Multi-Dimensional Stability of Travelling Waves through Rectangular Lattices

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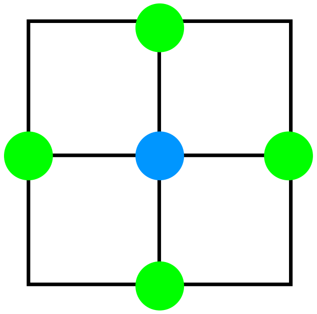
(Joint work with E. van Vleck and A. Hoffman)

2d Lattice Differential Equation

Focus in this talk: lattice differential equation (LDE)

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

- Often called: discrete Nagumo equation.
- Two dimensional spatial lattice: $(i, j) \in \mathbb{Z}^2$.
- Nonlinearity g is **bistable**.
- Discrete Laplacian Δ^+ mixes nearest neighbours:

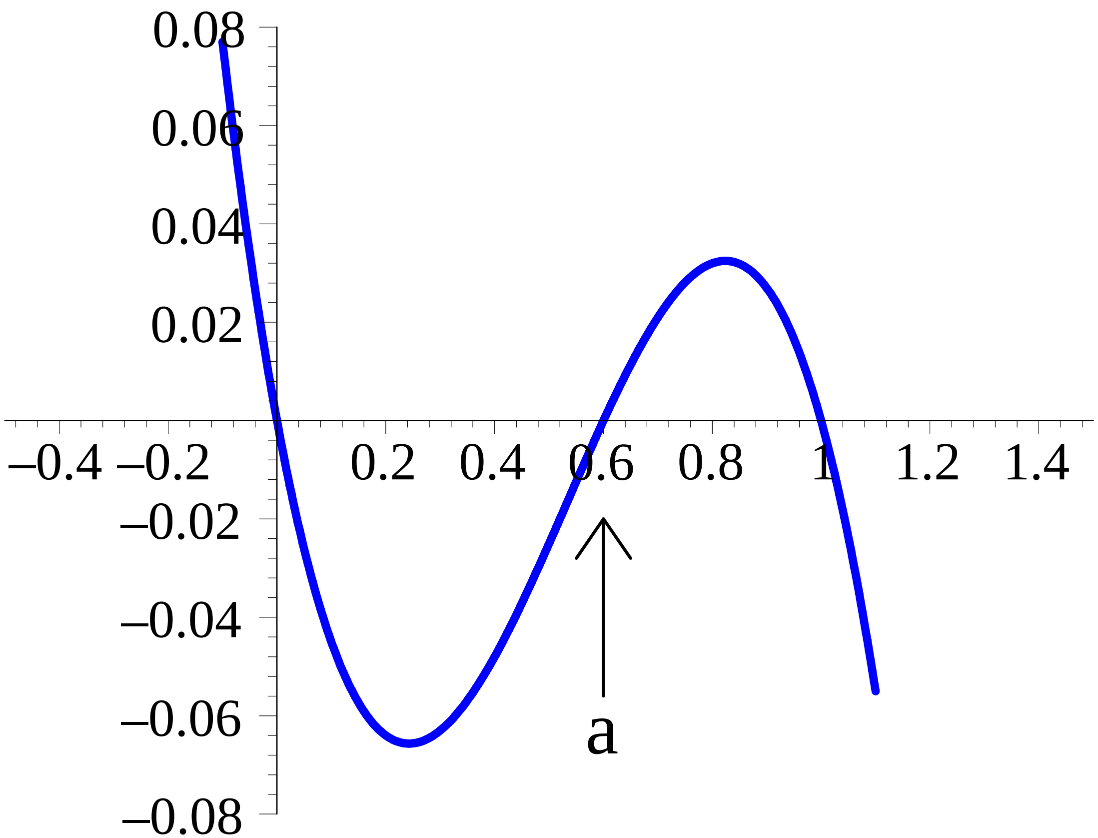


$$[\Delta^+ u]_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}.$$

2d LDE: Nonlinearity

Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$



Bistable nonlinearity g given by

$$g(u; a) = u(a - u)(u - 1).$$

Two **stable** equilibria $u = 0$ and $u = 1$.
One **unstable** equilibrium $u = a$.

Lattice equations: Travelling Waves

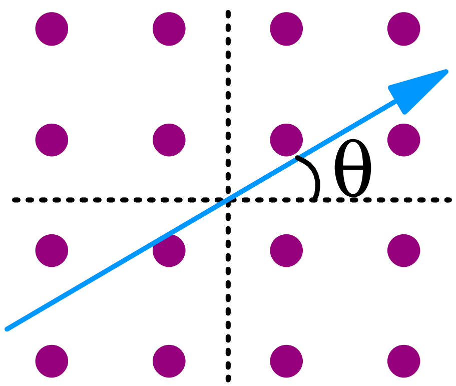
Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

The nonlinearity g 'pulls' u towards either $u = 0$ or $u = 1$ [competition].

The discrete diffusion 'smooths' out any wrinkles in u.

Travelling waves: compromise between these two forces.



Travelling waves with **profile** Φ and **speed** c connecting $u = 0$ to $u = 1$ in direction

$$\vec{k} = (\cos\theta, \sin\theta).$$

$$u_{i,j}(t) = \Phi((\cos\theta, \sin\theta) \cdot (i, j) + ct), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1.$$

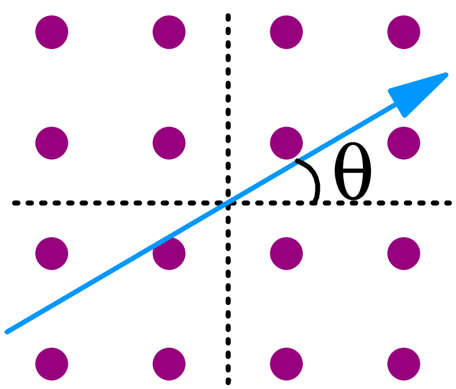
Lattice equations: Travelling Waves

Recall the dynamics:

$$\dot{u}_{i,j}(t) = [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

- Travelling waves connecting $u \equiv 0$ to $u \equiv 1$ must satisfy

$$\begin{aligned} c\Phi'(\xi) &= \Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi) \\ &\quad + g(\Phi(\xi); a) \end{aligned}$$



This is a mixed type functional differential equation (MFDE).

Direction θ explicitly appears in wave equation.

Lattice equations: Travelling Waves

Recall the dynamics:

$$\dot{u}_{i,j}(t) \quad = \quad [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

Existence of travelling waves For each $a \in (0,1)$ and $\theta \in [0, 2\pi]$ there exists a travelling wave.

Speed $c(a, \theta)$ is **unique**.

If $c \neq 0$, then wave profile Φ is **unique** and also **monotone**, i.e. $\Phi' > 0$.

[Mallet-Paret]

Dependence of c on angle θ and detuning parameter a very delicate. [Aaron Hoffman's talk]

In this talk: we fix (a, θ) and **assume** that $c \neq 0$.

Goal: understand **stability** of the travelling wave.

Stability - Coordinate System

Assumption: we have a wave solution (c, Φ) travelling $(c \neq 0)$ in **rational** direction $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$.

Naive Ansatz

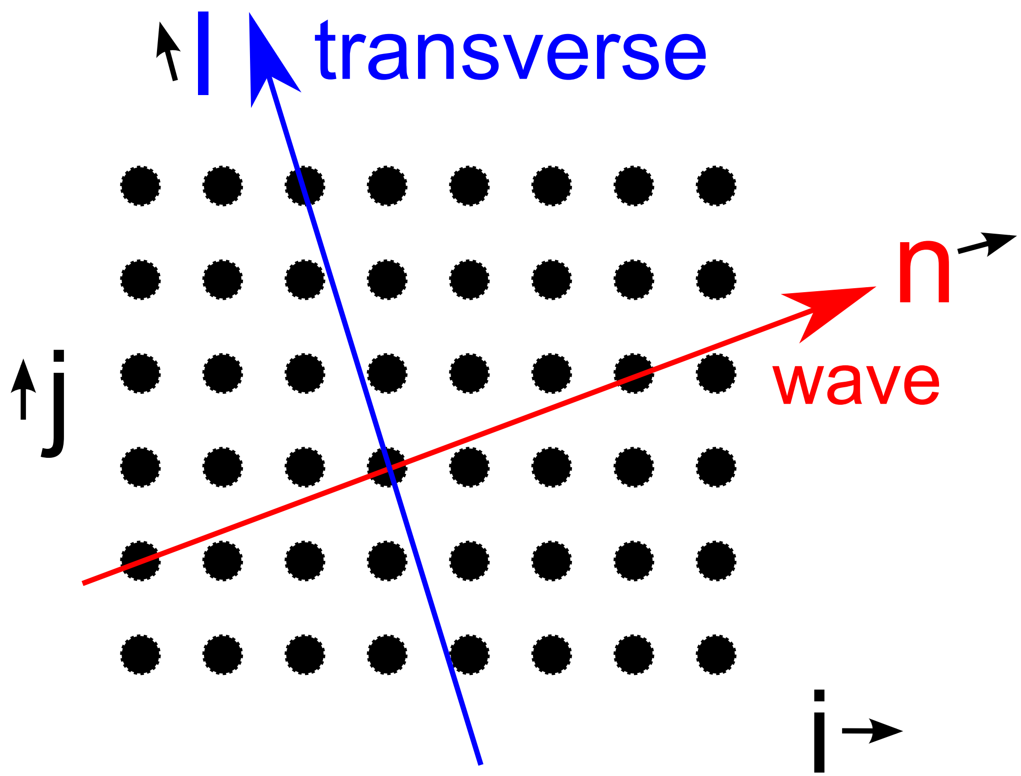
$$u_{ij}(t) = \Phi(i\sigma_1 + j\sigma_2 + ct) + v_{ij}(t).$$

Need to understand behaviour of perturbation $v(t)$.

First step: want natural coordinates **parallel** and **perpendicular** to propagation of wave.

$$\begin{array}{rcll} n & = & i\sigma_1 + j\sigma_2 & \text{parallel} \\ l & = & i\sigma_2 - j\sigma_1 & \text{transverse.} \end{array}$$

Stability - Coordinate System



New coordinates:

$$\begin{aligned} n &= i\sigma_1 + j\sigma_2 && \text{parallel} \\ l &= i\sigma_2 - j\sigma_1 && \text{transverse.} \end{aligned}$$

Old coordinates:

$$\begin{aligned} i &= [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_1 + l\sigma_2] \\ j &= [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_2 - l\sigma_1] \end{aligned}$$

Equation only posed on sublattice of $(n, l) \in \mathbb{Z}^2$ in new coordinates.

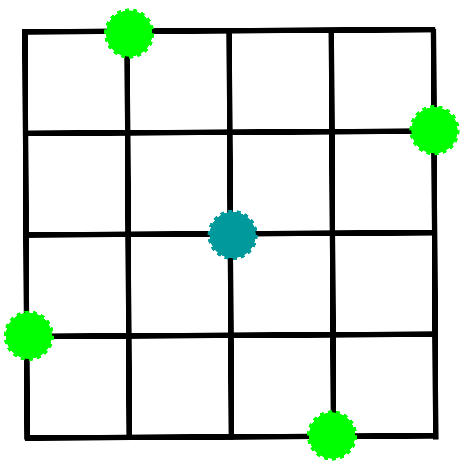
Remember: $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$.

Stability - Coordinate System

In new coordinates, LDE becomes

$$\dot{u}_{nl}(t) = [\Delta^{\times} u(t)]_{nl} + g\big(u_{nl}(t)\big).$$

The discrete operator $\Delta^{\times}$ now acts as



$$\begin{aligned}[\Delta^{\times} u]_{n,l} &= u_{n+\sigma_1,l+\sigma_2} + u_{n+\sigma_2,l-\sigma_1} \\ &\quad + u_{n-\sigma_1,l-\sigma_2} + u_{n-\sigma_2,l+\sigma_1} \\ &\quad - 4u_{n,l}.\end{aligned}$$

All geometrical information encoded in $\Delta^{\times}$.

Travelling wave becomes: $u_{nl}(t) = \Phi(n + ct)$

Special cases $(\sigma_1, \sigma_2) = (1, 0)$ or $(0, 1)$ (horizontal or vertical waves): $\Delta^{\times} = \Delta^{+}$.

Stability - Perturbation

Substituting naive perturbation Ansatz

$$u_{nl}(t) = \Phi(n+ct) + v_{nl}(t)$$

into LDE we obtain

$$\begin{array}{rcl}\dot{v}_{nl}(t) & = & [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big) v_{nl}(t) \\ & & + O\big(|v_{nl}(t)|^2 \big).\end{array}$$

(L) Need to understand growth rate of linear system

$$\dot{v}_{nl}(t) = [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big) v_{nl}(t).$$

In general, since we are in 2d, expect something algebraic.

(NL) Quadratic nonlinearities combined with slow algebraic decay spell trouble.

$$\int_0^t \underbrace{(1+t-t_0)^{-1/2}}_{\text{Linear decay}} \underbrace{[(1+t_0)^{-1/2}]^2}_{\text{nonlinearity}} \, dt_0 \sim \ln(1+t)(1+t)^{-1/2}.$$

Stability - Linear System

Focus on linear LDE posed on $\mathbb{Z}^2$:

$$\dot{v}_{nl}(t) = [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big)v_{nl}(t).$$

Observe: transverse coordinate l does **not** appear in coefficients.

Ideal for Fourier transform in transverse direction.

Write, for $\omega \in [-\pi, \pi]$:

$$\widehat{v}_n(\omega) = \sum_{l \in \mathbb{Z}} v_{nl} e^{-i\omega l}.$$

Inverse transformation:

$$v_{nl} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{il\omega} \widehat{v}_n(\omega)\, d\omega.$$

Stability - Linear System

Focus on linear LDE posed on $\mathbb{Z}^2$:

$$\dot{v}_{nl}(t) = [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big)v_{nl}(t).$$

Observe: transverse coordinate l does **not** appear in coefficients.

Ideal for Fourier transform in transverse direction.

System is decoupled into

$$\frac{d}{dt}\widehat{v}_n(\omega,t) = [\widehat{\Delta}^{\times}(\omega)\widehat{v}(\omega,t)]_n + g'(\Phi(n+ct))\widehat{v}_n(\omega,t),$$

with

$$[\widehat{\Delta}^{\times}(\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.$$

In other words, for each frequency ω we have an LDE posed on a 1d lattice (in parallel coordinate n).

Stability - Linear System

Recall decoupled LDE

$$\frac{d}{dt}\widehat{v}_n(\omega, t) = [\widehat{\Delta}^{\times}(\omega)\widehat{v}(\omega, t)]_n + g'(\Phi(n + ct))\widehat{v}_n(\omega, t),$$

with

$$[\widehat{\Delta}^{\times}(\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.$$

Special case $\omega = 0$. Write $w_n(t) = \widehat{v}_n(0, t)$. We get

$$\frac{d}{dt}w_n(t) = [\widehat{\Delta}^{\times}(0)w(t)]_n + g'(\Phi(n + ct))w_n(t),$$

with

$$[\widehat{\Delta}^{\times}(0)w]_n = w_{n+\sigma_1} + w_{n+\sigma_2} + w_{n-\sigma_1} + w_{n-\sigma_2} - 4w_n.$$

Stability - Linear System

In special case $\omega = 0$, writing $w_n(t) = \widehat{v}_n(0,t)$, we hence have:

$$\begin{aligned} \frac{d}{dt} w_n(t) &= w_{n+\sigma_1}(t) + w_{n+\sigma_2}(t) + w_{n-\sigma_1}(t) + w_{n-\sigma_2}(t) - 4w_n(t) \\ &\quad + g'(\Phi(n+ct))w_n(t). \end{aligned}$$

Notice that $w_n(t) = \Phi'(n+ct)$ is a solution.

Indeed: wave profile Φ had to satisfy

$$c\Phi'(\xi) = \Phi(\xi+\sigma_1) + \Phi(\xi+\sigma_2) + \Phi(\xi-\sigma_1) + \Phi(\xi-\sigma_2) - 4\Phi(\xi) + g(\Phi(\xi)).$$

The zero-frequency component is hence the usual linearization around the travelling wave, just like in 1d.

Stability - 1d Linear Systems

Need to understand 1d LDE's, e.g.

$$\dot{U}_j(t) = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t)), \qquad j \in \mathbb{Z}.$$

Write as

$$\dot{U}(t) = \mathcal{F}\big(U(t)\big),$$

with $\mathcal{F} : \ell^\infty(\mathbb{Z};\mathbb{R}) \to \ell^\infty(\mathbb{Z};\mathbb{R})$.

View as ODE posed on sequence space $\ell^\infty(\mathbb{Z};\mathbb{R})$.

Suppose we have a wave solution $\overline{U}_j(t) = \Phi(j + ct)$ with $c > 0$, with

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi(\xi) = 1.$$

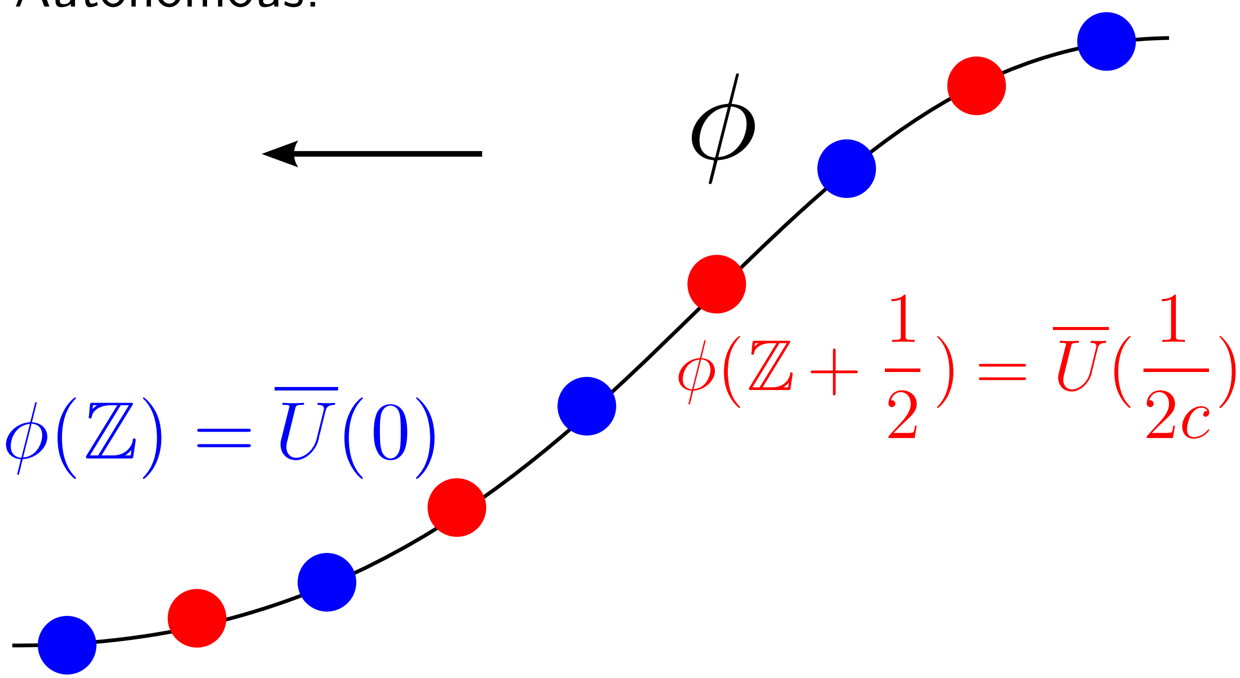
Want to understand linear behaviour of $U(t) = \overline{U}(t) + V(t)$.

Stability - 1d Linear Systems

Linear dynamics for $V(t) = U(t) - \overline{U}(t)$:

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t), \qquad V(t) \in \ell^\infty(\mathbb{Z};\mathbb{R}).$$

Problem: Non-Autonomous!



Remember: $\overline{U}_j(t) = \Phi(j + ct)$. We DO have shift-periodicity

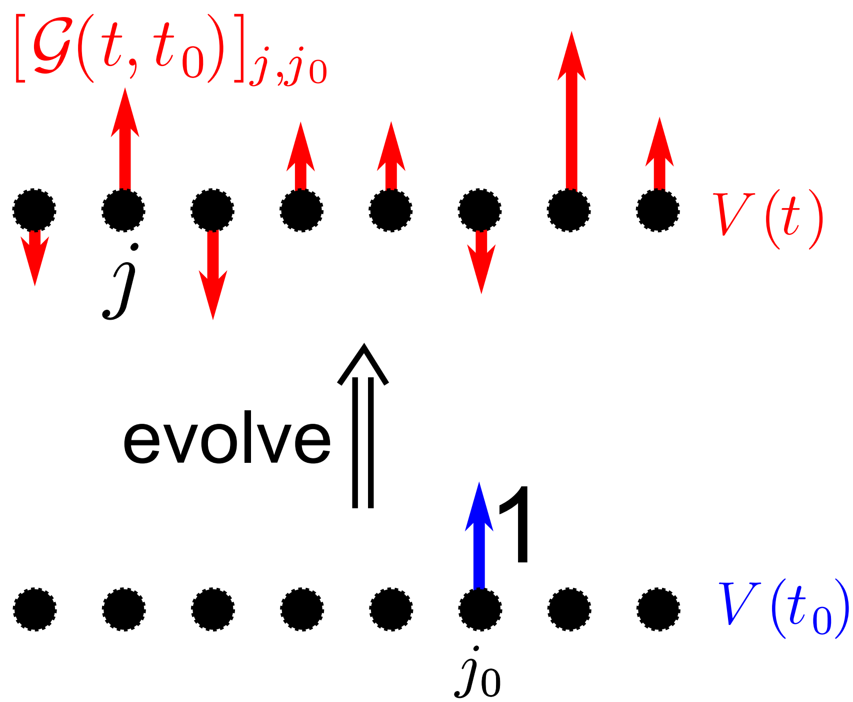
$$\overline{U}_j\big(t + 1/c\big) = \overline{U}_{j+1}(t) \qquad \Big(= \Phi(j+1)\Big).$$

Stability - 1d Linear Systems

Linear behaviour $V(t) = U(t) - \overline{U}(t)$:

Green's function $[\mathcal{G}(t, t_0)]_{jj_0}$ is value of $V_j(t)$ for unique solution to linearized LDE

$$\begin{aligned} \dot{V}(t) &= D\mathcal{F}\Big(\overline{U}(t)\Big)V(t) \\ V_{j'}(t_0) &= \delta_{j',j_0}. \end{aligned}$$



Stability - 1d Linear Systems

Linear behaviour $V(t) = U(t) - \overline{U}(t)$:

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$$\begin{aligned} \dot{V}(t) &= D\mathcal{F}\Big(\overline{U}(t)\Big)V(t) \\ V_{j'}(t_0) &= \delta_{j',j_0}. \end{aligned}$$

For $V \in \ell^\infty(\mathbb{Z}; \mathbb{R})$, write $\mathcal{G}(t, t_0)V$ for sequence

$$[\mathcal{G}(t, t_0)V]_j = \sum_{j_0 \in \mathbb{Z}} [\mathcal{G}(t, t_0)]_{j,j_0} V_{j_0}$$

(convolution).

All information (time + space) on linear system encoded in $\mathcal{G}(t, t_0)$.

Stability - 1d Linear Systems

To understand $\mathcal{G}(t, t_0)$ must solve

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t).$$

[Chow, Mallet-Paret, Shen] Can exploit shift-periodicity to develop shift-periodic Floquet theory.

Problem: must analyze 'monodromy map' $\mathcal{G}(t_0 + \frac{1}{c}, t_0)$ 'by hand'. Heavily dependent on ad-hoc arguments e.g. comparison principles. All arguments in sequence space $\ell^\infty(\mathbb{Z}; \mathbb{R})$.

Nevertheless, authors managed to understand discrete Nagumo equation.

Our goal: Make connection with highly developed nonlinear stability theory for PDEs [Zumbrun, Howard, ...].

Stability - 1d Linear Systems

Recall linear problem on $\ell^\infty(\mathbb{Z};\mathbb{R})$:

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t),$$

which for discrete Nagumo LDE is:

$$\dot{V}_j(t) = V_{j+1}(t) + V_{j-1}(t) - 2V_j(t) + g'\big(\Phi(j+ct)\big)V_j(t).$$

We 'fill in the gaps' between lattice points and look for solutions

$$V_j(t) = e^{\lambda t}v(j+ct).$$

Here $\lambda \in \mathbb{C}$ is spectral parameter and v must be bounded and solve

$$cv'(\xi) + \lambda v(\xi) = v(\xi-1) + v(\xi+1) - 2v(\xi) + g'\big(\Phi(\xi)\big)v(\xi)$$

in comoving frame $\xi = j + ct$. Write as $\mathcal{L}v = \lambda v$ with

$$[\mathcal{L}v](\xi) = -cv'(\xi) + v(\xi-1) + v(\xi+1) - 2v(\xi) + g'\big(\Phi(\xi)\big)v(\xi).$$

Fundamental relation

Reminder: **Green's function** $[\mathcal{G}(t,t_0)]_{jj_0}$ is value of $V_j(t)$ for unique solution to linearized LDE

$$\begin{array}{rcl} \dot{V}(t) & = & D\mathcal{F}\Big(\overline{U}(t)\Big)V(t) \\ V_{j'}(t_0) & = & \delta_{j',j_0}. \end{array}$$

Thm. [Benzoni-Gavage, Huot, Rousset] For $\gamma \gg 1$ and $t > t_0$,

$$[\mathcal{G}(t,t_0)]_{jj_0} = \frac{-1}{2\pi i}\int_{\gamma - i\pi c}^{\gamma + i\pi c} e^{\lambda(t-t_0)} G_\lambda(j+ct, j_0+ct_0)d\lambda.$$

Resolvent kernel $G_\lambda(\xi,\xi_0)$ is unique solution [if defined] to

$$(\mathcal{L}-\lambda)G_\lambda(\cdot,\xi_0) = \delta(\xi-\xi_0).$$

Stability

Recall identity ($\gamma \gg 1$ and $t > t_0$)

$$[\mathcal{G}(t,t_0)]_{jj_0} = \frac{-1}{2\pi i}\int_{\gamma - i\pi c}^{\gamma + i\pi c} e^{\lambda(t-t_0)} G_\lambda(j+ct, j_0+ct_0) d\lambda.$$

Can view this as refined version of meta-identity

$$e^{t\mathcal{L}} = -\frac{1}{2\pi i}\int e^{\lambda t}[\mathcal{L}-\lambda]^{-1}\, d\lambda.$$

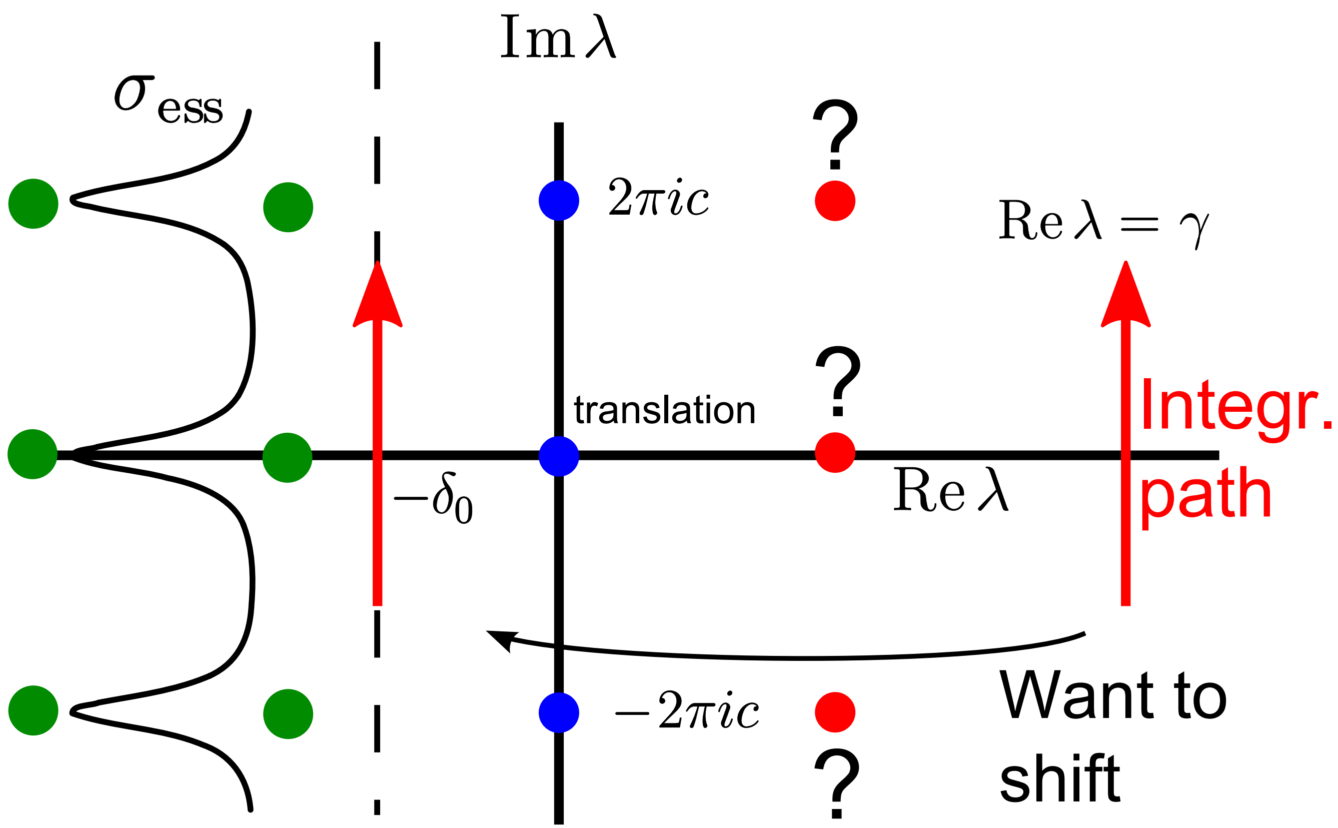
Have to worry about invertibility of $\mathcal{L}-\lambda$, i.e. study spectrum of $\mathcal{L}$.

For example $\mathcal{L}\Phi' = 0$ (translational invariance), so $\lambda = 0$ in spectrum.

Stability

Recall identity ($\gamma \gg 1$ and $t > t_0$)

$$[\mathcal{G}(t,t_0)]_{jj_0} = \frac{-1}{2\pi i}\int_{\gamma-i\pi c}^{\gamma+i\pi c} e^{\lambda(t-t_0)} G_\lambda(j+ct, j_0+ct_0)\,d\lambda.$$



Stability - 1d Linear Systems

Recall identity ($\gamma \gg 1$ and $t > t_0$)

$$[\mathcal{G}(t,t_0)]_{jj_0} = \frac{-1}{2\pi i}\int_{\gamma - i\pi c}^{\gamma + i\pi c} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0)d\lambda.$$

Main goal: construct expressions for $G_\lambda(\xi, \xi_0)$ that can be extended meromorphically in λ near poles of $[\mathcal{L} - \lambda]^{-1}$.

Can do this if translational eigenvalue $\lambda = 0$ is a simple eigenvalue [H. + Sandstede]. In particular, if $\mathrm{Ker}\mathcal{L} = \mathrm{span}\{\Phi'\}$ and $\Phi' \notin \mathrm{Range}\mathcal{L}$.

One obtains

$$G_\lambda(\xi, \xi_0) = \lambda^{-1}\Phi'(\xi)\Psi(\xi_0) + O(e^{-\nu|\xi - \xi_0|}),$$

where we have

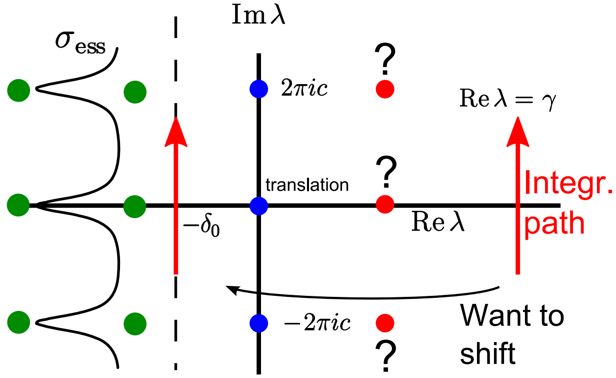
$$\mathrm{Ker}\,\mathcal{L}^* = \mathrm{span}\{\Psi\},$$

with $\mathcal{L}^*$ the formal adjoint of $\mathcal{L}$.

Stability - 1d Linear Systems

Recall identity ($\gamma \gg 1$ and $t > t_0$)

$$[\mathcal{G}(t,t_0)]_{jj_0} = \frac{-1}{2\pi i}\int_{\gamma - i\pi c}^{\gamma + i\pi c} e^{\lambda(t-t_0)} G_\lambda(j+ct, j_0+ct_0)d\lambda.$$



Using meromorphic form

$$G_\lambda(\xi,\xi_0) = \lambda^{-1}\Phi'(\xi)\Psi(\xi_0) + O(e^{-\nu|\xi-\xi_0|}),$$

we now obtain the key result

$$\begin{aligned}[\mathcal{G}(t,t_0)]_{jj_0} &= \Phi(j+ct)\Psi(j_0+ct_0) \\ &\quad + O\big(e^{-\nu(t-t_0)}e^{-\nu|j+ct-j_0-ct_0|}\big)\end{aligned}$$

In particular, Green's function for 1d lattice system can be 'read-off' from well-behaved spectral pictures.

Stability - back to 2d

Remember: for $\omega = 0$, writing $w_n(t) = \widehat{v}_n(0,t)$, we had:

$$\begin{aligned}\frac{d}{dt}w_n(t) &= w_{n+\sigma_1}(t) + w_{n+\sigma_2}(t) + w_{n-\sigma_1}(t) + w_{n-\sigma_2}(t) - 4w_n(t) \\ &\quad + g'(\Phi(n+ct))w_n(t).\end{aligned}$$

In this case, the relevant linear operator is:

$$[\mathcal{L}_0 w](\xi) = -cw'(\xi) + w(\xi \pm \sigma_1) + w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

Remember $\mathcal{L}_0\Phi' = 0$. We also have: $\mathcal{L}_0^*\Psi = 0$ for the adjoint Ψ which has $\Psi(\xi) > 0$ [Mallet-Paret].

For the Green's function we hence get

$$\begin{aligned}[\mathcal{G}_{\omega=0}(t,t_0)]_{nn_0} &= \Phi'(n+ct)\Psi(n_0+ct_0) \\ &\quad + O\big(e^{-\nu(t-t_0)}e^{-\nu|n+ct-n_0-ct_0|}\big).\end{aligned}$$

Note: no temporal decay.

Stability - Linear System

Back to $\omega \neq 0$. Recall decoupled LDE

$$\frac{d}{dt}\widehat{v}_n(\omega, t) = [\widehat{\Delta}^{\times}(\omega)\widehat{v}(\omega, t)]_n + g'(\Phi(n + ct))\widehat{v}_n(\omega, t),$$

with

$$[\widehat{\Delta}^{\times}(\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.$$

Relevant operator now is:

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2}w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1}w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

Need to understand spectrum of this operator.

What happens to zero eigenvalue for $\omega \approx 0$?

Stability - Linear System

Recall ω-dependent linear operators

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

There exists a branch

$$\omega \mapsto (\lambda_\omega, \phi_\omega, \psi_\omega)$$

for $\omega \approx 0$ with

$$[\mathcal{L}_\omega - \lambda_\omega]\phi_\omega = 0, \qquad [\mathcal{L}_\omega^* - \lambda_\omega^*]\psi_\omega = 0$$

Of course, $\lambda_0 = 0$, $\phi_0 = \Phi'$ and $\psi_0 = \Psi$.

Key assumption:

$$\mathrm{Re}\,\lambda_\omega \leq -\kappa\omega^2, \qquad \omega \approx 0, \qquad \kappa > 0$$

For general directions $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$, can only establish this with numerics.

Stability - Linear System

Recall ω-dependent linear operators

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

In special case $(\sigma_1, \sigma_2) = (1, 0)$ we get

$$\begin{aligned} [\mathcal{L}_\omega w](\xi) &= -cw'(\xi) + w(\xi \pm 1) + 2\cos\omega w(\xi) - 4w(\xi) + g'(\Phi(\xi))w(\xi) \\ &= [\mathcal{L}_0 w](\xi) + 2(\cos\omega - 1)w(\xi). \end{aligned}$$

This immediately gives $\lambda_\omega = 2(\cos\omega - 1)$ and $\phi_\omega = \Phi'$.

Eigenfunctions ϕ_ω now **independent** of ω.

Stability - Linear System

Recall ω-dependent linear operators

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

In special case $(\sigma_1, \sigma_2) = (1, 1)$ we get

$$[\mathcal{L}_\omega w](\xi) \quad = \quad -cw'(\xi) + (2\cos\omega)w(\xi \pm 1) - 4w(\xi) + g'(\Phi(\xi))w(\xi)$$

This gives $[\partial_\omega \lambda_\omega]_{\omega=0} = 0$ and $[\partial_\omega \phi_\omega]_{\omega=0} = 0$.

Eigenfunctions ϕ_ω now **dependent** on ω. But everything is quadratic in ω.

Stability - Linear System

Recall decoupled LDE

$$\frac{d}{dt}\widehat{v}_n(\omega,t) = [\widehat{\Delta}^{\times}(\omega)\widehat{v}(\omega,t)]_n + g'(\Phi(n+ct))\widehat{v}_n(\omega,t).$$

For the Green's function we get

$$\begin{aligned}[\mathcal{G}_\omega(t,t_0)]_{nn_0} &= e^{\lambda_\omega(t-t_0)}\phi_\omega(n+ct)\psi^*_\omega(n_0+ct_0) \\ &\quad + O\big(e^{-\nu(t-t_0)}e^{-\nu|n+ct-n_0-ct_0|}\big).\end{aligned}$$

Note: temporal decay of order $O(e^{-\kappa\omega^2\Delta t})$ since $\mathrm{Re}\,\lambda_\omega \leq -\kappa\omega^2$.

In particular, expect heat-kernel type decay in transverse direction.

Stability - Linear System

Return to full 2d linear system

$$\dot{v}_{nl}(t) = [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big)v_{nl}(t).$$

Look at initial condition

$$v_{nl}(0) = v^0_{nl} = (v^0_n)_l$$

with $v^0 \in \ell^\infty(\mathbb{Z}; \ell^1(\mathbb{Z}; \mathbb{R}))$.

Norm on v^0: ℓ^∞ in direction parallel to wave and ℓ^1 in direction transverse to wave.

We get for ℓ^2 norm in transverse direction:

$$\|v(t)\|_{\ell^\infty(\mathbb{Z};\ell^2(\mathbb{Z};\mathbb{R}))} \sim (1+t)^{-1/4} \left\|v^0\right\|_{\ell^\infty(\mathbb{Z};\ell^1(\mathbb{Z};\mathbb{R}))}.$$

For ℓ^∞ norm in transverse direction get extra decay:

$$\|v(t)\|_{\ell^\infty(\mathbb{Z};\ell^\infty(\mathbb{Z};\mathbb{R}))} \sim (1+t)^{-1/2} \left\|v^0\right\|_{\ell^\infty(\mathbb{Z};\ell^1(\mathbb{Z};\mathbb{R}))}.$$

Stability - Naive Ansatz

Substituting naive perturbation Ansatz

$$u_{nl}(t) = \Phi(n+ct) + v_{nl}(t)$$

led to

$$\begin{aligned} \dot{v}_{nl}(t) &= [\Delta^{\times} v(t)]_{nl} + g'\big(\Phi(n+ct)\big) v_{nl}(t) \\ &\quad + O\big(\, |v_{nl}(t)|^2 \,\big). \end{aligned}$$

Linear decay of $t^{-1/4}$ much too weak to close nonlinear argument.

However, we understand precisely the terms in Green's function leading to slow decay:

$$[\mathcal{G}_\omega(t,t_0)]_{nn_0} \sim e^{\lambda_\omega (t-t_0)} \phi_\omega(n+ct) \psi^*_\omega(n_0+ct_0).$$

Since $\phi_0 = \Phi'$, deformations in wave profile are the main culprit of slow decay.

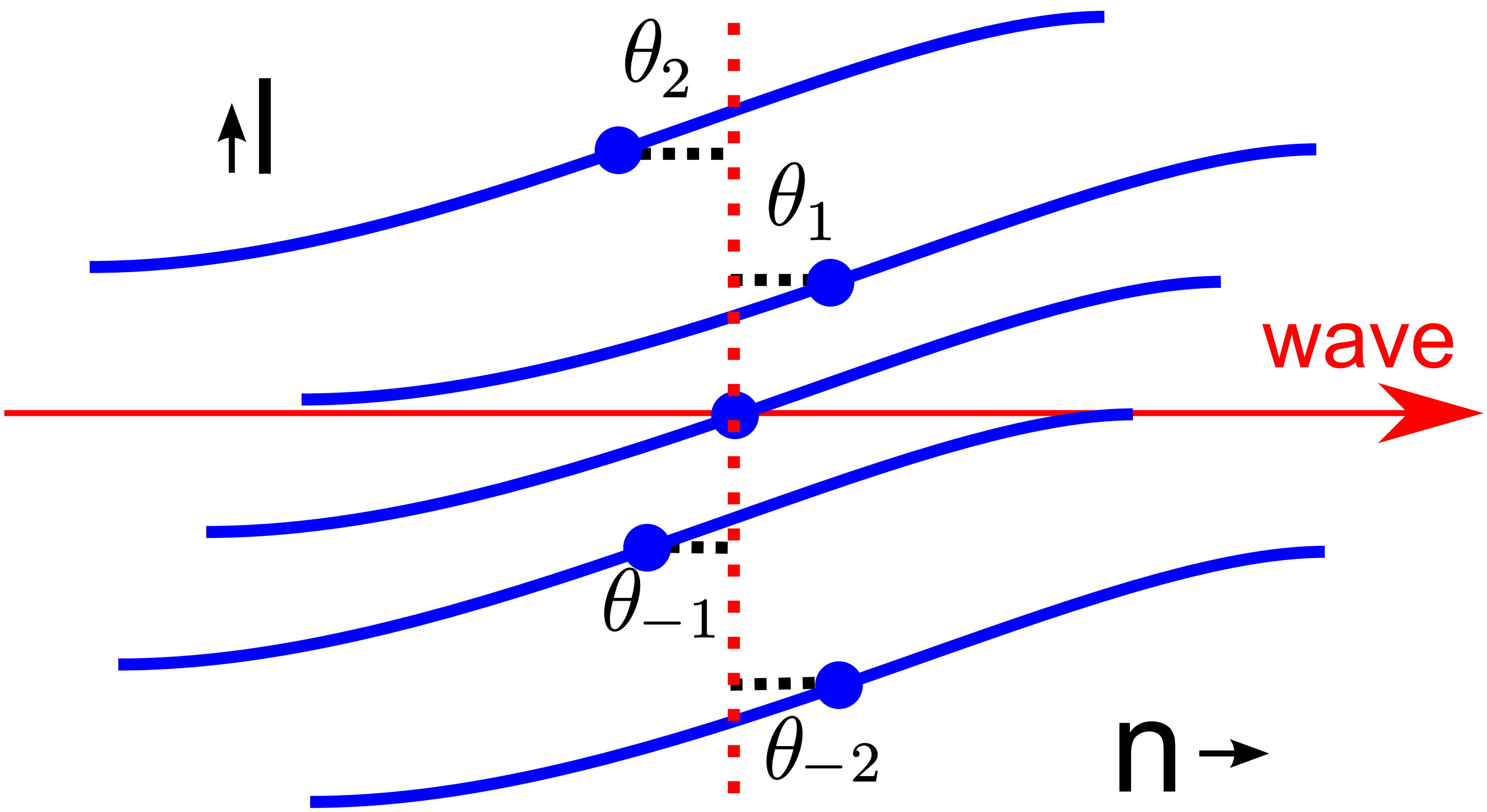
Stability - Refined Ansatz

Refined perturbation Ansatz

$$u_{nl}(t) = \Phi\big(n + ct + \theta_l(t)\big) + v_{nl}(t).$$

Here $\theta_l(t)$ measures deformation of wave profile (expect slow decay).

Remainder included in $v(t)$ (expect faster decay).



Stability - Refined Ansatz

Refined perturbation Ansatz

$$u_{nl}(t) = \Phi\big(n + ct + \theta_l(t)\big) + v_{nl}(t).$$

Normalization conditions:

$$\sum_{n \in \mathbb{Z}} \Psi(n + ct) v_{nl}(t) = 0, \qquad \text{for all } l \in \mathbb{Z}.$$

Let us shorten this to:

$$Q_{ct} v(t) = 0 \in \ell^\infty(\mathbb{Z}; \mathbb{R}).$$

Reminder: we had $\mathcal{L}_0 \Phi' = 0$ and $\mathcal{L}_0^* \Psi = 0$ with

$$\sum_{n \in \mathbb{Z}} \Psi(n + ct) \Phi'(n + ct) = 1.$$

Stability - Refined Ansatz

Need notation:

$$\theta_l^{\diamond} = \big(\theta_{l+\sigma_2} - \theta_l, \theta_{l-\sigma_2} - \theta_l, \theta_{l-\sigma_1} - \theta_l, \theta_{l+\sigma_1} - \theta_l\big).$$

This expression contains only **differences** in θ. Fourier symbol for difference: $e^{\pm i\omega\sigma_i} - 1 = O(\omega)$.

Linear evolution for θ can be written as:

$$\dot{\theta}_l(t) = Q_{ct}L(ct+\theta)v(t) + Q_{ct}M(ct+\theta)\theta^{\diamond}(t) + cQ'_{ct}v(t)$$

Here we have [Very similar to naive linearization]:

$$[L(ct+\theta)v]_{nl} = [\Delta^{\times}v]_{nl} + g'\big(\Phi(n+ct+\theta_l)\big)v_{nl}.$$

New term [Measures effect of profile mismatches]:

$$\begin{aligned}[M(ct+\theta)\theta^{\diamond}]_{nl} \quad &= \quad \Phi'(n+ct+\theta_l \pm \sigma_1)[\theta_{l\pm\sigma_2} - \theta_l] \\ &\qquad + \Phi'(n+ct+\theta_l \pm \sigma_2)[\theta_{l\mp\sigma_1} - \theta_l].\end{aligned}$$

Stability - Refined Ansatz

Recall linear evolution for θ:

$$\dot{\theta}_l(t) = Q_{ct}L(ct+\theta)v(t) + Q_{ct}M(ct+\theta)\theta^{\diamond}(t) + cQ'_{ct}v(t)$$

with mismatch term

$$\begin{aligned}[M(ct+\theta)\theta^{\diamond}]_{nl} &= \Phi'(n+ct+\theta_l \pm \sigma_1)[\theta_{l\pm\sigma_2} - \theta_l] \\ &\qquad + \Phi'(n+ct+\theta_l \pm \sigma_2)[\theta_{l\mp\sigma_1} - \theta_l].\end{aligned}$$

Special case $(\sigma_1, \sigma_2) = (1, 0)$:

$$\begin{aligned}[M(ct+\theta)\theta^{\diamond}]_{nl} &= \Phi'(n+ct+\theta_l)[\theta_{l+1} + \theta_{l-1} - 2\theta_l] \\ &= [\widetilde{M}(ct+\theta)\theta^{\diamond\diamond}]_{nl}\end{aligned}$$

with **second-difference** operator

$$\theta_l^{\diamond\diamond} = (\theta_{l+1} + \theta_{l-1} - 2\theta_l).$$

Similar reduction to **second differences** also possible for $(\sigma_1, \sigma_2) = (1, 1)$.

Stability - Refined Ansatz

Recall linear evolution for θ:

$$\dot{\theta}_l(t) = Q_{ct} L(ct+\theta) v(t) + Q_{ct} M(ct+\theta)\theta^{\diamond}(t) + cQ'_{ct} v(t).$$

Write $L_{ct} = L(ct+0)$ and $M_{ct} = M(ct+0)$. Now obtain

$$\dot{\theta}_l(t) = Q_{ct} L_{ct} v(t) + Q_{ct} M_{ct} \theta^{\diamond}(t) + cQ'_{ct} v(t) + h.o.t.$$

Worst higher order terms given by θv and $\theta\theta^{\diamond}$.

In special directions $(1,0)$ and $(1,1)$, worst higher order terms given by θv, $\theta\theta^{\diamond\diamond}$ and $(\theta^{\diamond})^2$. **No** $\theta\theta^{\diamond}$ term.

Stability - Refined Ansatz

Full linear system for v and θ:

$$\begin{array}{rcl} \dot{v}(t) & = & [I - P_{ct}]L_{ct}v(t) + [I - P_{ct}]M_{ct}\theta^{\diamond} - cP'_{ct}v(t), \\ \dot{\theta}(t) & = & Q_{ct}L_{ct}v(t) + Q_{ct}M_{ct}\theta^{\diamond}(t) + cQ'_{ct}v(t), \end{array}$$

with $P_{ct} = \Phi'(\cdot + ct)Q_{ct}$. Note $P_{ct}^2 = P_{ct}$.

Write $\mathcal{G}(t, t_0)$ for Green's function. Also write $\overline{\mathcal{G}}(t, t_0)$ for Green's function for:

$$\dot{w}_{nl}(t) = [L_{ct}w(t)]_{nl} = [\Delta^{\times} w(t)]_{nl} + g'\big(\Phi(n + ct)\big)w_{nl}(t)$$

[We have already studied this system].

We then have:

$$\mathcal{G}(t, t_0) = \left(\begin{array}{cc} [I - P_{ct}]\overline{\mathcal{G}}(t, t_0)[I - P_{ct_0}] & [I - P_{ct}]\overline{\mathcal{G}}(t, t_0)\Phi'(\cdot + ct_0) \\ Q_{ct}\overline{\mathcal{G}}(t, t_0)[I - P_{ct_0}] & Q_{ct}\overline{\mathcal{G}}(t, t_0)\Phi'(\cdot + ct_0) \end{array} \right).$$

Stability - Refined Ansatz

Recall Green's function:

$$\mathcal{G}(t, t_0) = \begin{pmatrix} [I - P_{ct}]\overline{\mathcal{G}}(t, t_0)[I - P_{ct_0}] & [I - P_{ct}]\overline{\mathcal{G}}(t, t_0)\Phi'(\cdot + ct_0) \\ Q_{ct}\overline{\mathcal{G}}(t, t_0)[I - P_{ct_0}] & Q_{ct}\overline{\mathcal{G}}(t, t_0)\Phi'(\cdot + ct_0) \end{pmatrix}.$$

We know the **slow** parts of $\overline{\mathcal{G}}(t, t_0)$. In Fourier space tbese are given by

$$[\overline{\mathcal{G}}_\omega(t, t_0)]_{nn_0} \sim e^{\lambda_\omega(t-t_0)}\phi_\omega(n + ct)\psi^*_\omega(n_0 + ct_0).$$

Now, $[I - P_{ct}]$ projects away $\phi_0(n + ct)$. In addition, $\psi_0(n_0 + ct_0)$ can be seen as Q_{ct_0}, and we have $Q_{ct_0}[I - P_{ct_0}] = 0$.

Roughly speaking, in Fourier space:

$$\mathcal{G}_\omega(t, t_0) = \begin{pmatrix} \omega^2 e^{-\kappa\omega^2(t-t_0)} & \omega e^{-\kappa\omega^2(t-t_0)} \\ \omega e^{-\kappa\omega^2(t-t_0)} & e^{-\kappa\omega^2(t-t_0)} \end{pmatrix}.$$

Stability - Refined Ansatz

In special direction and $(1,1)$ we have better expansion:

$$\mathcal{G}_\omega(t,t_0) = \begin{pmatrix} \omega^4 e^{-\kappa\omega^2(t-t_0)} & \omega^2 e^{-\kappa\omega^2(t-t_0)} \\ \omega^2 e^{-\kappa\omega^2(t-t_0)} & e^{-\kappa\omega^2(t-t_0)} \end{pmatrix}.$$

Each ω gives $t^{-1/2}$ extra decay. We hence expect, for initial condition (v^0, θ^0) that are ℓ^1 in transverse direction:

$$\begin{array}{lll} \|\theta(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} & \sim & (1+t)^{-1/4} \\ \|\theta^\diamond(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} & \sim & (1+t)^{-3/4} \\ \|\theta^{\diamond\diamond}(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} & \sim & (1+t)^{-5/4} \\ \|v(t)\|_{\ell^\infty(\mathbb{Z};\ell^2(\mathbb{Z};\mathbb{R}))} & \sim & (1+t)^{-5/4}, \end{array}$$

Since **worst** nonlinear terms are θv, $\theta\theta^{\diamond\diamond}$ and $(\theta^\diamond)^2$, which all decay in ℓ^1 as $(1+t)^{-3/2}$, a nonlinear argument closes easily.

Situation for $(1,0)$ is even better, since $\phi_\omega = \Phi'$ for all ω.

Stability - Refined Ansatz

Recall rough expansion

$$\mathcal{G}_\omega(t, t_0) = \begin{pmatrix} \omega^2 e^{-\kappa\omega^2 t} & \omega e^{-\kappa\omega^2 t} \\ \omega e^{-\kappa\omega^2 t} & e^{-\kappa\omega^2 t} \end{pmatrix}.$$

Each ω gives $t^{-1/2}$ extra decay. We hence expect, for initial condition (v^0, θ^0) that are ℓ^1 in transverse direction:

$$\begin{aligned} \|\theta(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} &\sim (1+t)^{-1/4} \\ \|\theta^\diamond(t)\|_{\ell^2(\mathbb{Z};\mathbb{R})} &\sim (1+t)^{-3/4} \\ \|v(t)\|_{\ell^\infty(\mathbb{Z};\ell^2(\mathbb{Z};\mathbb{R}))} &\sim (1+t)^{-3/4}, \end{aligned}$$

Worst nonlinear terms now $v\theta$ and $\theta\theta^\diamond$. Both are $O(t^{-1})$ in ℓ^1-transverse.

Need delicate non-linear argument.

Stability - Refined Ansatz

Need to deal with $\theta\theta^\diamond$ and $v\theta$ terms.

Key trick:

$$\theta_l(\theta_{l+1} - \theta_l) = \frac{1}{2}\big(\theta_{l+1}^2 - \theta_l^2 - (\theta_{l+1} - \theta_l)^2\big).$$

This is discrete version of

$$uu_x = \frac{1}{2}(u^2)_x,$$

heavily exploited in study of conservation laws.

Key point: $(\theta_{l+1} - \theta_l)^2$ decays very fast ($t^{-3/2}$). Difference $\theta_{l+1}^2 - \theta_l^2$ decays very slow ($t^{-1/2}$), but gives an extra ω in Fourier space which leads to more decay on Green's function ($t^{-3/4}$ instead of $t^{-1/4}$).

$$\int_0^t (1+t-t_0)^{-1/4}(1+t_0)^{-1}\,dt_0 \sim \ln(1+t)(1+t)^{-1/4} \quad BAD$$
$$\int_0^t (1+t-t_0)^{-3/4}(1+t_0)^{-1/2}\,dt_0 \sim (1+t)^{-1/4} \quad GOOD.$$

Stability - Refined Ansatz

Final term to deal with: θv.

Key trick: isolate slowest decaying part of v from Taylor expansion of Fourier symbol. Taylor expansion not in ω but in $e^{i\omega} - 1$ in order to exploit difference structure!

Slowest decaying part of v directly proportional to slowest decaying part of $\theta^\diamond$. Can decompose:

$$v_{nl}(t) = w_{nl}(t) - i[I - P_{ct}][\partial_\omega \phi(\cdot + ct)]_{\omega=0}\big(\theta_{l+1}(t) - \theta_l(t)\big).$$

New variable $w(t)$ decays **faster** than v, at rate $t^{-5/4}$.

Slow part of $v(t)$ proportional to $\theta^\diamond$. Can treat in same way as $O(\theta\theta^\diamond)$ term!

Notice that in special directions $(1,0)$ and $(1,1)$, we have $v(t) = w(t)$.

Stability in 2d

Recall Ansatz

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t).$$

Thm. [H., Hoffman, Van Vleck, 2012] Travelling wave ($c \neq 0$) in any rational direction is nonlinearly stable under small perturbations

$$\begin{array}{rcl} \sum_{l \in \mathbb{Z}} |\theta_l(0)| & \ll & 1 \\ \sup_{n \in \mathbb{Z}} [\sum_{l \in \mathbb{Z}} |v_{nl}(0)|] & \ll & 1. \end{array}$$

Note: perturbations need to be summable in transverse direction.

We have $\theta_l(t) \to 0$ and $v_{nl}(t) \to 0$ as $t \to \infty$.

In other words, deformations of interface diffuse in transverse direction.

It does NOT lead to a shift in the wave.

Stability in 2d

Recall Ansatz

$$u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t).$$

Algebraic decay rates depend on direction of propagation!

Horizontal waves ($\theta = 0$):

$$\theta_l(t) \sim t^{-1/2}, \qquad v_{nl}(t) \sim t^{-7/4}.$$

Diagonal waves ($\theta = \frac{\pi}{4}$):

$$\theta_l(t) \sim t^{-1/2}, \qquad v_{ij}(t) \sim t^{-3/2}.$$

Other rational directions: (very slow decay - delicate nonlinear analysis needed)

$$\theta_l(t) \sim t^{-1/2}, \qquad v_{ij}(t) \sim t^{-1}.$$

Summary

- Obtained stability in 2d for rational directions
- Only spectral conditions imposed on wave.
- Works even in absence of comparison principles.

Outlook:

- What about irrational directions ?
- What about standing waves ($c = 0$) ?