Well-posedness of Initial Value Problems for Functional Differential-Algebraic Equations of Mixed Type

MIZZOU™

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(Joint work with E. Augeraud-Véron)

Initial value problems

Prototype initial value problem:

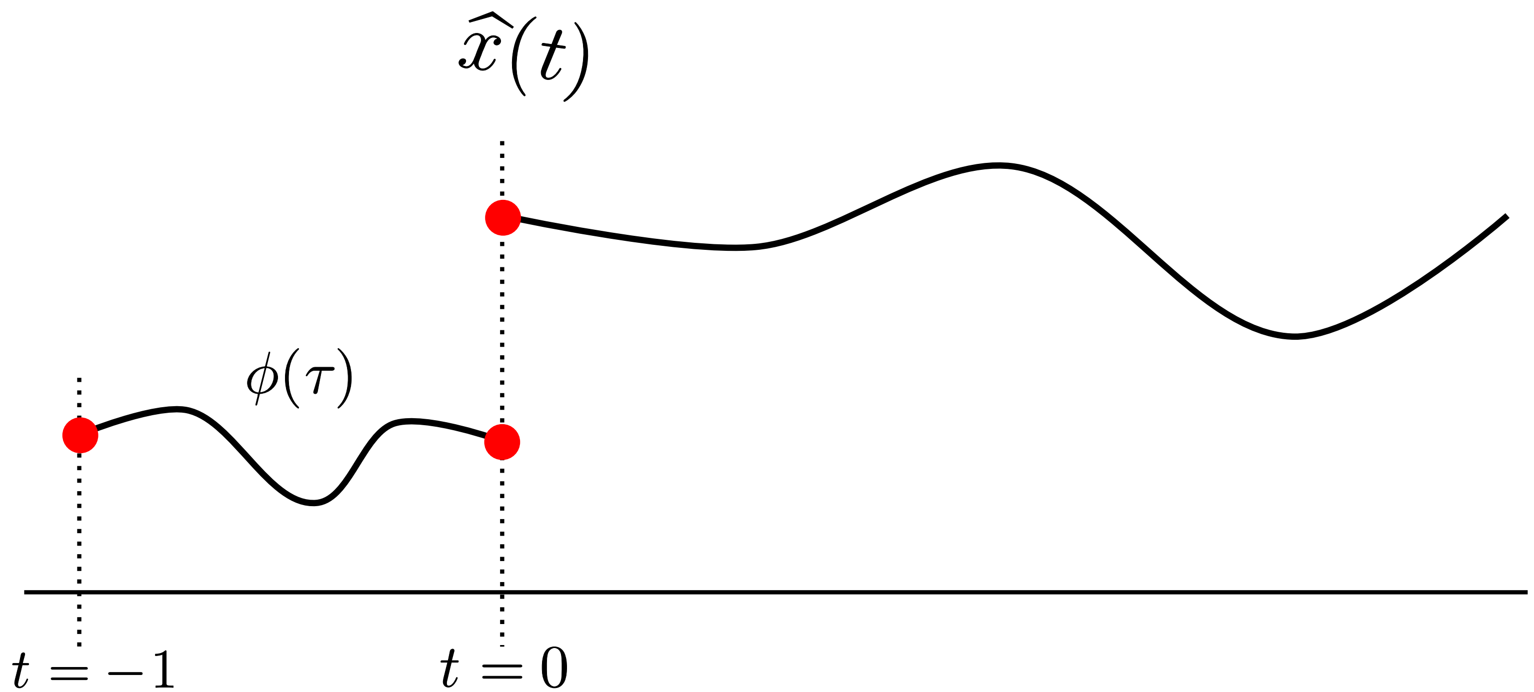
$$\begin{array}{rcll} \mathcal{I}x'(t) & = & x(t) + \int_{-1}^{1} x(t+\sigma)d\,\sigma + f\big(x(t)\big) & \text{for all } t \geq 0, \\ x(\tau) & = & \phi(\tau) & \text{for all } -1 \leq \tau \leq 0. \end{array}$$

- Matrix $\mathcal{I}$ diagonal: singular and invertible both allowed.
- Dependence both on 'past' and 'future' arguments of x
- If $\mathcal{I}$ invertible: mixed type functional differential equation (MFDE).
- If $\mathcal{I}$ singular: mixed type differential-algebraic equation (MFDAE)
- Does every initial condition ϕ lead to a (bounded) solution?
- Uniqueness of solution for given ϕ?

Initial value problems

Recall prototype initial value problem:

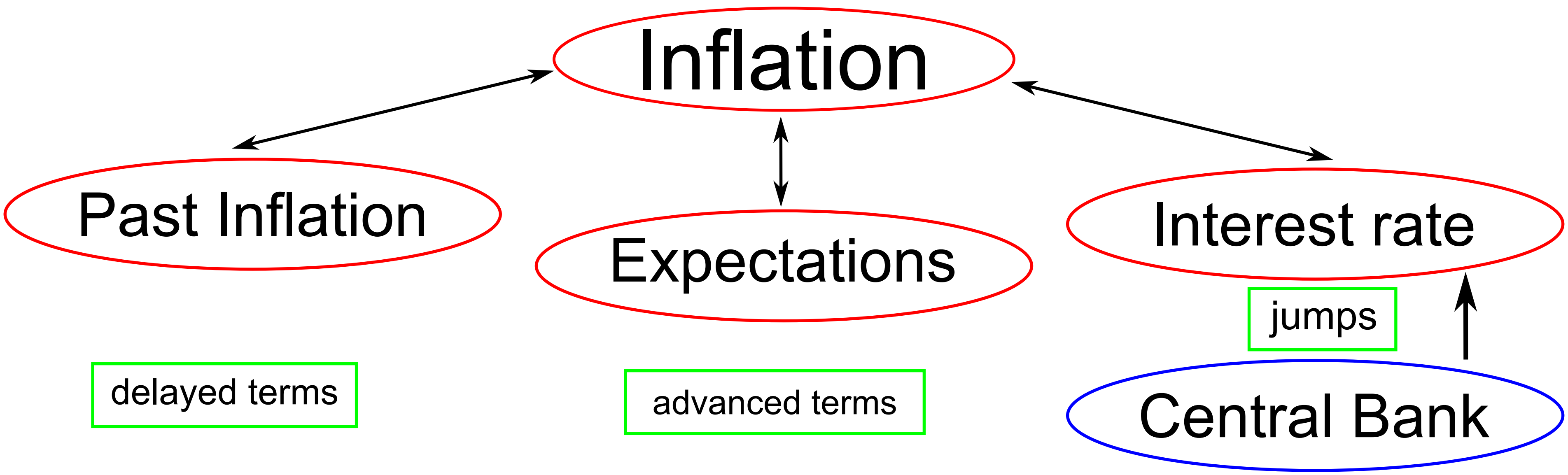
$$\begin{aligned} \mathcal{I}x'(t) &= x(t) + \int_{-1}^{1} x(t+\sigma)\,d\sigma + f\big(x(t)\big) && \text{for all } t \geq 0, \\ x(\tau) &= \phi(\tau) && \text{for all } -1 \leq \tau \leq 0. \end{aligned}$$



- Solution $x = \widehat{x}(t)$ may have jump at $t = 0$; similar to impulsive equations [Liu, Ballinger]

Motivation

Direct motivation comes from economic modelling.



- Can central bank stabilize inflation by jump in interest rate?
- Do multiple self-fulfilling paths exist?

Model Equations

Model system given by

$$\begin{aligned}
\Lambda(R(t))R'(t) &= \pi(t) + r - R(t) \\
\pi^b(t) &= \int_{-1}^{0} e^{\beta^b \sigma} \pi(\sigma) d\sigma \\
\pi^f(t) &= \int_{0}^{1} e^{\beta^f \sigma} \pi(\sigma) d\sigma \\
\pi(t) &= f(R(t)) - \pi^b(t) - \pi^f(t)
\end{aligned}$$

Differential equation coupled with algebraic equations.

- Interest rate: $R(t)$.
- Inflation rate: $\pi(t)$.
- Past inflation: $\pi^b(t)$.
- Inflation expectation: $\pi^f(t)$.

Variable $\pi(t)$ can be eliminated, leaving three independent variables.

Initial values for R, π^b and π^f given on $[-1, 0]$.

Delay Equations

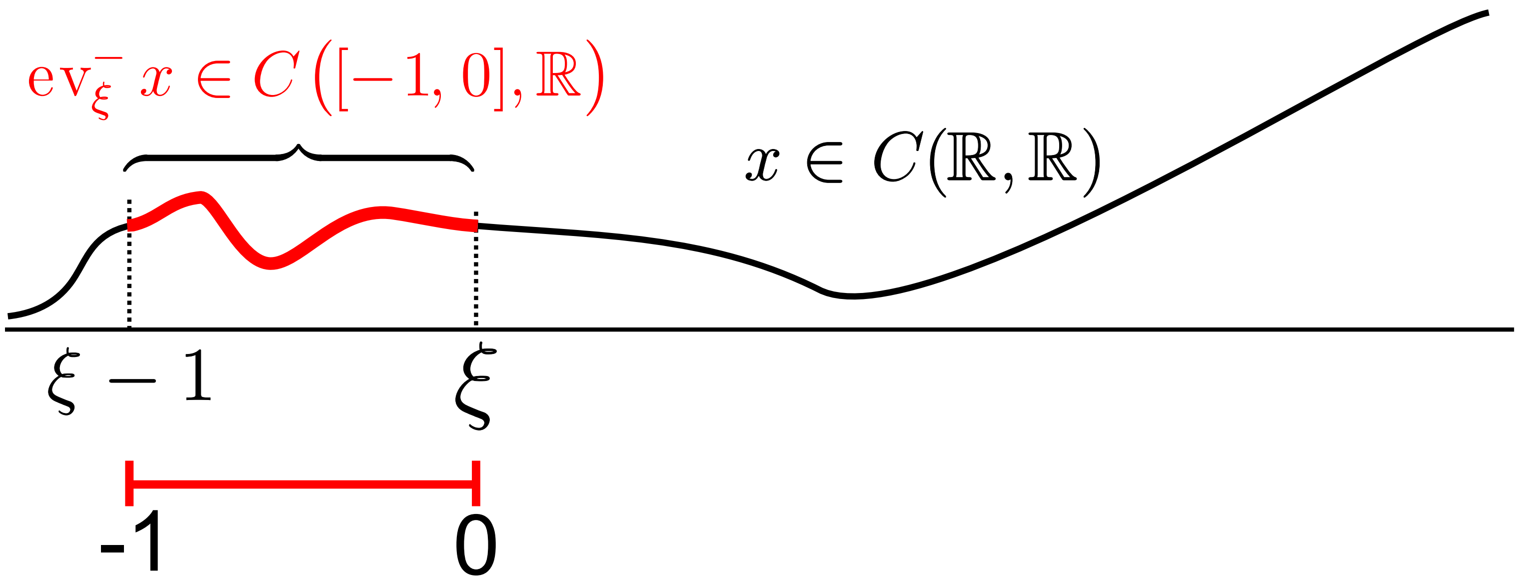
Consider first the linear delay differential equation

$$x'(\xi) = L_- \, \mathrm{ev}_\xi^- \, x,$$

where $L_- : C([-1,0],\mathbb{R}) \to \mathbb{R}$.

Characteristic function given by

$$\Delta_{L_-}(z) = z - L_- e^{z\cdot}$$



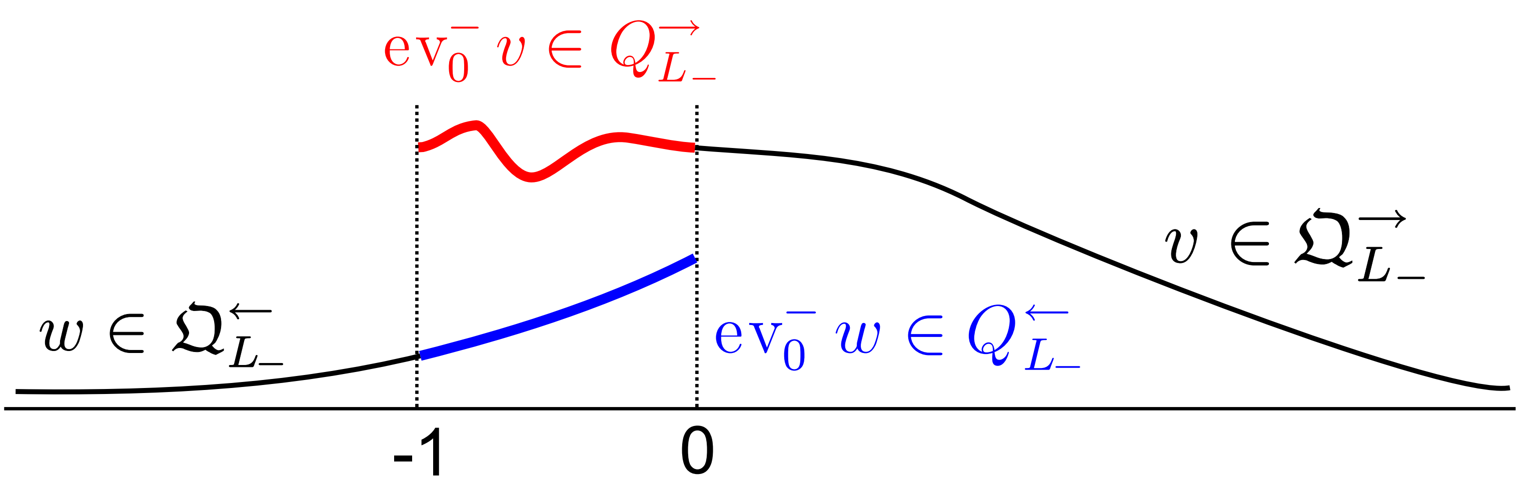
Delay Equations

We are interested in solution spaces

$$\begin{aligned}\mathfrak{Q}^{\leftarrow}_{L_-} &= \{w \in BC((-\infty, 0], \mathbb{R}) : w'(\xi) = L_- \mathrm{ev}^-_{\xi} w \text{ for all } \xi \leq 0\}, \\ \mathfrak{Q}^{\rightarrow}_{L_-} &= \{v \in BC([-1, \infty), \mathbb{R}) : v'(\xi) = L_- \mathrm{ev}^-_{\xi} v \text{ for all } \xi \geq 0\}.\end{aligned}$$

We also use 'initial segment' spaces

$$Q^{\leftarrow}_{L_-} = \mathrm{ev}^-_0 \left(\mathfrak{Q}^{\leftarrow}_{L_-}\right), \qquad Q^{\rightarrow}_{L_-} = \mathrm{ev}^-_0 \left(\mathfrak{Q}^{\rightarrow}_{L_-}\right).$$

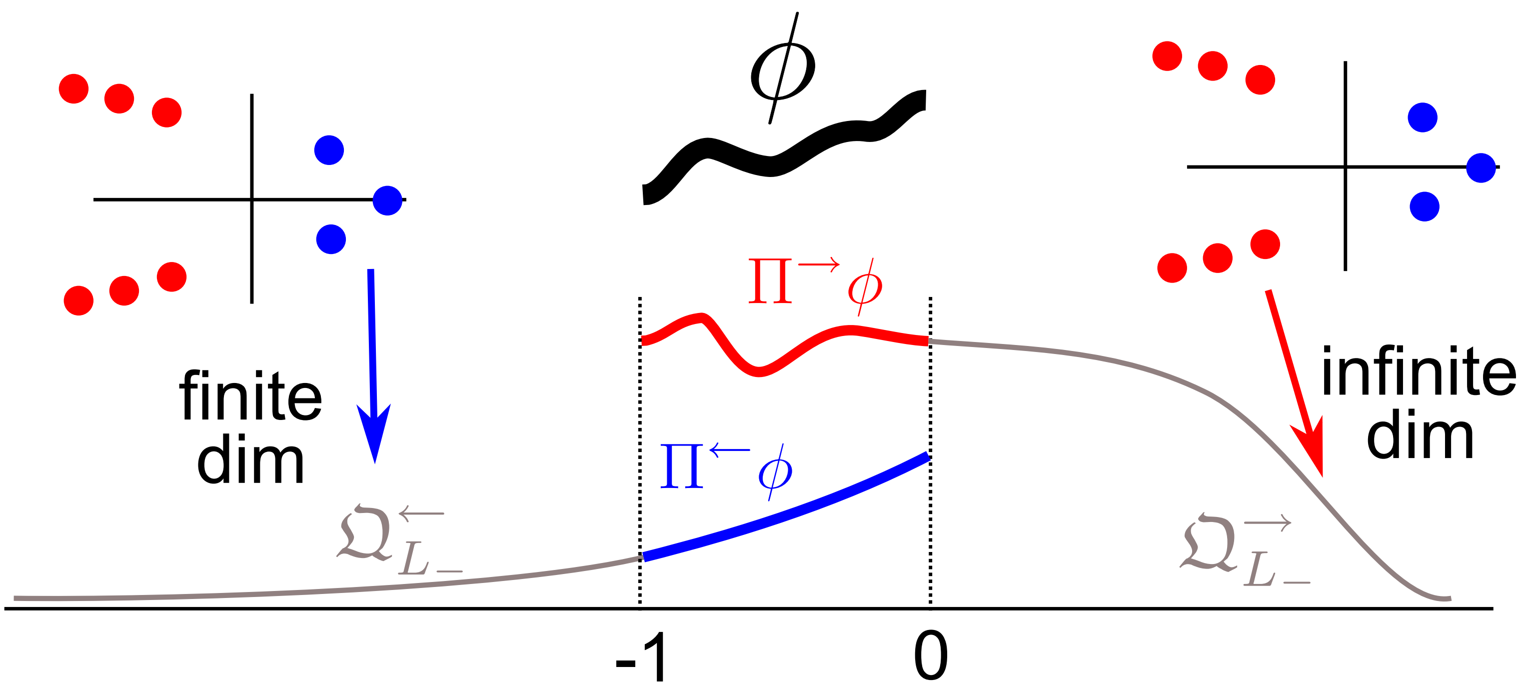


Delay Equations

Thm. If $\Delta_{L_-}(z) = 0$ has no roots on imag. axis, then

$$C([-1,0],\mathbb{C}) = Q^{\leftarrow}_{L_-} \oplus Q^{\rightarrow}_{L_-}.$$

$Q^{\leftarrow}_{L_-}$ is finite dimensional, spanned by eigenfunctions for roots $\Delta_{L_-}(z) = 0$ with $\operatorname{Re} z > 0$.



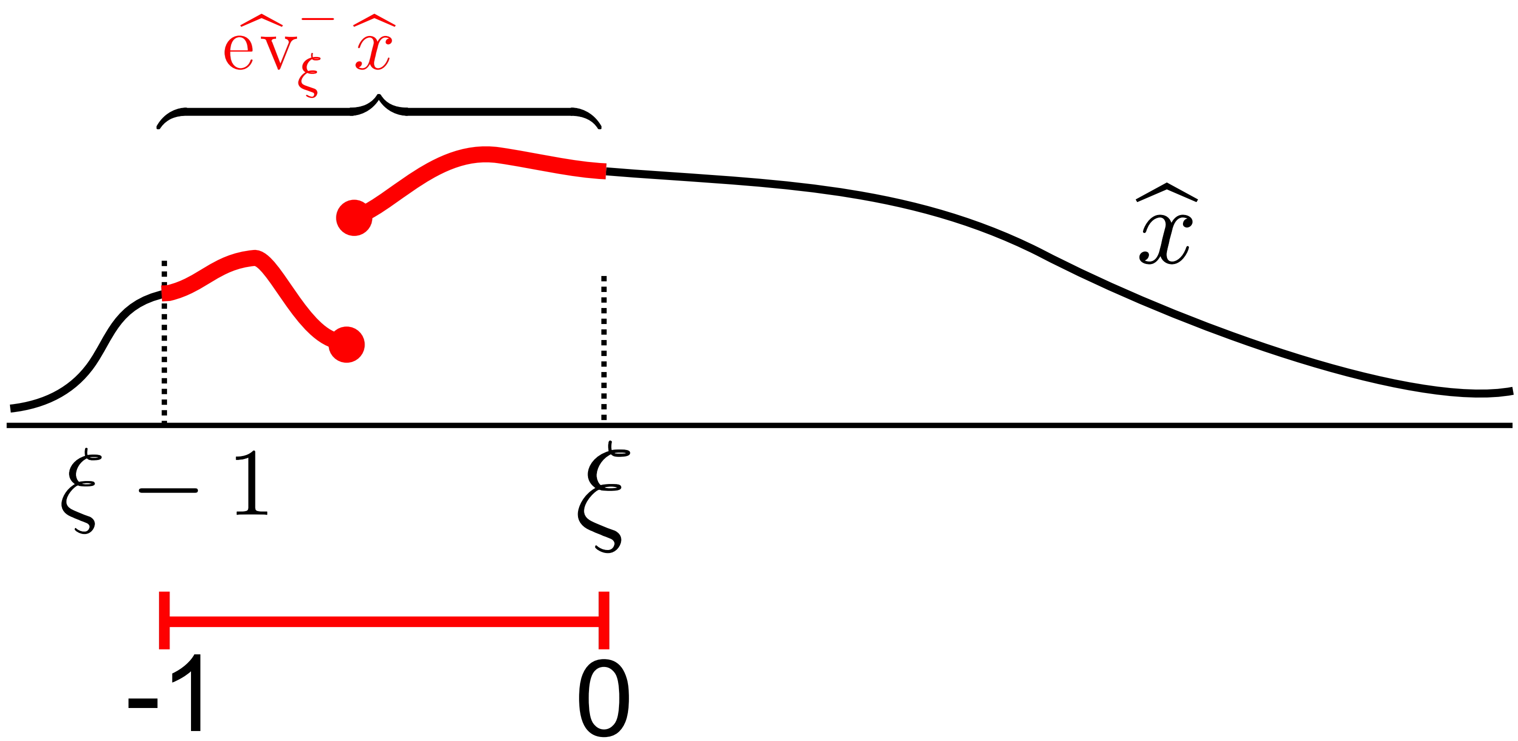
Spectral projections explicitly give projections $\Pi^{\rightarrow}$ and $\Pi^{\leftarrow}$.

Delay Equations - Jumps

Consider now the linear delay differential equation

$$x'(\xi) = L_- \,\widehat{\mathrm{ev}}_\xi^- \, x,$$

where $L_- : \mathrm{PC}([-1,0],\mathbb{R}) \to \mathbb{R}$.



Look for solutions $\widehat{x}$ with single discontinuity at $\xi = 0$.

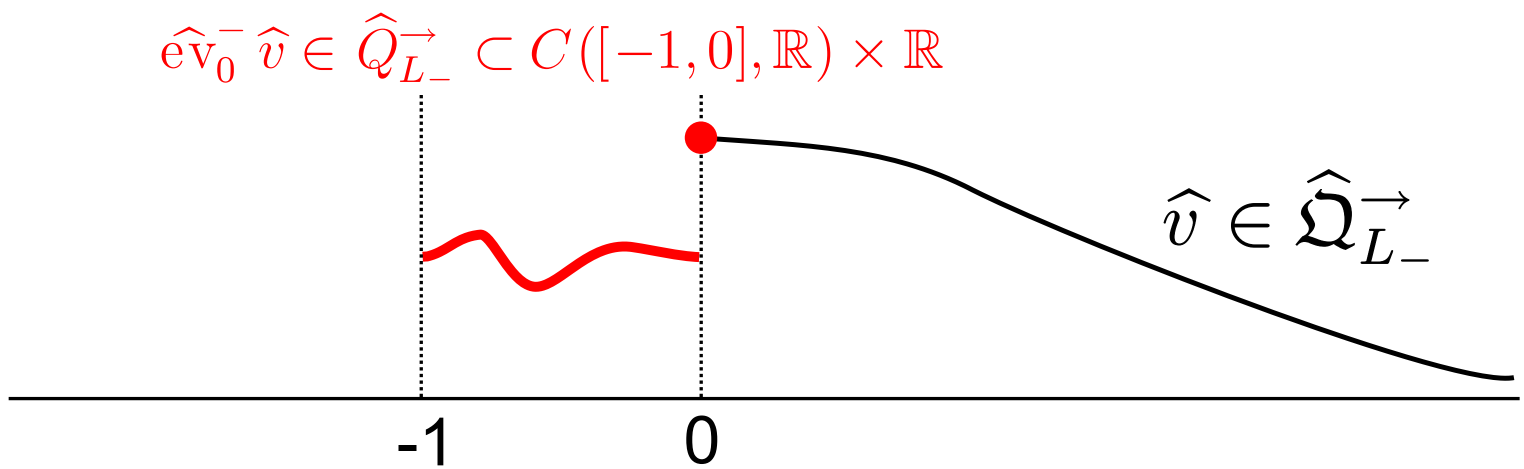
Delay Equations

Interested in solution spaces

$$\widehat{\mathfrak{Q}}_{\overrightarrow{L_-}} = \{\widehat{v} \in C([-1,0],\mathbb{R}) \times C([0,\infty),\mathbb{R}) : \widehat{v}'(\xi) = L_- \,\widehat{\mathrm{ev}}_\xi^- \,\widehat{v} \text{ for almost all } \xi \geq 0\}.$$

We also use 'initial segment' space

$$\widehat{Q}_{\overrightarrow{L_-}} = \widehat{\mathrm{ev}}_0^- \left(\mathfrak{Q}_{\overrightarrow{L_-}}\right) \subset C([-1,0],\mathbb{R}) \times \mathbb{R}.$$

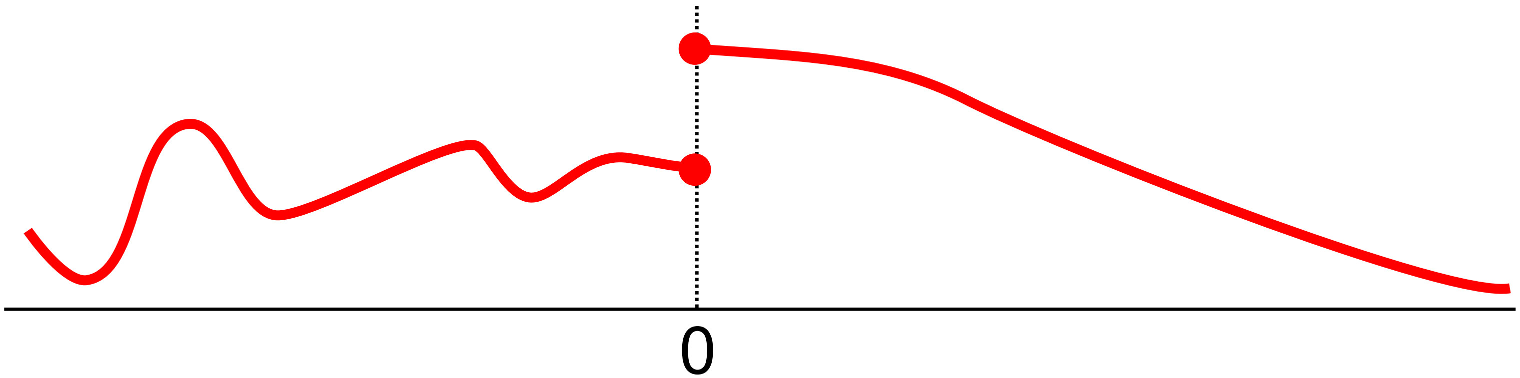


Delay Equations

Question: how to link jump-solutions $\widehat{\mathfrak{Q}}^{\rightarrow}_{L_-}$ with continuous solutions $\mathfrak{Q}^{\rightarrow}_{L_-}$.

Answer: Green's function $\widehat{G}_{L_-}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\nu\xi}\Delta_{L_-}(i\nu)^{-1}\, d\nu.$

$\widehat{G}_{L_-} \in C(\mathbb{R}_-, \mathbb{R}) \times C(\mathbb{R}_+, \mathbb{R})$



$\widehat{G}_{L_-}$ continuous except for discontinuity at $\xi = 0$; solves

$$\widehat{G}'_{L_-}(\xi) = L_- \,\widehat{\mathrm{ev}}_\xi^- \,\widehat{G}_{L_-} + \delta(\xi)$$

This gives us

$$\widehat{\mathfrak{Q}}^{\rightarrow}_{L_-} = \mathfrak{Q}^{\rightarrow}_{L_-} \oplus \mathrm{span}\{\widehat{G}_{L_-}\}.$$

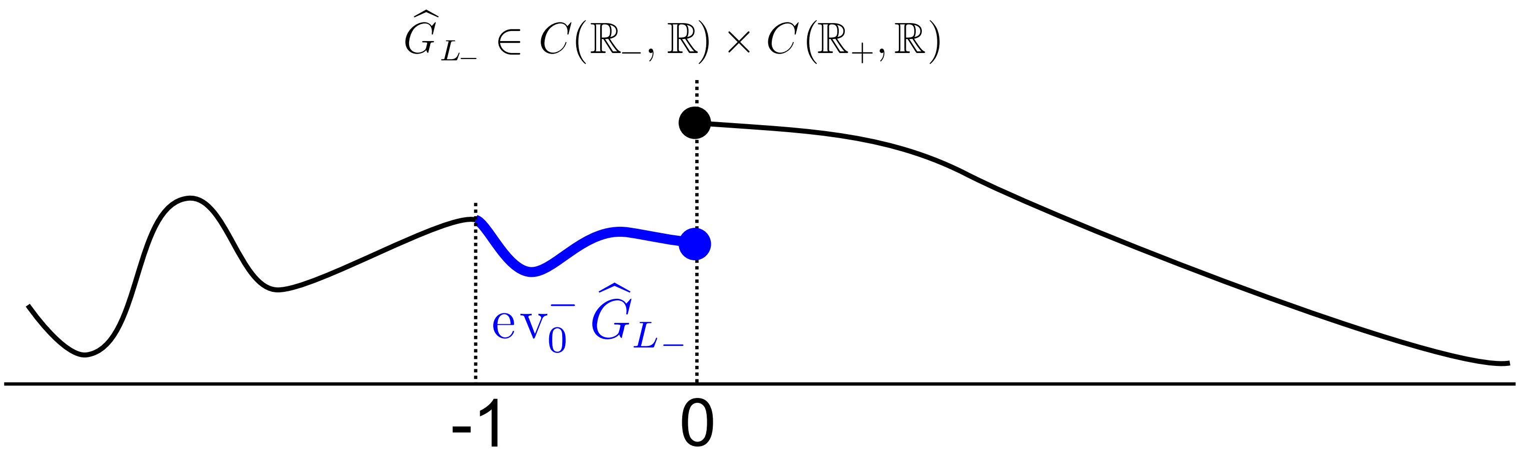
Delay Equations

Starting from

$$\widehat{\mathfrak{Q}}^{\to}_{L_-} = \mathfrak{Q}^{\to}_{L_-} \oplus \operatorname{span}\{\widehat{G}_{L_-}\},$$

what are consequences for initial segments

$$\widehat{Q}^{\to}_{L_-} \qquad \text{vs} \qquad Q^{\to}_{L_-}$$



Need to understand relation

$$\mathrm{ev}_0^- \widehat{G}_{L_-} \qquad \text{vs} \qquad Q^{\to}_{L_-}.$$

Delay Equations

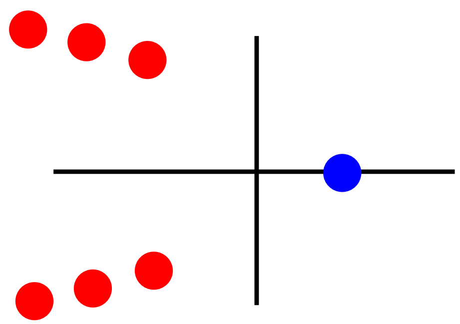
Use spectral projections to understand relation

$$\mathrm{ev}_0^- \, \widehat{G}_{L_-} \qquad \text{vs} \qquad Q^{\rightarrow}_{L_-}.$$

Prop. If $\Delta_{L_-}(z_*) = 0$ with $\mathrm{Re}\, z_* > 0$, then we have

$$\Pi^{\mathrm{sp}}(z_*)\, \mathrm{ev}_0^- \, \widehat{G}_{L_-} = -\mathrm{Res}_{z=z_*} e^{z\cdot} \Delta_{L_-}(z)^{-1}.$$

Example



We have $Q^{\rightarrow}_{L_-} \neq C([-1,0],\mathbb{R})$. But: any $\phi \in C([-1,0],\mathbb{R})$ can be extended to a bounded solution $\widehat{x}$, where the jump at zero depends directly (and explicitly) on ϕ.

Delay Equations

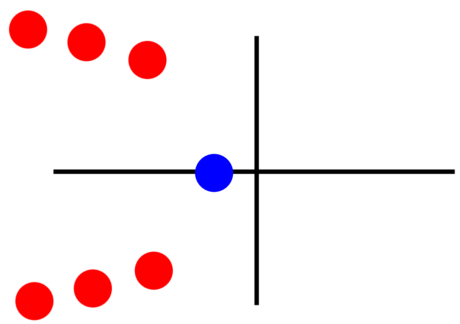
Use spectral projections to understand relation

$$\mathrm{ev}_0^- \widehat{G}_{L_-} \qquad \text{vs} \qquad Q^{\rightarrow}_{L_-}.$$

Prop. If $\Delta_{L_-}(z_*) = 0$ with $\mathrm{Re}\, z_* > 0$, then we have

$$\Pi^{\mathrm{sp}}(z_*)\, \mathrm{ev}_0^- \widehat{G}_{L_-} = -\mathrm{Res}_{z=z_*} e^{z\cdot} \Delta_{L_-}(z)^{-1}.$$

Example 2

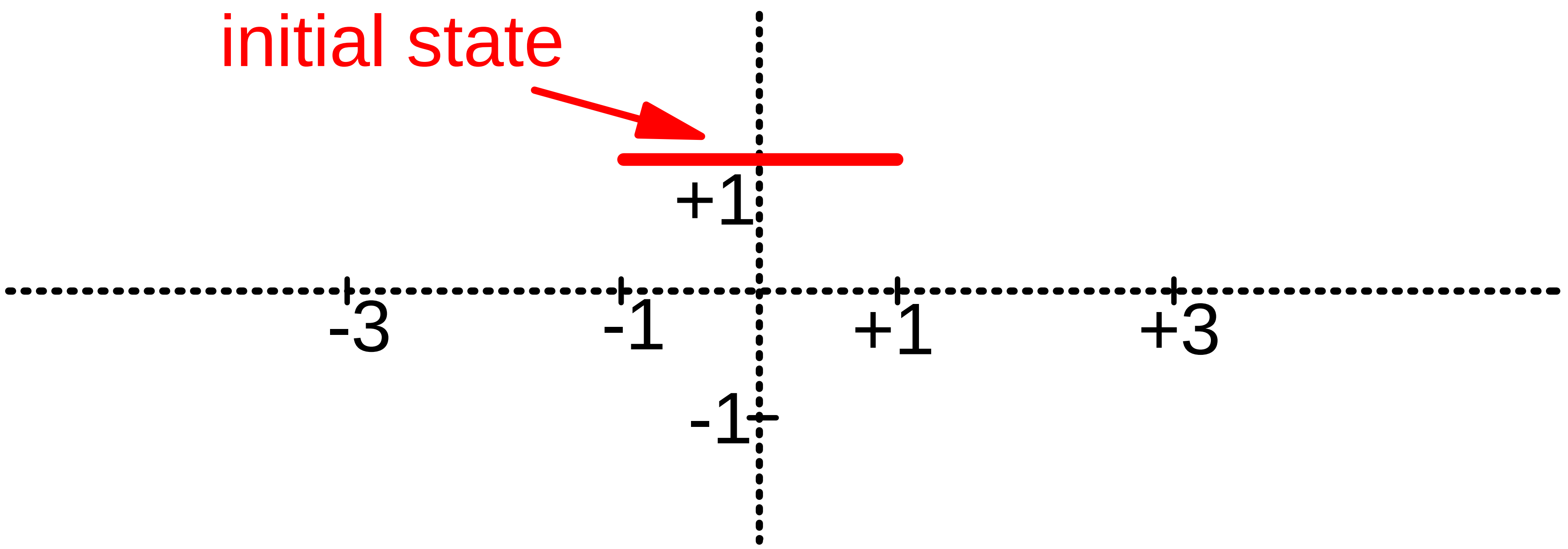


We have $Q^{\rightarrow}_{L_-} = C([-1,0],\mathbb{R})$. Any $\phi \in C([-1,0],\mathbb{R})$ can be extended **in multiple ways** to a bounded solution $\widehat{x}$, where the jump at zero can be chosen arbitrarily.

MFDE - Ill-posedness

Moving on to mixed type equations, consider the MFDE

$$v'(\xi) = v(\xi - 1) + v(\xi + 1).$$

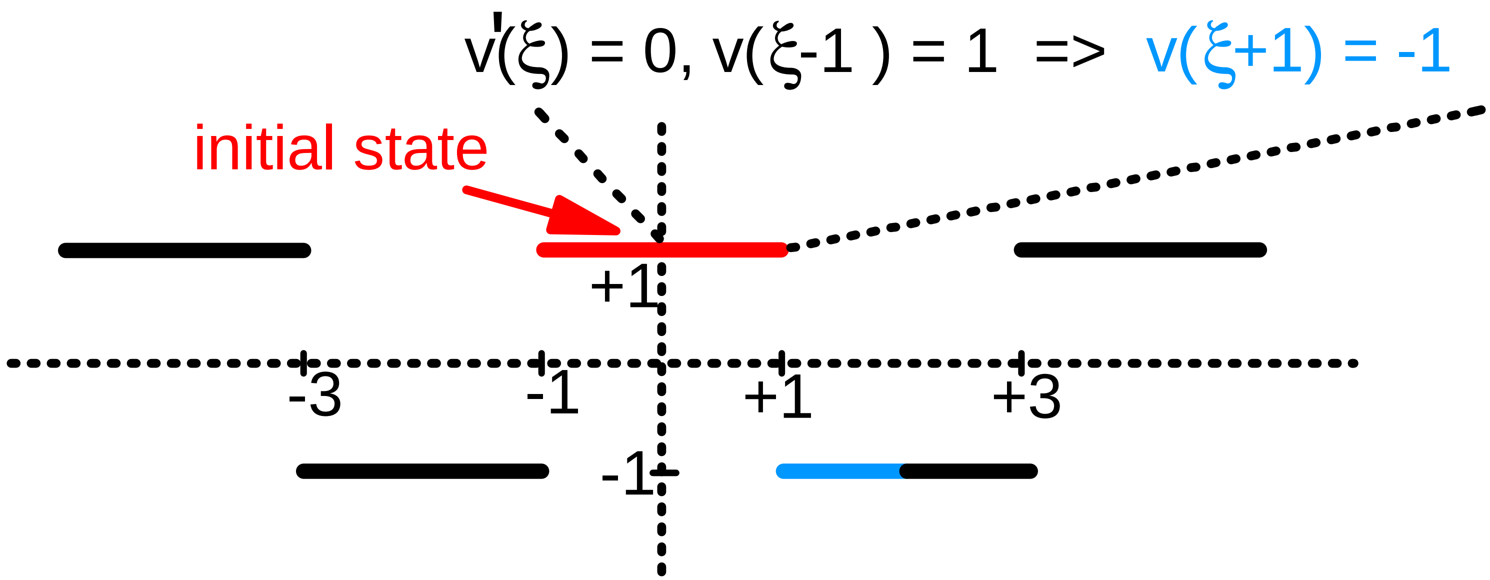


(Example due to Rustichini)

Ill-posedness

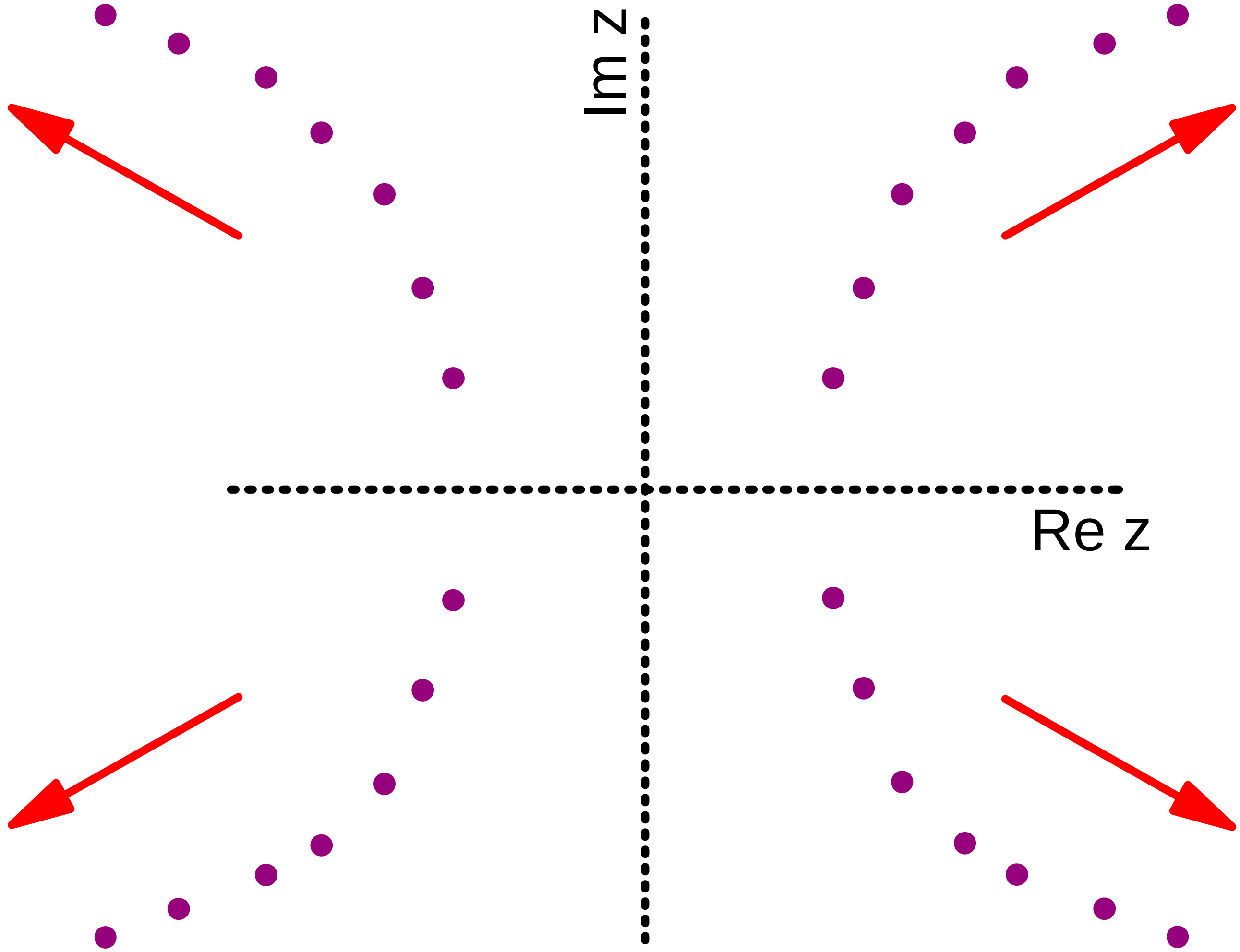
Moving on to mixed type equations, consider the MFDE

$$v'(\xi) = v(\xi - 1) + v(\xi + 1).$$



- Continuity lost $\implies$ ill-posed as an initial value problem with initial conditions in the 'mathematical' state space $C([-1, 1], \mathbb{R})$.

Ill-posedness: What is going on?



Substitution of $e^{z\xi}$ into

$$v'(\xi) = v(\xi - 1) + v(\xi + 1),$$

yields the characteristic equation

$$\Delta(z) := z - e^{-z} - e^{z} = 0.$$

- No exponential bound possible for solutions, at both $\pm\infty$ (unlike delay equations)!

MFDEs

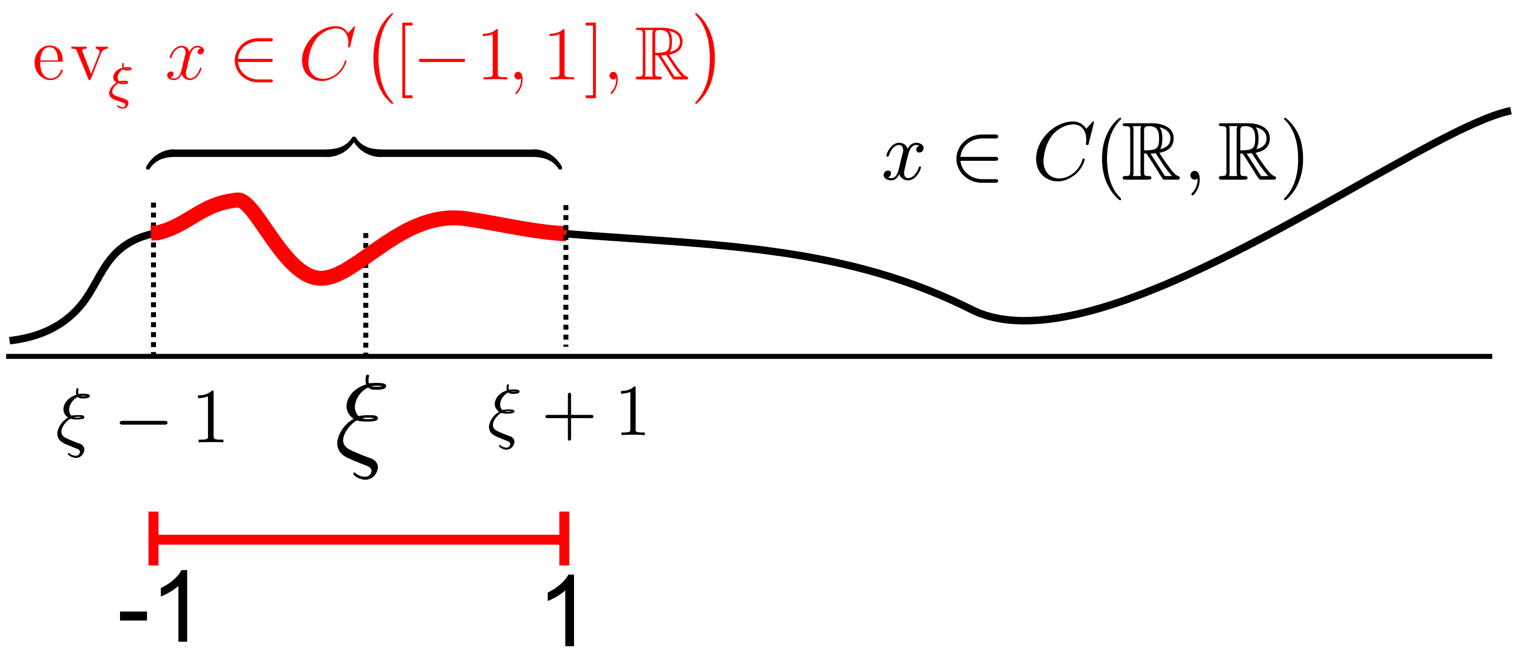
Consider the linear mixed-type equation (MFDE)

$$x'(\xi) = L\,\mathrm{ev}_\xi\, x,$$

where $L : C([-1,1],\mathbb{R}) \to \mathbb{R}$.

Characteristic function given by:

$$\Delta_L(z) = z - Le^{z\cdot}$$



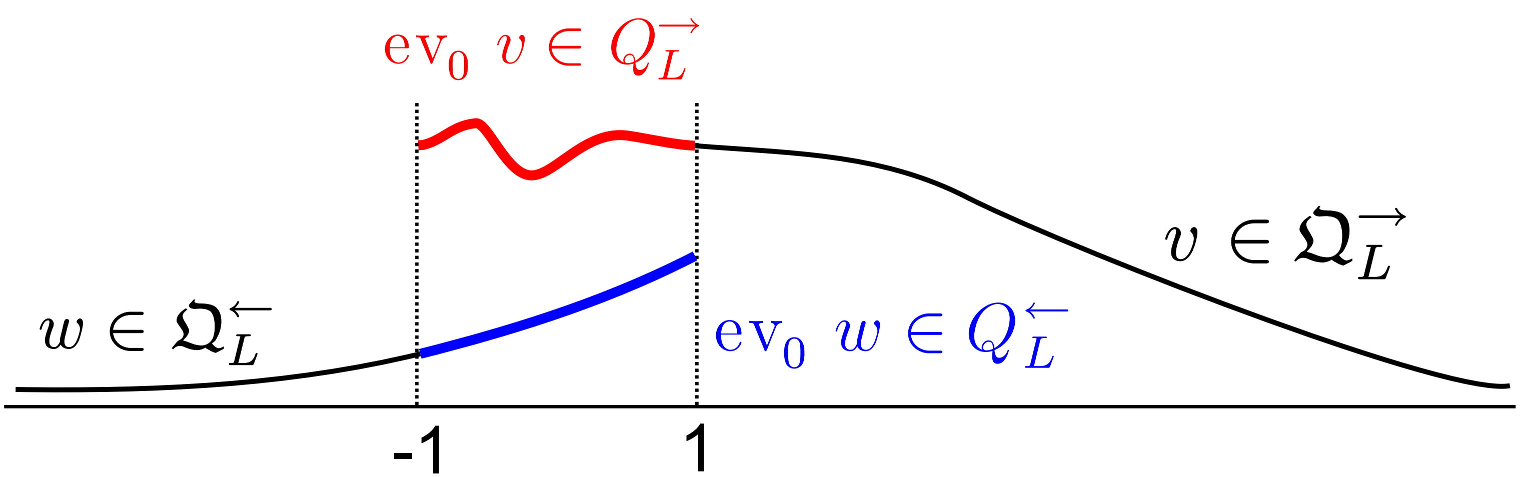
MFDEs

We are interested in solution spaces

$$\begin{aligned} \mathfrak{Q}_L^{\leftarrow} &= \{w \in BC((-\infty, 1], \mathbb{R}) : w'(\xi) = L\,\mathrm{ev}_\xi\, w \text{ for all } \xi \leq 0\}, \\ \mathfrak{Q}_L^{\rightarrow} &= \{v \in BC([-1, \infty), \mathbb{R}) : v'(\xi) = L\,\mathrm{ev}_\xi\, v \text{ for all } \xi \geq 0\}. \end{aligned}$$

We also use 'initial segment' spaces

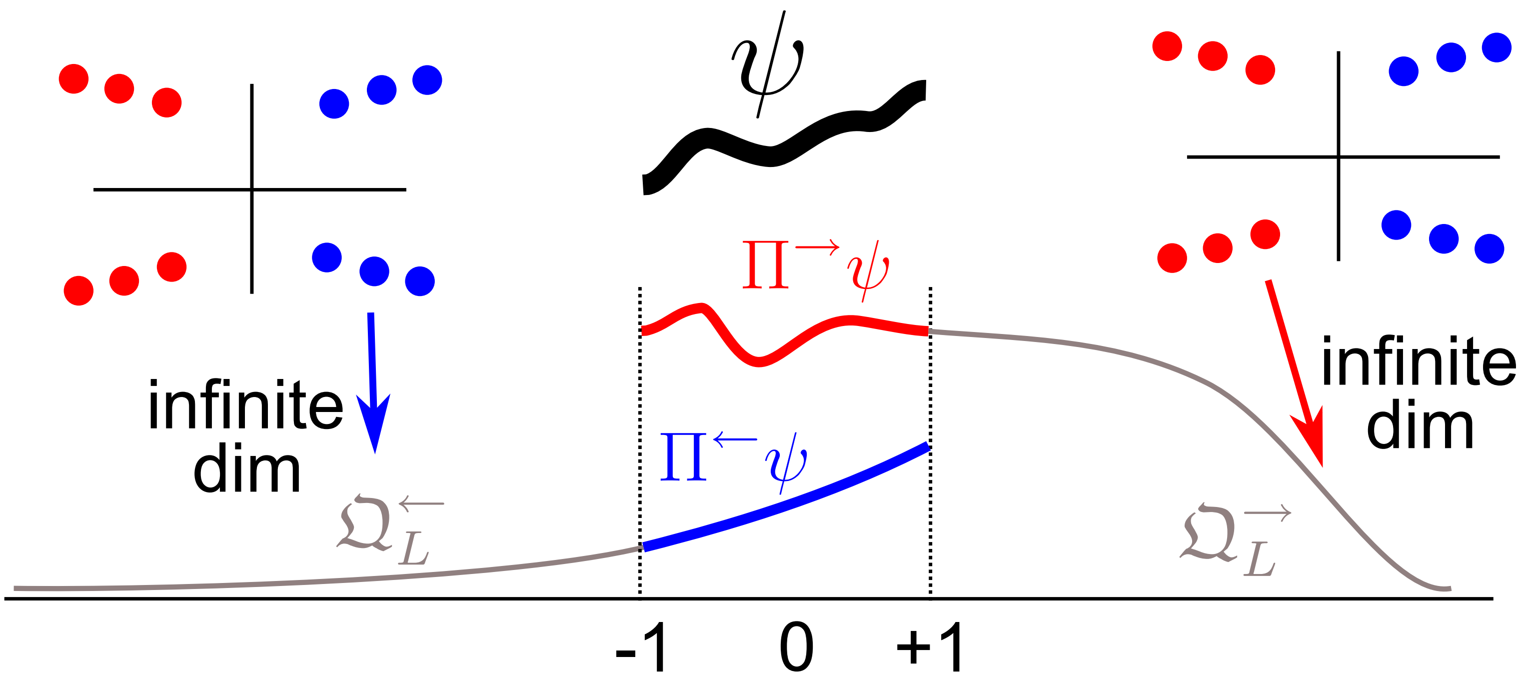
$$Q_L^{\leftarrow} = \mathrm{ev}_0\big(\mathfrak{Q}_L^{\leftarrow}\big), \qquad Q_L^{\rightarrow} = \mathrm{ev}_0\big(\mathfrak{Q}_L^{\rightarrow}\big).$$



MFDEs

Thm. [Verduyn-Lunel+Mallet-Paret, Rustichini] If $\Delta_L(z) = 0$ has no roots on imag. axis, then

$$C([-1,1],\mathbb{C}) = Q_L^{\leftarrow} \oplus Q_L^{\rightarrow}.$$



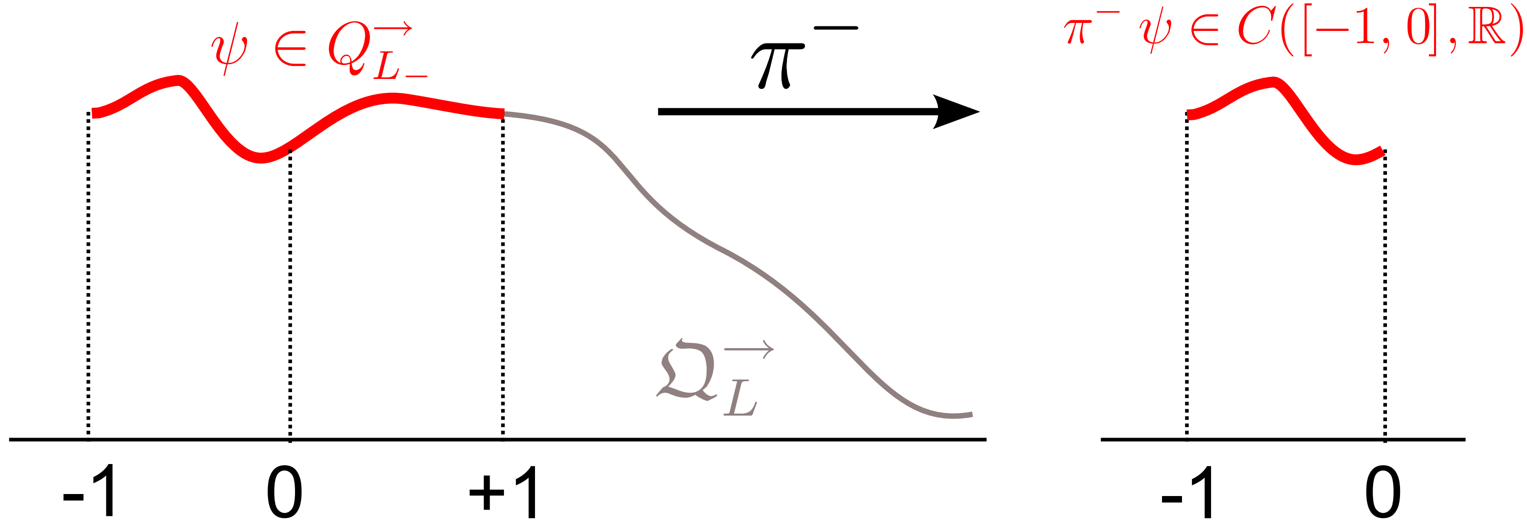
Can no longer use spectral projections to define projections $\Pi^{\leftarrow}$ and $\Pi^{\rightarrow}$.

MFDEs

'Mathematical' state space $C([-1,1],\mathbb{R})$, but 'modelling' state space $C([-1,0],\mathbb{R})$.

Restriction operators:

$$\pi^- : Q_L^{\rightarrow} \to C([-1,0],\mathbb{R}), \qquad \psi \mapsto \mathrm{ev}_0^- \psi = \psi_{[-1,0]}$$



Thm. [Verduyn-Lunel+Mallet-Paret] π^- is Fredholm.

MFDEs

Recall Fredholm restriction operator:

$$\pi^- : Q_L^{\rightarrow} \to C([-1,0],\mathbb{R}), \qquad \psi \mapsto \mathrm{ev}_0^- \psi = \psi_{[-1,0]}$$

and write

$$R = \mathrm{Range}\,\pi^- \subset C([-1,0],\mathbb{R}), \qquad K = \mathrm{Ker}\,\pi^- \subset C([-1,1],\mathbb{R})$$

R has finite codimension and determines **possibility** of extending initial condition $\phi \in C([-1,0],\mathbb{R})$.

K has finite dimension and determines **uniqueness** of such an extension.

Unfortunately, no direct way to characterize R and K.

MFDEs - Wiener-Hopf Factorization

Thm. [Verduyn-Lunel+Mallet-Paret; slightly generalized by H.+Augeraud-Veron]
Pick $\alpha > 0$. There exist (non-unique) linear operators

$$L_- : C([-1,0],\mathbb{C}) \to \mathbb{C}, \qquad L_+ : C([0,1],\mathbb{C}) \to \mathbb{C}$$

such that

$$\Delta_{L_-}(z)\Delta_{L_+}(z) = (z+\alpha)\Delta_L(z).$$

The integer

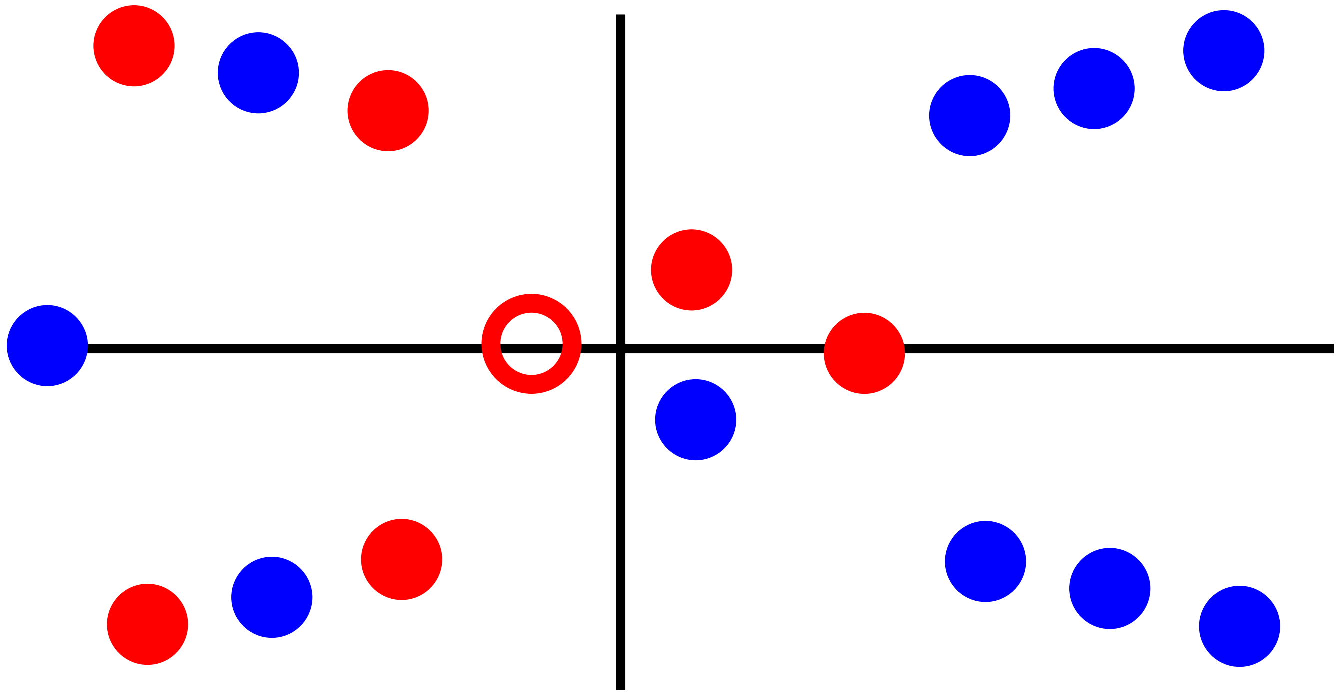
$$n_L^\sharp = \{z : \Delta_{L_+}(z) = 0 \text{ and } \mathrm{Re}\, z < 0\} - \{z : \Delta_{L_-}(z) = 0 \text{ and } \mathrm{Re}\, z > 0\}$$

does not depend on specific pair L_-, L_+. Finally,

$$\operatorname{codim} R = \max\{1 - n_L^\sharp, 0\}, \qquad \dim K = \max\{n_L^\sharp - 1, 0\}$$

MFDEs - Wiener-Hopf Factorization

Integer $n_L^\sharp$ counts roots of $\Delta_{L_-}(z) = 0$ and $\Delta_{L_+}(z) = 0$ that are on 'wrong' side of imaginary axis.



$\Delta_{L_-} = 0$	$\Delta_{L_+} = 0$	$\Delta_L = 0$

$$(z + \alpha)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z)$$

MFDEs - Wiener-Hopf Factorization

In practice, finding a Wiener-Hopf factorization is intractable.

However, suppose once has a special reference system L_{ref} that one **can** factorize (easier to find).

Construct a path

$$\Gamma : [0,1] \to \mathcal{L}\big(C([-1,1],\mathbb{C}),\mathbb{C}\big)$$

that connects L_{ref} to L,

$$\Gamma(0) = L_{\mathrm{ref}}, \qquad \Gamma(1) = L.$$

Thm. [H., Augeraud-Veron] Under some nondegeneracy conditions on the path Γ,

$$n^{\sharp}_{L} = n^{\sharp}_{L_{\mathrm{ref}}} - \mathrm{cross}(\Gamma),$$

where $\mathrm{cross}(\Gamma)$ is number of roots that cross imaginary axis from left to right as μ increases from zero to one.

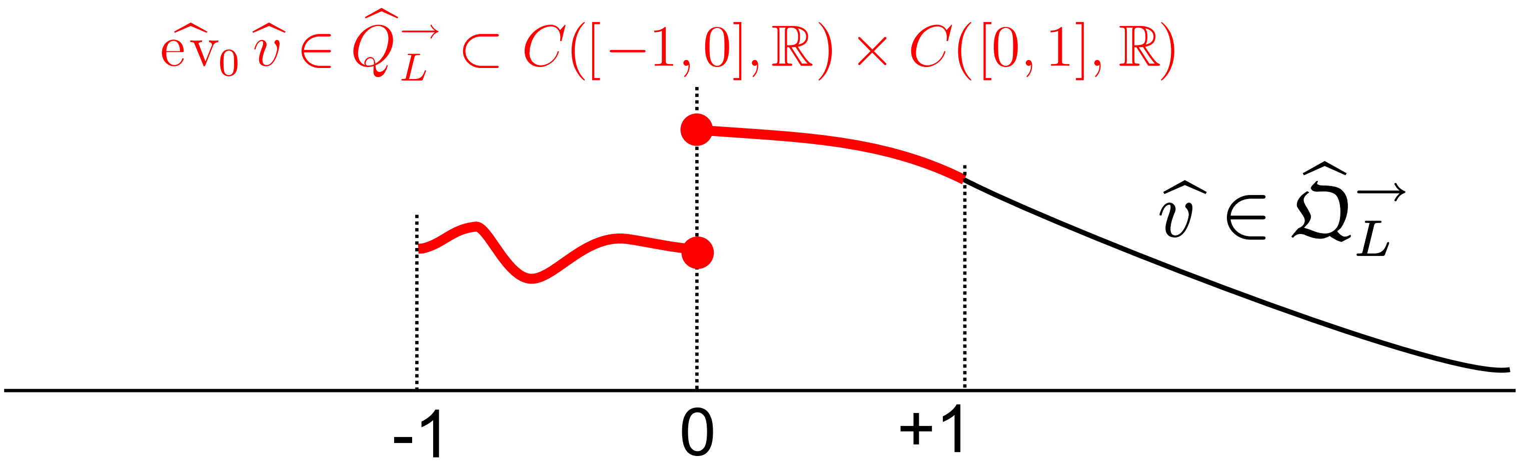
MFDEs with Jumps

Interested in solution spaces

$$\widehat{\mathfrak{Q}}_L^{\rightarrow} = \{\widehat{v} \in C([-1,0],\mathbb{R}) \times C([0,\infty),\mathbb{R}) : \widehat{v}'(\xi) = L\,\widehat{\mathrm{ev}}_\xi\,\widehat{v} \text{ for almost all } \xi \geq 0\}.$$

We also use 'initial segment' space

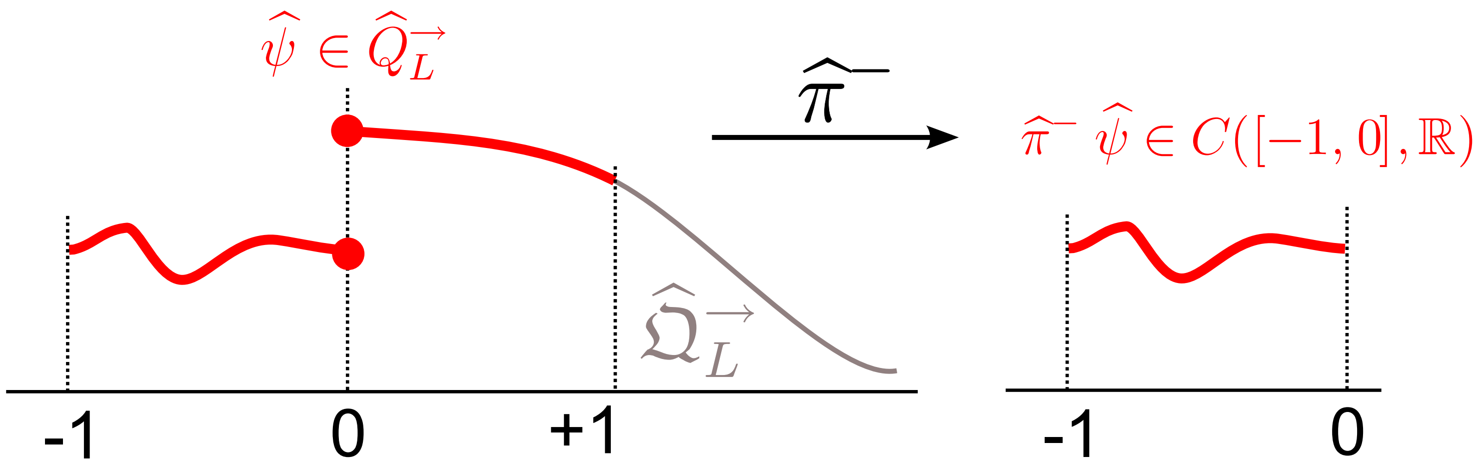
$$\widehat{Q}_L^{\rightarrow} = \widehat{\mathrm{ev}}_0\big(\mathfrak{Q}_L^{\rightarrow}\big) \subset C([-1,0],\mathbb{R}) \times C([0,1],\mathbb{R}).$$



MFDEs with Jumps

Again, introduce restriction

$$\widehat{\pi}^- : \widehat{Q}_L^{\rightarrow} \to C([-1,0],\mathbb{R}), \qquad \widehat{\psi} \mapsto \mathrm{ev}_0^- \widehat{\psi} = \widehat{\psi}_{[-1,0]}$$



Also introduce:

$$\widehat{R} = \mathrm{Range}\,\widehat{\pi}^- \in C([-1,0],\mathbb{R}), \qquad \widehat{K} = \mathrm{Ker}\,\widehat{\pi}^- \in C([-1,0],\mathbb{R}) \times C([0,1],\mathbb{R})$$

$\widehat{R}$ determines **possibility** of extending initial condition $\phi \in C([-1,0],\mathbb{R})$ to bounded solution **with jump**.

$\widehat{K}$ determines **uniqueness** of such an extension.

MFDEs - Comparison

Recall **non-jump** setting:

$$\begin{aligned} &\pi^- : Q_L^{\rightarrow} \to C([-1,0],\mathbb{R}), \qquad \psi \mapsto \psi_{[-1,0]} \\ &R = \operatorname{Range} \pi^- \subset C([-1,0],\mathbb{R}), \qquad K = \operatorname{Ker} \pi^- \in C([-1,1],\mathbb{R}) \end{aligned}$$

with dimensions

$$\operatorname{codim} R = \max\{1 - n_L^\sharp, 0\}, \qquad \dim K = \max\{n_L^\sharp - 1, 0\}$$

Recall **jump** setting:

$$\begin{aligned} &\widehat{\pi}^- : \widehat{Q}_L^{\rightarrow} \to C([-1,0],\mathbb{R}), \qquad \widehat{\psi} \mapsto \mathrm{ev}_0^- \widehat{\psi} = \widehat{\psi}_{[-1,0]} \\ &\widehat{R} = \operatorname{Range} \widehat{\pi}^- \in C([-1,0],\mathbb{R}), \qquad \widehat{K} = \operatorname{Ker} \widehat{\pi}^- \in C([-1,0],\mathbb{R}) \times C([0,1],\mathbb{R}) \end{aligned}$$

Thm.[H., Augeraud-Veron] The operator $\widehat{\pi}^-$ is Fredholm and we have

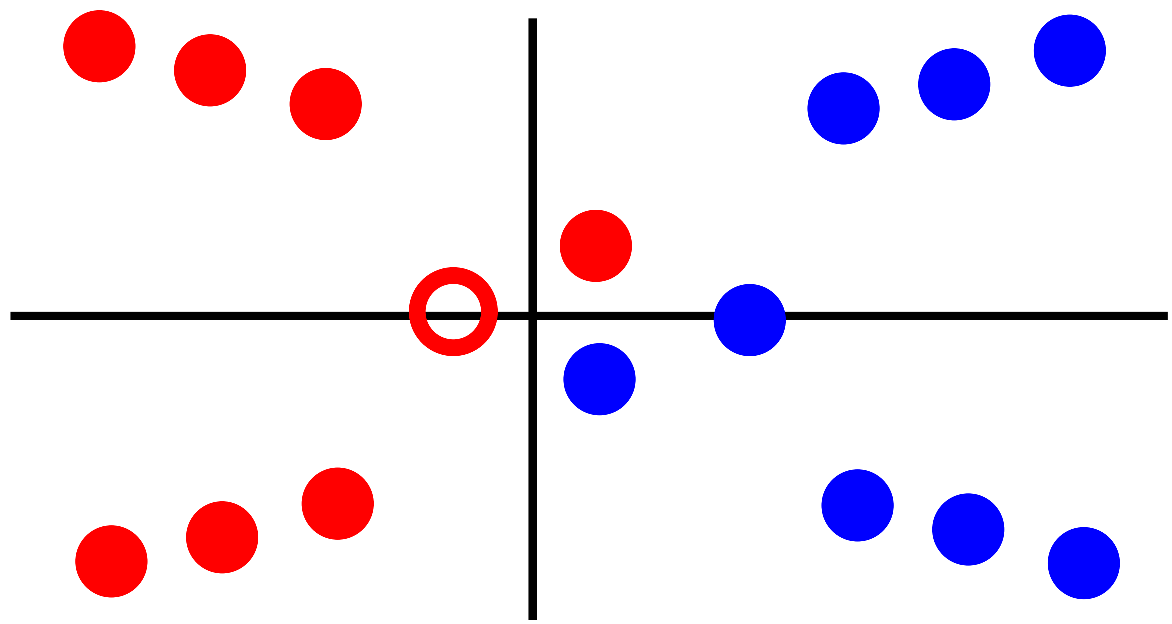
$$\operatorname{codim} \widehat{R} = \max\{-n_L^\sharp, 0\}, \qquad \dim \widehat{K} = \max\{n_L^\sharp, 0\}$$

Conclusion: presence of jump introduces one extra d.o.f. for initial value problem.

MFDEs - Sketch of Proof

We again have $$\widehat{\mathfrak{Q}}_L^{\rightarrow} = \mathfrak{Q}_L^{\rightarrow} \oplus \text{span}\{\widehat{G}_L\},$$

with $\widehat{G}_L$ the Green's function for MFDE. Use special ordered Wiener-Hopf factorization:



$\Delta_{L_-} = 0$	$\Delta_{L_+} = 0$	$\Delta_L = 0$

In this case, relation between $\text{ev}_0^- \widehat{G}_L$ and $\pi^-(Q_L^{\rightarrow})$ can be determined by using spectral projections **of operator** L_-.

Differential-Algebraic Equations

We know turn to the differential-algebraic equation

$$\mathcal{I}x'(\xi) = M\mathrm{ev}_\xi x,$$

where $M : C([-1,1],\mathbb{R}^n) \to \mathbb{R}^n$ and $\mathcal{I}$ is diagonal and typically singular.

Characteristic equation given by

$$\delta_{\mathcal{I},M}(z) = \mathcal{I}z - M\, e^{z\cdot}.$$

Main Assumption: There exists $L : C([-1,1],\mathbb{R}^n) \to \mathbb{R}^n$ so that solutions satisfy

$$x'(\xi) = L\,\mathrm{ev}_\xi\, x.$$

Differential-Algebraic Equations - Example

Example: Consider algebraic equation

$$0 = -x(\xi) + \int_{-1}^{1} x(\xi + \sigma)\, d\sigma, \tag{1}$$

which after a single differentiation yields the MFDE

$$x'(\xi) = x(\xi + 1) - x(\xi - 1) \tag{2}$$

Vice versa, if x solves (2) **and** has $x(0) = \int_{-1}^{1} x(\sigma) d\sigma$, then x solves (1).

Characteristic function for (1) given by

$$\delta(z) = 1 - \int_{-1}^{1} e^{z\sigma} d\sigma = 1 - \frac{1}{z}(e^{z} - e^{-z})$$

Characteristic function for (2) given by

$$\Delta(z) = z - (e^{z} - e^{-z})$$

Notice $z\delta(z) = \Delta(z)$.

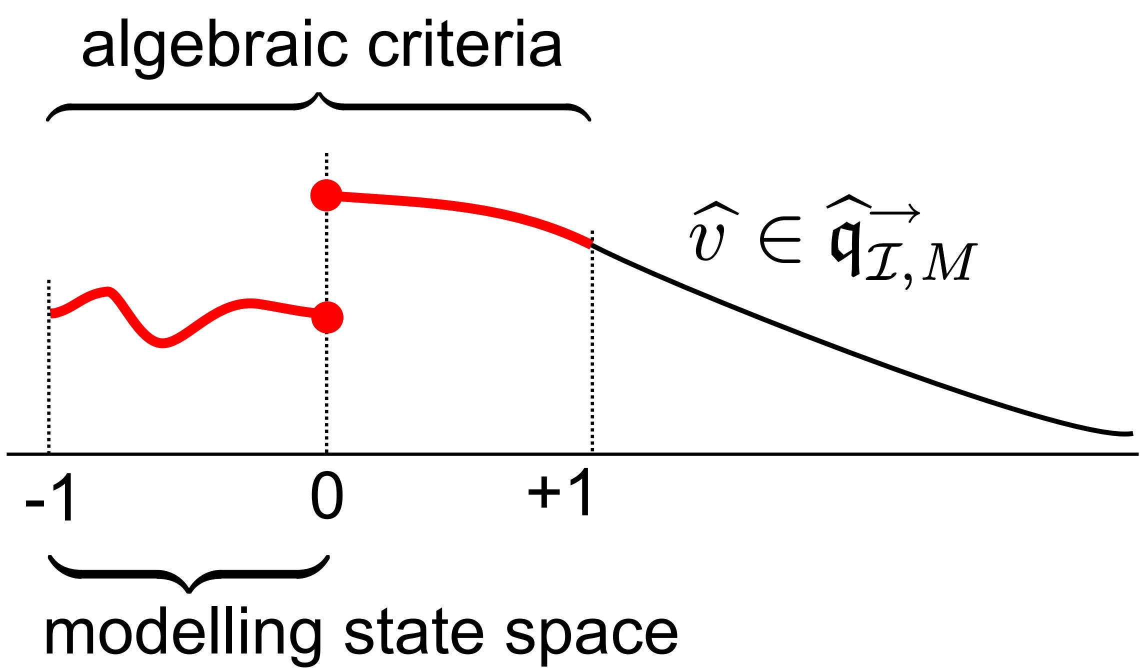
Differential-Algebraic Equations with Jump

Recall 'smooth' differential-algebraic equation (posed on $\mathbb{R}^n$)

$$\mathcal{I}x'(\xi) = M\mathrm{ev}_\xi x.$$

Interested in solution spaces

$$\begin{aligned} \widehat{\mathfrak{q}}_{\overrightarrow{\mathcal{I},M}} &= \{\widehat{v} \in C([-1,0],\mathbb{R}^n) \times C([0,\infty),\mathbb{R}^n) : \\ &\qquad \mathcal{I}\widehat{v}'(\xi) = M\,\widehat{\mathrm{ev}}_\xi\,\widehat{v} \text{ for 'almost' all } \xi \geq 0\}. \end{aligned}$$



Differential-Algebraic Equations

Recall 'smooth' differential-algebraic equation (posed on $\mathbb{R}^n$)

$$\mathcal{I}x'(\xi) = M\mathrm{ev}_{\xi}x.$$

Thm. [H. + Augeraud-Veron] Pick any $\gamma > 0$. There exist non-negative integers $\ell_1, \ldots, \ell_n$ and an operator $L(\gamma) : C([-1,1], \mathbb{R}^n) \to \mathbb{R}^n$ such that

$$\begin{pmatrix} (z-\gamma)^{\ell_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (z-\gamma)^{\ell_n} \end{pmatrix} \delta_{\mathcal{I},M}(z) = \Delta_{L(\gamma)}(z).$$

In addition, we have

$$\widehat{\mathfrak{q}}_{\overrightarrow{\mathcal{I}},M} = \widehat{\mathfrak{Q}}_{\overrightarrow{L(\gamma)}}.$$

Conclusion: Can use prior results to study $\widehat{\mathfrak{q}}_{\overrightarrow{\mathcal{I}},M}$.

Outlook

- Linear results can be lifted to local nonlinear results.
- Jumps allow 'unstable' equilibria to be stabilized.
- Mixed type equations can have non-unique continuations of initial conditions.
- Mixed type equations in more than one dimension still elusive.