Propagation Failure In The Discrete Nagumo Equation

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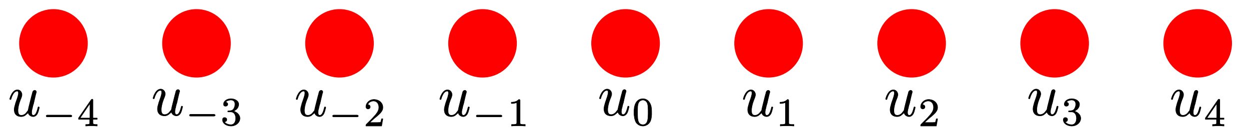
Lattice Differential Equations

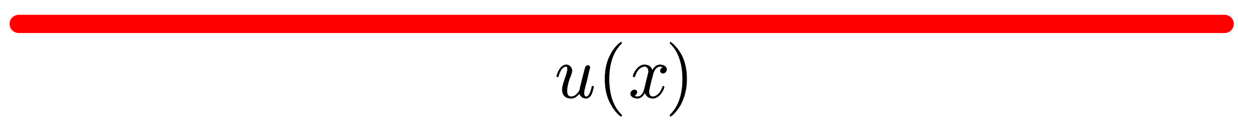
Lattice differential equations (LDEs) are ODEs indexed on a spatial lattice, e.g.

$$\frac{d}{dt}u_j(t) = \alpha\big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\big) + f\big(u_j(t)\big), \qquad j \in \mathbb{Z}.$$



Picking $\alpha = h^{-2} \gg 1$, LDE can be seen as discretization with distance h of PDE

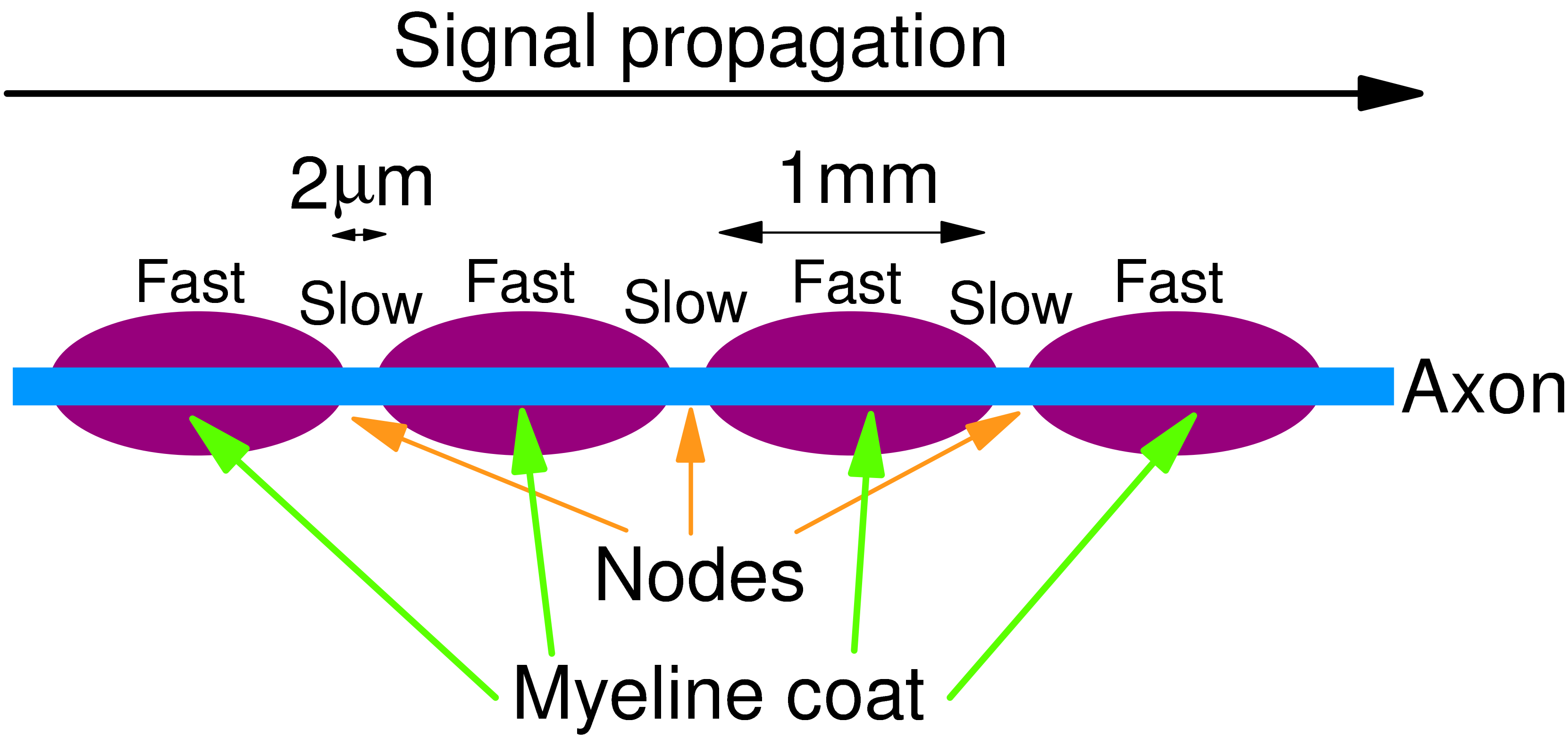
$$\partial_t u(t,x) = \partial_{xx}u(t,x) + f\big(u(t,x)\big), \qquad x \in \mathbb{R}.$$



- Many physical models have a discrete spatial structure $\rightarrow$ LDEs.
- No need for α to be large; some models even have $\alpha < 0$.
- Main theme: qualitative differences between PDEs and LDEs.

Signal Propagation through Nerves

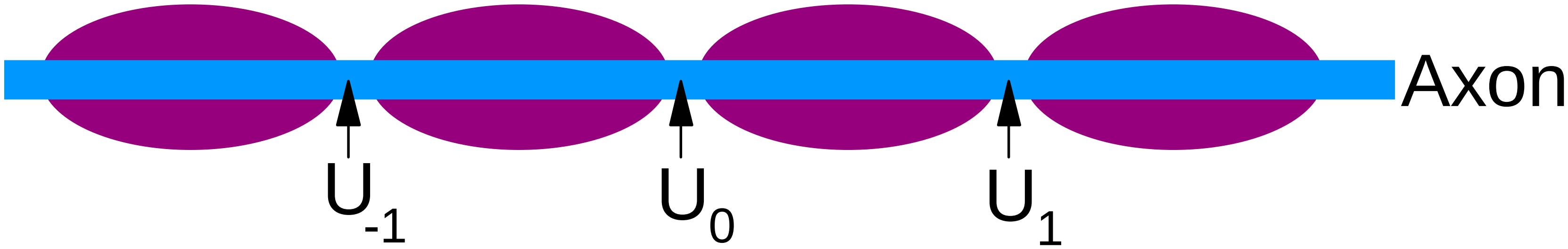
Nerve fibres carry signals over large distances (meter range).



- Fiber has myeline coating with periodic gaps called *nodes of Ranvier* .
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range)
- Slow propagation in gaps, but signal chemically reinforced.

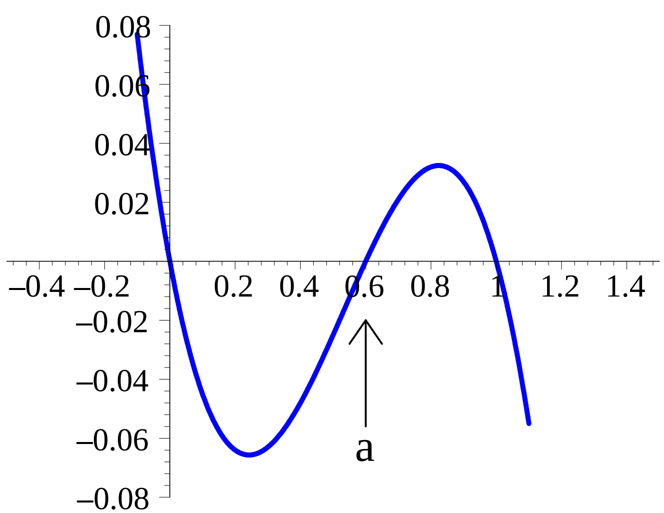
Signal Propagation: The Model

One is interested in the potential U_j at the node sites.



Signals appear to "hop" from one node to the next [Lillie, 1925].
Ignoring recovery, one arrives at the LDE [Keener and Sneyd, 1998]

$$\frac{d}{dt}U_j(t) \quad = \quad U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t);a), \qquad j \in \mathbb{Z}.$$



Bistable nonlinearity g given by

$$g(u;a) = u(a-u)(u-1).$$

Signal Propagation: PDE

In continuum limit: Nagumo LDE becomes Nagumo PDE

$$\partial_t u = \partial_{xx} u + u(a-u)(u-1).$$

Starting step [Fife, McLeod]: travelling waves.

Travelling wave $u(x,t) = \phi(x+ct)$ satisfies:

$$c\phi'(\xi) \quad = \quad \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

Interested in pulse solutions connecting 0 to 1, i.e.

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \phi(\xi) = 1.$$

Signal Propagation: PDE

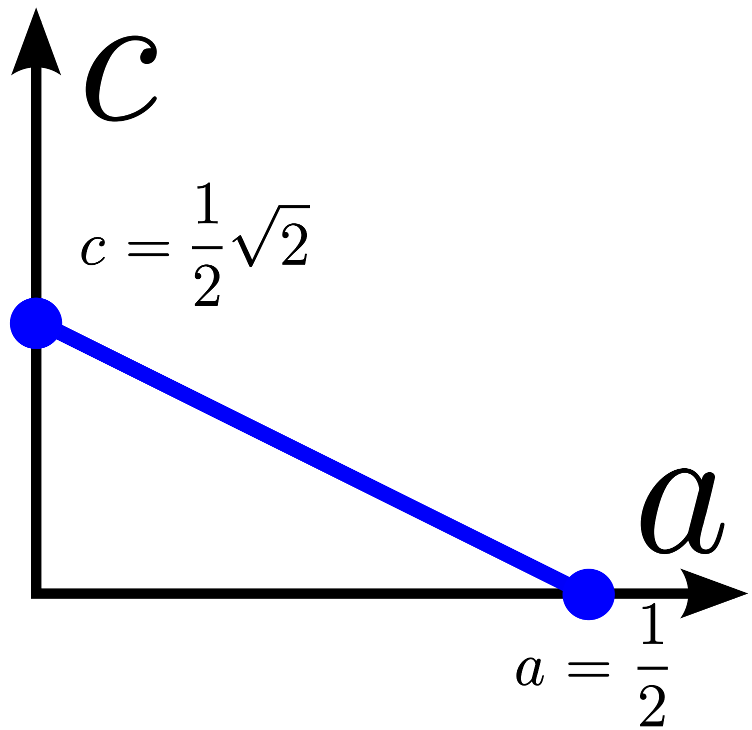
Recall travelling wave ODE

$$c\phi'(\xi) = \phi''(\xi) + \phi(\xi)\big(a - \phi(\xi)\big)\big(\phi(\xi) - 1\big).$$

$$\begin{aligned} \lim_{\xi \to -\infty} \phi(\xi) &= 0, \\ \lim_{\xi \to +\infty} \phi(\xi) &= 1. \end{aligned}$$

Explicit solutions available:

$$\begin{aligned} \phi(\xi) &= \tfrac{1}{2} + \tfrac{1}{2} \tanh\big(\tfrac{1}{4}\sqrt{2}\,\xi\big), \\ c(a) &= \tfrac{1}{\sqrt{2}}(1 - 2a). \end{aligned}$$



Signal Propagation: LDE

Recall the Nagumo LDE

$$\frac{d}{dt}U_j(t) \quad = \quad \frac{1}{h^2}[U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)] + g(U_j(t); a), \qquad j \in \mathbb{Z}.$$

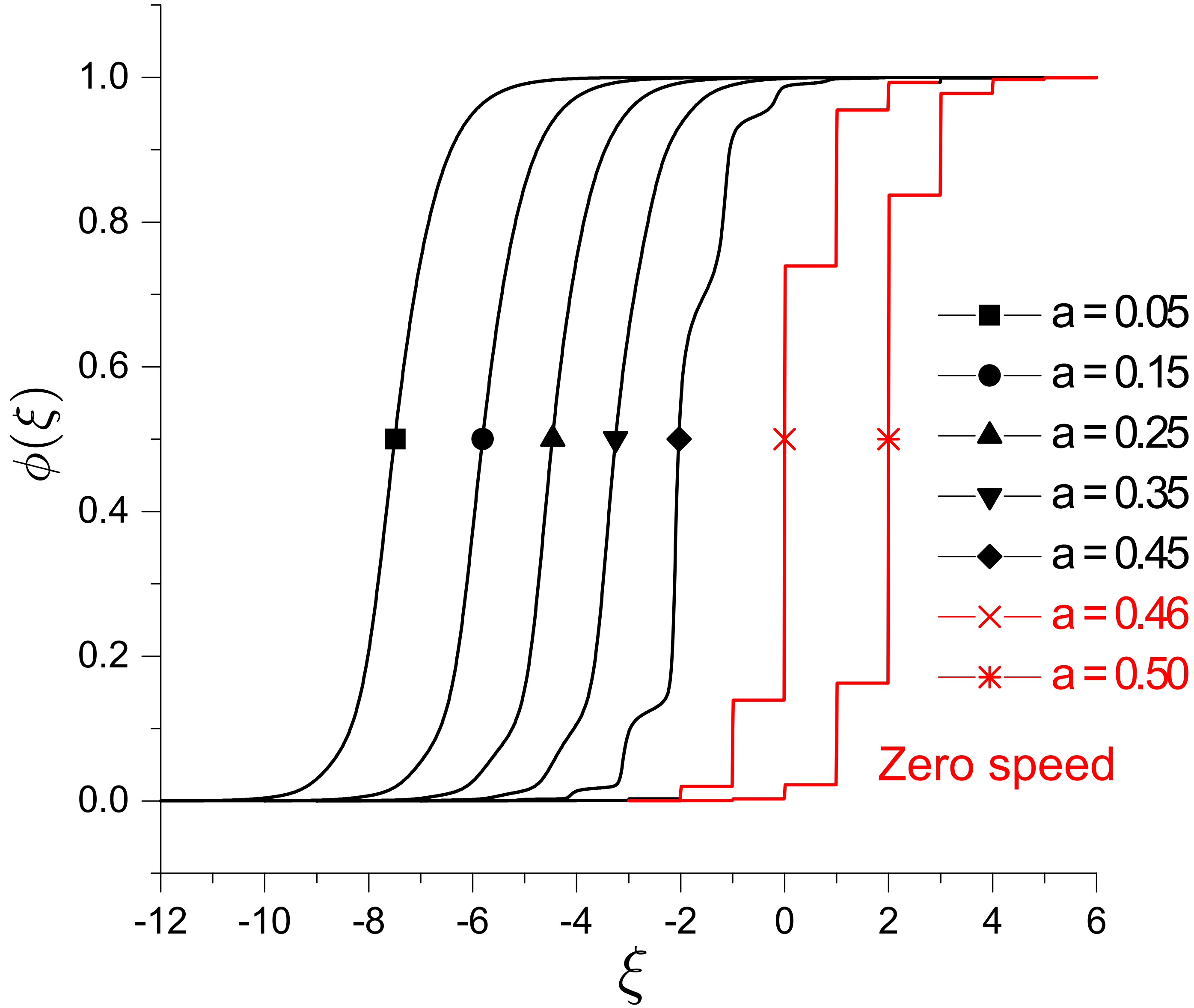
Travelling wave profile $U_j(t) = \phi(j + ct)$ must satisfy:

$$\begin{aligned} c\phi'(\xi) &= \frac{1}{h^2}[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] + g\big(\phi(\xi); a\big) \\ \lim_{\xi \to -\infty} \phi(\xi) &= 0, \\ \lim_{\xi \to +\infty} \phi(\xi) &= 1. \end{aligned}$$

- Notice that wave speed c enters in singular fashion.
- When $c \neq 0$, this is a functional differential equation of mixed type (MFDE).
- When $c = 0$, this is a difference equation.

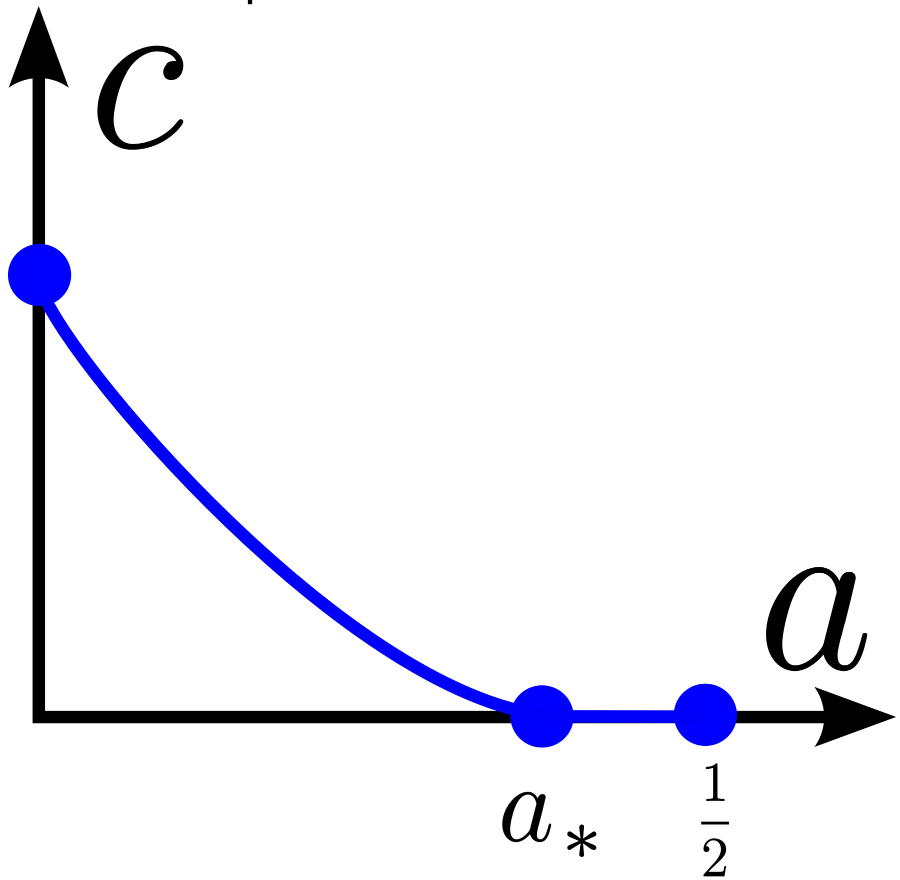
Discrete Nagumo LDE - Propagation failure

Travelling waves for the discrete Nagumo LDE connecting $0 \to 1$.



Propagation

Typical wave speed c versus a plot for discrete reaction-diffusion systems:



In principle, can have $a_* = \frac{1}{2}$ or $a_* < \frac{1}{2}$.

In case $a_* < \frac{1}{2}$, then we say that LDE suffers from propagation failure.

Propagation failure widely studied; pioneered by [Keener].

Signal Propagation: Comparison

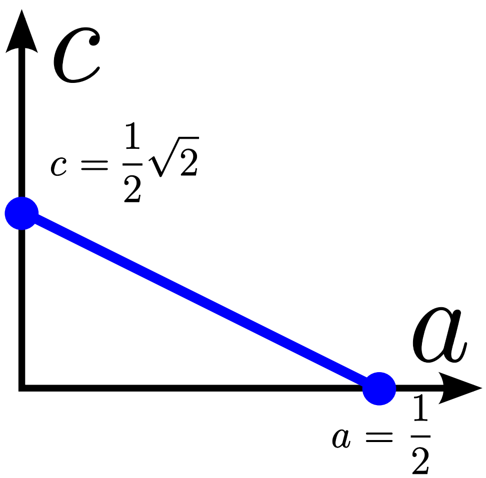
PDE

$$\partial_t u = \partial_{xx} u + g(u, a)$$

Travelling wave $u = \phi(x + ct)$ satisfies:

$$c\phi'(\xi) \quad = \quad \phi''(\xi) + g\big(\phi(\xi); a\big)$$

Travelling waves connecting 0 to 1:



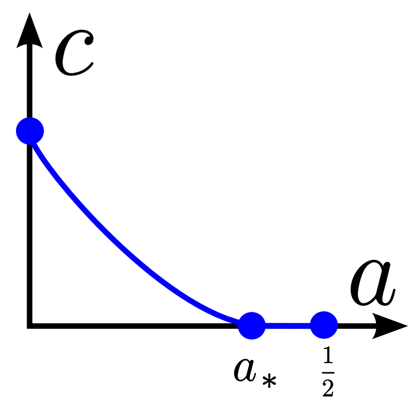
LDE

$$\frac{d}{dt} U_j = U_{j+1} + U_{j-1} - 2U_j + g(U_j; a)$$

Travelling wave $U_j = \phi(j + ct)$ satisfies:

$$\begin{aligned} c\phi'(\xi) \quad = \quad & \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) \\ & + g\big(\phi(\xi); a\big) \end{aligned}$$

Travelling waves connecting 0 to 1:



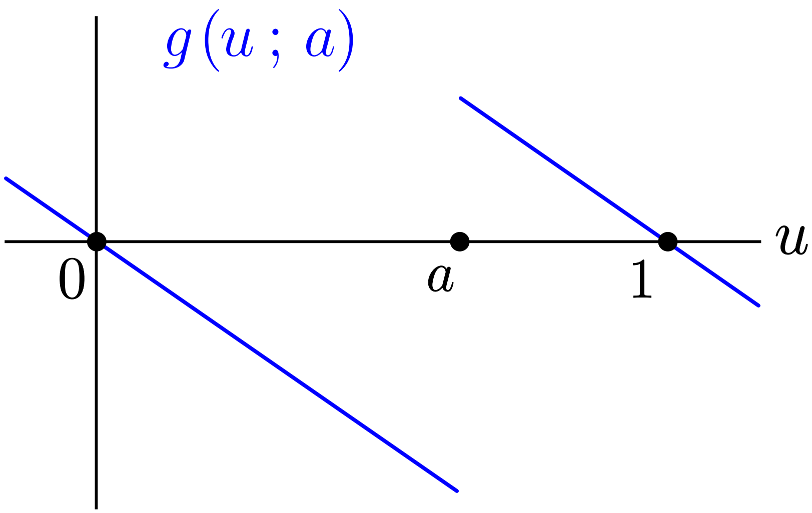
Propagation failure if $a_* < \frac{1}{2}$.

Propagation failure

Consider travelling wave MFDE with saw-tooth nonlinearity

$$\begin{aligned} c\phi'(\xi) &= \frac{1}{h^2}[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] + g\big(\phi(\xi); a\big) \\ \lim_{\xi \to -\infty} \phi(\xi) &= 0, \\ \lim_{\xi \to +\infty} \phi(\xi) &= 1. \end{aligned}$$

Thm. [Cahn, Mallet-Paret, Van Vleck]:
Propagation failure for all $h > 0$ (1999).



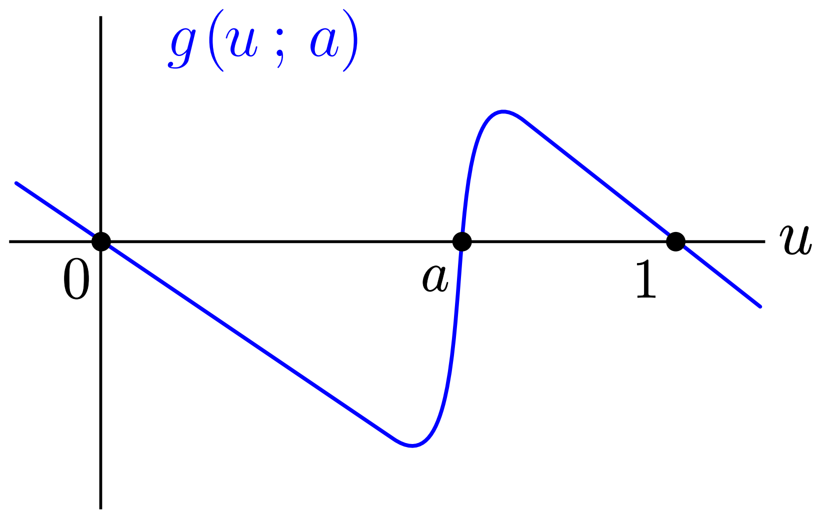
Linear analysis with Fourier series.

Propagation failure

Consider travelling wave MFDE with near-saw-tooth nonlinearity

$$
\begin{aligned}
c\phi'(\xi) &= \tfrac{1}{h^2}[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)] + g\big(\phi(\xi);a\big) \\
\lim_{\xi\to-\infty}\phi(\xi) &= 0, \\
\lim_{\xi\to+\infty}\phi(\xi) &= 1.
\end{aligned}
$$

Thm. [Mallet-Paret]: Propagation failure when g sufficiently close to saw-tooth.



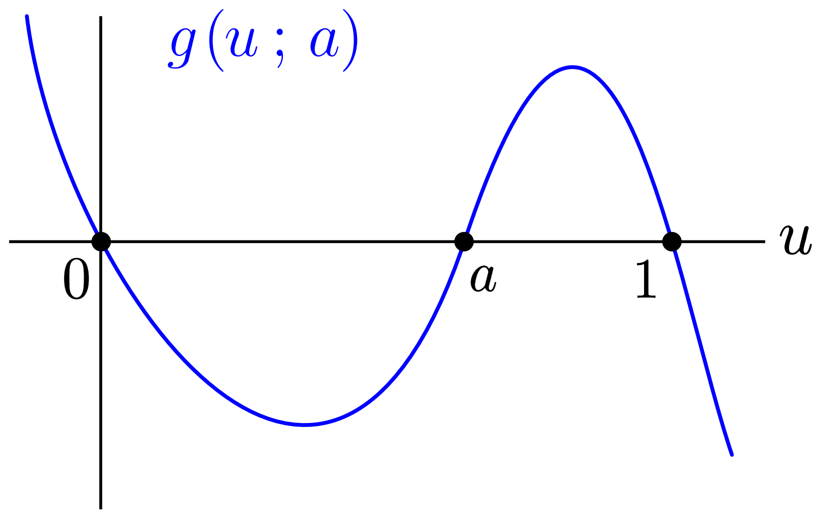
Propagation failure

Consider travelling wave MFDE with generic bistable nonlinearity

$$\begin{aligned} c\phi'(\xi) &= \frac{1}{h^2}[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)] + g\big(\phi(\xi);a\big) \\ \lim_{\xi\to-\infty}\phi(\xi) &= 0, \\ \lim_{\xi\to+\infty}\phi(\xi) &= 1. \end{aligned}$$

Thm. [Hoffman, Mallet-Paret]: Generic condition on g guarantees propagation failure.

Unknown if cubic satisfies this condition for all $h > 0$.

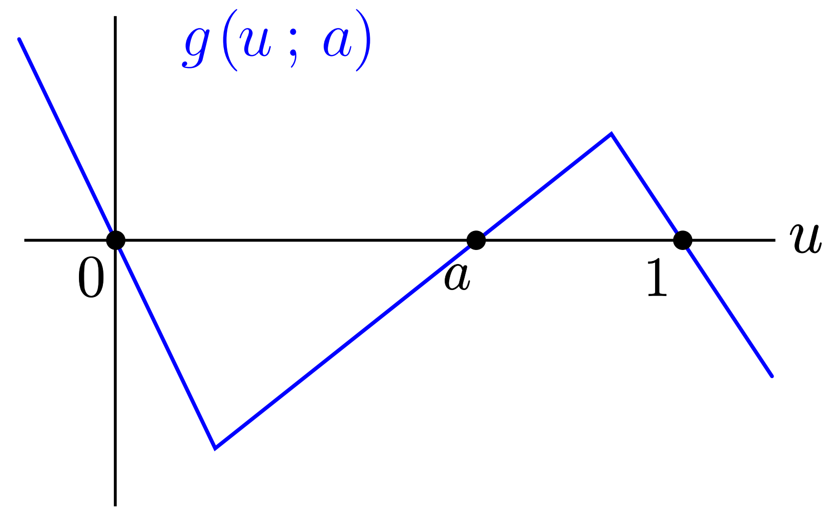


Propagation failure

Consider travelling wave MFDE with zig-zag bistable nonlinearity

$$
\begin{aligned}
c\phi'(\xi) &= \tfrac{1}{h^2}[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)] + g\big(\phi(\xi);a\big) \\
\lim_{\xi\to-\infty}\phi(\xi) &= 0, \\
\lim_{\xi\to+\infty}\phi(\xi) &= 1.
\end{aligned}
$$

Thm. [Elmer]: There exist countably many h for which there is no propagation failure.



Propagation Failure

Recall travelling wave MFDE:

$$\begin{array}{rcl} c\phi'(\xi) & = & \frac{1}{h^2}[\phi(\xi+1)+\phi(\xi-1)-2\phi(\xi)] + g_{\mathrm{cub}}\big(\phi(\xi);a\big) \\ \lim_{\xi\to-\infty}\phi(\xi) & = & 0, \\ \lim_{\xi\to+\infty}\phi(\xi) & = & 1. \end{array}$$

When $c = 0$, can restrict to $\xi \in \mathbb{Z}$: recurrence relation!

With $p_j = \phi(j)$ and $r_j = \phi(j+1)$, we find

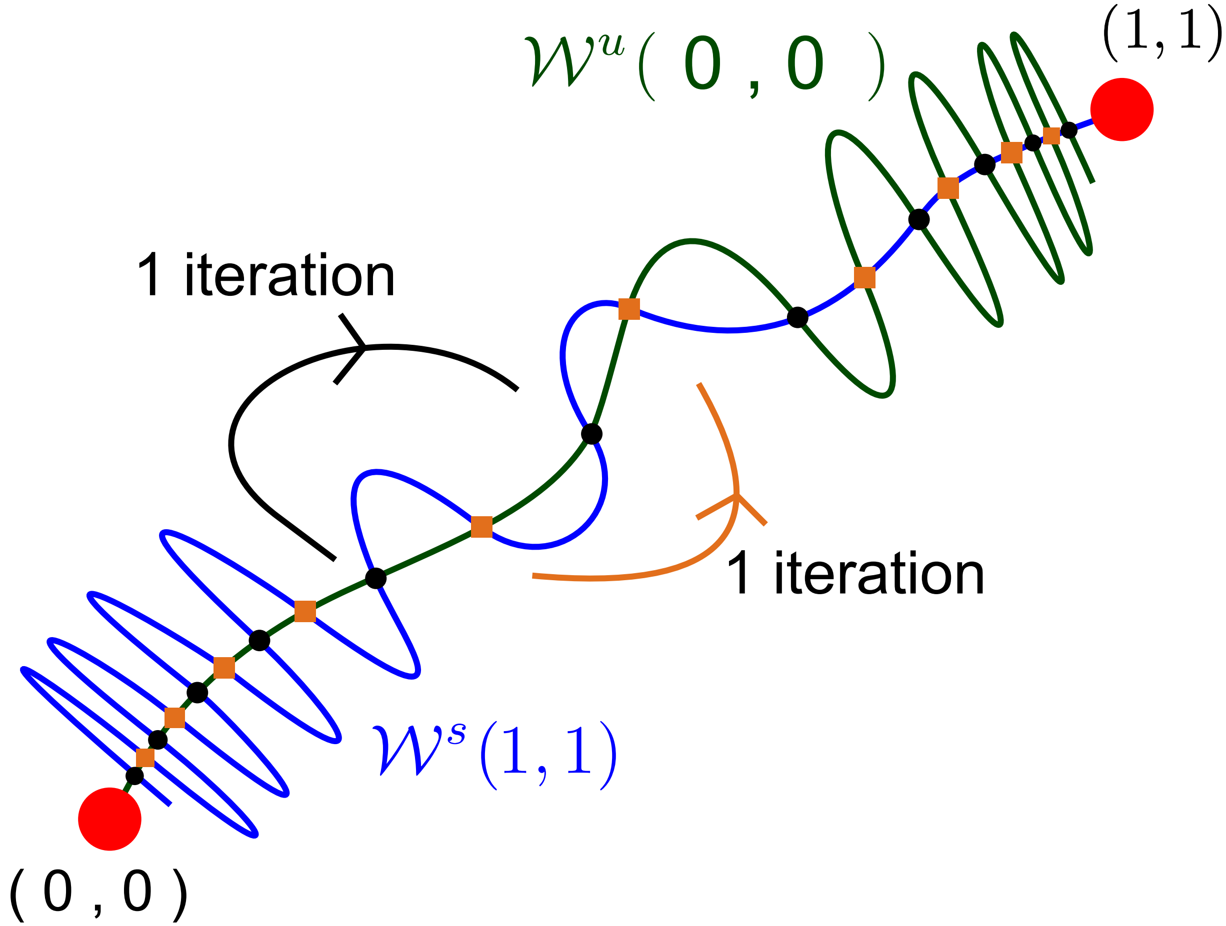
$$\begin{array}{rcl} p_{j+1} & = & r_j \\ r_{j+1} & = & -p_j + 2r_j - h^2 r_j(r_j - a)(1 - r_j). \end{array}$$

Saddles $(0,0)$ and $(1,1)$.

Propagation Failure

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - \alpha^{-1} r_j (r_j - a)(1 - r_j). \end{aligned}$$

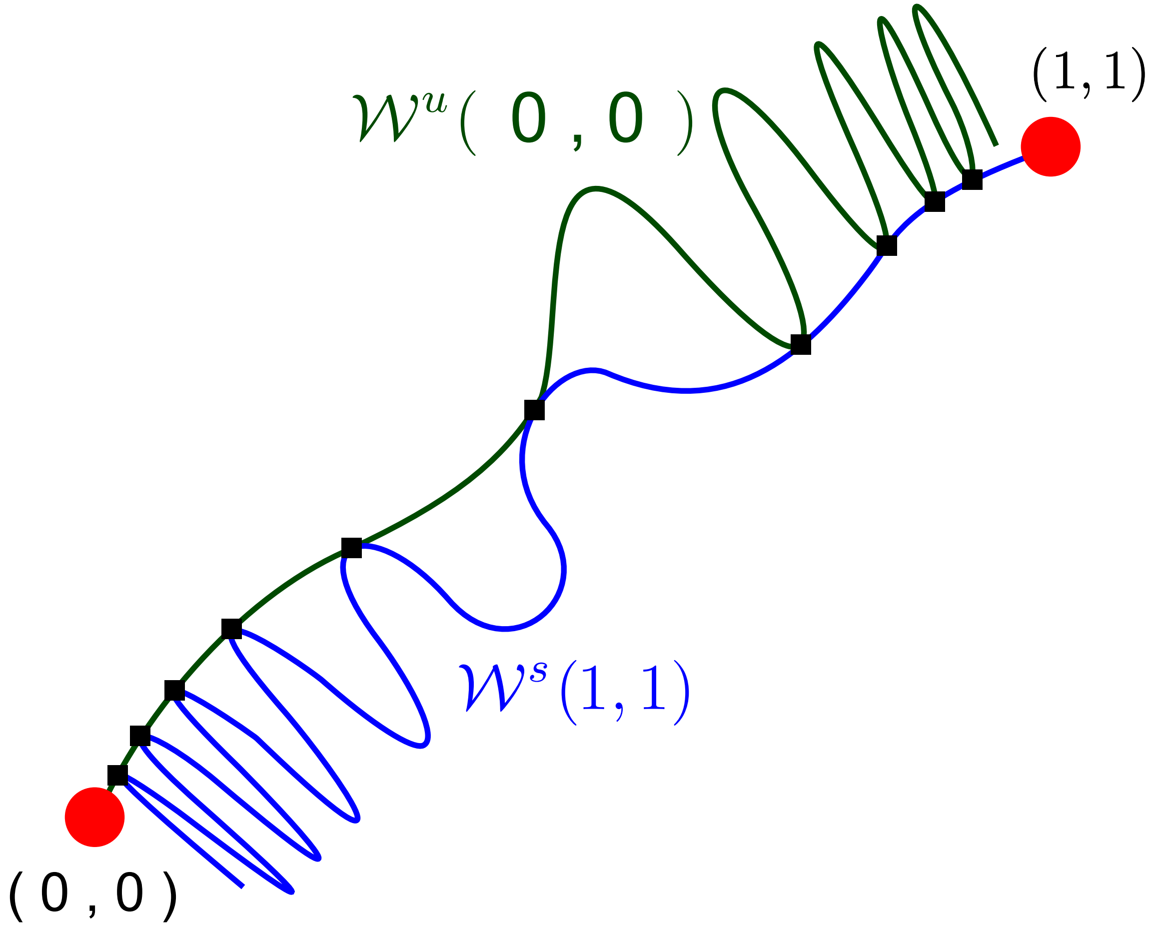
For $a = \frac{1}{2}$, site-centered (orange) and bond-centered (black) solutions. Generically:



Propagation Failure

$$\begin{aligned} p_{j+1} &= r_j \\ r_{j+1} &= -p_j + 2r_j - \alpha^{-1} r_j (r_j - a)(1 - r_j). \end{aligned}$$

Two branches coincide and annihilate at $a = a_*$.



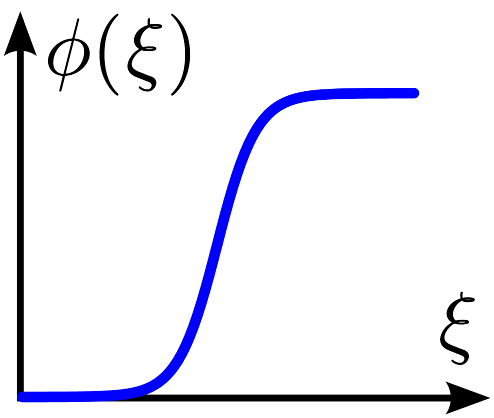
Propagation Failure

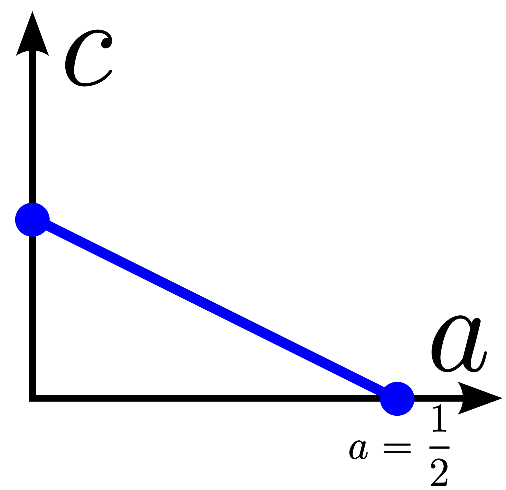
Discretizations of cubic may also involve multiple lattice sites:

$$\frac{d}{dt}U_j = \frac{1}{h^2}[U_{j-1} + U_{j+1} - 2U_j] + \frac{1}{2}U_j(U_{j+1} + U_{j-1} - 2a)(1 - U_j).$$

Explicit solutions available:

$$U_j(t) = \frac{1}{2} + \frac{1}{2}\tanh\Big(\text{arcsinh}(\frac{1}{4}\sqrt{2}\,h)(j + ct)\Big), \qquad c(a) = \frac{(1-2a)}{4\,\text{arcsinh}(\frac{1}{4}\sqrt{2}\,h)}.$$

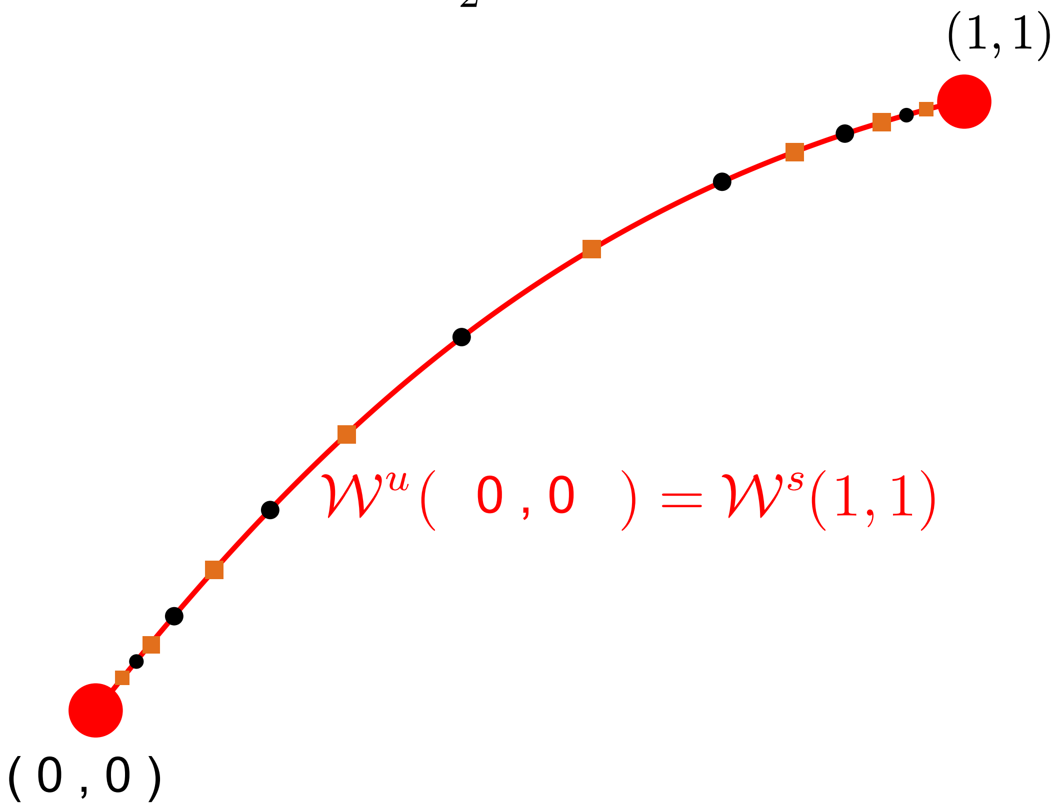




No propagation failure; smooth wave profile.

Propagation Failure - Discrete map

Smooth standing wave profile at $a = \frac{1}{2}$ gives:



Site centered and bond centered solutions now connected by continuous branch of standing waves.

Q: What happens to manifolds when $a \neq \frac{1}{2}$?

Do intersections disappear (no prop failure) or survive (prop failure)?

Lattice point of view

Let us write LDE as:

$$\frac{d}{dt}U(t) = \mathcal{F}(U(t)\,;\,a),$$

with $U(t) \in \ell^\infty$ and $\mathcal{F} : \ell^\infty \times [0,1] \to \ell^\infty$.

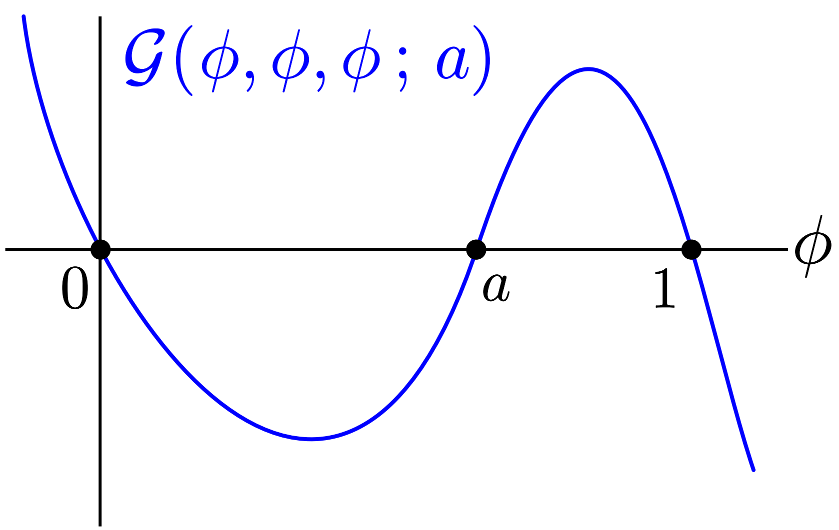
Travelling waves $U_j(t) = \phi(j + ct)$ satisfy some MFDE

$$c\phi'(\xi) = \mathcal{G}\Big(\phi(\xi - 1), \phi(\xi), \phi(\xi + 1)\,;\,a\Big).$$

- Assumption: We have

$$\begin{aligned} \partial_{\phi(\xi-1)}\mathcal{G} &> 0, \\ \partial_{\phi(\xi+1)}\mathcal{G} &> 0. \end{aligned}$$

- Assumption: The function $\mathcal{G}(\phi, \phi, \phi\,;\,a)$ is bistable. In special case $a = \frac{1}{2}$, it is symmetric.



Lattice point of view

Recall LDE as:

$$\frac{d}{dt}U(t) = \mathcal{F}(U(t)\,;\,a),$$

and travelling wave MFDE

$$c\phi'(\xi) = \mathcal{G}\Big(\phi(\xi-1), \phi(\xi), \phi(\xi+1)\,;\,a\Big)$$

Suppose at $a = \frac{1}{2}$ we have a smooth solution $p(\xi)$ to

$$0 = \mathcal{G}\Big(p(\xi-1), p(\xi), p(\xi+1)\,;\,a\Big), \qquad \xi \in \mathbb{R}.$$

Then for every $\vartheta \in \mathbb{R}$, we have equilibrium solution $p^{(\vartheta)} \in \ell^\infty$ to our LDE:

$$\mathcal{F}(p^{(\vartheta)}\,;\,\frac{1}{2}) = 0, \qquad p_j^{(\vartheta)} = p(\vartheta + j).$$

Invariant Manifold

Recall $p^{(\vartheta)} \in \ell^\infty$ with $p_j^{(\vartheta)} = p(\vartheta + j)$.

Notice that

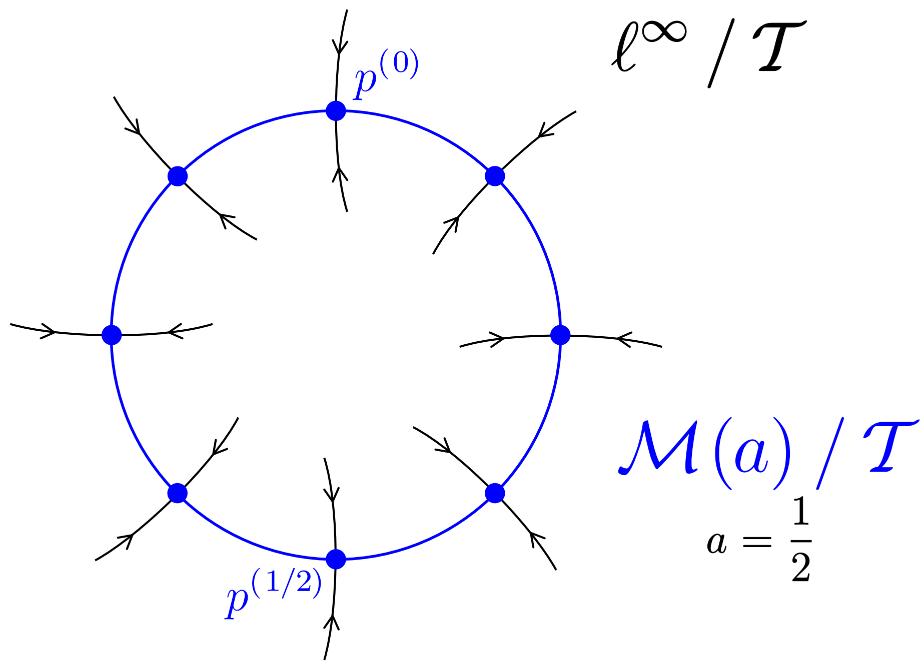
$$p^{(\vartheta)} = \mathcal{T} p^{(\vartheta+1)},$$

where $\mathcal{T} : \ell^\infty \to \ell^\infty$ is right-shift operator $(\mathcal{T}u)_j = u_{j-1}$.

Combining these equilibria gives a smooth manifold

$$\mathcal{M}(a = \frac{1}{2}) = \{p^{(\vartheta)}\}_{\vartheta \in \mathbb{R}}.$$

After dividing out $\mathcal{T}$, we get a ring!

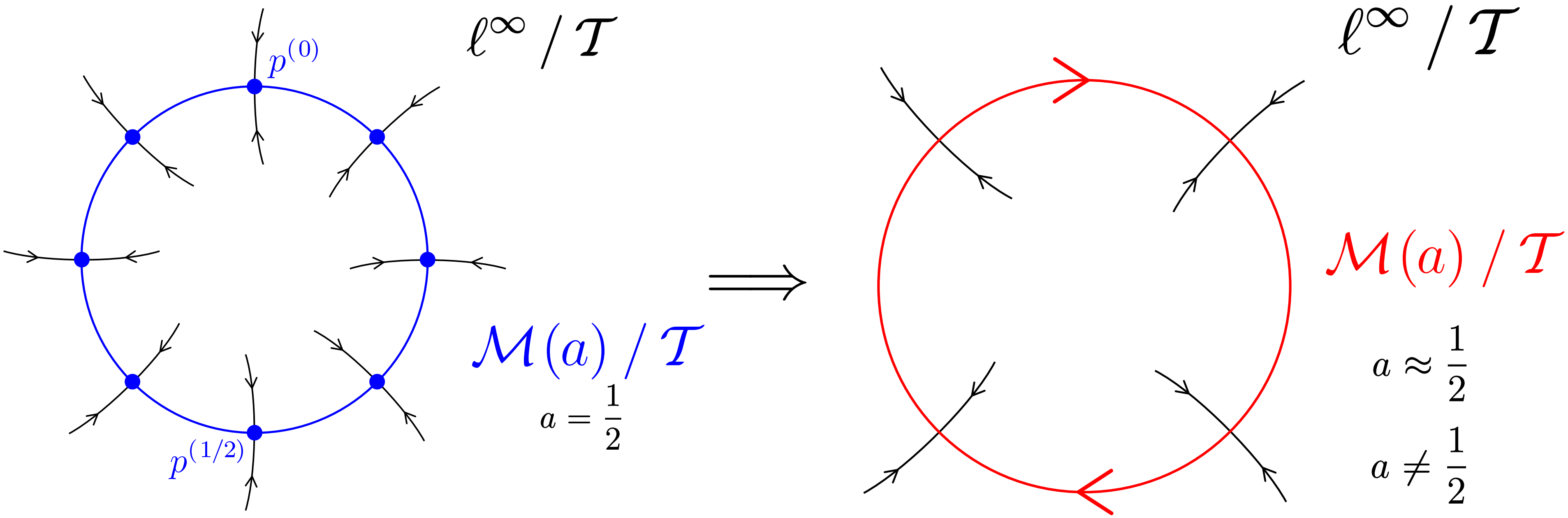


Invariant Manifold - Scenario #1

Based on spectral stability of equilibria $p^{(\vartheta)}$ [Chow, Mallet-Paret, Shen, 1998] and comparison principles can prove:

Prop: The manifold $\mathcal{M}(a = \frac{1}{2})$ is normally hyperbolic.

Possible scenario #1 for persistence of $\mathcal{M}(a)$ with $a \neq \frac{1}{2}$:



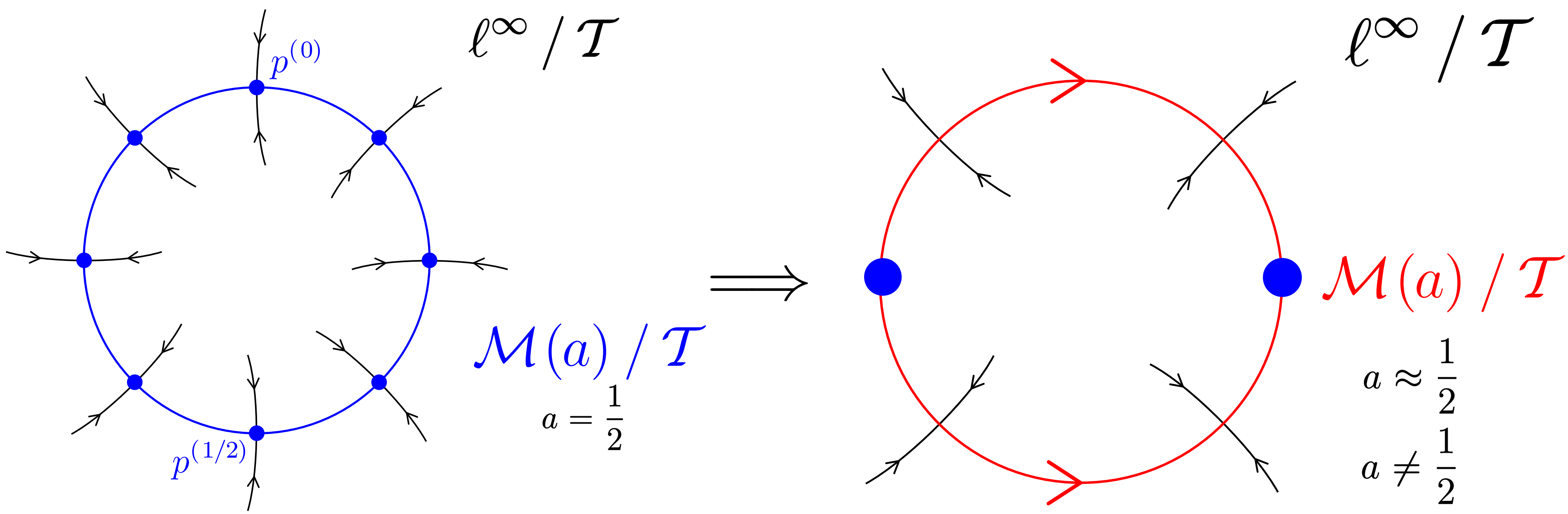
No equilibria survive; $\mathcal{M}(a)$ is orbit of travelling wave. No Propagation Failure.

Invariant Manifold - Scenario #2

Based on spectral stability of equilibria $p^{(\vartheta)}$ [Chow, Mallet-Paret, Shen, 1998] and comparison principles can prove:

Prop: The manifold $\mathcal{M}(a = \frac{1}{2})$ is normally hyperbolic.

Possible scenario #2 for persistence of $\mathcal{M}(a)$ with $a \neq \frac{1}{2}$:



One or more equilibria survive. Propagation Failure*.

*Certain terms and conditions apply...

Dynamics near $\mathcal{M}$

Angular coordinate θ measures position along $\mathcal{M}(a)$. Dynamics given by

$$\frac{d}{dt}\theta = (a - \frac{1}{2})\Psi(\theta) + O\Big(\Big|a - \frac{1}{2}\Big|^2 \Big),$$

in which $\Psi(\theta)$ given by

$$\Psi(\vartheta) = \sum_{j \in \mathbb{Z}} q_j^{(\vartheta)} \partial_a \mathcal{G}\Big(p_{j-1}^{(\vartheta)}, p_j^{(\vartheta)}, p_{j+1}^{(\vartheta)} \,;\, a = \frac{1}{2}\Big).$$

Here $q^{(\vartheta)}$ is adjoint eigenvector; i.e. solves $L^{(\vartheta)*}q^{(\vartheta)} = 0$ with

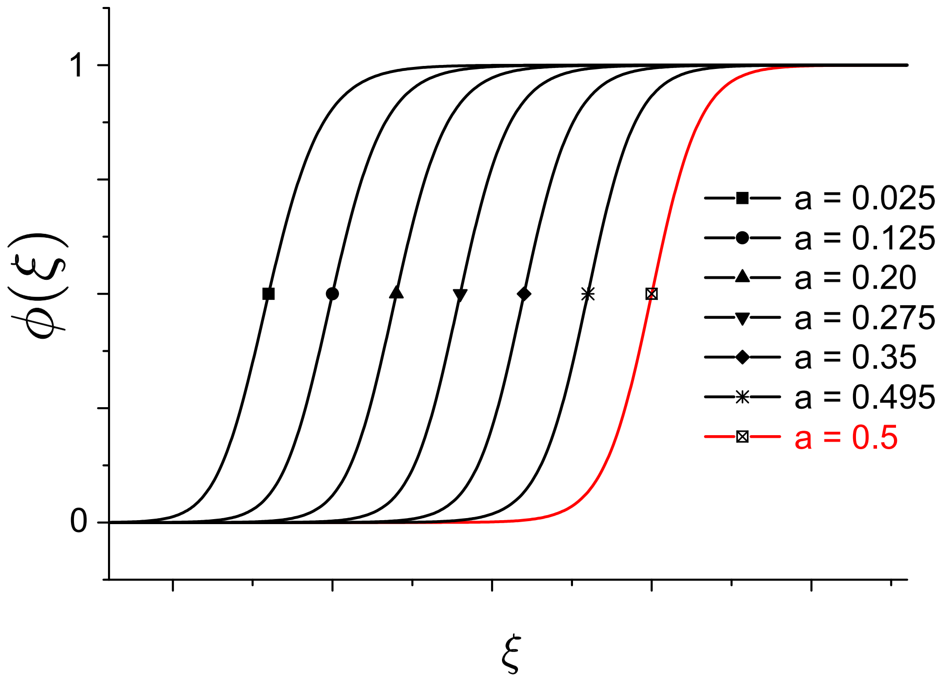
$$\begin{aligned} (L^{(\vartheta)*}w)_j \quad = \quad & \partial_{\phi(\xi-1)}\mathcal{G}\big(p_j^{(\vartheta)}, p_{j+1}^{(\vartheta)}, p_{j+2}^{(\vartheta)} \,;\, \tfrac{1}{2}\big) w_{j+1} \\ & + \partial_{\phi(\xi)}\mathcal{G}\big(p_{j-1}^{(\vartheta)}, p_j^{(\vartheta)}, p_{j+1}^{(\vartheta)} \,;\, \tfrac{1}{2}\big) w_j \\ & + \partial_{\phi(\xi+1)}\mathcal{G}\big(p_{j-2}^{(\vartheta)}, p_{j-1}^{(\vartheta)}, p_j^{(\vartheta)} \,;\, \tfrac{1}{2}\big) w_{j-1}. \end{aligned}$$

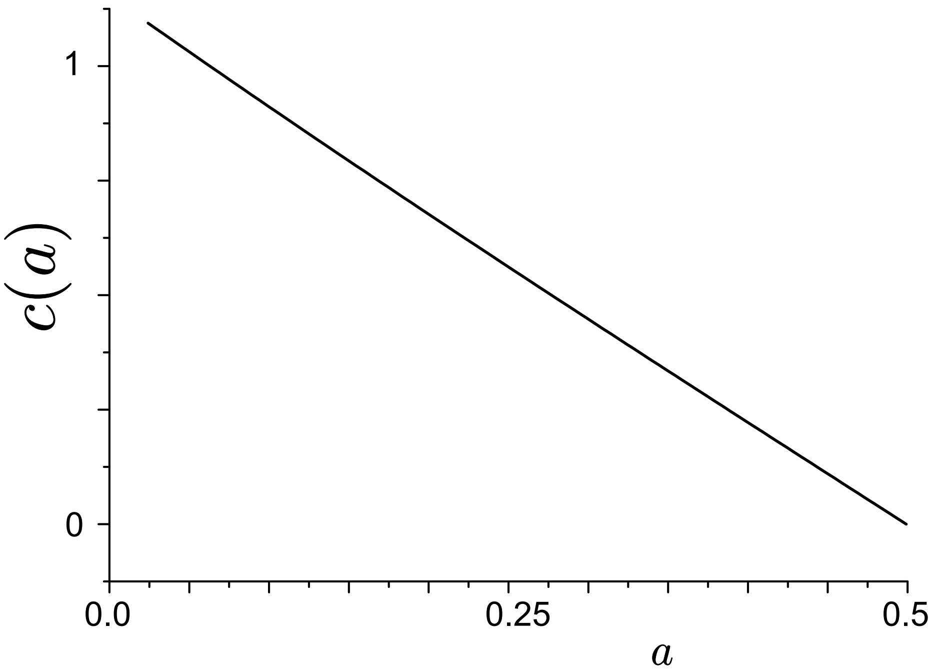
Known: $q_j^{(\vartheta)} > 0$ for all $j \in \mathbb{Z}$ and $\vartheta \in \mathbb{R}$. So $\partial_a \mathcal{G} < 0$ guarantees no prop failure.

Propagation Failure

Thm. [H., Sandstede, Pelinovsky] No prop failure for LDE

$$\frac{d}{dt}u_j \quad = \quad u_{j-1} + u_{j+1} - 2u_j + (u_j - a)\Big(u_{j-1}(1 - u_{j+1}) + u_{j+1}(1 - u_{j-1})\Big)$$

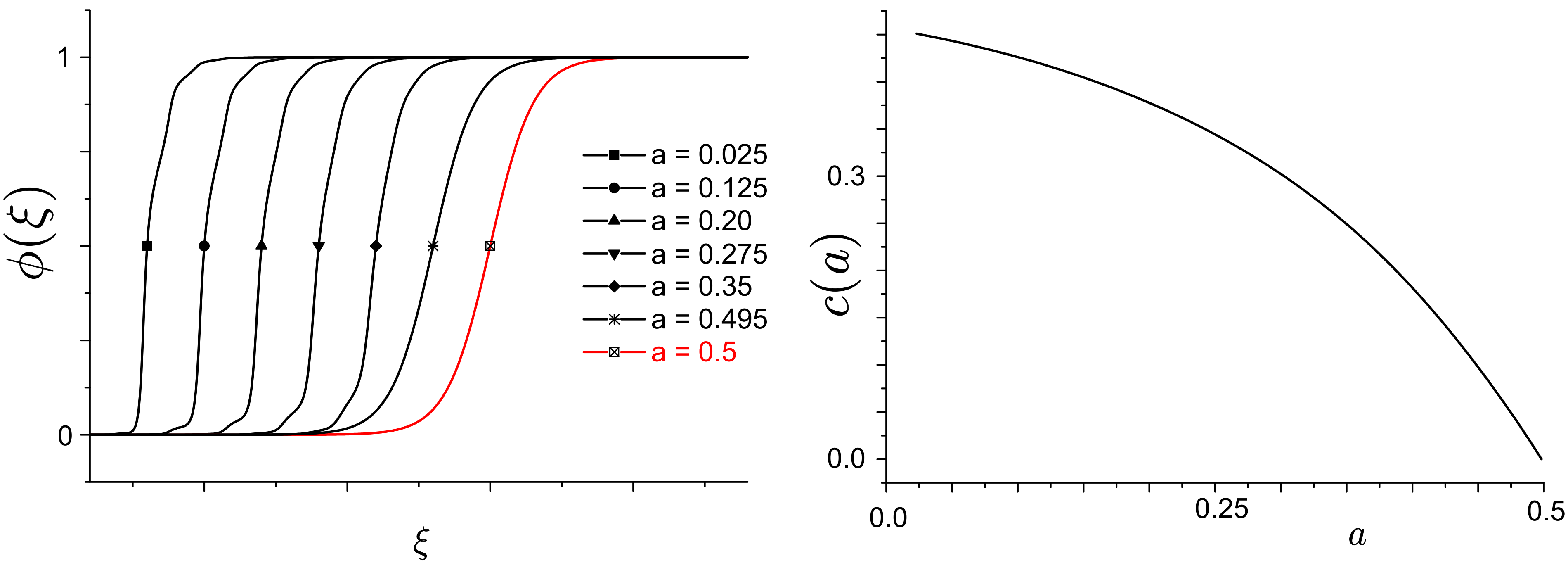




Propagation Failure

Thm. [H., Sandstede, Pelinovsky] No prop failure for LDE

$$\frac{d}{dt}u_j = u_{j-1} + u_{j+1} - 2u_j + (u_j - a)\Big(u_{j-1}(1-u_{j+1}) + u_{j+1}(1-u_{j-1})\Big) - \tfrac{5}{4}(a - \tfrac{1}{2})\sin(2\pi u_j).$$

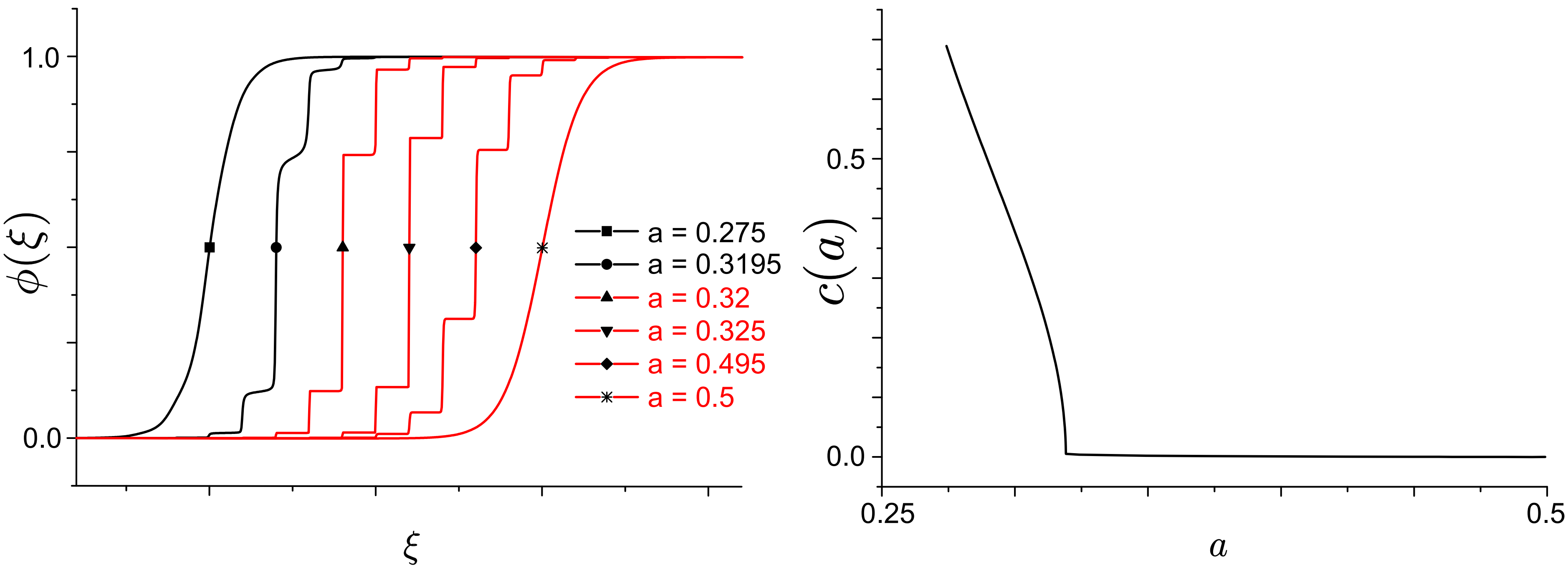


Here $\partial_a \mathcal{G}$ may have both signs, but (numerically) $\Psi(\theta) < 0$ for all θ.

Propagation Failure

Thm. [H., Sandstede, Pelinovsky] Do have prop failure for LDE

$$\frac{d}{dt}u_j = u_{j-1} + u_{j+1} - 2u_j + 4u_j(1-u_j)(u_{j-1}+u_{j+1}-2a)$$
$$-5(a-\tfrac{1}{2})\sin(2\pi u_j)(\tfrac{6}{5}+\tfrac{8}{5}u).$$



Numerically computed: $\Psi(\theta=0) < 0 < \Psi(\theta=\frac{1}{2})$.

Discussion

Recall PDE $u_t = u_{xx} + g(u\,;\,a)$.

- Active interest in multi-lattice-site discretizations of g that admit continuous branch of stationary solutions [Barashenkov, Oxtoby, Pelinovsky, Dmitriev, Kevrekidis, Yoshikawa].

- One generally expects size of propagation failure interval to be exponentially small in h.

- For higher dimensional problems, indications are that using 'small enough' $h > 0$ to reduce influence of propagation failure can hurt numerical performance [Beyn, Speight].