Travelling Pulses for the Discrete FitzHugh-Nagumo System

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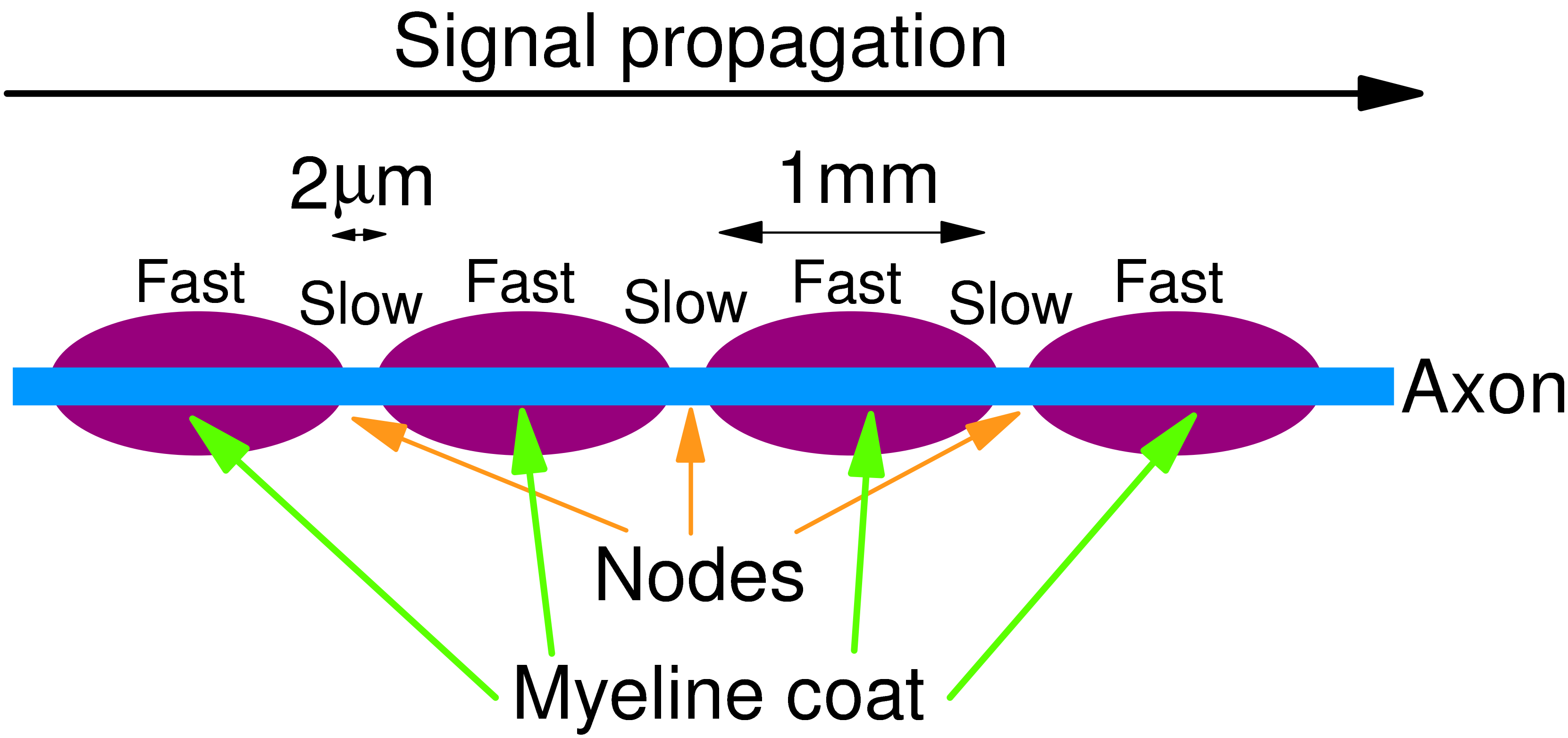
Signal Propagation through Nerves

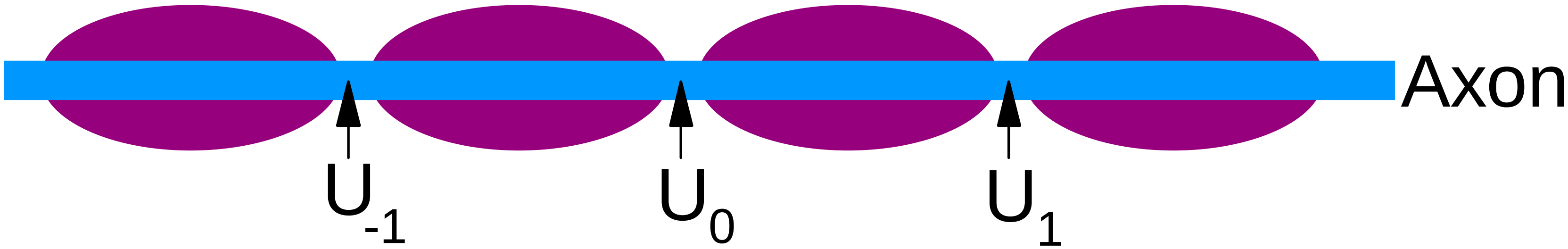
Nerve fibres carry signals over large distances (meter range).



- Fiber has myeline coating with periodic gaps called *nodes of Ranvier* .
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range)
- Slow propagation in gaps, but signal chemically reinforced.

Signal Propagation: The Model

One is interested in the potential U_j at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].

Ionic current has sodium and potassium component.

Electro-chemical analysis leads to the two component LDE [Keener and Sneyd, 1998]

$$\begin{aligned} \dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)], \end{aligned}$$

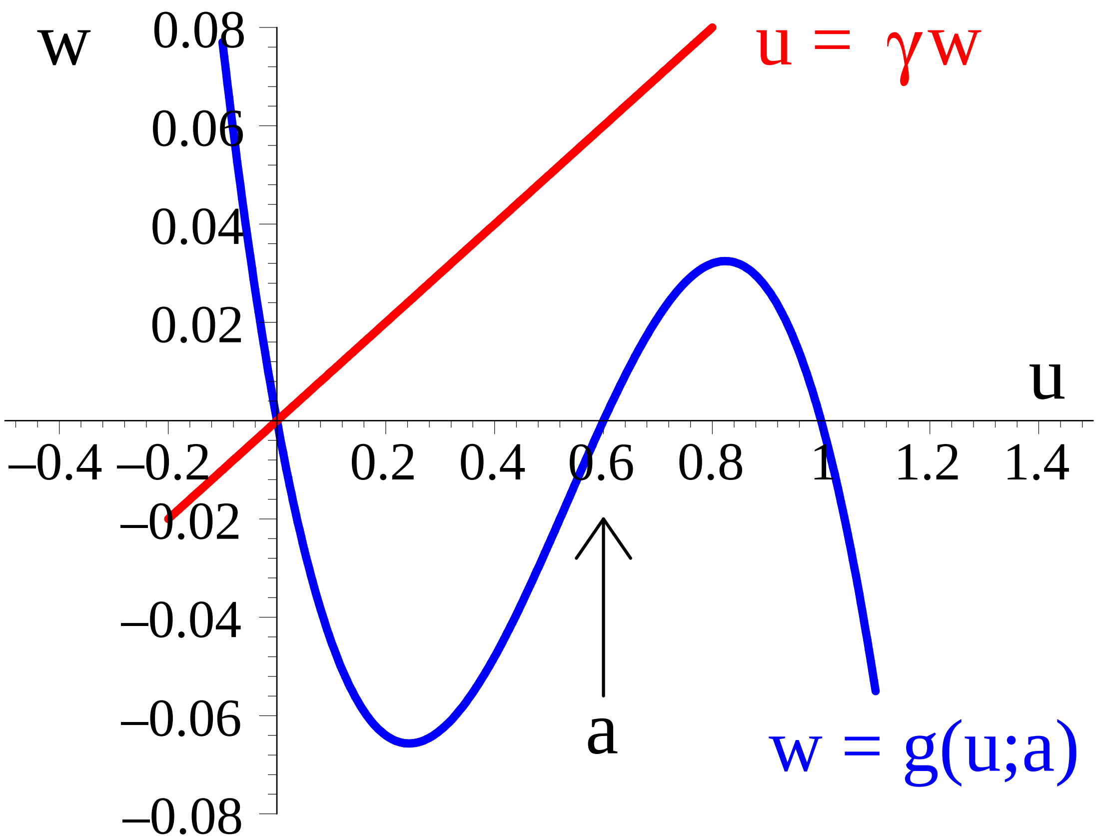
posed on a 1-dimension lattice, i.e. $j \in \mathbb{Z}$.

Potassium recovery encoded in second equation.

Signal Propagation: Nonlinearity

Recall the dynamics:

$$\begin{aligned}\dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)].\end{aligned}$$



Bistable nonlinearity g given by

$$g(u; a) = u(a - u)(u - 1).$$

Parameter $\gamma > 0$ small so

$$w \neq g(\gamma w; a)$$

for $w \neq 0$.

Signal Propagation: FitzHugh-Nagumo PDE

The discrete FitzHugh-Nagumo system arises by discretizing the FH-N PDE

$$\begin{aligned} U_t &= U_{xx} + g(U;a) - W, \\ W_t &= \epsilon[U - \gamma W]. \end{aligned}$$

- Many authors have studied this equation.
- Starting point: travelling wave Ansatz

$$(U, W)(x, t) = (u, w)(x + ct).$$

This Ansatz yields the ODE

$$\begin{aligned} u' &= v, \\ v' &= cv - g(u;a) + w, \\ w' &= \tfrac{\epsilon}{c}(u - \gamma w). \end{aligned}$$

This slow-fast system has served as a prototype for development of geometric singular perturbation theory.

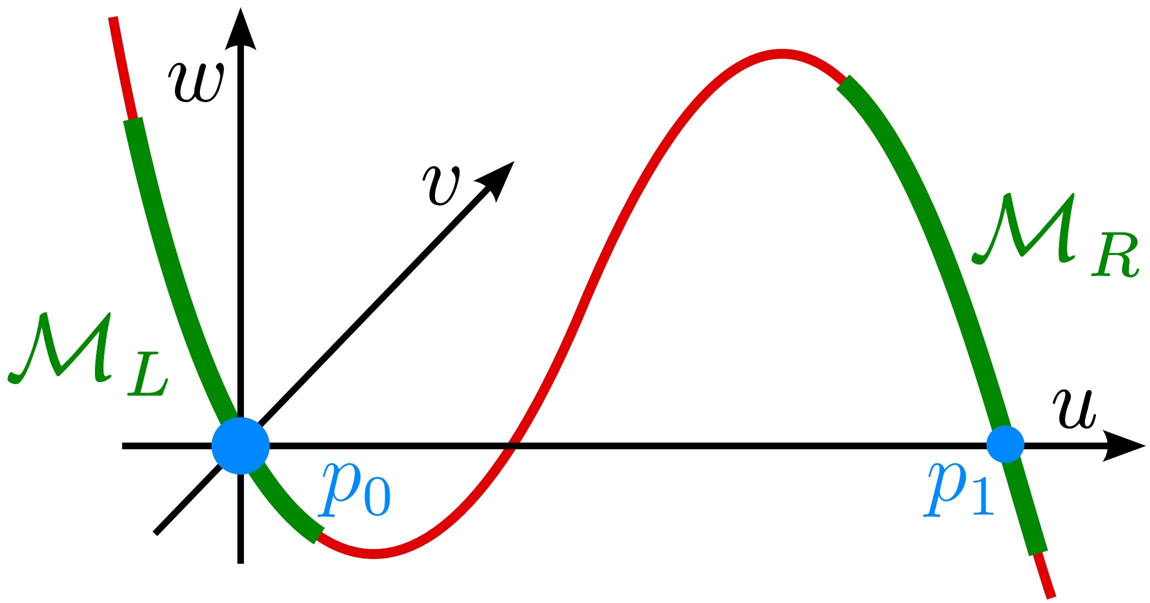
Signal Propagation: FitzHugh-Nagumo PDE

Choosing $\epsilon = 0$, we find

$$\begin{aligned} u' &= v, \\ v' &= cv - g(u;a) + w, \\ w' &= 0, \end{aligned}$$

admitting an equilibria-manifold $\mathcal{M} = (u, 0, g(u;a))$.

Write $p_0 = (0,0,0)$ and $p_1 = (1,0,0)$ and choose $p_0 \in \mathcal{M}_L \subset \mathcal{M}$ and $p_1 \in \mathcal{M}_R \subset \mathcal{M}$; avoiding knees of the cubic.



Signal Propagation: FitzHugh-Nagumo PDE

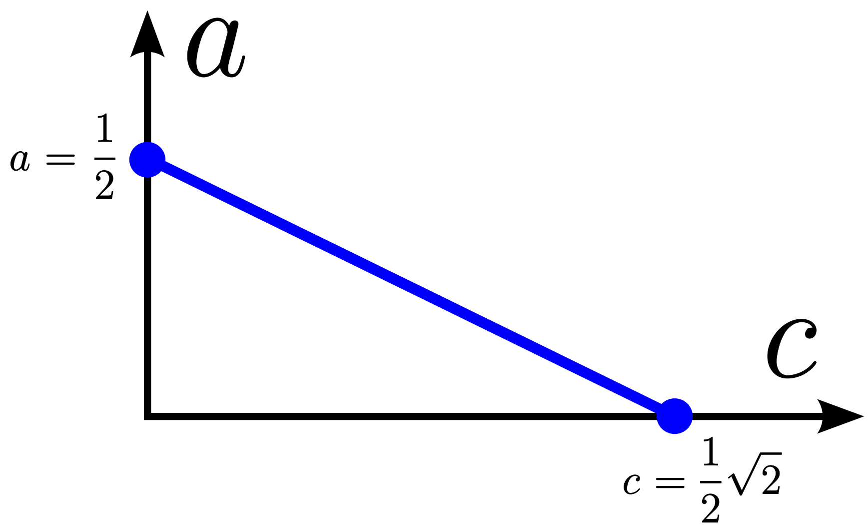
Heteroclinics $p_0 \to p_1$ must solve

$$\begin{aligned} u' &= v, \\ v' &= cv - g(u;a), \end{aligned}$$

and satisfy $u(-\infty) = 0$ and $u(+\infty) = 1$.

These correspond to travelling pulses of the Nagumo PDE

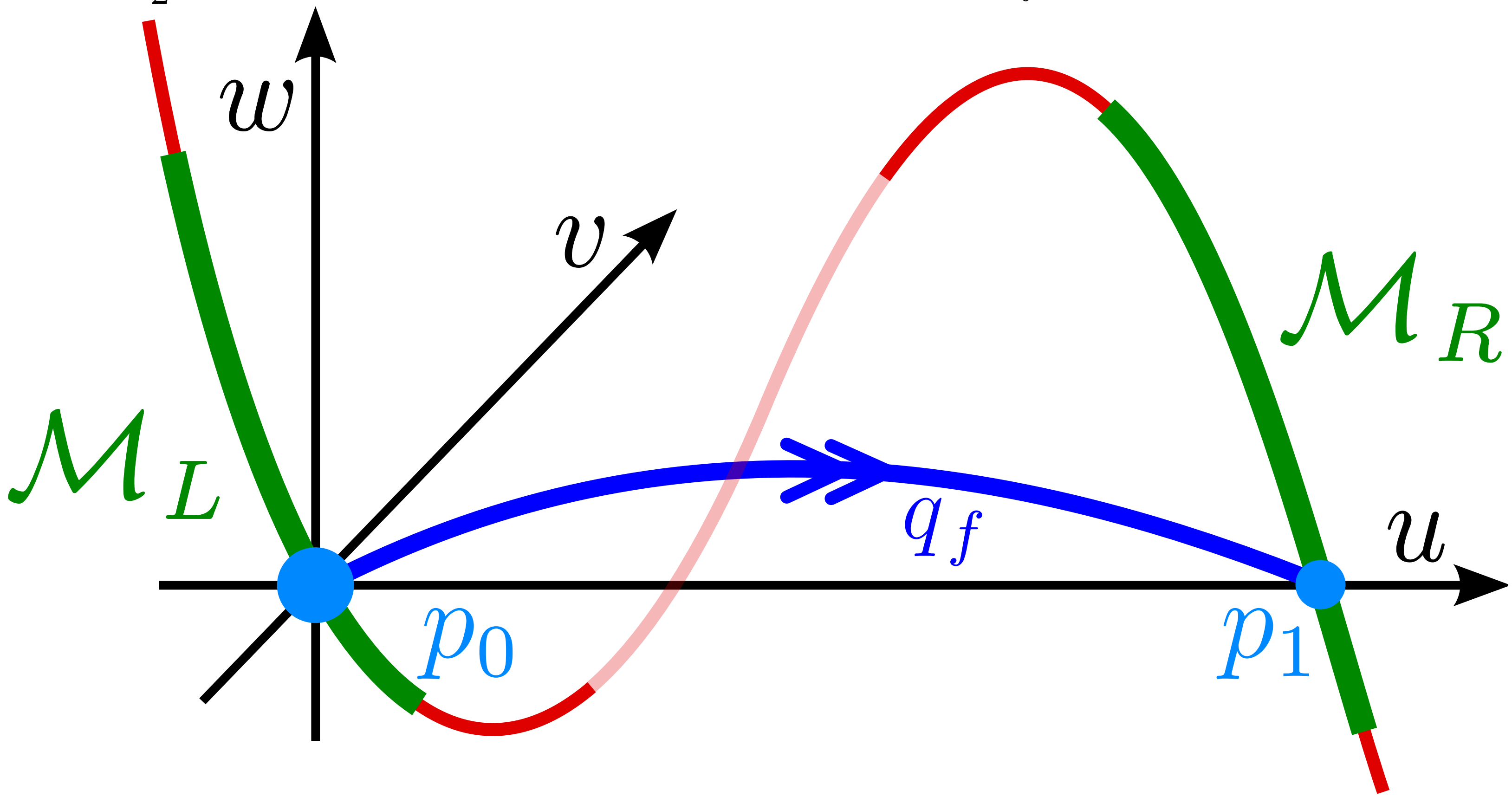
$$U_t = U_{xx} + g(U;a).$$



Existence of such pulses is well-known; explicit calculations are possible for the cubic g.

Signal Propagation: FitzHugh-Nagumo PDE

Fix $0 < a < \frac{1}{2}$; there exists wave speed c_* and front q_f:

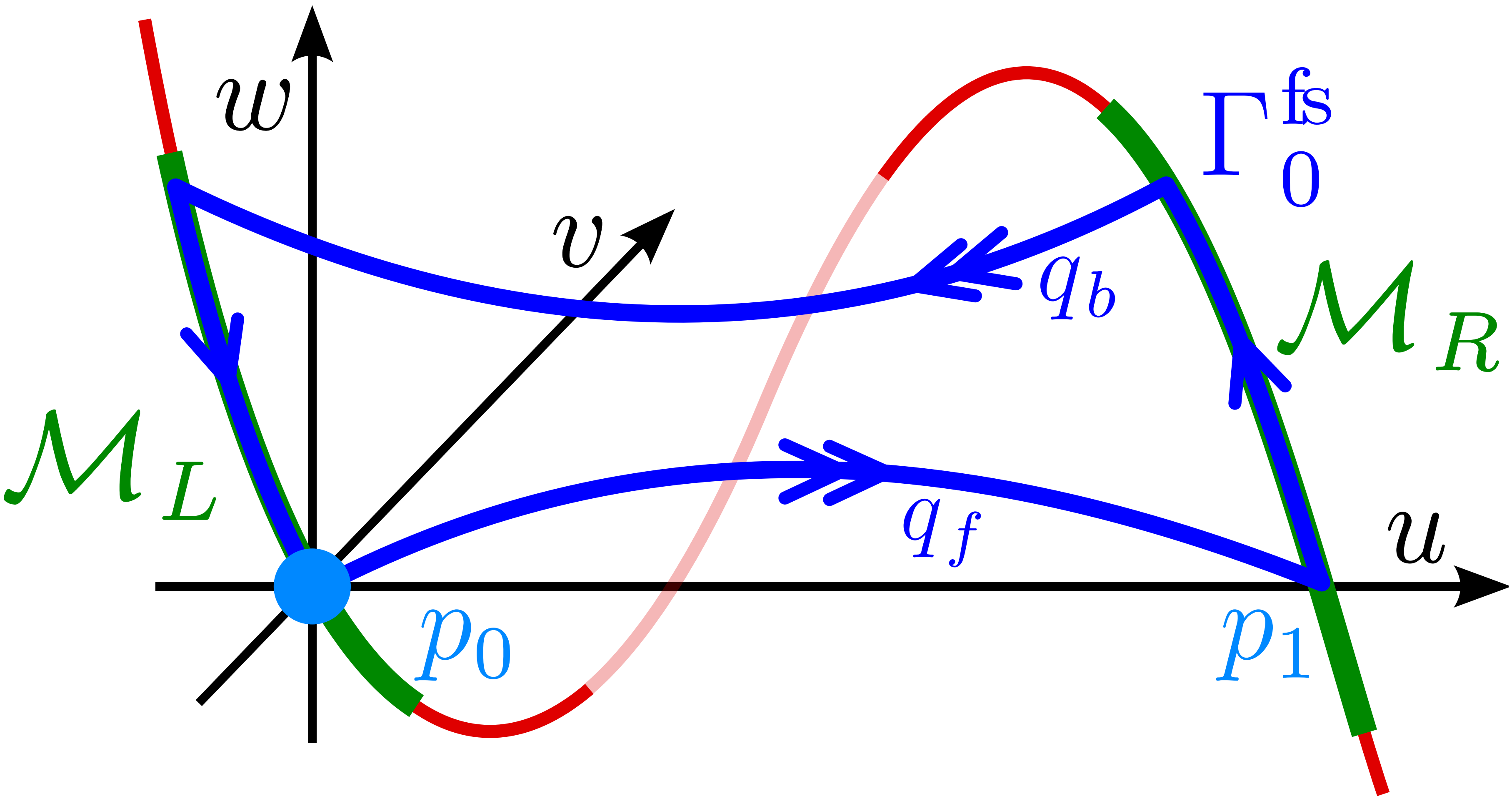


We now need to go back from $\mathcal{M}_R$ to $\mathcal{M}_L$.

Cubic is mirror symmetric around inflection point $\longrightarrow$ there exists w_* and profile q_b connecting $\mathcal{M}_R$ to $\mathcal{M}_L$ for same wave speed $c = c_*$.

Signal Propagation: FitzHugh-Nagumo PDE

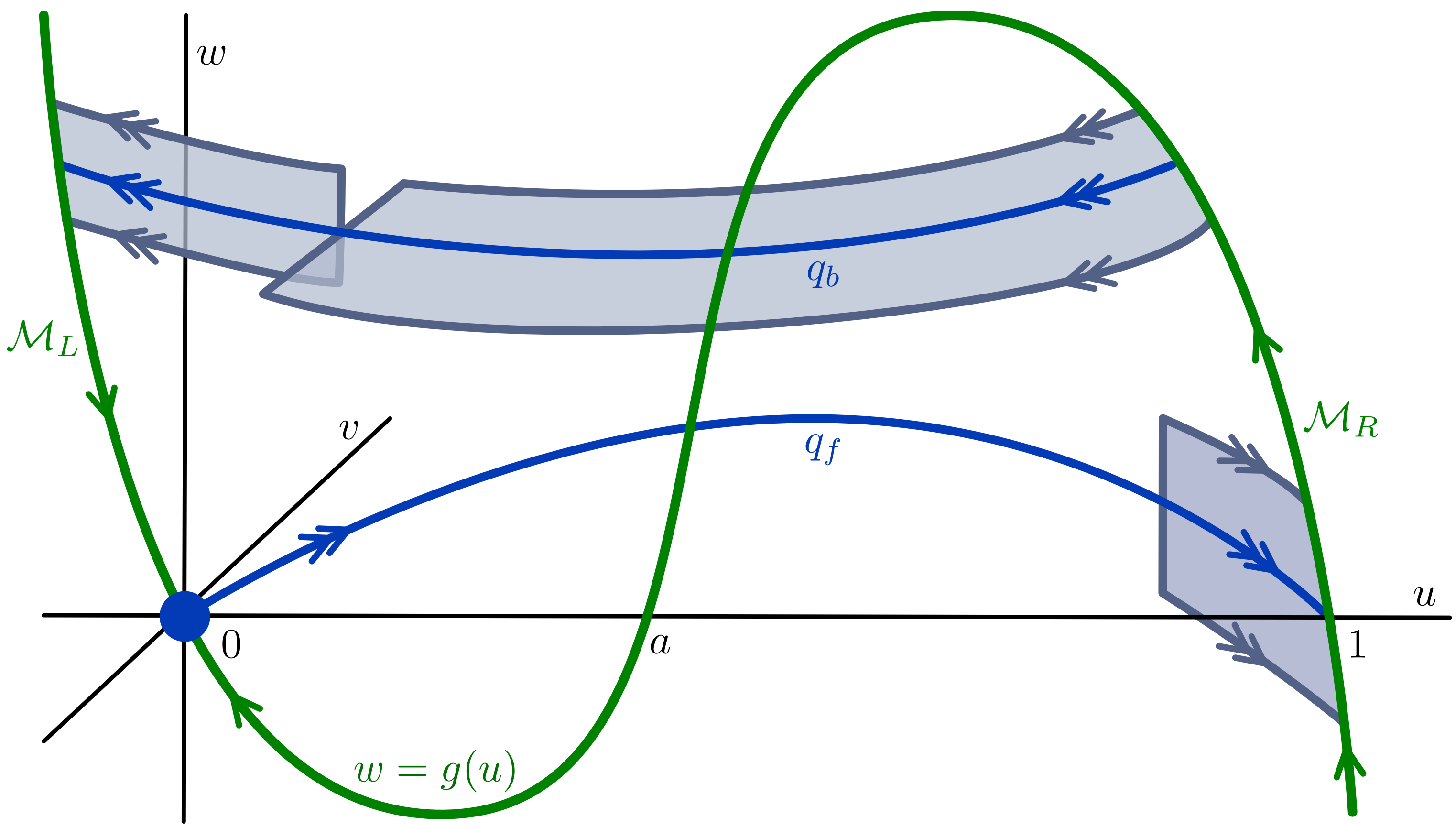
Connecting the pieces we find a fast $[c_* > 0]$ singular homoclinic orbit Γ_0^{fs}.



Classic Theorem: For sufficiently small $\epsilon > 0$, there is a [locally unique] travelling pulse solution to FH-N PDE that winds around Γ_0^{fs} once, with wavespeed $c < c_*$.

Signal Propagation: FitzHugh-Nagumo PDE

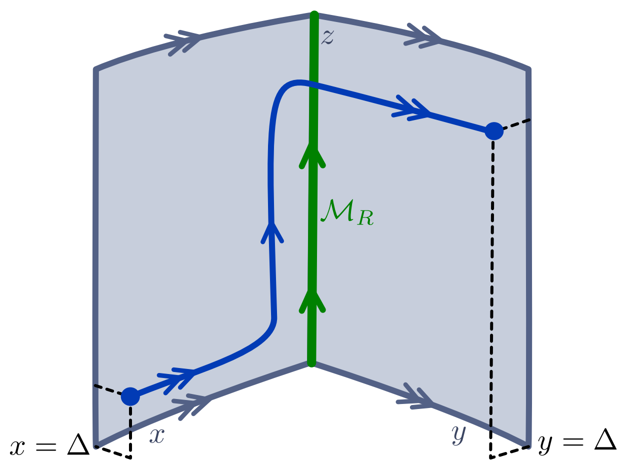
- First proofs given by Carpenter and Hastings [1976].
- 'Modern' proof developed by Jones and coworkers based on transverse intersection of manifolds $\mathcal{W}^u(0)$ and $\mathcal{W}^s(\mathcal{M}_L)$.



Main difficulty: track $\mathcal{W}^u(0)$ as it spends time $O(\epsilon^{-1})$ near $\mathcal{M}_R$.

FitzHugh-Nagumo PDE: Exchange Lemma

Exchange Lemma is key tool to track $\mathcal{W}^u(0)$ near $\mathcal{M}_R$.



Fenichel coordinates:

$$\begin{aligned} x' &= -A^s(x,y,z)x \\ y' &= A^u(x,y,z)y \\ z' &= \epsilon[1 + B(x,y,z)xy], \end{aligned}$$

with $A^s, A^u > \eta > 0$; A^s, A^u, B smooth and bounded.

- Fix small $\Delta > 0$.
- Pick $z_0 \in \mathbb{R}$, T large and $\epsilon > 0$ small
- Find solution with

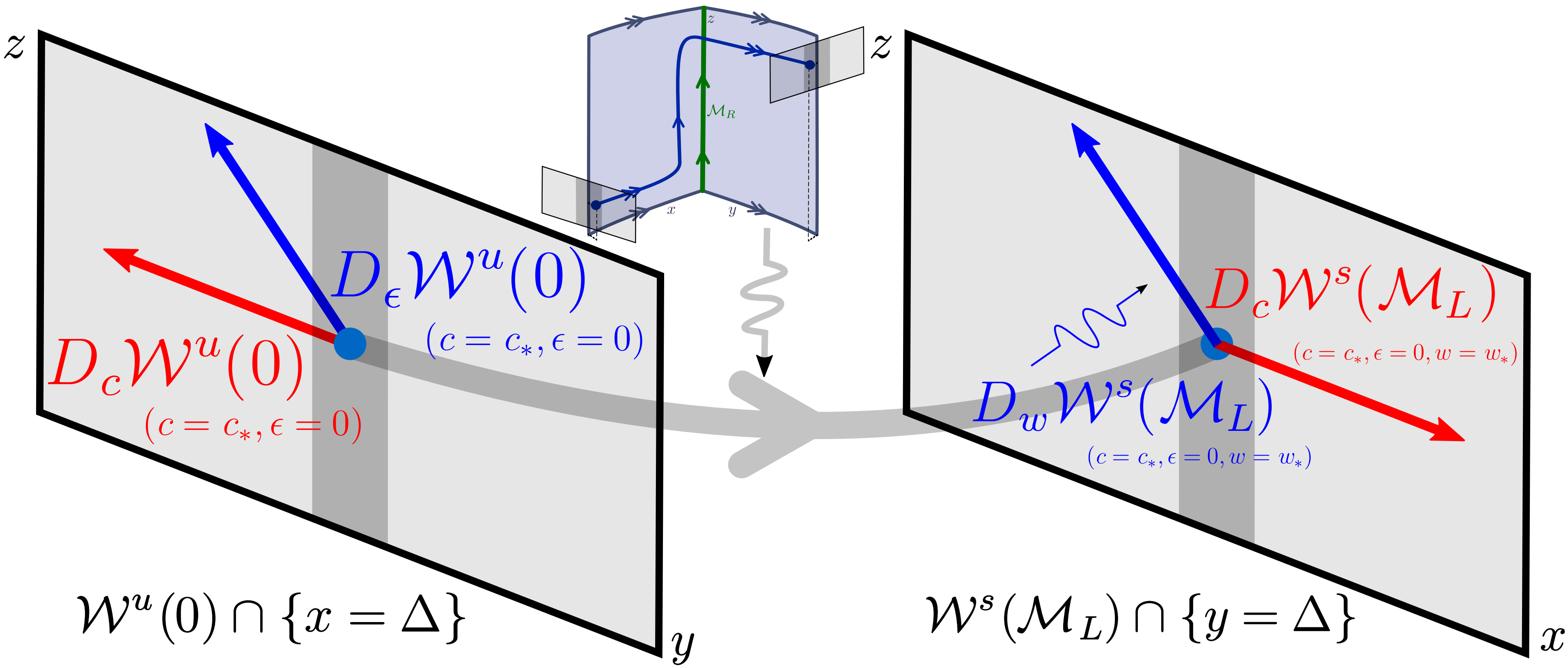
$$x(0) = \Delta, \qquad z(0) = z_0, \qquad y(T) = \Delta$$

- Exchange Lemma: unique solution exists, bounds:

$$|y(0)| + |x(T)| + |z(T) - z_0 - \epsilon T| = O(e^{-\eta T})$$

FitzHugh-Nagumo PDE: Exchange Lemma

The problem can now be decomposed into two parts:



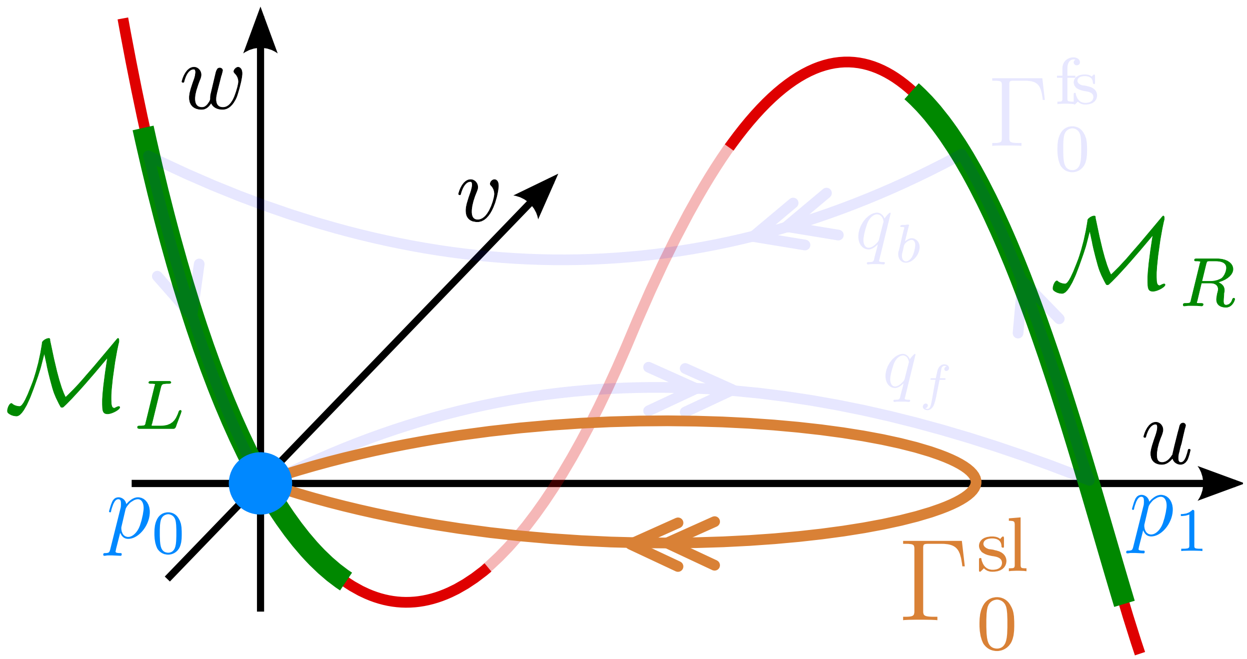
- Intersection $\mathcal{W}^u(0) \cap \{x = \Delta\}$ can be studied separately from intersection $\mathcal{W}^s(\mathcal{M}_L) \cap \{y = \Delta\}$.
- Melnikov identities yield signs of $D_c \mathcal{W}^u(0)$ etc.
- Exchange Lemma used to link pieces together.

FitzHugh-Nagumo PDE: Slow Pulses

Recall the travelling wave ODE

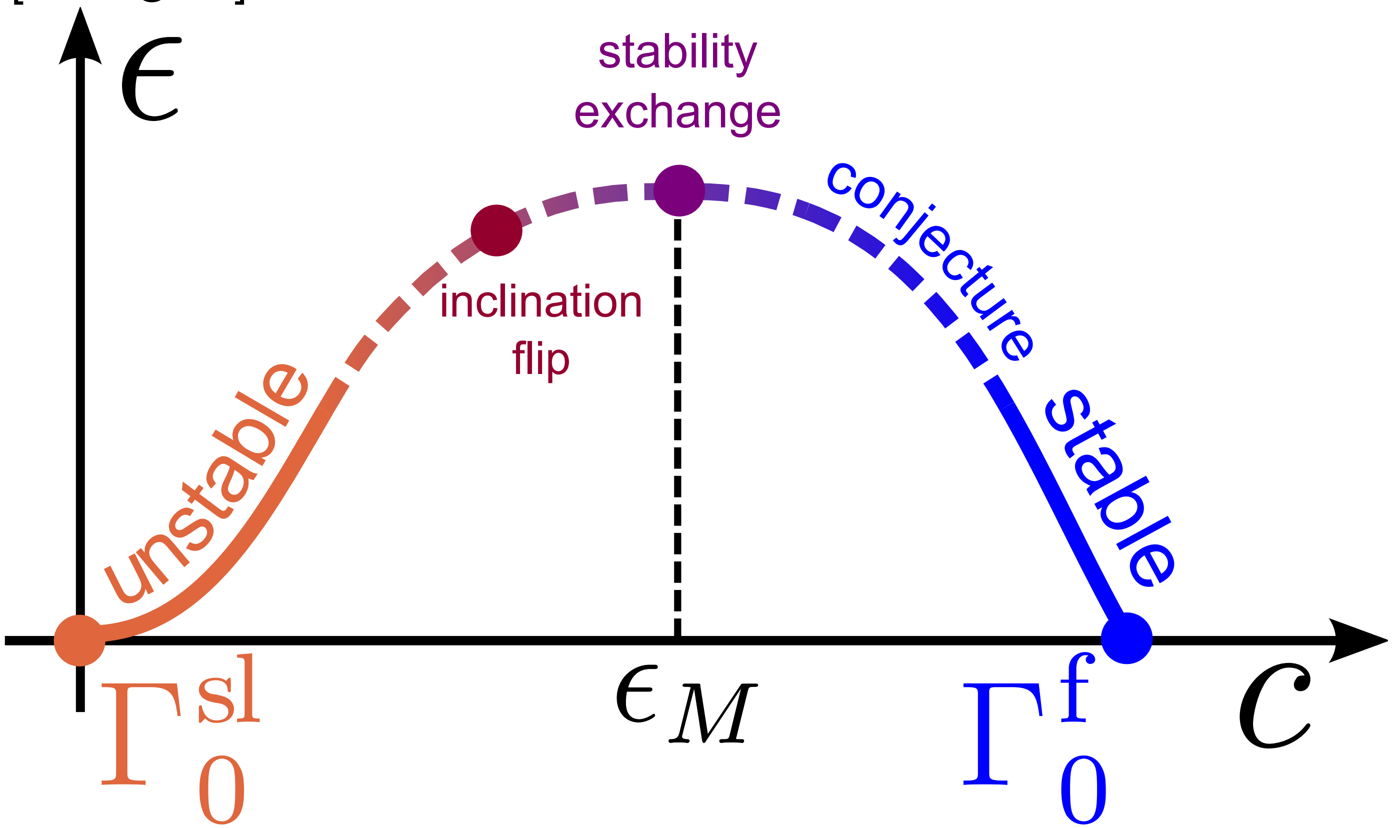
$$\begin{aligned} u' &= v, \\ v' &= cv - g(u;a) + w, \\ w' &= \frac{\epsilon}{c}(u - \gamma w). \end{aligned}$$

In the singular limit $c \longrightarrow 0$ and $\frac{\epsilon}{c} \longrightarrow 0$, one finds an additional slow-singular orbit Γ_0^{sl}.



FitzHugh-Nagumo PDE: Status

Conjecture [Yanagida]: fast and slow branches are connected.



- Sandstede, Krupa, Szmolyan (1997): for $a \approx \frac{1}{2}$, conjecture is true. Inclination-flip somewhere along connecting curve.
- Jones, Yanagida (1984): fast waves are asymptotically stable for full PDE.
- Flores (1991): slow waves are unstable.
- Sandstede: stability change at maximum of curve.

Discrete FitzHugh-Nagumo LDE

We return to the discrete FitzHugh-Nagumo system

$$\begin{aligned} \dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)]. \end{aligned}$$

Travelling wave Ansatz $(U_j, W_j)(t) = (u, w)(j + ct)$ leads to

$$\begin{aligned} cu'(\xi) &= u(\xi + 1) + u(\xi - 1) - 2u(\xi) + g(u(\xi); a) - w(\xi), \\ cw'(\xi) &= \epsilon[u(\xi) - \gamma w(\xi)]. \end{aligned}$$

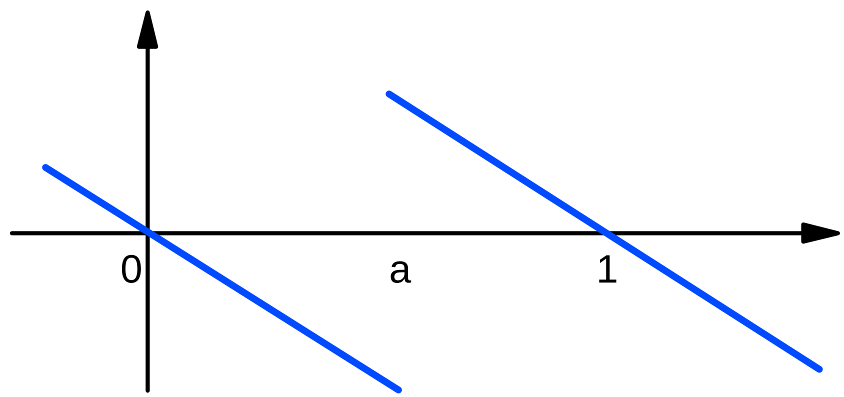
This is a singularly perturbed functional differential equation of mixed type (MFDE).

Discrete FitzHugh-Nagumo - Previous work

Two main directions for previous work on discrete FitzHugh-Nagumo LDE

$$\begin{aligned}\dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t);a) - W_j(t),\\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)].\end{aligned}$$

- Rigorous results for specially prepared nonlinearities
  - Chen + Hastings: nonlinearity vanishes identically on critical regions of U and W.
  - Tonnelier; Elmer and Van Vleck: explicit calculations with Fourier series for McKean sawtooth caricature:



- Carpio and coworkers: formal results using asymptotic techniques.

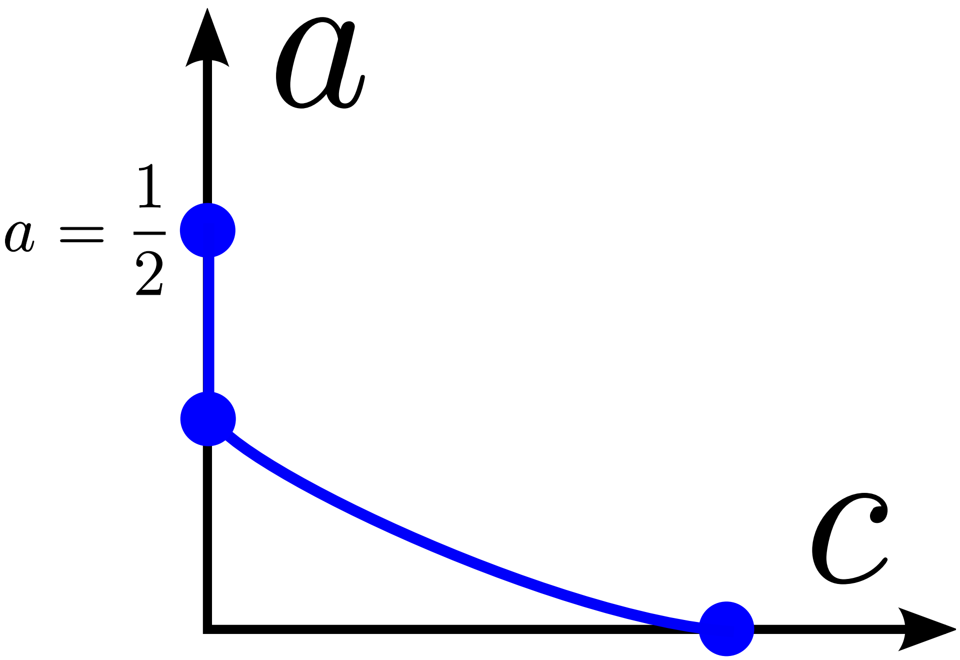
Discrete FitzHugh-Nagumo LDE

For $\epsilon = 0$ and $w = 0$, we obtain the discrete Nagumo LDE

$$\dot{U}_j(t) = \alpha[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + g(U_j(t); a),$$

with travelling pulse MFDE

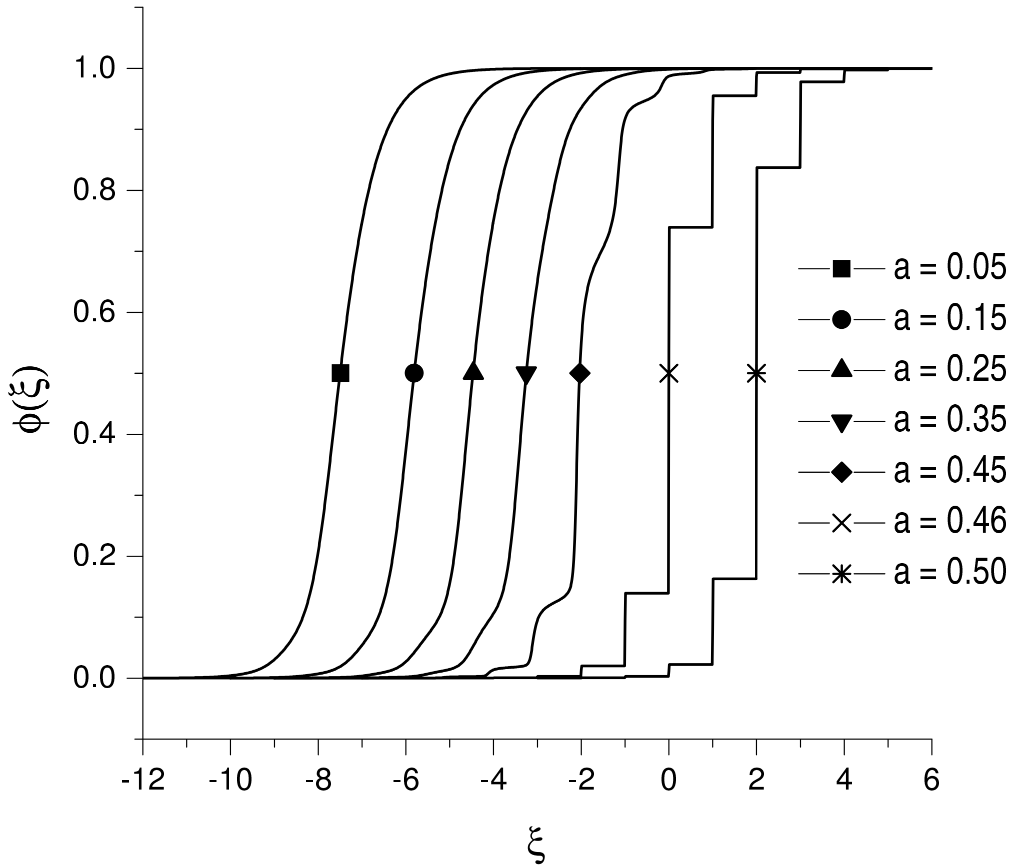
$$cu'(\xi) = \alpha[u(\xi + 1) + u(\xi - 1) - 2u(\xi)] + g(u(\xi); a).$$

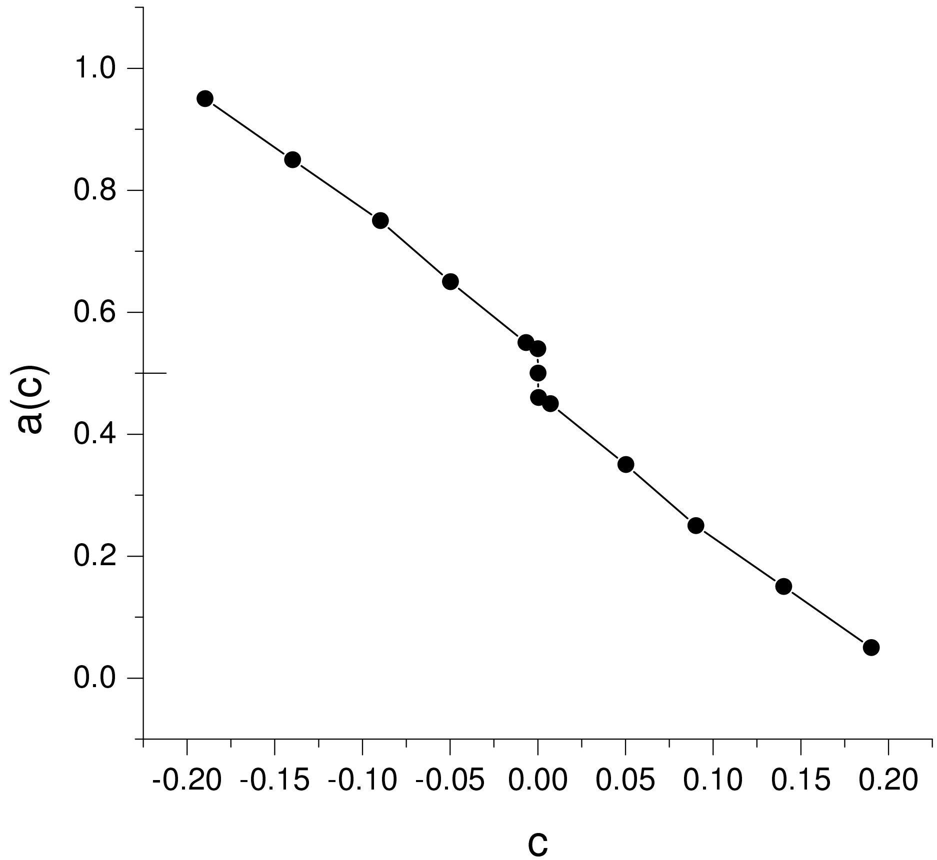


- This problem becomes singular in the $c \to 0$ limit, in contrast with the Nagumo PDE case.
- Keener (1987) + Mallet-Paret (1999): pick $\alpha > 0$ small; $c = 0$ for a in nonempty interval $[a_*, \frac{1}{2}]$.

Discrete FitzHugh-Nagumo LDE - Propagation failure

Travelling waves for the discrete Nagumo equation $[\alpha = 0.1]$ connecting $0 \to 1$.





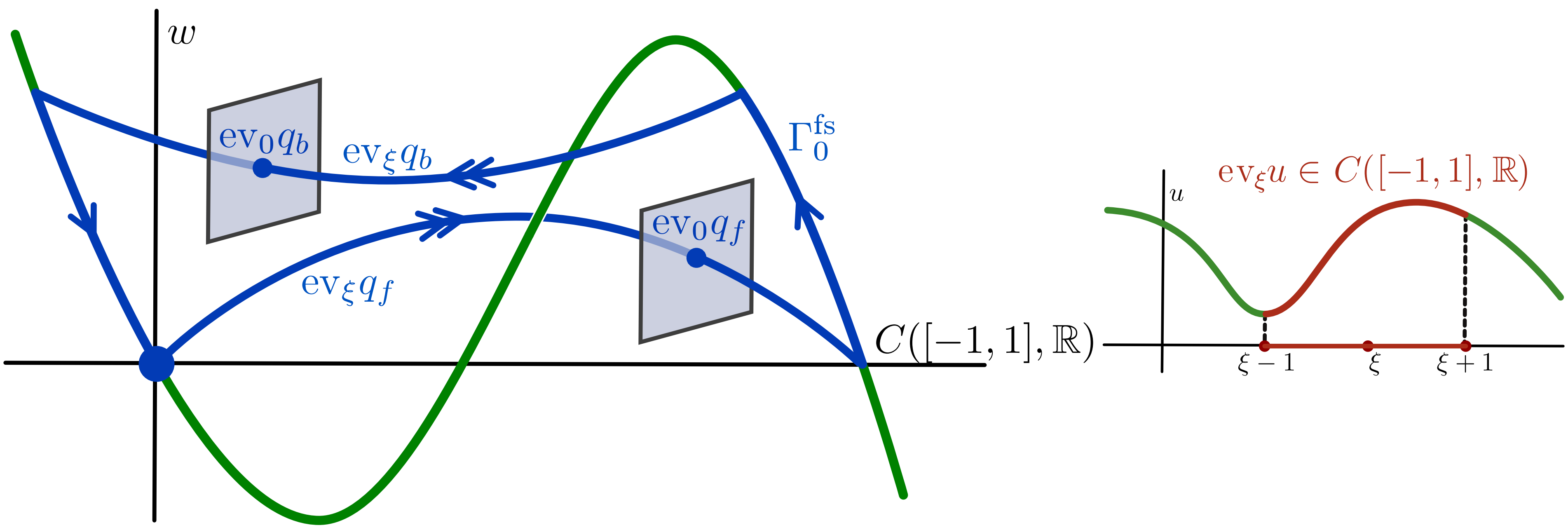
- Note that $c = 0$ for all $a \in [0.46, 0.54]$. Propagation failure!
- Observe the discontinuities in the wave profiles in this region.
- Gaps cause "energy barrier" that signal must overcome.

Discrete FitzHugh-Nagumo LDE - Fast Pulses

- Focus on fast-solutions to discrete FHN bifurcating from Γ_0^{fs},

$$\begin{aligned} cu'(\xi) &= u(\xi+1) + u(\xi-1) - 2u(\xi) + g(u(\xi);a) - w(\xi), \\ cw'(\xi) &= \epsilon[u(\xi) - \gamma w(\xi)]. \end{aligned}$$

- Unclear how to treat slow-solutions in propagation failure regime.



Mixed Type Functional Differential Equations (MFDEs)

Let us first study the Nagumo travelling wave MFDE

$$u'(\xi) = u(\xi + 1) + u(\xi - 1) - 2u(\xi) + g\big(u(\xi); a\big).$$

- Theory for MFDEs started developing $\sim$ 10 years ago.
- MFDEs generalize delay equations, e.g.

$$u'(\xi) = u(\xi - 1) + g\big(u(\xi)\big),$$

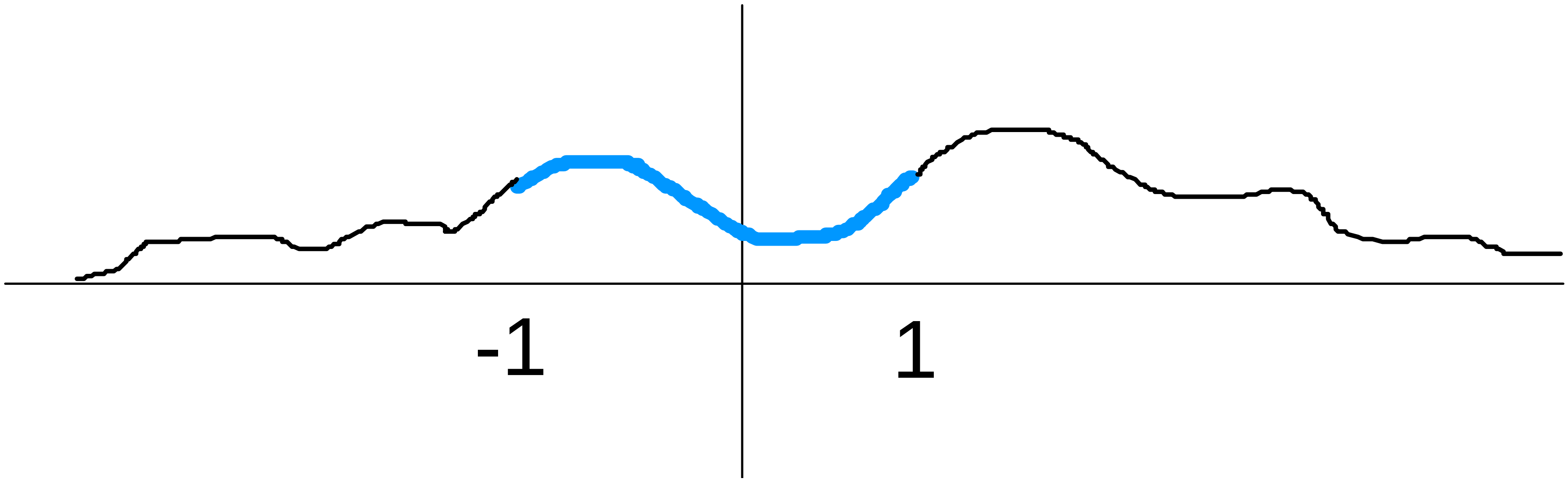
  which have been used for more than half a century.
- Time lags naturally in many modelling applications.
- Delay equations: functional-analytic setup developed in past three decades.

MFDEs

Recall our prototype MFDE

$$u'(\xi) = u(\xi + \mathbf{1}) + u(\xi - \mathbf{1}) - 2u(\xi) + g\big(u(\xi)\,;a\big).$$

Such equations differ from ODEs and delay equations in a fundamental way.

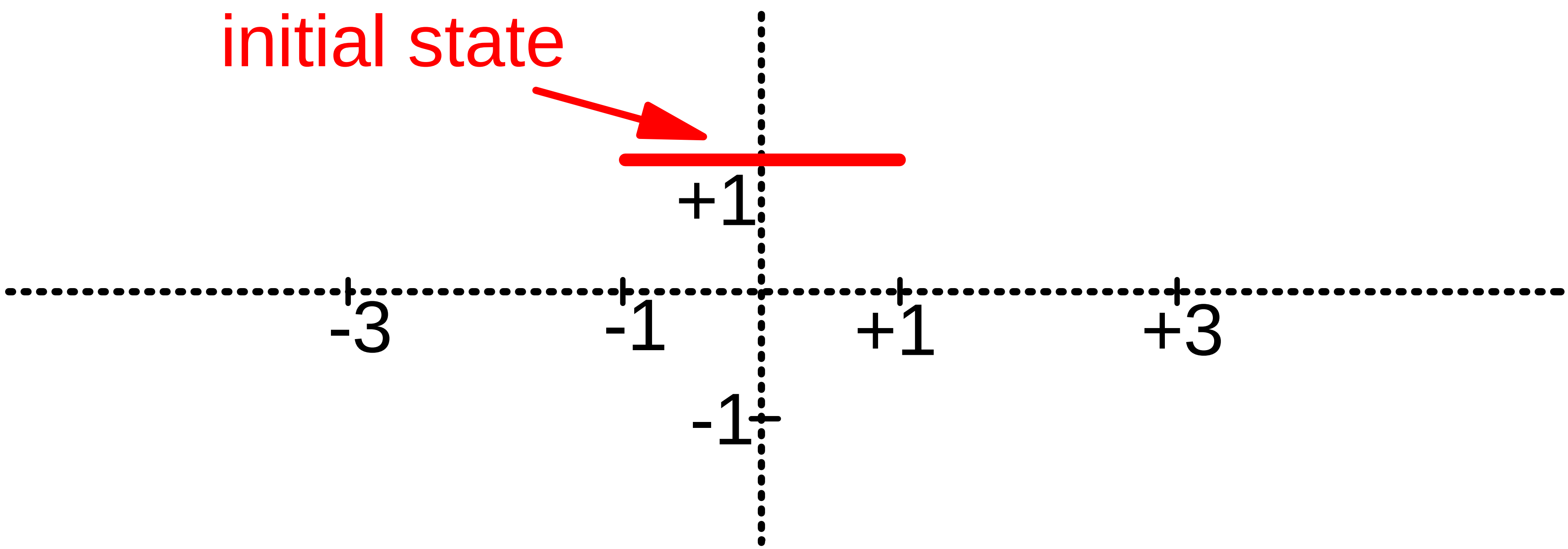


Problem I: Statespace is infinite dimensional: need to specify an initial function on $[-1, 1]$.

Problem II: Ill-posedness

Consider the homogeneous MFDE

$$v'(\xi) = v(\xi - 1) + v(\xi + 1).$$

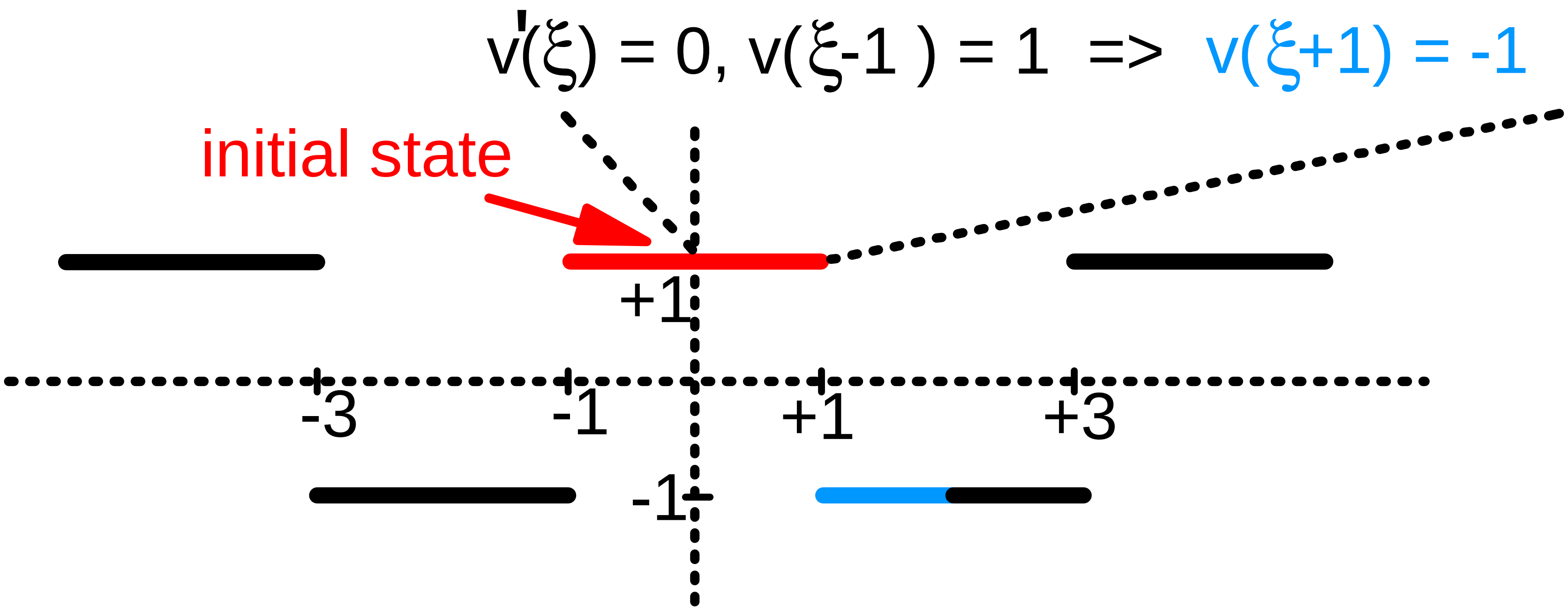


(Example due to Rustichini)

Problem II: Ill-posedness

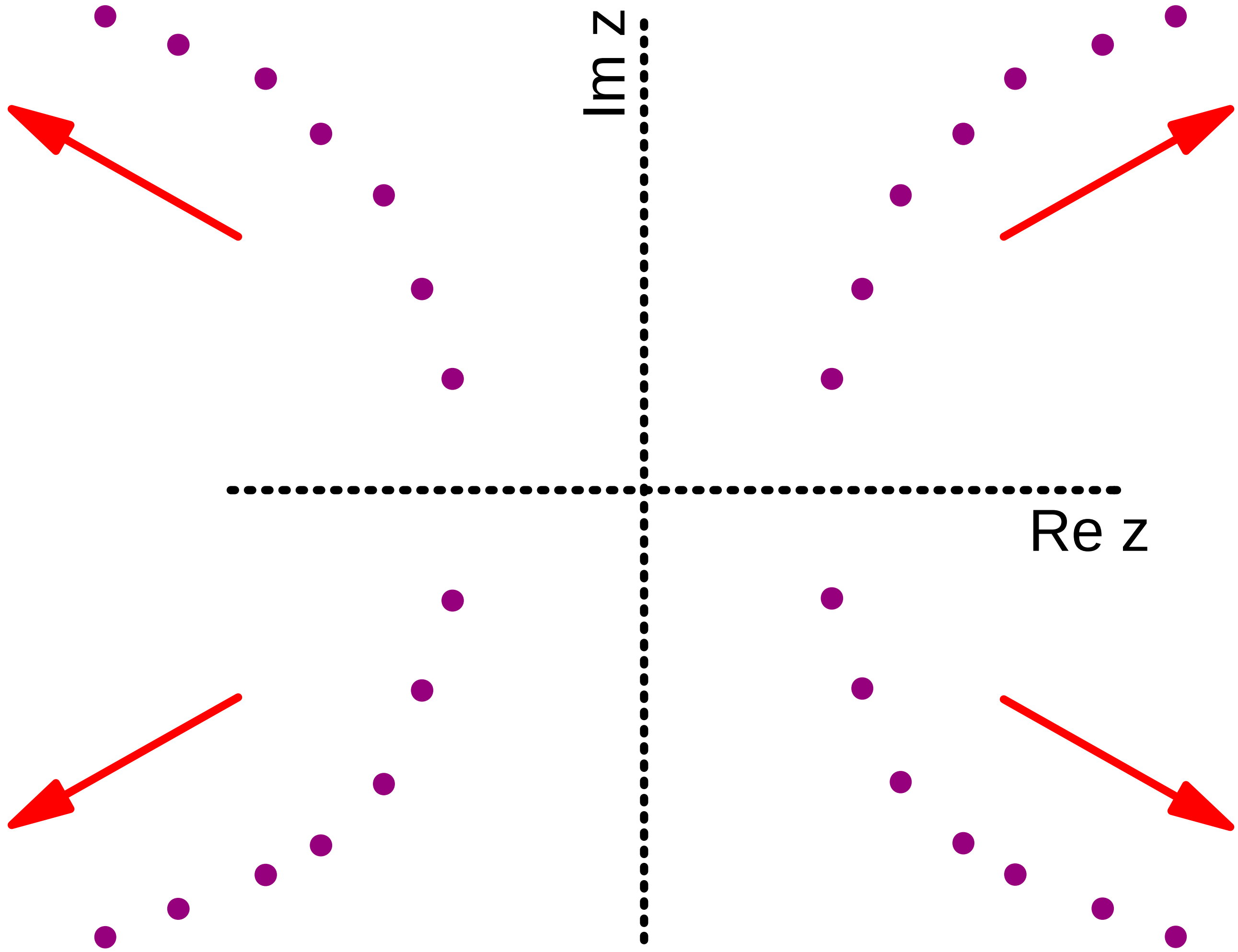
Consider the homogeneous MFDE

$$v'(\xi) = v(\xi - 1) + v(\xi + 1).$$



- Continuity lost $\Longrightarrow$ ill-defined as an initial value problem.

Ill-posedness: What is going on?



Substitution of $e^{z\xi}$ into

$$v'(\xi) = v(\xi - 1) + v(\xi + 1),$$

yields the characteristic equation

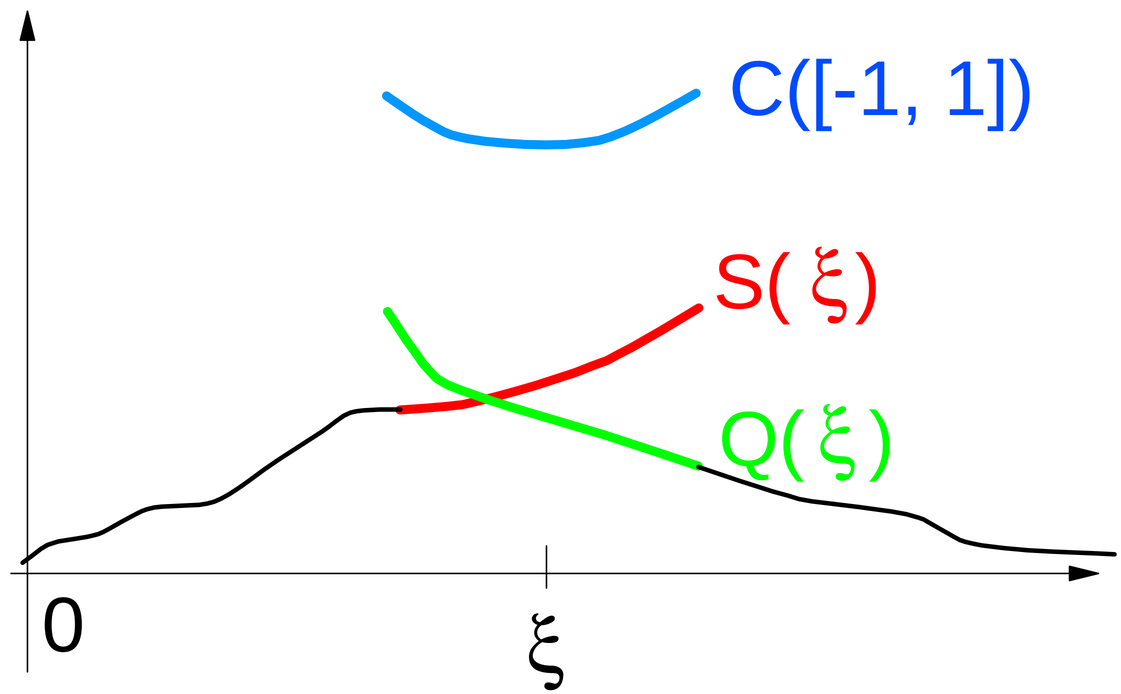
$$\Delta(z) := z - e^{-z} - e^{z} = 0.$$

- The problem is infinite dimensional (as for delay equations).
- There is no exponential bound possible for solutions, at both $\pm\infty$ (unlike delay equations)!

Exponential Dichotomies

Exponential dichotomies are the method of choice for ill-posed problems. Consider the linearization around some function q,

$$v'(\xi) = v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\big(q(\xi)\big)v(\xi).$$



H. + Verduyn Lunel (2008): For $\xi \geq 0$, we have $C([-1,1],\mathbb{R}) = Q(\xi) \oplus S(\xi)$.

Exponential decay for forward-solutions and backward-solutions.

Exponential Dichotomies - Inhomogeneous system

Consider the inhomogeneous system

$$v'(\xi) = v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'\big(q(\xi)\big)v(\xi) + f(\xi).$$

Recall the splitting $C([-1, 1], \mathbb{R}) = Q(\xi) \oplus S(\xi)$.

Usually, exponential dichotomies can be used to construct a variation-of-constants formula

$$v \sim \int_0^\xi T(\xi, \xi')\Pi_{Q(\xi')}f(\xi')d\xi' + \int_\infty^\xi T(\xi, \xi')\Pi_{S(\xi')}f(\xi')d\xi',$$

where T should be seen as an evolution operator.

However, since $f : \mathbb{R} \to \mathbb{C}^n$ does not map into the state space $C([-1, 1])$ complications arise.

- Delay equations: sun-star calculus based upon semigroup properties
- Mixed type equations: unclear how to mimic this construction for $C([-1, 1])$. Possibilities on space $L^2([-1, 1])$, but technical complications.

Inhomogeneous systems

Mallet-Paret (1998) considered operator $\Lambda : BC^1(\mathbb{R}, \mathbb{R}) \to BC(\mathbb{R}, \mathbb{R})$,

$$[\Lambda v](\xi) = v'(\xi) - [v(\xi + 1) + v(\xi - 1) - 2v(\xi)] - g'\big(q(\xi)\big)v(\xi).$$

- Λ is a Fredholm operator:
  - Kernel is finite dimensional
  - Range is closed and has finite dimensional codimension
- Range $\mathcal{R}(\Lambda)$ can be explicitly characterized:

$$\mathcal{R}(\Lambda) = \{f \in BC(\mathbb{R}, \mathbb{R}) \mid \int_{-\infty}^{\infty} d(\xi)^* f(\xi) d\xi = 0 \text{ for all } d \in \mathcal{K}(\Lambda^*)\},$$

with adjoint given by

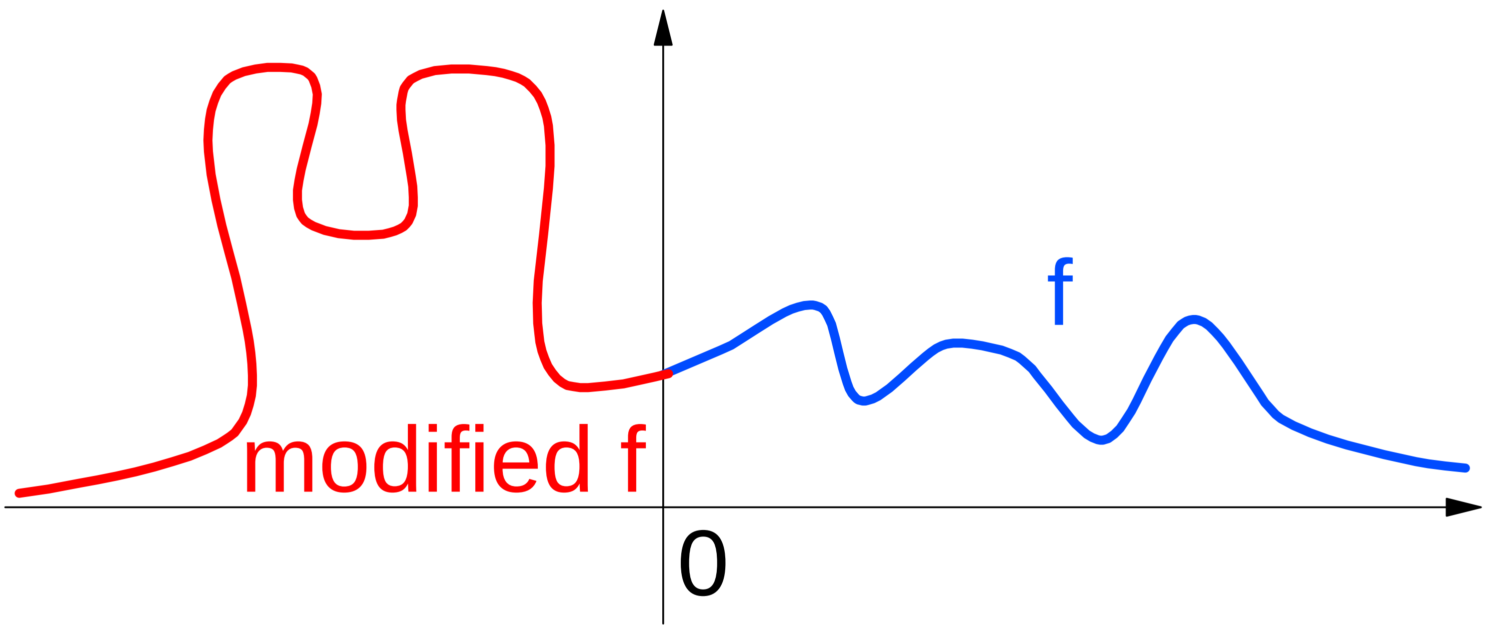
$$[\Lambda^* v](\xi) = v'(\xi) + [v(\xi + 1) + v(\xi - 1) - 2v(\xi)] + g'\big(q(\xi)\big)v(\xi).$$

Inhomogeneous systems - II

In general $\mathcal{R}(\Lambda) \neq BC(\mathbb{R}, \mathbb{R})$, with again

$$[\Lambda v](\xi) = v'(\xi) - [v(\xi+1) + v(\xi-1) - 2v(\xi)] - g'\big(q(\xi)\big)v(\xi).$$

Important property Any solution to $\Lambda^* v = 0$ with $\mathrm{ev}_\xi v = 0$ for some ξ, has $v = 0$ everywhere.



For any f, solve $\Lambda v = f$ on $[0, \infty)$, by modifying f on $\mathbb{R}_-$.

The program

Recall the singularly perturbed MFDE

$$\begin{aligned} cu'(\xi) &= u(\xi+1) + u(\xi-1) - 2u(\xi) + g(u(\xi);a) - w(\xi), \\ cw'(\xi) &= \epsilon[u(\xi) - \gamma w(\xi)]. \end{aligned}$$

Main goal: lift geometric singular perturbation theory to MFDEs.

- Persistence of slow manifold $\mathcal{M}_R$ for $\epsilon > 0$ relies on Fenichel's first thm.
- Almost every proof relies on geometric Hadamard-graph transform.
- Exchange Lemma: Fenichel coordinates unavailable in infinite dimensions.
- Unstable / stable manifolds will be infinite dimensional. How to track intersections?

Main ingredients:

- Isolate suitable finite dimensional subspaces of $C([-1,1],\mathbb{R})$.
- Provide firm analytical underpinning for geometrical constructions.

The program

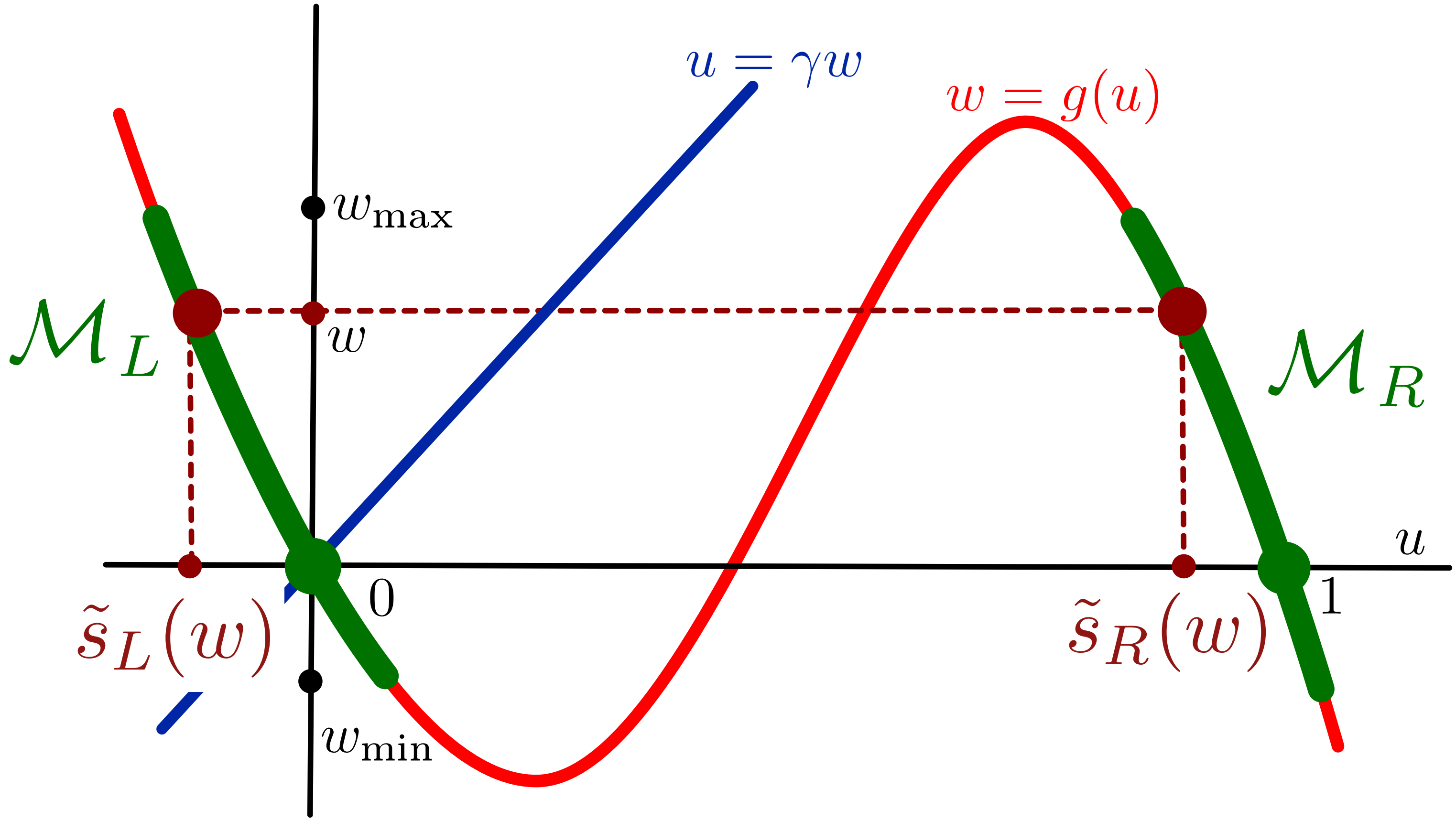
Recall the singularly perturbed MFDE

$$\begin{aligned} cu'(\xi) &= u(\xi+1) + u(\xi-1) - 2u(\xi) + g(u(\xi);a) - w(\xi), \\ cw'(\xi) &= \epsilon[u(\xi) - \gamma w(\xi)]. \end{aligned}$$

- Step 1: Persistence of $\mathcal{M}_L$ and $\mathcal{M}_R$ for $\epsilon > 0$.
- Step 2: How do q_f and q_b break as $\epsilon \approx 0$ and $c \approx c_*$?
- Step 3: Connect broken front and back solutions as they pass near $\mathcal{M}_R(c, \epsilon)$.
- Step 4: Set up and solve two-dimensional nonlinear bifurcation equations to repair front and backs and find $c(\epsilon)$.

The program: Step 1 - Persistence of Slow Manifolds

Introduce function $\tilde{s}_R$ such that $g(\tilde{s}_R(w)) = w$.



We have $\mathcal{M}_R = \{(\tilde{s}_R(w), w)\}$ for $w \in [w_{\min}, w_{\max}]$.

Goal: find functions $s_R(w, c, \epsilon)$ so that the manifold $\mathcal{M}_R(c, \epsilon) = \{(s_R(w, c, \epsilon), w)\}$ is invariant.

The program: Step 1 - Persistence of Slow Manifolds

Idea based upon Sakomoto (1990): find solution (u, w) with $w(0) = w_0$ and

$$u(\xi) = \widetilde{s}_R(w(\xi)) + v(\xi),$$

with **small** v and write $s_R(w_0, c, \epsilon) = u(0)$. Need to solve

$$\begin{aligned} cv'(\xi) &= L\Big(\widetilde{s}_R\big(w(\xi)\big)\Big)\mathrm{ev}_\xi v + \mathcal{R}_{\mathrm{nl}}(v, w, c, \epsilon)(\xi), \\ cw'(\xi) &= \epsilon[\widetilde{s}_R\big(w(\xi)\big) + v(\xi) - \gamma w(\xi)] \end{aligned}$$

with nonlinear $\mathcal{R}_{\mathrm{nl}}$ and linear operator $L(u) : C([-1, 1], \mathbb{R}) \to \mathbb{R}$ given by

$$L(u)\mathrm{ev}_\xi v = v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'(\mathbf{u})v(\xi).$$

Note that $w' = O(\epsilon)$, so linear part **varies slowly**.

- Equation for w with $w(0) = w_0$ can be solved $\longrightarrow W(v, c, \epsilon, w_0)$.
- Suppose that operator $\mathcal{K}(w, c)$ solves linear v-problem $\longrightarrow$ fixed point problem

$$v = \mathcal{K}(W(v, c, \epsilon, w_0), c)\mathcal{R}_{\mathrm{nl}}(v, W(v, c, \epsilon, w_0), c, \epsilon)$$

The program: Step 1 - Persistence of Slow Manifolds

Key ingredient is the construction of solution operator $\mathcal{K}(w, c)$ for linear systems

$$cv'(\xi) \quad = \quad L\Big(\widetilde{s}_R\big(w(\xi)\big)\Big)\mathrm{ev}_\xi v + f(\xi).$$

Use the fact that for each fixed $w_0 \in [w_{\min}, w_{\max}]$, the system

$$cv'(\xi) \quad = \quad L\Big(\widetilde{s}_R\big(w_0\big)\Big)\mathrm{ev}_\xi v + f(\xi),$$

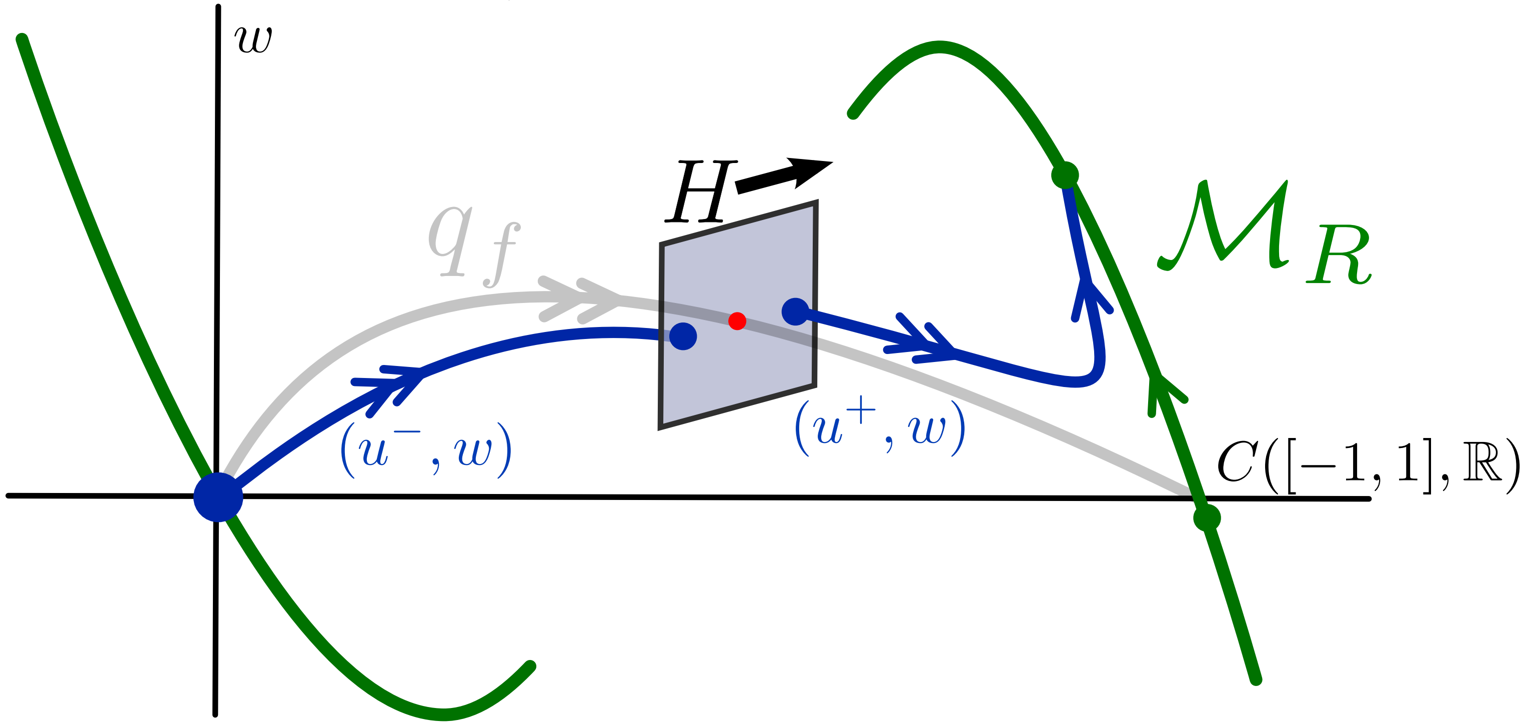
can be solved; $v = \mathcal{K}_{\mathrm{fx}}(w_0, c)f$ [Mallet-Paret 1998]. Can now define approximate solution operator

$$[\mathcal{K}_{\mathrm{apx}}(w, c)f](\xi) = \int_{\xi-\frac{1}{2}}^{\xi+\frac{1}{2}} [\mathcal{K}_{\mathrm{fx}}(w(\zeta), c)f](\xi) d\zeta.$$

If w' is small, the error is small and can be corrected; $\mathcal{K}_{\mathrm{apx}} \to \mathcal{K}$.

The program: Step 2 - Breaking the front

Varying ϵ and c breaks orbit q_f into two parts (u^-, w) and (u^+, w).



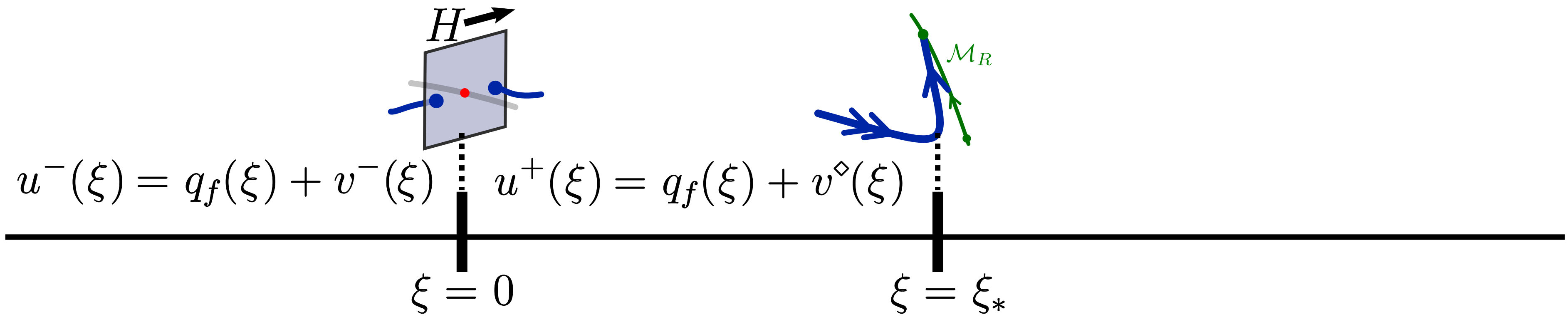
Hyperplane H transverse to orbit q_f at $\xi = 0$, i.e.,

$$C([-1,1],\mathbb{R}) = \text{ev}_0 q_f + H \oplus \text{span}\{\text{ev}_0 q_f'\}.$$

- Perturbation u^+ from q_f is large as $\xi \to \infty$.
- Hyperplane H is infinite dimensional

The program: Step 2 - Breaking the front

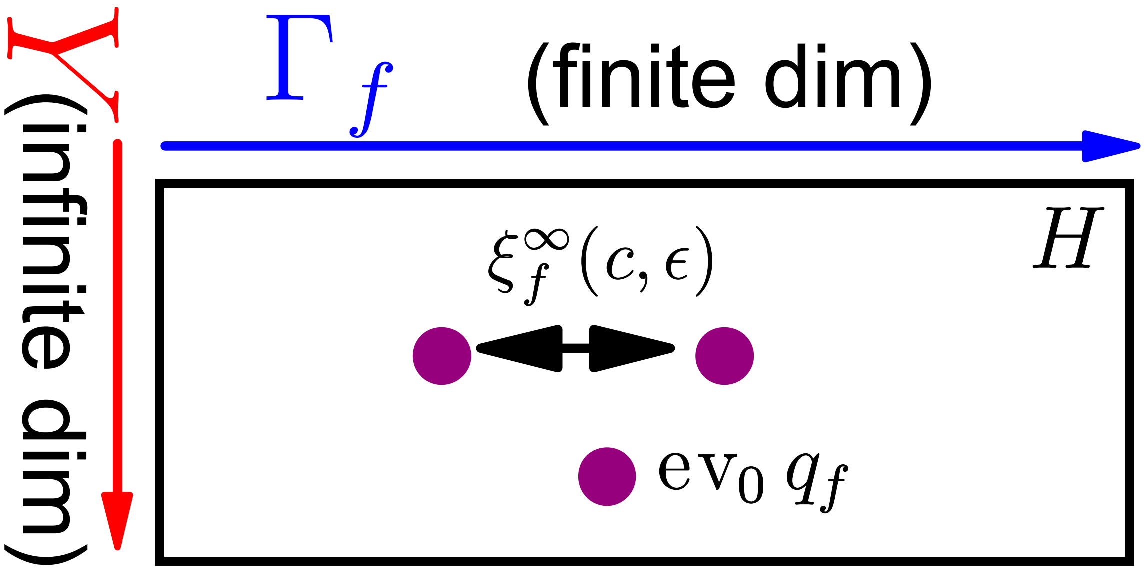
To control size of perturbation, split up real line into three separate parts.



- The functions v^-, $v^\diamond$ and $w_{|(-\infty,\xi_*]}$ are small.

The program: Step 2 - Breaking the front

We need to study the remaining gap in H. Call this gap $\xi_f^\infty(c, \epsilon)$.



Main Goal: Reduce problem to finite dimensions.

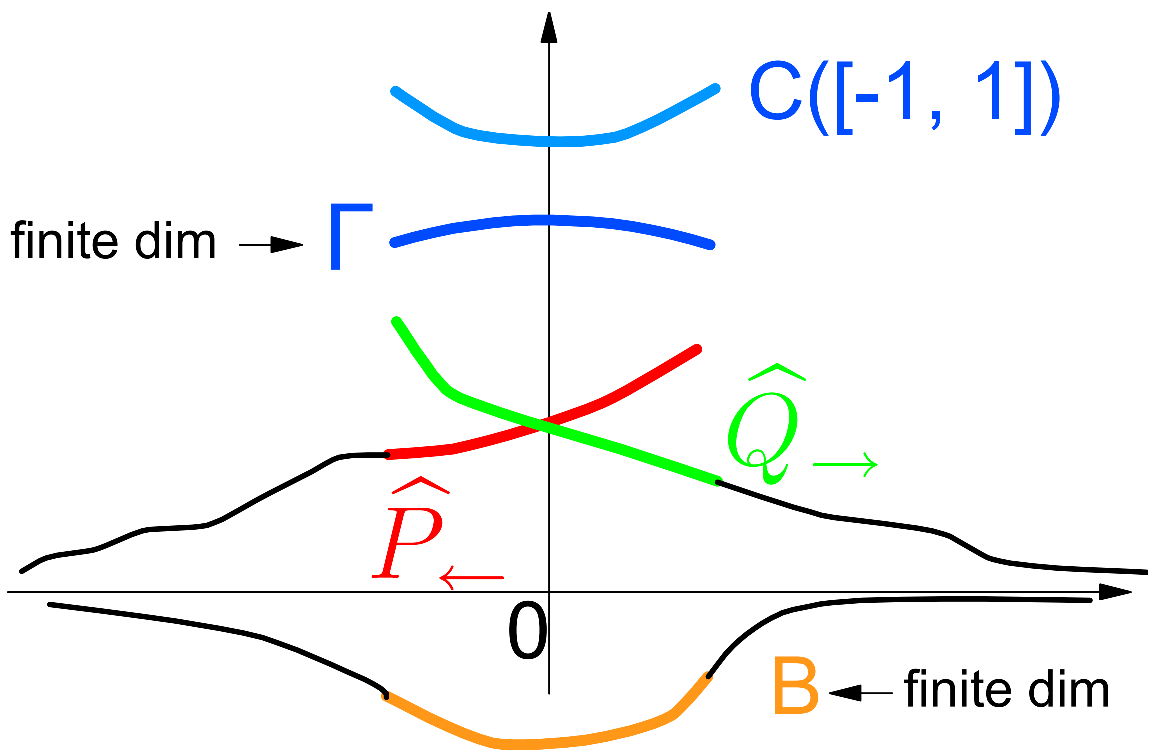
To do this, we will need to split $H = \mathrm{ev}_0\, q_f + Y \oplus \Gamma_f$, with Γ_f **finite** dimensional.

In addition, need to make sure that the "gaps" $\xi_f^\infty(c, \epsilon)$ are all in Γ.

The program: Step 2 - Breaking the front

Construction based upon exponential dichotomies on $\mathbb{R}$ for

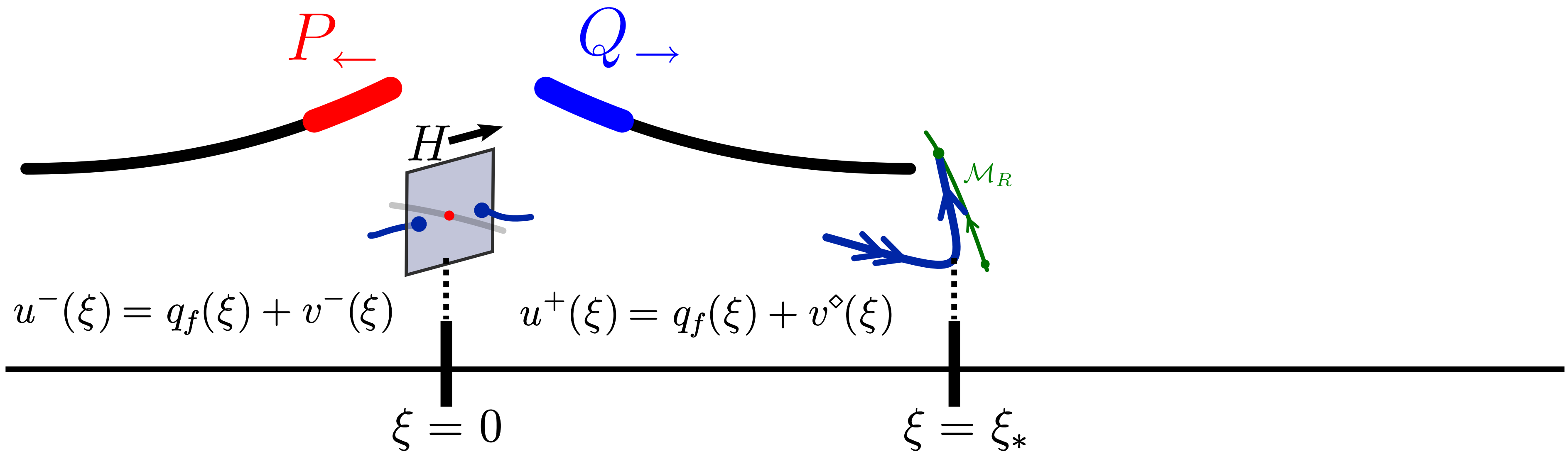
$$v'(\xi) = v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\big(q_f(\xi)\big)v(\xi).$$



Mallet-Paret + Verduyn Lunel (2001): $C([-1,1],\mathbb{R}) = \widehat{P}_{\leftarrow} \oplus \widehat{Q}_{\rightarrow} \oplus B \oplus \Gamma$.

We have $\mathrm{ev}_0\, q' \in B$. Can use $Y = \widehat{P}_{\leftarrow} \oplus \widehat{Q}_{\rightarrow}$. The space Γ can be explicitly characterized using special integral inner product (Hale inn. pr.).

The program: Step 2 - Breaking the front



Can use remaining freedom to ensure that gap is in Γ, since

$$C([-1,1],\mathbb{R}) = \mathrm{ev}_0 q_f + \widehat{P}_{\leftarrow} \oplus \widehat{Q}_{\rightarrow} \oplus \{\mathrm{ev}_0 q_f'\} \oplus \Gamma$$

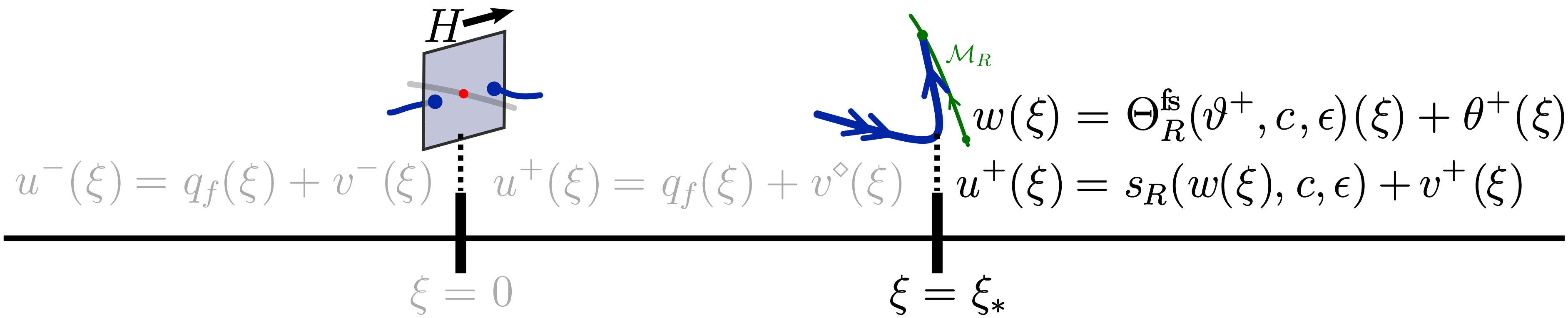
At $c = 0$ and $\epsilon = 0$, we have Melnikov identities such as

$$D_c \langle \mathrm{ev}_0 d, \xi_f^{\infty} \rangle_{\mathrm{Hale}} = -\int_{-\infty}^{\xi_*} d(\xi') q_f'(\xi) d\xi' + O(e^{-\eta_* \xi_*}),$$

for d that solves adjoint $-cd'(\xi) = \alpha[d(\xi+1) + d(\xi-1) - 2d(\xi)] + g'(q_f(\xi))v(\xi)$.

The program: Step 2 - Breaking the front

Now need to study part near $\mathcal{M}_R$.



- The functions v^+, θ^+ are small.
- The parameter ϑ^+ selects the fibre of $\mathcal{M}_R$ to which (u^+, w) converges as $\xi \to \infty$.
- The function $\Theta_R^{\mathrm{fs}}(\vartheta^+, c, \epsilon)$ is unique solution of ODE

$$\Theta'(\xi) = \epsilon[s_R(\Theta(\xi), c, \epsilon) - \gamma\Theta(\xi)], \qquad \Theta(0) = \vartheta^+,$$

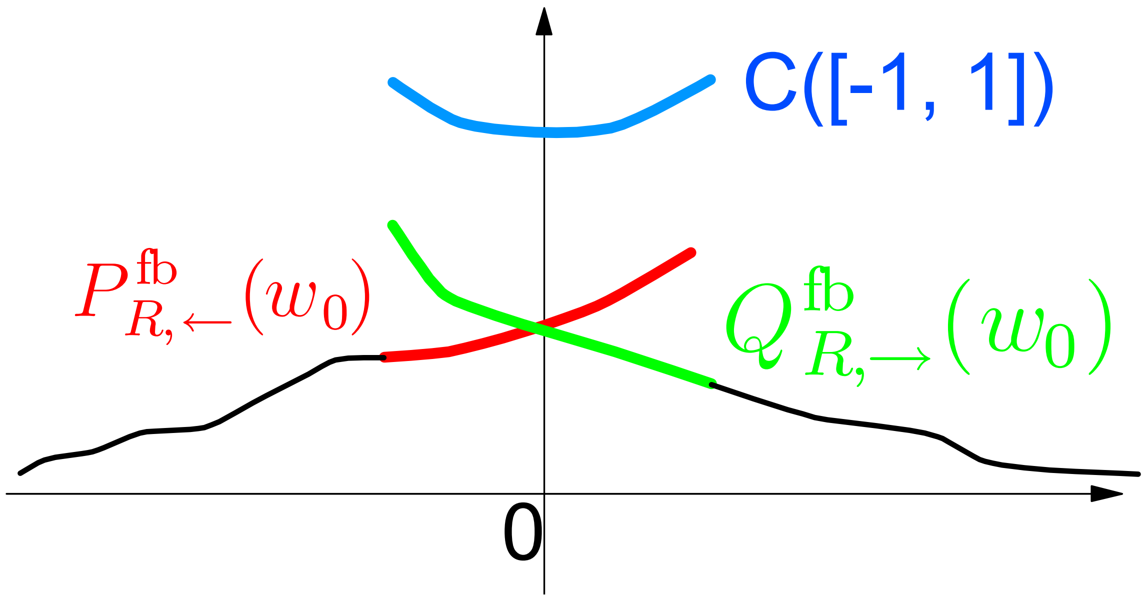
  which describes flow along $\mathcal{M}_R(c, \epsilon)$ in terms of fast time scale.

The program: Step 2 - Breaking the front

Must understand linearization near slow manifold $\mathcal{M}_R(c, \epsilon)$.

First fix $w_0 \in [w_{\min}, w_{\max}]$ and consider **constant coefficient** linearization

$$v'(\xi) = v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'\big(\widetilde{s}_R(w_0)\big)v(\xi)$$



Mallet-Paret + Verduyn Lunel (2001): $C([-1, 1], \mathbb{R}) = P^{\text{fb}}_{R,\leftarrow}(w_0) \oplus Q^{\text{fb}}_{R,\rightarrow}(w_0)$.

The program: Step 2 - Breaking the front

Now consider $w \in C^1(\mathbb{R}, [w_{\min}, w_{\max}])$ that has very small $\|w'\|_\infty$ and $w(0) = w_0$.

Consider linearization

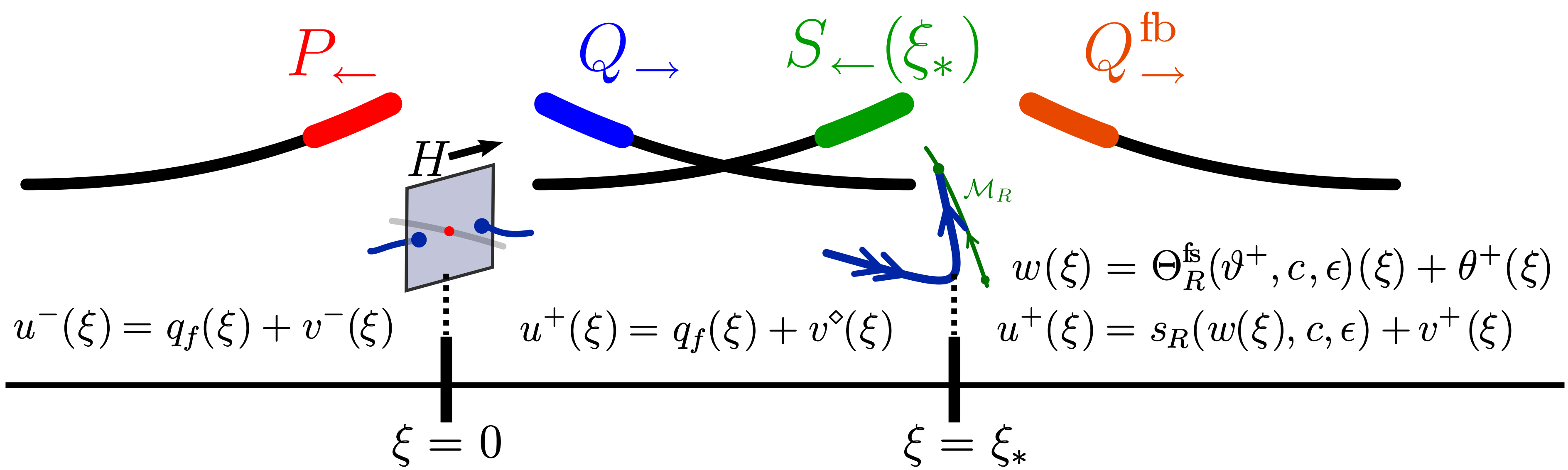
$$v'(\xi) = v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'\Big(\widetilde{s}_R\big(w(\xi)\big)\Big)v(\xi). \qquad (1)$$

Main idea:

- For any $\phi \in Q^{\mathrm{fb}}_{R,\rightarrow}(w_0)$, there exists $v \in C([-1, \infty), \mathbb{R})$ that solves (1) with $\Pi_{Q^{\mathrm{fb}}_{R,\rightarrow}(w_0)} \mathrm{ev}_0 v = \phi$.

- Any bounded solution to (1) can be written in this form.

The program: Step 2 - Breaking the front



Gap at $\mathcal{M}_R$ can be completely closed, since

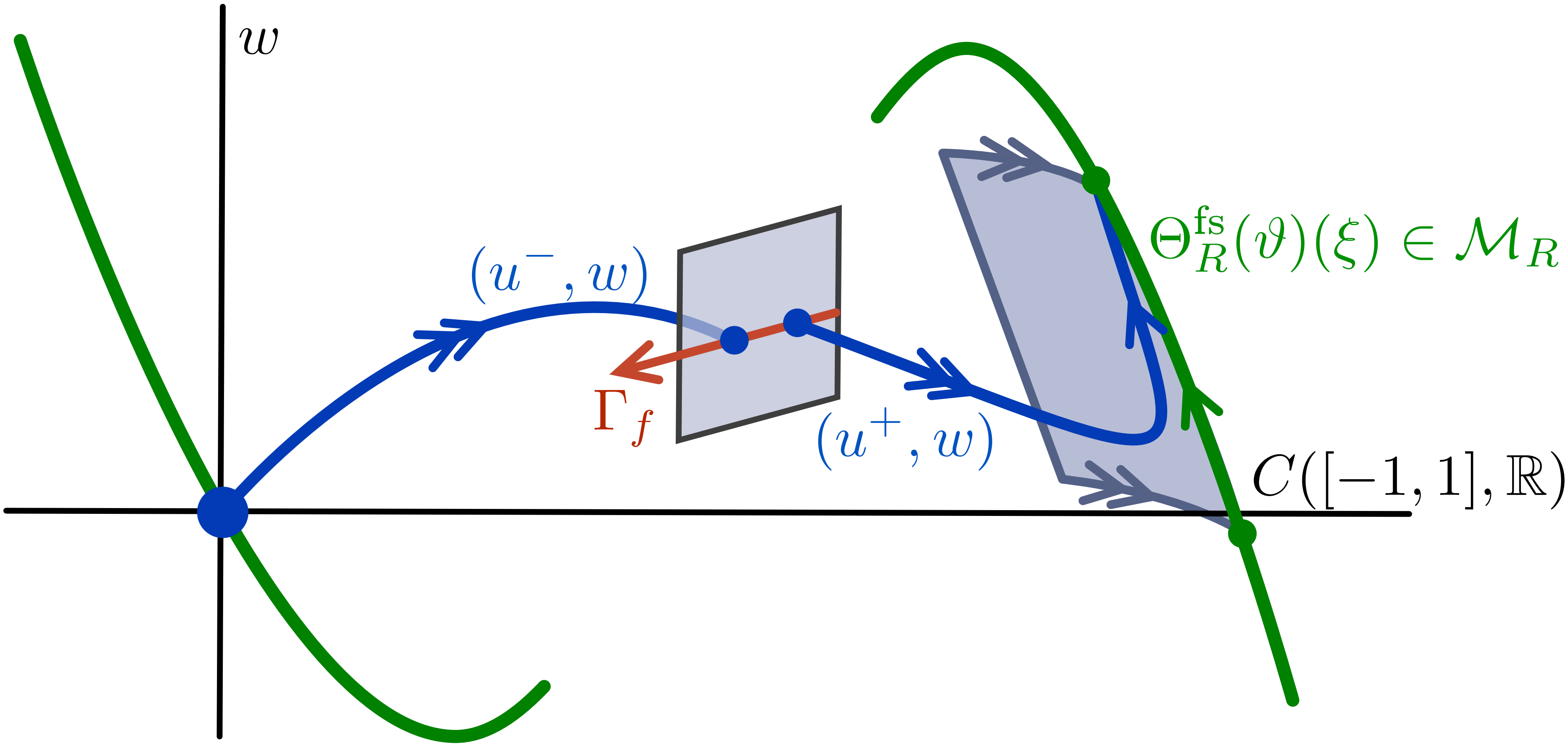
$$S_{\leftarrow}(\xi_*) \approx P^{fb}_{R,\leftarrow}(0)$$

and

$$C([-1,1],\mathbb{R}) = P^{fb}_{R,\leftarrow}(0) \oplus Q^{fb}_{R,\rightarrow}(0).$$

The program: Step 2 - Breaking the front

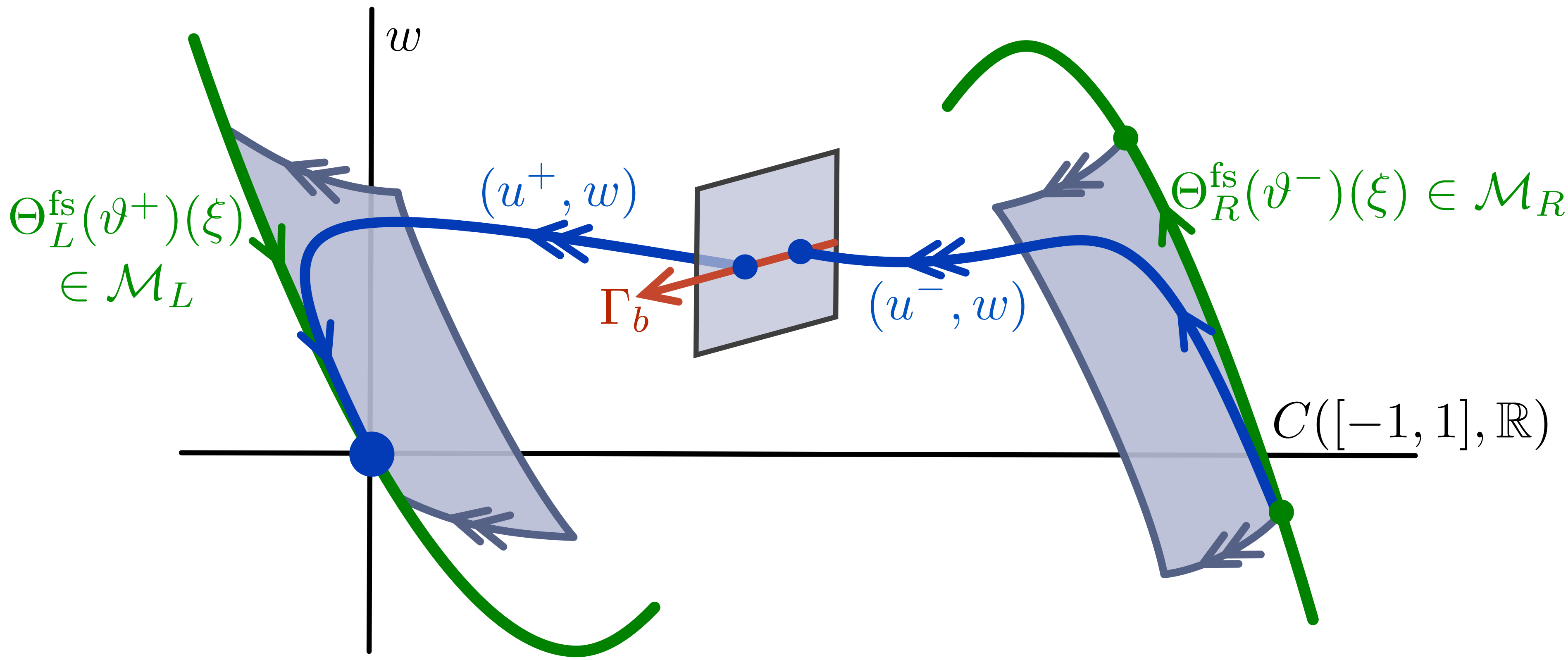
In summary, we have constructed **quasi-front** solutions to the travelling wave equation for $\epsilon \approx 0$ and $c \approx c_*$.



The program: Step 2 - Breaking the back

Similarly, can construct **quasi-back** solutions to the travelling wave equation for $\epsilon \approx 0$, $c \approx c_*$ and **extra** degree of freedom $w_0 \approx w_*$.

This extra d.o.f. used to specify $w(0) = w_0$ (lift quasi-back up and down).



The program: Step 3 - Exchange Lemma

The quasi-fronts and quasi-backs need to be tied together near $\mathcal{M}_R(c, \epsilon)$.

Primary parameter: time T that solution spends near $\mathcal{M}_R(c, \epsilon)$.

Note that $\epsilon = 0$ is not a useful parameter, since quasi-fronts and quasi-backs do not connect when $\epsilon = 0$.

Write $\Theta_R^{\mathrm{sl}}(\vartheta, c, \epsilon)$ for unique solution of ODE

$$\Theta'(\zeta) = [s_R(\Theta(\zeta), c, \epsilon) - \gamma\Theta(\zeta)], \qquad \Theta(0) = \vartheta,$$

which describes flow along $\mathcal{M}_R(c, \epsilon)$ in terms of **slow** time scale.

Slow time T_*^{sl} uniquely defined by

$$\Theta_R^{\mathrm{sl}}(0, c_*, 0)(T_*^{\mathrm{sl}}) = w_*$$

We will need $\epsilon T \approx T_*^{\mathrm{sl}}$; introduce new variable $T^{\mathrm{sl}} = \epsilon T$.

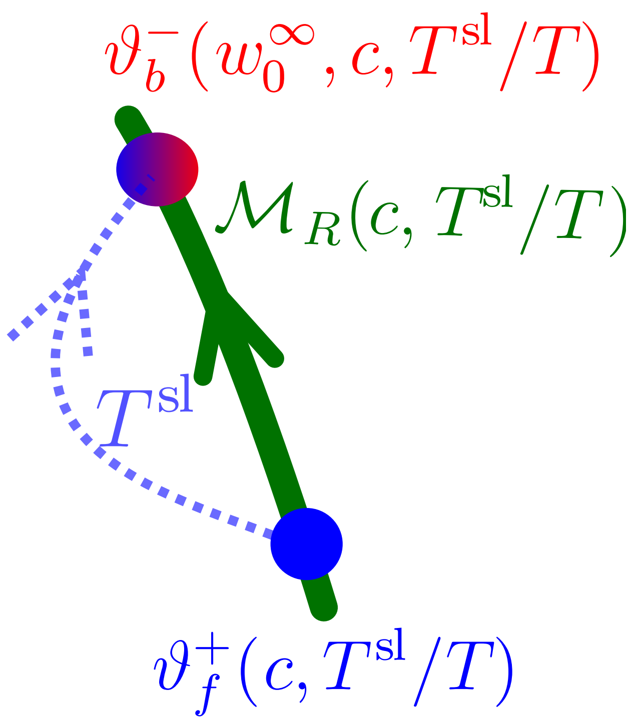
Independent parameters are now (c, T^{sl}, T) taken near $(c_*, T_*^{\mathrm{sl}}, \infty)$.

The program: Step 3 - Exchange Lemma

Recall the fibre $\vartheta_f^+(c,\epsilon)$ that was selected by the quasifront.

Recall also the fibre $\vartheta_b^-(w_0,c,\epsilon)$ selected by the quasiback.

Want to make sure fibres match.



- Define $w_0^\infty(c,T^{\mathrm{sl}},T)$ by the following identity:

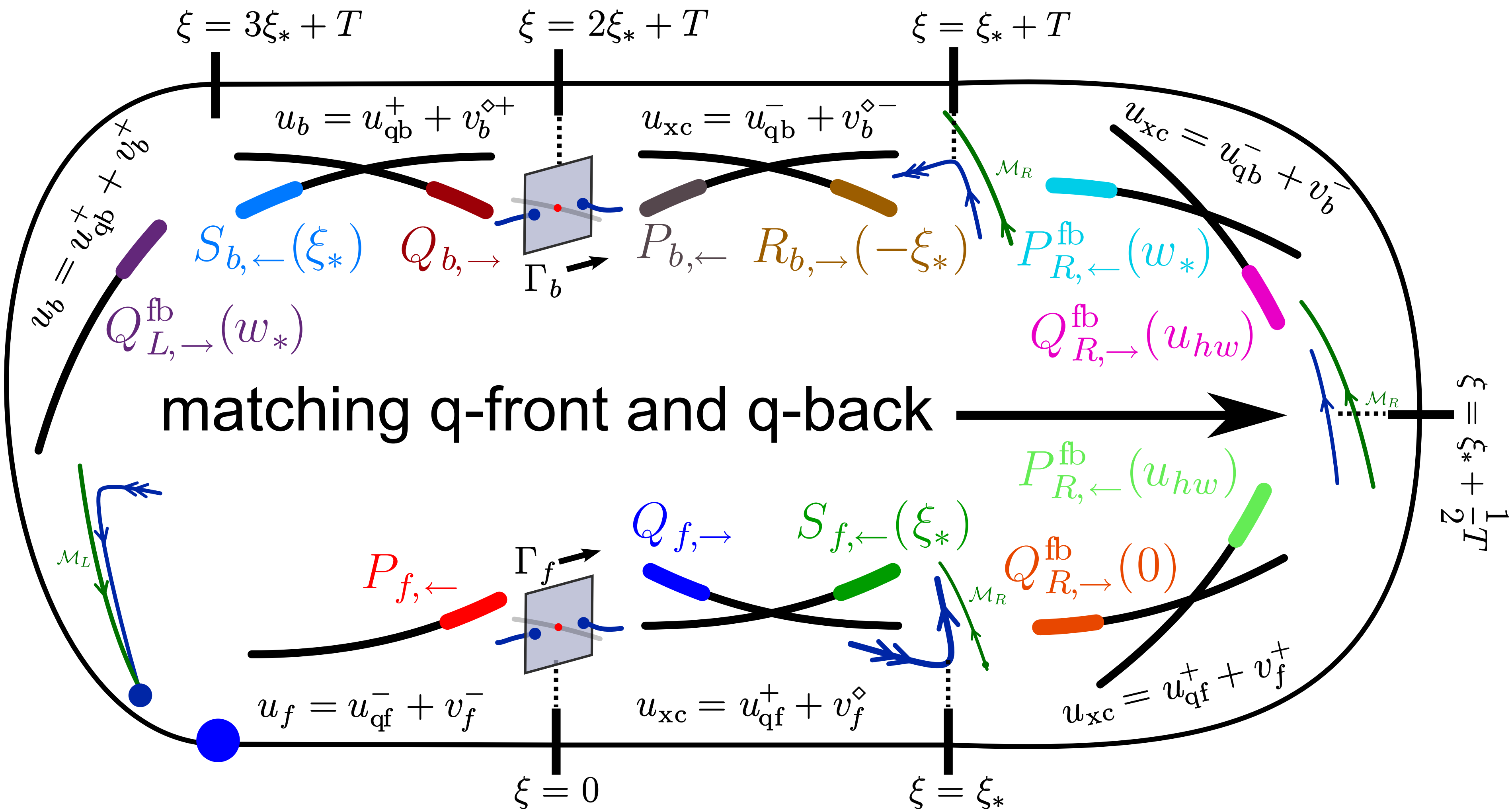
$$\vartheta_b^-(w_0^\infty, c, T^{\mathrm{sl}}/T) = \Theta_R^{\mathrm{sl}}(\vartheta_f^+(c,T^{\mathrm{sl}}/T), c, T^{\mathrm{sl}}/T)(T^{\mathrm{sl}})$$

for $(c,T^{\mathrm{sl}},T) \approx (c_*, T_*^{\mathrm{sl}}, \infty)$.

Consequence: at "half-way" point, quasi-front and quasi-back miss each other by $O(e^{-\frac{1}{2}\eta_* T})$!

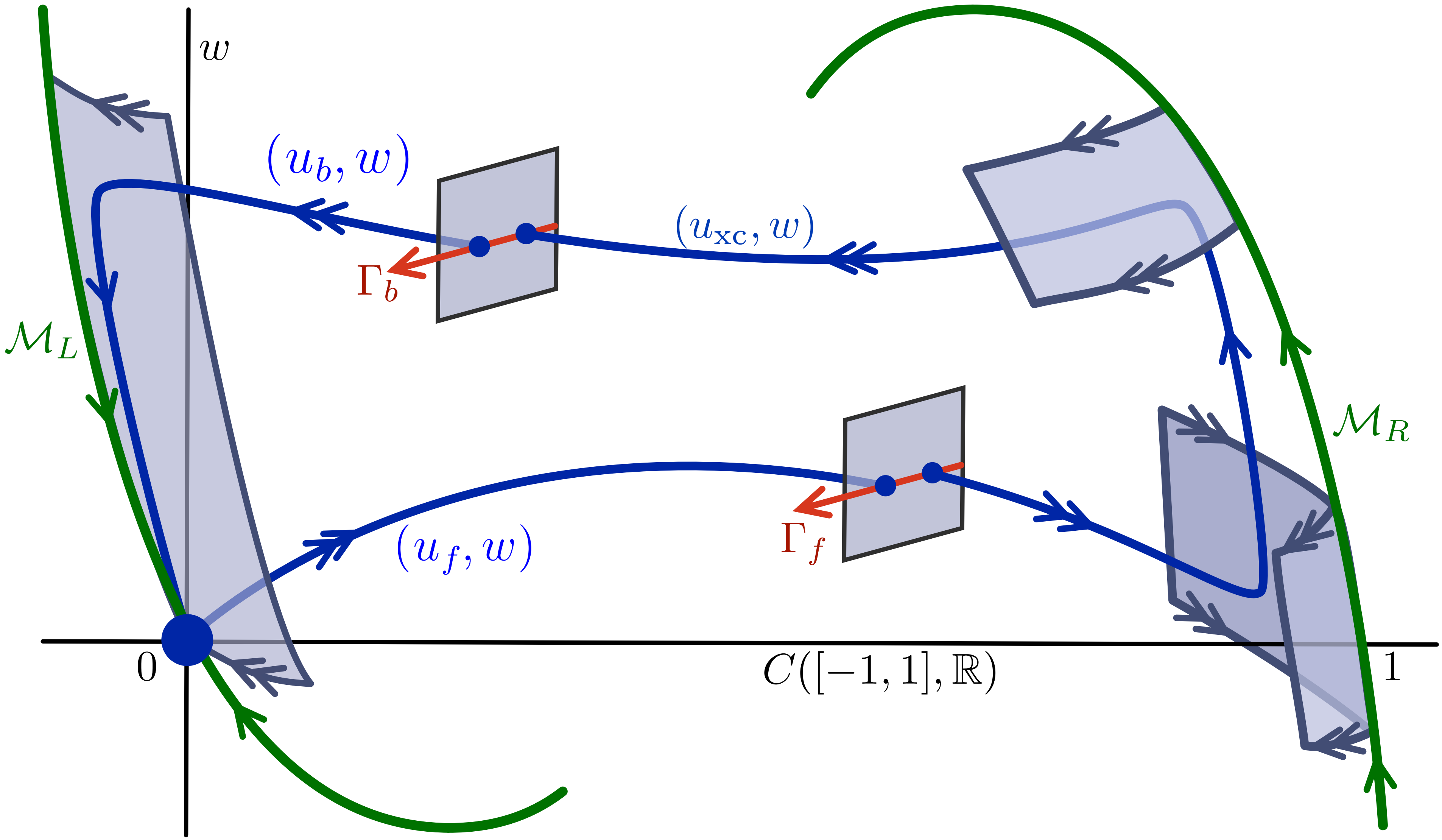
The program: Step 3 - Exchange Lemma

Match the quasi-front and quasi-back at halfway-point along $\mathcal{M}_R$. Split into seven distinct intervals.



The program: Step 3 - Exchange Lemma

Quasi-front and quasi-back can be matched up to two one-dimensional jumps.



The program: Step 4 - Bifurcation equations

The independent parameters are (c, T^{sl}, T) taken near $(c_*, T_*^{\mathrm{sl}}, \infty)$.

The jumps in Γ_f and Γ_b can be split into two parts:

- Construction of quasi-fronts and quasi-backs
- Modification due to Exchange Lemma

The Exchange Lemma contribution $+$ derivatives are of order $O(e^{-\eta_* T})$.

System to solve is hence, to first order,

$$\begin{aligned} M_c^f(c - c_*) &= -M_\epsilon^f T^{\mathrm{sl}}/T \\ M_c^b(c - c_*) &= -M_w^b(T^{\mathrm{sl}} - T_*^{\mathrm{sl}}) - M_\epsilon^b T^{\mathrm{sl}}/T \end{aligned}$$

The sign of the M-constants can be read off from Melnikov integrals.

Three unknowns; two equations $\longrightarrow$ curve of solutions $\big(\epsilon, c(\epsilon)\big)$.

Outlook

Recall FHN-LDE:

$$\begin{aligned} \dot{U}_j(t) &= \alpha[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + g(U_j(t); a) - W_j(t), \\ \dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)]. \end{aligned}$$

Number of issues open to explore:

- Stability of the fast pulses: same singular perturbation setup should yield results.
- What happens to fast pulses as propagation failure region is encountered?
- For $a \approx \frac{1}{2}$, can one Taylor expand in the Exchange Lemma and connect slow and fast pulses [as in Krupa, Sandstede, Szmolyan (1997)]?
- Multi-pulses, homoclinic blow-up etc in other singularly perturbed lattice problems.