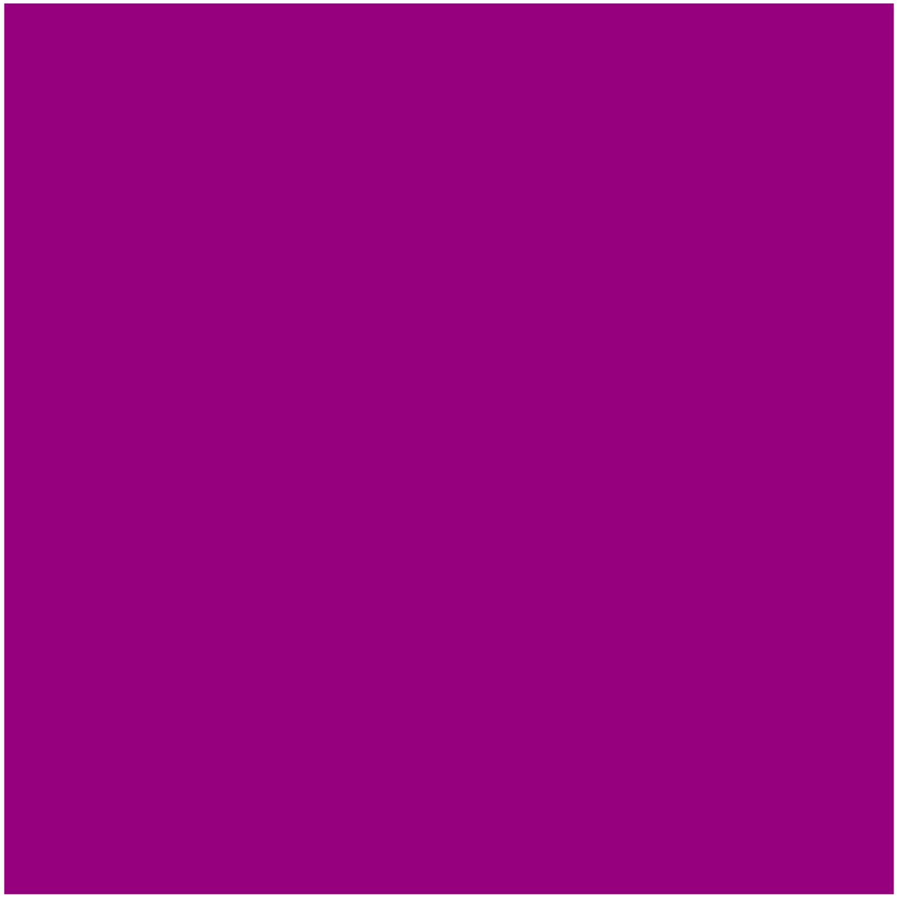
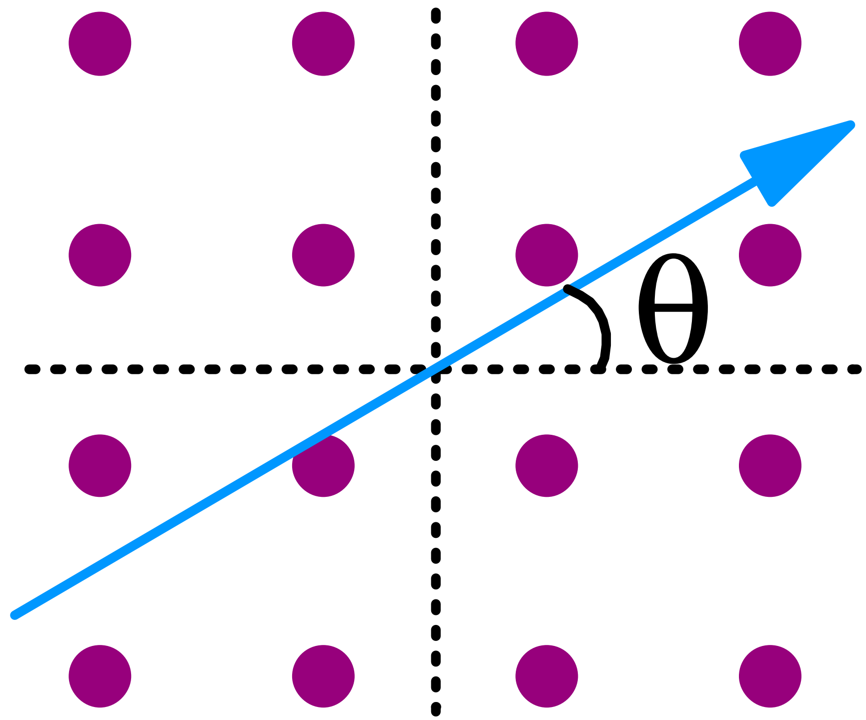
Lattice equations



Continuous media (PDE)



Discrete media (Lattice Equation)

- Fruitful to include structure of underlying space into models.
- Differential equations on lattices (LDEs) are becoming increasingly popular.

LDE Applications

Lattice equations have arisen in many disciplines.

- Image processing
  - Chua and Roska (1993): cellular networks for recognizing edges / outlines in pictures
- Biology
  - Dauxois, Peyrard and Bishop (1993): Denaturation of DNA
  - Keener and Sneed (1998): signal propagation through nerves with discrete gaps
- Material science
  - Kontorova and Frenkel (1938): Deformation of crystals
  - Fermi, Pasta, Ulam (1955): Acoustics in strings of particles
  - Bates and Chmaj (1999): Ising model for phase transitions
- Chemical reaction theory
  - T. Erneux and G. Nieolis (1993): Bistable reaction-diffusion systems

Interesting dynamical features such as lattice anisotropy, propagation failure and thermalization barriers are chief motivations from modelling perspective.

Mixed Type Functional Differential Equations (MFDEs)

Looking for travelling wave solutions for LDEs, one immediately encounters

$$x'(\xi) = G(x_\xi). \tag{1}$$

- x is a continuous function with $x(\xi) \in \mathbb{R}^n$.

- $x_\xi \in C([-1,1])$ is the **state** of x at ξ, i.e.,

$$x_\xi(\theta) = x(\xi + \theta), \qquad \theta \in [-1,1] \ .$$

- $G : C([-1,1]) \to \mathbb{R}^n$ is sufficiently smooth.

Note that $x'(\xi)$ depends on both past and future values of x.

Eq. (1) is called a functional differential equation of mixed type (MFDE).

The program

Recall the MFDE

$$x'(\xi) = G(x_\xi). \qquad (2)$$

Main theme: lift ODE techniques and constructions to the infinite dimensional setting of (2).

First step Interested in solutions to (2) near equilibria $\overline{x}$.

Flow cannot be defined for (2). Mielke and Kirchgässner faced with similar problem when considering elliptic PDEs, but still managed to construct a CM.

(2006) H. + VL: All solutions to (2) sufficiently close to equilibrium $\overline{x}$ lie on a finite dimensional center manifold. *J. Dyn. Diff. Eqns 19, 497-560.*

The flow on this CM is described by an ODE.

Allows analysis of Hopf bifurcation for (2).

Applications

Lattice differential equations are not the only application of MFDEs.

- Solving optimal control problems with delays.

  Hughes (1968): Euler Lagrange equations for such problems are MFDEs.

  Benhabib & Nishimura (1979): introduced high dimensional economic growth optimal control model. Periodic orbits established.

  Rustichini (1989): Added delay into framework. Even scalar model now yields the desired periodic orbits, using Hopf bifurcation theorem with the CM-reduction.

- Recent models in economic theory lead directly to algebraic MFDEs (H. d'Albis and E. Augeraud-Veron),

$$Ax'(\xi) = G(x_\xi), \quad A \text{ singular matrix.}$$

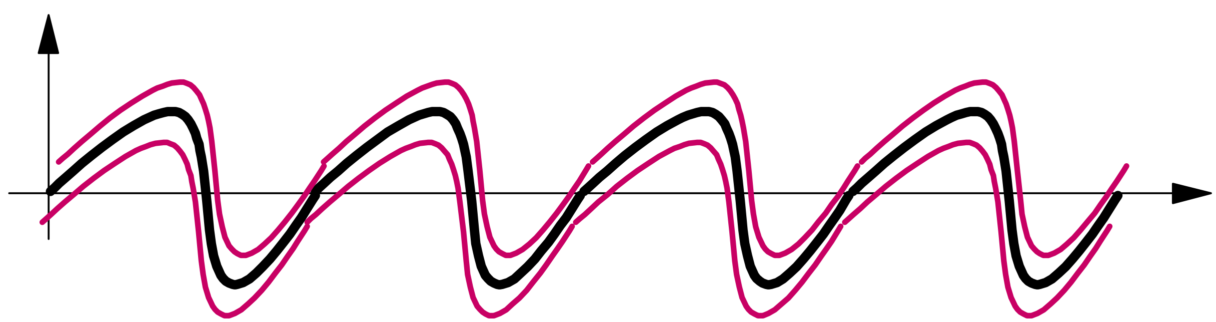
(2008) H. + Augeraud-Veron + VL: Hopf bifurcation for algebraic MFDEs *J. Diff. Eqns 244, 803-835.*

Floquet theory

Next Step Interested in solutions to (3) near periodic solutions $x = p$.

$$x'(\xi) = G(x_\xi). \tag{3}$$

(2007) H. + VL: All solutions to (3) sufficiently close to periodic solution p lie on a finite dimensional center manifold (under discreteness condition on the Floquet spectrum). *J. Diff. Eq.; in press.*



Again, behaviour on CM is described by ODE, which can be analyzed with standard techniques.

Computation of Floquet exponents is still hard. Traditional monodromy-approach does not work.

Bifurcations from periodic travelling waves

Consider the Frenkel-Kontorova model

$$\ddot{x}_j(t) + \gamma \dot{x}_j(t) = x_{j-1}(t) + x_{j+1}(t) - 2x_j(t) - d \sin x_j(t) + F.$$

Known: Uniform Sliding States: travelling wave solutions with profile Φ and

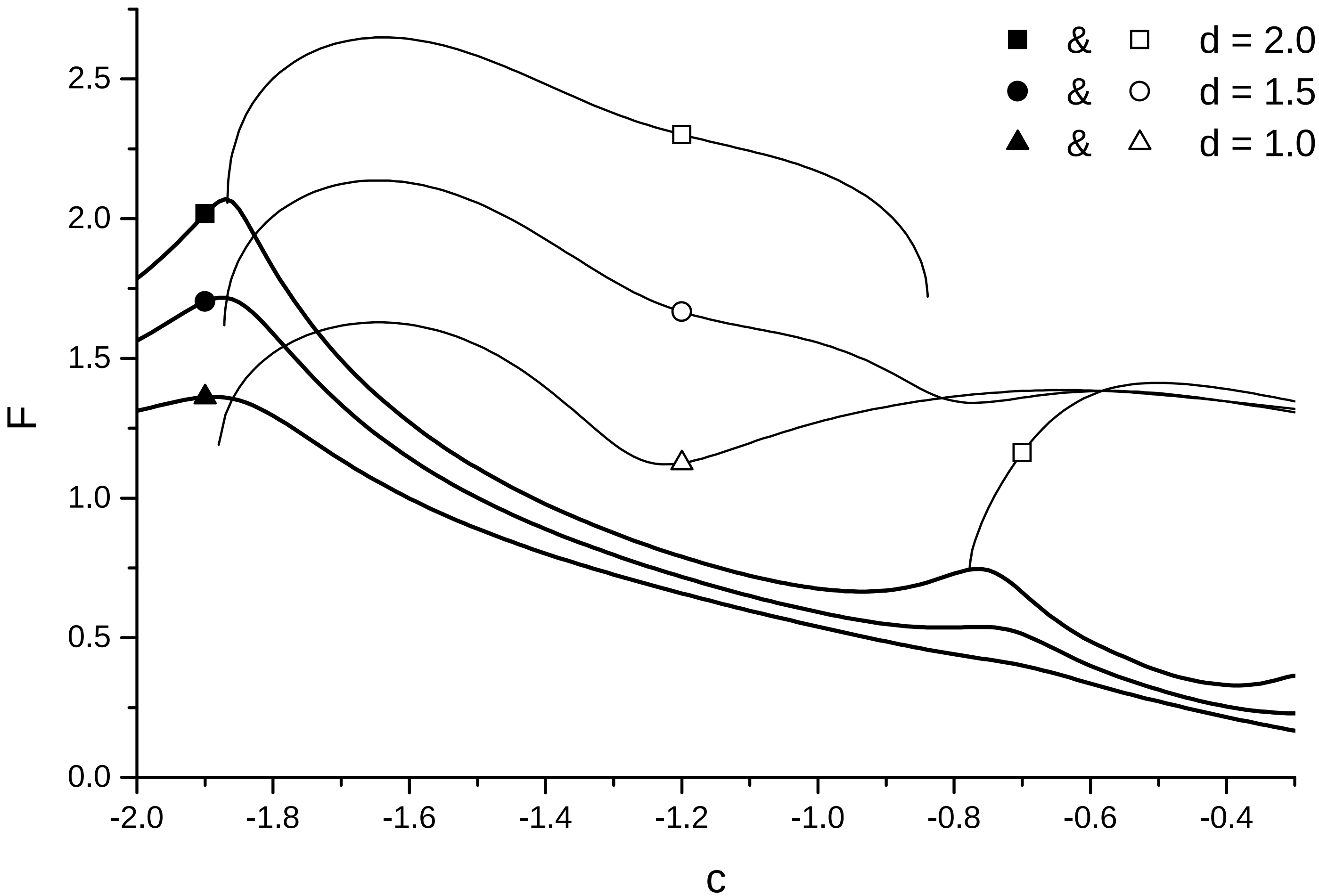
$$x_{j+N}(t) = x_j(t) + 2\pi M.$$

In terms of dynamic hull function $\Phi(\xi) = \xi + \Psi(\xi)$, we get MFDE

$$\begin{cases} c^2\Psi''(\xi) - \gamma c(1 + \Psi'(\xi)) & = & \Psi(\xi + \frac{2\pi M}{N}) + \Psi(\xi - \frac{2\pi M}{N}) - 2\Psi(\xi) \\ & & -d \sin(\xi + \Psi) + F \\ & & \\ \Psi(-\pi) & = & \Psi(\pi) \end{cases}$$

For any force F, can find matching Ψ and c. (Strunz and Elmer, 1998)
Question: bifurcations as F varies?

Frenkel-Kontorova



Thick lines: characteristics F-c at fixed γ (Elmer + Van Vleck 2003).
Thin lines: Flip-bifurcation lines (γ free). Presence of Floquet multiplier -1.

Cusps associated to period doubling?

Homoclinic bifurcations

Consider the MFDE

$$x'(\xi) = G(x_\xi, \mu),$$

and let h be a homoclinic orbit at $\mu = \mu_0$.

Main question: behaviour as parameter μ is varied.

- Homoclinic $\rightarrow$ Homoclinic bifurcations (homoclinic doubling, ...)
- Homoclinic $\rightarrow$ Periodic bifurcations (blue-sky catastrophe, ...)

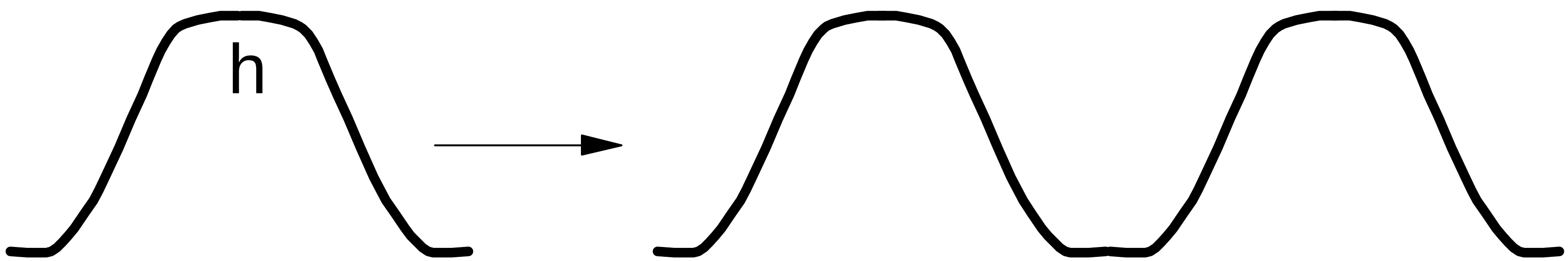
Bifurcations from heteroclinic orbits

Homoclinic bifurcations analyzed by Lin's Method. Suppose we have an MFDE

$$x'(\xi) = G(x_\xi, \mu)$$

that for $\mu = \mu_0$ has a homoclinic orbit h with $\lim_{\xi \to \pm\infty} h(\xi) = 0$. We require 0 to be a **hyperbolic** equilibrium.

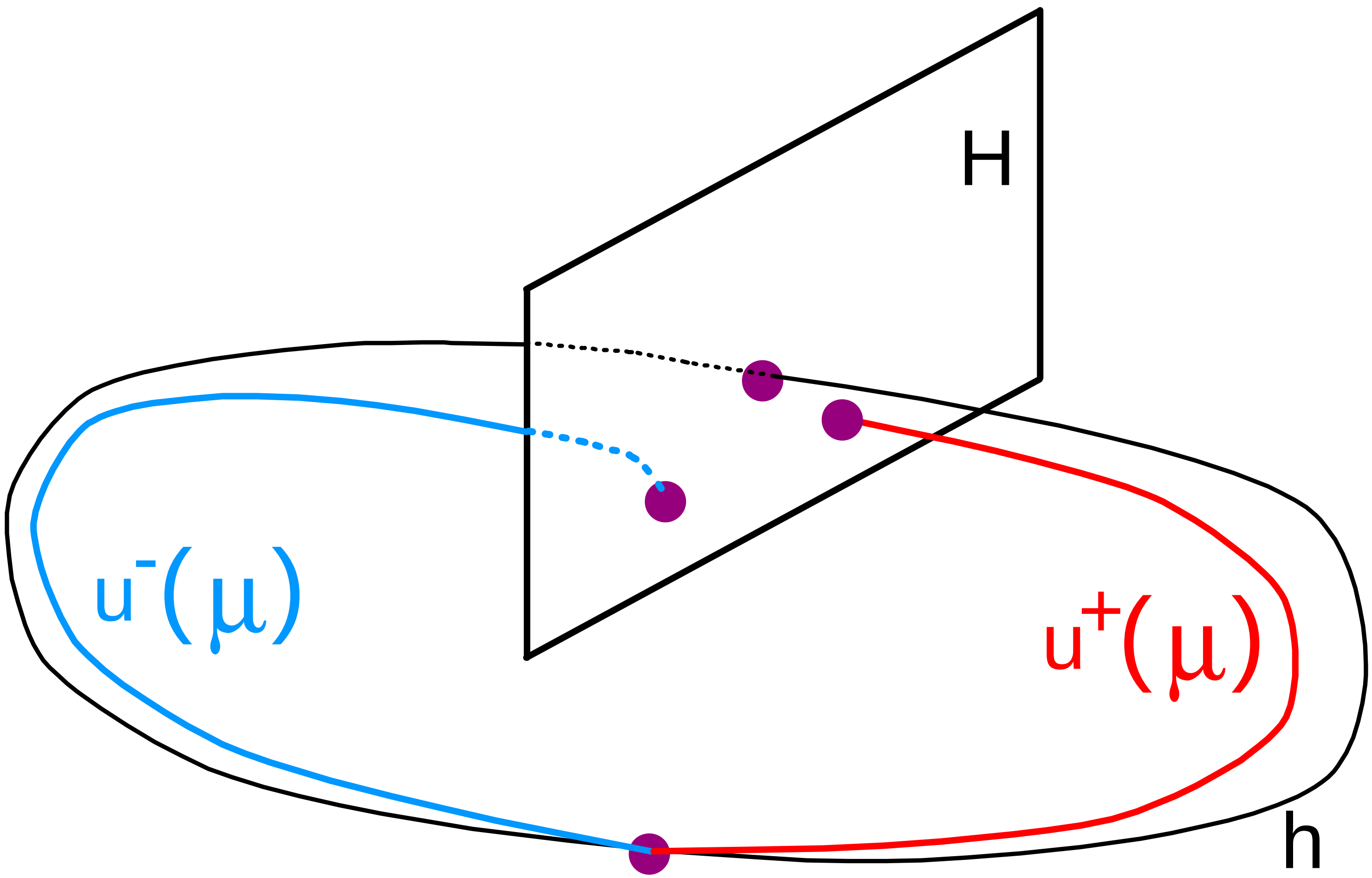
Idea: look for homoclinic / periodic solutions that wind around h a specified number of times before converging to equilibria / repeating their pattern.



Presentation here based upon work by Sandstede (1993) for ODEs.

Lin's Method - Step I

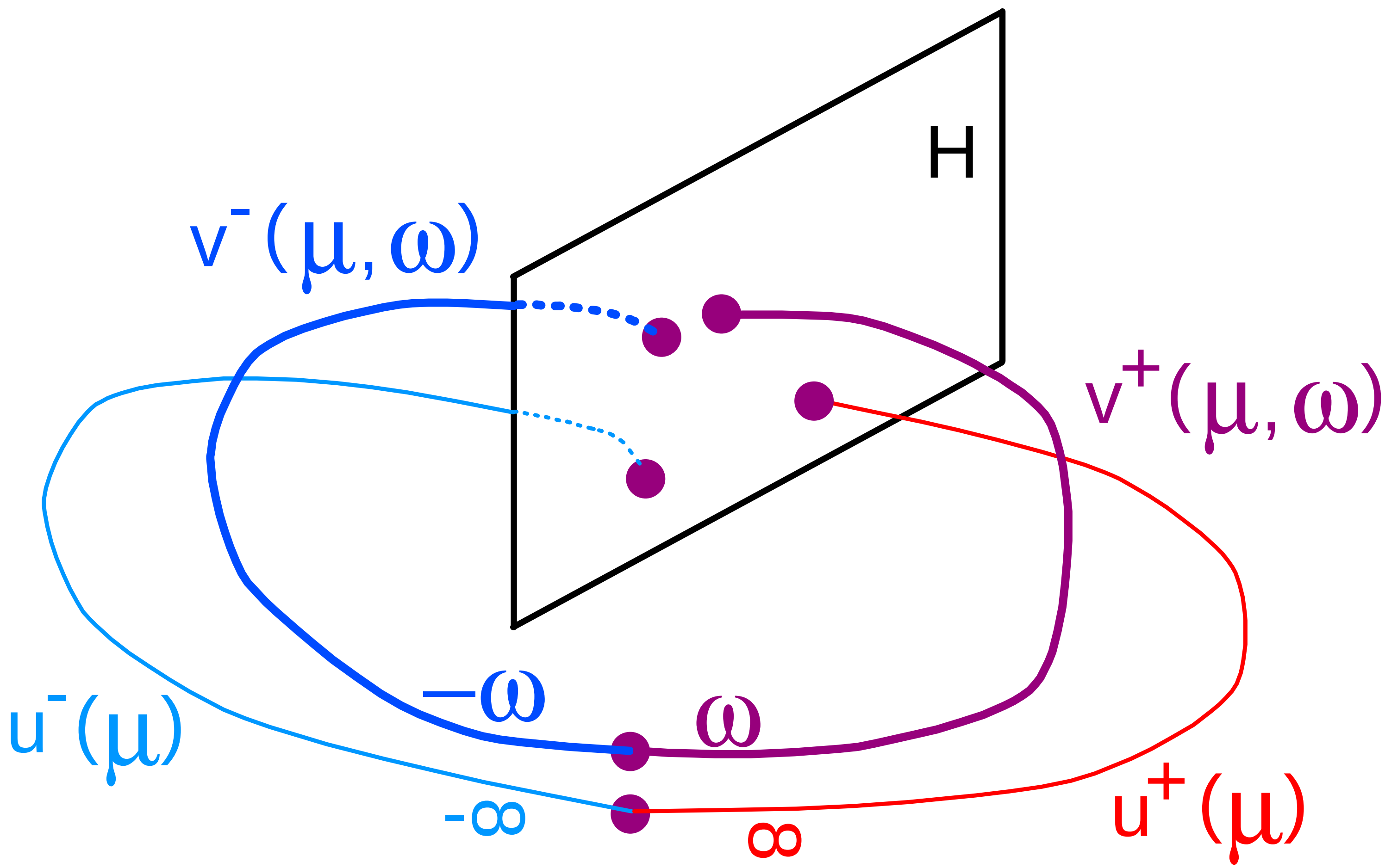
First step is to construct stable and unstable manifolds around h_0 and intersect with Poincare section H.



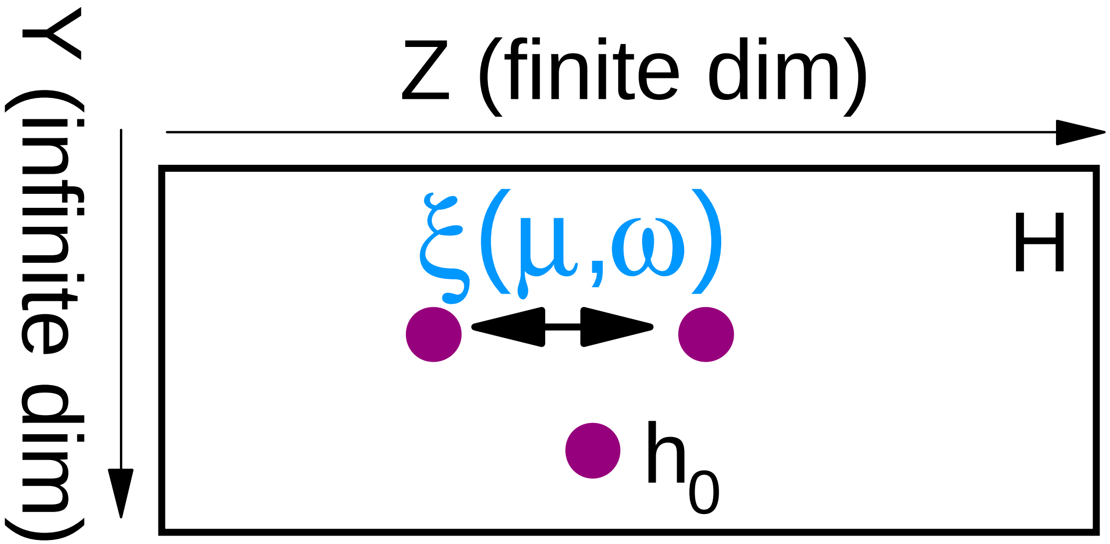
Georgie (2008): bifurcations from symmetric homoclinic orbit in reversible systems.

Lin's Method - Step II

In order to consider periodic orbits, we add a perturbation $v^{\pm}$ in order to connect the orbits at $\pm\omega$.



Lin's Method - Step III



Hyperplane $H \subset C([-1,1],\mathbb{C}^n)$ is "transverse" to h at h_0, but is **infinite** dimensional.

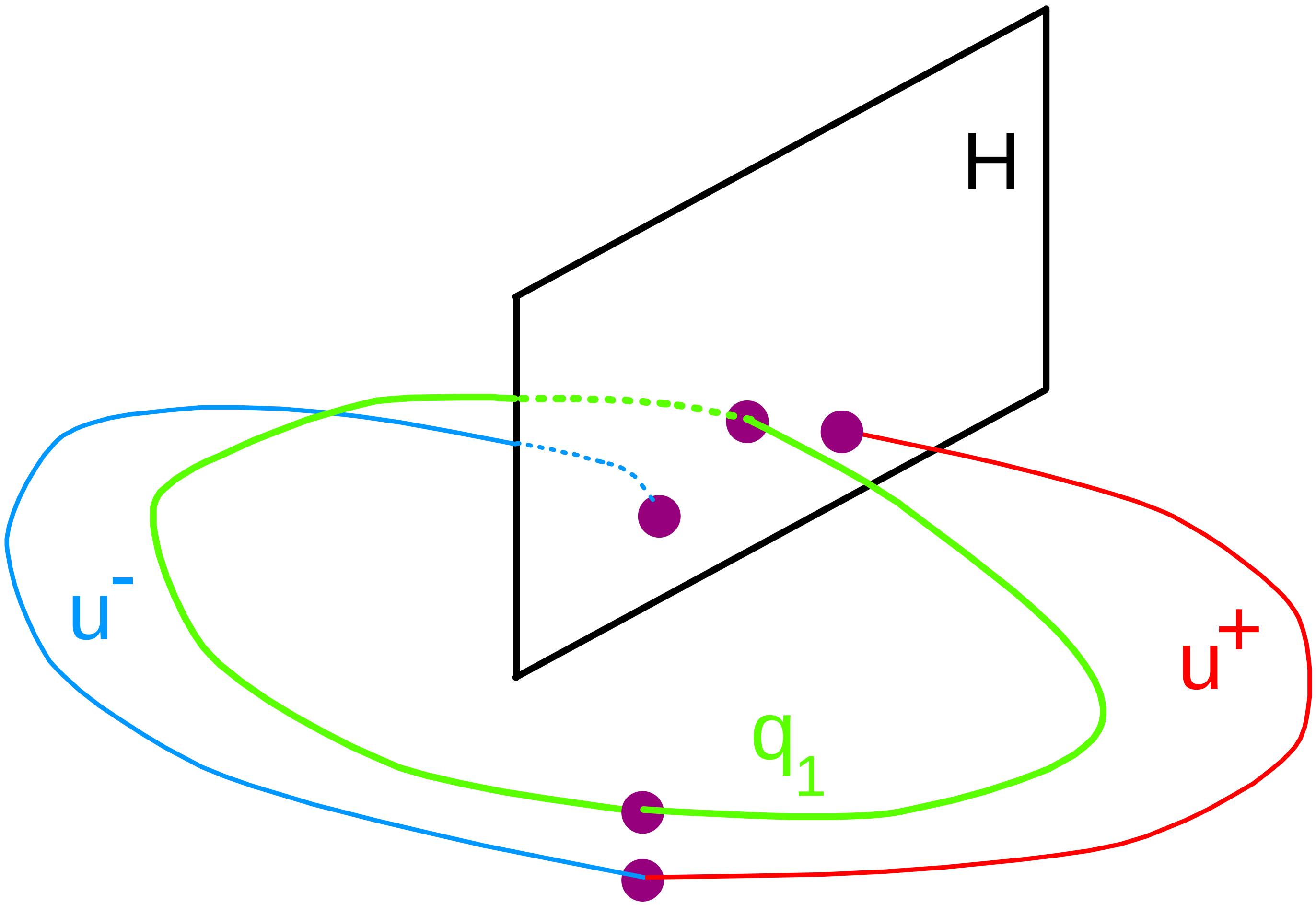
Main Goal: Reduce problem to finite dimensional bifurcation equations.

To do this, we will need to split $H = h_0 + Y \oplus Z$, with Z **finite** dimensional.

In addition, need to make sure that the "gaps" $\xi(\mu,\omega)$ are all in Z.

Lin's Method - Step IV

Next step is to analyze the bifurcation equations and find pairs (ω, μ) that close the gap in H, to find 1-periodic or 1-homoclinic orbits q_1.



The construction

This construction is based upon exponential splitting of the state space

$$C([-1,1],\mathbb{C}^n) = \widehat{P} \oplus \widehat{Q} \oplus B \oplus Z,$$

obtained by Mallet-Paret and Verduyn-Lunel for the non-autonomous MFDE

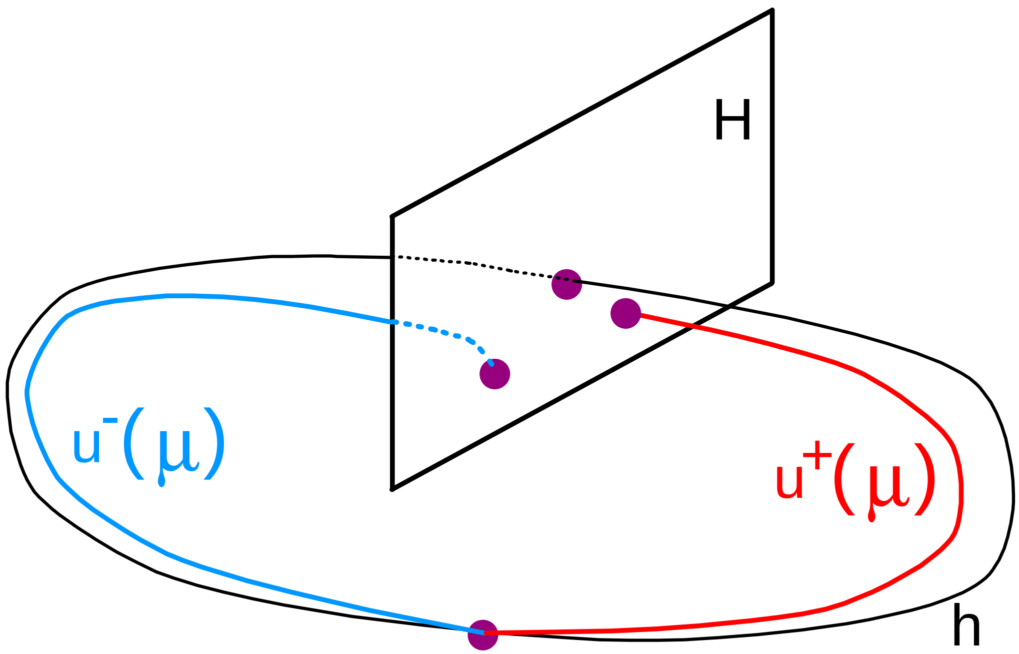
$$\dot{x}(\xi) = L(\xi)x_\xi. \quad (4)$$

- $P = \{x_0 \mid x$ solves (4) on $(-\infty, 0]\}$ (initial conditions for solutions for $\xi \leq 0$).
- $Q = \{x_0 \mid x$ solves (4) on $[0, \infty)\}$ (initial conditions for solutions for $\xi \geq 0$).
- $B = \{x_0 \mid x$ solves (4) on $\mathbb{R}\}$ (initial conditions for solutions for $\xi \in \mathbb{R}$).
- Z is a finite dimensional complement, that can be characterized using the adjoint of (4) and the Hale inner product.
- $\widehat{P} \subset P$ and $\widehat{Q} \subset Q$ normalized so that $\widehat{P} \cap B = \emptyset$ and $\widehat{Q} \cap B = \emptyset$.

The construction

Recall the exponential splitting

$$C([-1,1],\mathbb{C}^n) = \widehat{P} \oplus \widehat{Q} \oplus B \oplus Z,$$



We will take $Y = \widehat{P} \oplus \widehat{Q}$ for the infinite dimensional part of H.

Idea: Write stable manifold as graph over $\widehat{Q}$ and unstable manifold as graph over $\widehat{P}$. This freedom allows us to obtain $u^+(\mu)_0 - u^-(\mu)_0 \in Z$.

Ingredients

To make these constructions precise, we need the following ingredients.

- For all $\xi \geq 0$, need to have parameter-dependent exponential splittings

$$C([-1,1],\mathbb{C}^n) = Q(\xi,\mu) \oplus S(\xi,\mu),$$

  in which $\phi \in Q(\xi,\mu)$ can be extended to solution $E\phi$ of homogeneous system

$$\dot{x}(\xi) = L(\mu)(\xi)x_\xi$$

  on $[\xi,\infty)$, while $\psi \in S(\xi,\mu)$ can be extended to a solution $E\psi$ on $[0,\xi]$.

- Need precise estimates on convergence rates $Q(\xi,\mu) \to Q(\infty)$

- Need to solve linear inhomogeneous systems

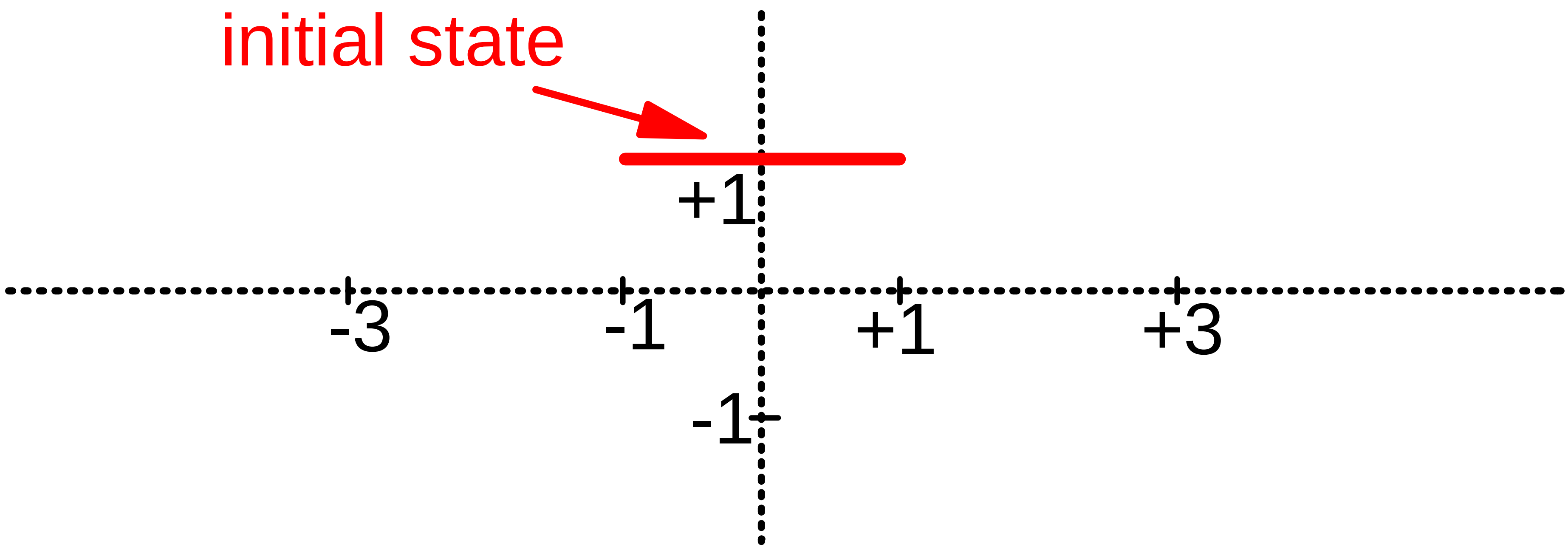
$$\dot{x}(\xi) = L(\mu)(\xi)x_\xi + f(\xi) \tag{5}$$

  on the half-lines $(-\infty,0]$ and $[0,\infty)$.

Obstacles - I

The most important problem is that MFDEs are ill-posed. Consider the homogeneous MFDE

$$\dot{x}(t) = x(t-1) + x(t+1).$$



(Example due to Härterich, Sandstede, Scheel (2002))

Obstacles - I

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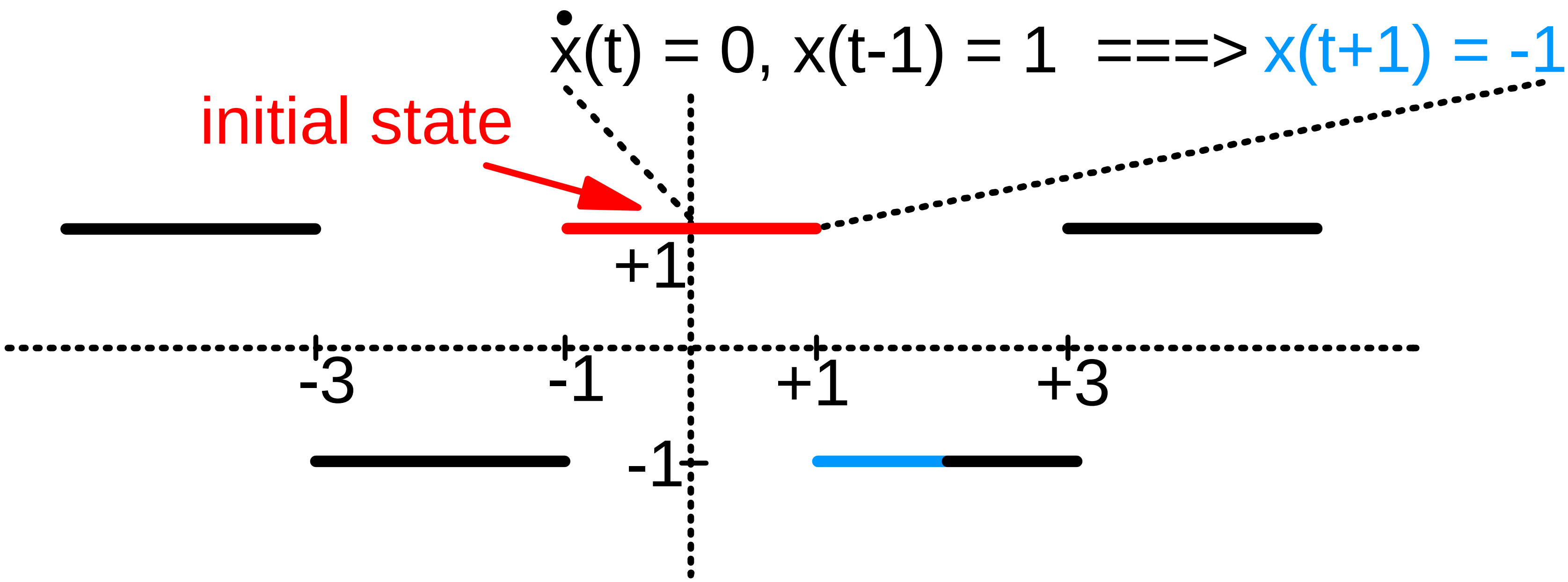


- Continuity lost $\Longrightarrow$ ill-defined as an initial value problem.

Exponential Dichotomies

Exponential dichotomies are the method of choice for ill-posed problems. Consider the system

$$x'(\xi) = L(\xi)x_\xi + f(\xi).$$

Suppose we have for $\xi \geq 0$ the splitting $C([-1,1], \mathbb{C}^n) = Q(\xi) \oplus S(\xi)$, where $\phi \in Q(\xi)$ can be extended to the right and $\psi \in S(\xi)$ can be extended to the left, both with $f = 0$.

Usually, exponential dichotomies can be used to construct a variation-of-constants formula

$$x \sim \int_0^\xi T(\xi,\xi')\Pi_{Q(\xi')} f(\xi')d\xi' + \int_\infty^\xi T(\xi,\xi')\Pi_{S(\xi')} f(\xi')d\xi',$$

where T should be seen as an evolution operator. However, since $f : \mathbb{R} \to \mathbb{C}^n$ does not map into the state space $C([-1,1])$ complications arise.

- Delay equations: sun-star calculus based upon semigroup properties
- Mixed type equations: unclear how to mimic this construction

Obstacles - II

Up to now, for fixed parameter μ_0 the splitting

$$C([-1,1]) = Q(\xi, \mu_0) \oplus S(\xi, \mu_0)$$

has only been obtained in a Hilbertspace setting (with $L^2([-1,1])$), by Härterich, Sandstede, Scheel (2002). Work of Mallet-Paret and Verduyn-Lunel needs to be (slightly) extended.

Second problem arises when attempting to define the perturbed exponential dichotomies

$$C([-1,1]) = Q(\xi, \mu) \oplus S(\xi, \mu),$$

which should depend smoothly on μ.

Robustness for exponential dichotomies for ODEs proved by means of variation-of-constants argument (eg Coppel, 1978).

Inhomogeneous systems

Recall Mallet-Paret result (1998) on $\Lambda : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^\infty(\mathbb{R}, \mathbb{C}^n)$,

$$[\Lambda x](\xi) = x'(\xi) - L(\xi)x_\xi = x'(\xi) - \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j).$$

- Λ is a Fredholm operator.
- Range $\mathcal{R}(\Lambda)$ given by

$$\mathcal{R}(\Lambda) = \{f \in L^\infty(\mathbb{R}, \mathbb{C}^n) \mid \int_{-\infty}^{\infty} d(\xi)^* f(\xi) d\xi = 0 \text{ for all } d \in \mathcal{K}(\Lambda^*)\},$$

with adjoint given by

$$[\Lambda^* x](\xi) = x'(\xi) + \sum_{j=0}^{N} A_j(\xi - r_j)^* x(\xi - r_j).$$

Inhomogeneous systems - II

We thus have $\mathcal{R}(\Lambda) \neq L^\infty(\mathbb{R}, \mathbb{C}^n)$, with again

$$[\Lambda x](\xi) = x'(\xi) - L(\xi)x_\xi = x'(\xi) - \sum_{j=0}^{N} A_j(\xi)x(\xi + r_j).$$

Suppose shifts are ordered, $r_0 < \ldots < r_N$.

Important property If $\det A_0(\xi) \neq 0$ and $\det A_N(\xi) \neq 0$, then any solution to $x'(\xi) = L(\xi)x_\xi$ with $x_\xi = 0$ for some ξ has $x \equiv 0$.

Consider a basis $\{d^i\}_{i=1}^{n_d}$ for $\mathcal{K}(\Lambda^*)$. Can now find functions $\{g^i\}_{i=1}^{n_d}$ with

$$\int_{-\infty}^{\infty} d^i(\xi)^* g^i(\xi) d\xi = \delta_{ij}$$

and $\operatorname{supp} g^i \subset [-4, -2]$.

Using these functions can define inverse Λ_+^- for Λ on the half-line $[0, \infty)$, by adding appropriate multiples of g^i to the inhomogeneity.

Parameter-dependent Exponential Dichotomies

Idea: construct $Q(\xi, \mu)$ as graph over $Q(\xi, \mu_0)$.

Use $\mathcal{G}(\mu) \in \mathcal{L}\big(BC_{-\epsilon}([r_{\min} + \xi, \infty), \mathbb{C}^n)\big)$,

$$\mathcal{G}(\mu)u = \Lambda_+^{-1}[L(\mu) - L(\mu_0)]u - E\Pi_{Q(\xi)}\mathrm{ev}_\xi\Lambda_+^{-1}[L(\mu) - L(\mu_0)]u.$$

For any $\phi \in Q(\xi, \mu_0)$, any u that satisfies

$$u = \mathcal{G}(\mu)u + E\phi$$

will have $u_\xi \in Q(\xi, \mu)$ with $\Pi_{Q(\xi,\mu_0)}u_\xi = \phi$.

This fixed point problem can be solved for μ close to μ_0, simultaneously for all $\xi \geq 0$, yielding a family $u^*_{Q(\xi)}(\mu) : Q(\xi, \mu_0) \to Q(\xi, \mu)$.

Exponential estimates follow from the weighted norm in the space $BC_{-\epsilon}$.

Smoothness of $\mu \mapsto u^*_{Q(\xi)}(\mu)$ follows from smoothness of $\mu \mapsto L(\mu)$.

Main Results

- Lin's method can be extended to MFDE.
- Bifurcation equations for the gaps $\xi(\mu, \omega)$ are finite-dimensional.
- Gaps $\xi(\mu, \omega)$ depend smoothly on parameters μ and ω.
- Asymptotic form of gap function $\xi(\mu, \omega)$ and first derivatives $D_\omega \xi$, $D_\mu \xi$ as $\omega \to \infty$ are same as those for ODEs.
- Bridge for lifting ODE bifurcation results to MFDEs.
- Results also hold for solutions that wind around a primary pulse multiple times.

Example: Orbit-flip bifurcation as stated by Sandstede for ODEs (1993) can be lifted to MFDEs.

H + VL (2008), submitted, available online.

Orbit-Flip Bifurcation

Consider the MFDE

$$x'(\xi) = G(x_\xi, \mu) = G(x(\xi + r_0), \dots, x(\xi + r_N), \mu),$$

with x scalar, $\mu \in \mathbb{R}^2$ and G at least C^4-smooth. Suppose that there is a homoclinic solution q at $\mu = 0$ with $\lim_{\xi \to \pm\infty} q(\xi) = 0$.

Consider the characteristic function associated to equilibrium at zero,

$$\Delta(z) = z - \sum_{j=0}^{N} D_j G(0,0) e^{z r_j}.$$

Suppose that $\Delta(\lambda) = 0$ has the roots $\lambda_\pm$ and λ_-^f with

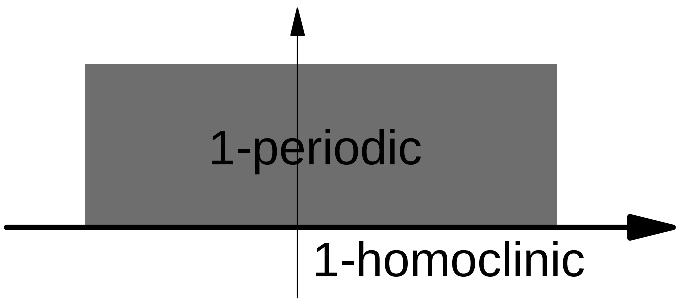
$$\eta_-^f < \lambda_-^f < \lambda_- < 0 < \lambda_+ < \eta_+.$$

Also no other roots with $\operatorname{Re} \lambda \in [\eta_-^f, \eta_+]$.

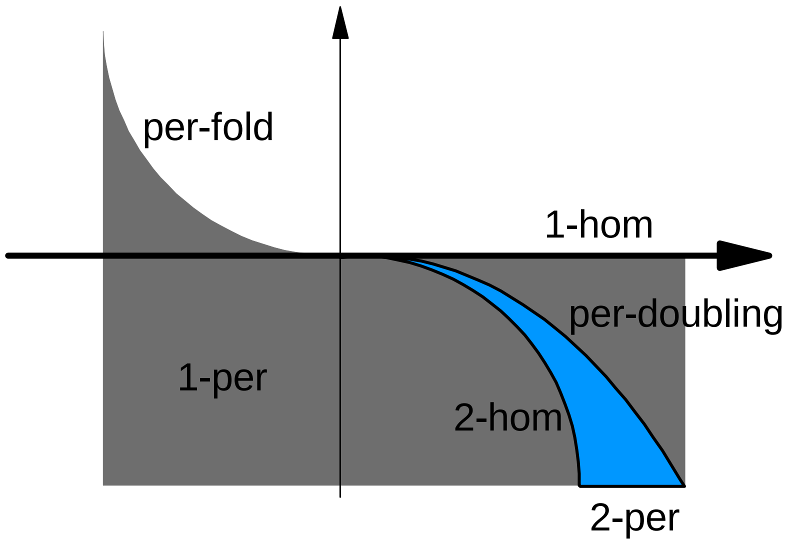
Suppose that q decays as $q(\xi) \sim e^{\lambda_-^f \xi}$ as $\xi \to \infty$.

Orbit-Flip Bifurcation - II

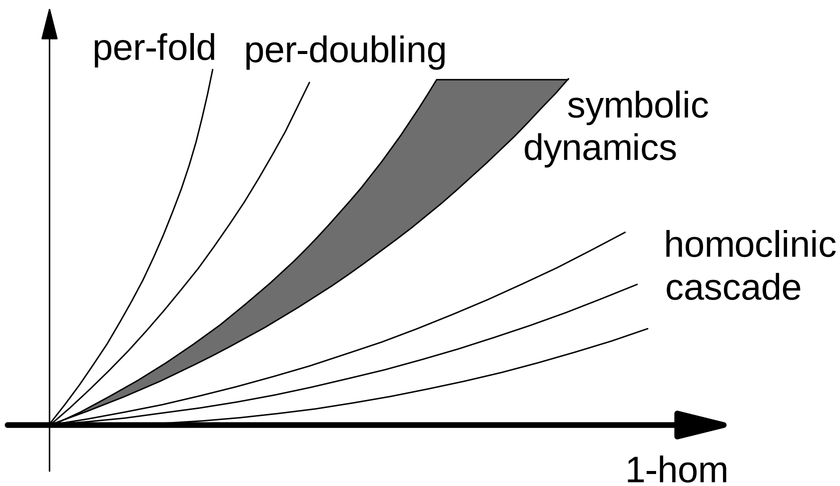
Under generic conditions, one of the following three options holds.



Homoclinic continuation ($\lambda^+ < -\lambda_-$).



Homoclinic doubling ($-\lambda_- < \lambda^+ < -\lambda_-^f$).



Homoclinic cascade ($\lambda^+ > -\lambda_-^f$).