Stability of Travelling Waves for Reaction-Diffusion Equations with Multiplicative Noise

C. H. S. Hamster [a,*], H. J. Hupkes [b],

[a] *Mathematisch Instituut - Universiteit Leiden*
P.O. Box 9512; 2300 RA Leiden; The Netherlands
Email: c.h.s.hamster@math.leidenuniv.nl

[b] *Mathematisch Instituut - Universiteit Leiden*
P.O. Box 9512; 2300 RA Leiden; The Netherlands
Email: hhupkes@math.leidenuniv.nl

Abstract

We consider reaction-diffusion equations that are stochastically forced by a small multiplicative noise term. We show that spectrally stable travelling wave solutions to the deterministic system retain their orbital stability if the amplitude of the noise is sufficiently small.

By applying a stochastic phase-shift together with a time-transform, we obtain a semilinear sPDE that describes the fluctuations from the primary wave. We subsequently develop a semigroup approach to handle the nonlinear stability question in a fashion that is closely related to modern deterministic methods.

AMS 2010 Subject Classification: 35K57, 35R60 .

Key words: travelling waves, stochastic forcing, nonlinear stability, stochastic phase shift.

1 Introduction

In this paper we consider stochastically perturbed versions of a class of reaction-diffusion equations that includes the bistable Nagumo equation

$$u_t = u_{xx} + f_{\mathrm{cub}}(u) \tag{1.1}$$

and the Fitzhugh-Nagumo equation

$$\begin{aligned} u_t &= u_{xx} + f_{\mathrm{cub}}(u) - v, \\ v_t &= v_{xx} + \varrho[u - \gamma v]. \end{aligned} \tag{1.2}$$

Here we take $\varrho > 0$, $\gamma > 0$ and consider the standard bistable nonlinearity

$$f_{\mathrm{cub}}(u) = u(1-u)(u-a). \tag{1.3}$$

*Corresponding author.

It is well-known [14, 34] that (1.1) admits spectrally stable travelling front solutions

$$u(x,t) = \frac{1}{2}\big[1 + \tanh(\frac{1}{2}\sqrt{2}(x - ct))\big] \tag{1.4}$$

that travel with speed

$$c = \sqrt{2}(a - \frac{1}{2}). \tag{1.5}$$

In addition, the existence of travelling pulse solutions to (1.2) with $0 < \varrho \ll 1$ was established recently [7] using variational methods. Using the Maslov index, a proof for the spectral stability of these waves has recently been obtained in [9, 10].

Our main results show that these spectrally stable wave solutions survive in a suitable sense upon adding a small pointwise multiplicative noise term to the underlying PDE. This noise-term is assumed to be globally Lipschitz and must vanish for the asymptotic values of the waves. For example, our results cover the sPDE

$$dU = \big[U_{xx} + f_{\mathrm{cub}}(U)\big]dt + \sigma\chi(U)U(1-U)d\beta_t \tag{1.6}$$

for small $|\sigma|$, in which (β_t) is a Brownian motion and $\chi(U)$ is a cut-off function with $\chi(U) = 1$ for $|U| \leq 2$. The presence of this cut-off is required to enforce the global Lipschitz-smoothness of the noise term. In this regime, one can think of (1.6) as a version of the Nagumo PDE (1.1) where the parameter a is replaced by $a + \sigma\dot{\beta}$.

Noisy patterns Stochastic forcing of PDEs has become an important tool for modellers in a large number of fields, ranging from medical applications such as neuroscience [5, 6] and cardiology [39] to finance [12] and meteorology [15]. While a rather general existence theory for solutions to sPDEs has been developed over the past decades [8, 17, 31, 32], the study of patterns such as stripes, spots and waves in such systems is less well-developed.

Preliminary results for specific equations such as Ginzburg-Landau [4, 16] and Swift-Hohenberg [25] are available. Kuehn and Gowda [18] analyzed both these equations in the linear regime before the onset of the Turing bifurcation. They obtained scaling laws for the natural covariance operators that can be used as early-warnings signs to predict the appearance of patterns.

In addition, several numerical studies have been initiated to study the impact of noise on patterns, see e.g. [28, 35, 38]. The results in [28] relating to (1.6) are particularly interesting from our perspective. Indeed, they clearly show that travelling wave solutions persist under the stochastic forcing, but the speed decreases linearly in σ^2 and the wave becomes steeper.

Rigorous results concerning the impact of stochastic forcing on deterministic waves are still relatively scarce, but some important contributions have already been made. Indeed, Stannat [36, 37] obtained results for a class of systems including (1.6) by coupling an ODE to track the position of the wave via a gradient-descent technique. Similar wave-tracking procedures using sODEs are used in [21, 26], where the authors consider systems with additive noise. Using a multi-scale approach, Lang [26] expands the phase and the shape of the distorted wave in powers of the noise strength on finite time intervals. Comparable results are obtained in [21] using a renormalization method. We emphasize that the latter paper also includes results that are valid on infinite time-scales.

The common feature in all these approaches is that the underlying linear flow is required to be contractive in the directional orthogonal to the translational eigenfunction, which corresponds with the derivative of the travelling wave. While this can be explicitly checked in some situations, such as (1.6) [36], it is typically a challenging task to do so.

Semigroup approach In this paper we take a step towards harnessing the power of modern deterministic nonlinear stability techniques for use in the stochastic setting. In particular, inspired by the informative expository paper [40], we abandon any attempt to describe the phase of the wave via a-priori geometric conditions. Instead, we initiate a semigroup approach based on the stochastic variation of constants formula. This leads to a stochastic evolution equation for the phase that follows naturally from technical considerations. More specifically, we use the phase to neutralize the dangerous non-decaying terms in our evolution equation.

The main advantage of our approach is that it provides orbital stability results without requiring the contractivity condition described above. As a bonus feature, we are also able to obtain exponential stability in special situations where the noise term is specially tailored to the travelling waves. For example, we are able to rigorously understand the changes to the waveprofile and speed that were numerically observed for (1.6) in [28]. In general, if the $\mathbb{R}^n$-orbit of the travelling wave of an n-component reaction-diffusion equation contains no self-intersections, our results allow special forcing terms to be constructed for which the modified waves remain exponentially stable.

However, the need to use stochastic calculus causes several delicate technical complications that are not observed in the deterministic setting. For example, the Ito-isometry is based on L^2 norms. At times, this forces us to square the natural semigroup decay rates, which leads to short-term regularity issues. Indeed, the heat semigroup $S(t)$ behaves as $\|S(t)\|_{\mathcal{L}(L^2;H^1)} \sim t^{-1/2}$, which is in $L^1(0,1)$ but not in $L^2(0,1)$. This precludes us from obtaining supremum control on the H^1-norm of our solutions. Instead, we obtain bounds on square integrals of the H^1-norm. For this reason, we need to carefully track how the cubic behaviour of $f_{\mathrm{cub}}(u)$ propagates through our arguments.

A second major complication is that stochastic phase-shifts lead to extra nonlinear diffusive terms. By contrast, deterministic phase-shifts lead to extra convective terms, which are of lower order and hence less dangerous. As a consequence, we encounter quasi-linear equations in our analysis that do not immediately fit into a semigroup framework. We solve this problem by using a suitable stochastic time-transform to scale out the extra diffusive terms. The fact that we need the diffusion coefficients in (1.2) to be identical is a direct consequence of this procedure.

Outlook Let us emphasize that we view the present paper merely as a proof-of-concept result for a pure semigroup-based approach. For example, we are in the process of removing the severe restriction on the diffusion coefficients of (1.2) by exploiting the block-structure of the semigroup.

In addition, our results here use the variational framework developed by Liu and Rockner [27] in order to ensure that our sPDE has a well-defined global weak solution. In future work, we intend to replace this procedure by constructing local mild solutions directly based on fixed-point arguments.

Finally, we are interested in more delicate spectral stability scenarios, which allow one or more branches of essential spectrum to touch the imaginary axis in a quadratic tangency. Situations of this type are encountered when analyzing the two-dimensional stability of travelling planar waves [3, 19, 20, 24] or when studying viscuous shocks in the context of conservation laws [1, 2, 30].

Organization This paper is organized as follows. We formulate our phase-tracking mechanism and state our main results in §2. In §3 we obtain preliminary estimates on our nonlinearities, which are used in §4 to fit our coupled sPDE into the theory outlined in [27, 32]. This guarantees that our sPDE has well-defined solutions, to which we apply a stochastic phase-shift in §5 followed by a stochastic time-transform in §6. These steps lead to a stochastic variation of constants formula.

In §7 we develop two fixed-point arguments that capture the modifications to the waveprofile and speed that arise from the stochastic forcing. These modifications allow us to obtain suitable estimates on the nonlinearities in the variation of constants formula in §8, which allow us to pursue a nonlinear-stability argument in §9.

Acknowledgements. Hupkes acknowledges support from the Netherlands Organization for Scientific Research (NWO) (grant 639.032.612). Both authors wish to thank O. van Gaans for helpful discussions during the conception and writing of this paper.

2 Main results

In this paper we are interested in the stability of travelling wave solutions to sPDEs of the form

$$dU = \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t. \tag{2.1}$$

Here we take $U = U(x,t) \in \mathbb{R}^n$ with $x \in \mathbb{R}$ and $t \geq 0$.

In §2.1 we formulate several conditions on the nonlinearity f and the diffusion operator A_*, which imply that in the deterministic case $\sigma = 0$ the system (2.1) has a variational structure and admits a spectrally stable travelling wave solution. In §2.2 we impose several standard conditions on the noise term in (2.1), which guarantee that (2.1) is covered by the variational framework developed in [27]. In addition, we couple an extra sODE to our sPDE that will serve as a phase-tracking mechanism. Finally, in §2.3 we formulate our main results concerning the impact of the noise term on the deterministic travelling wave solutions.

2.1 Deterministic setup

We start here by stating our conditions on the form of A_* and f. These conditions require A_* to be a diffusion operator with identical diffusion coefficients and restrict the growth-rate of f to be at most cubic.

(HA) For any $u \in C^2(\mathbb{R};\mathbb{R}^n)$ we have $A_* u = \rho I_n u_{xx}$, in which $\rho > 0$ and I_n is the $n \times n$-identity matrix.

(Hf) We have $f \in C^3(\mathbb{R}^n;\mathbb{R}^n)$ and there exist $u_\pm \in \mathbb{R}^n$ for which $f(u_-) = f(u_+) = 0$. In addition, there exists a constant $K_f > 0$ so that the bound

$$\big|D^3 f(u)\big| \leq K_f \tag{2.2}$$

holds for all $u \in \mathbb{R}^n$.

We now demand that the deterministic part of (2.1) has a travelling wave solution that connects the two equilibria $u_\pm$ (which are allowed to be equal). This travelling wave should approach these equilibria at an exponential rate.

(HTw) There exists a waveprofile $\Phi_0 \in C^2(\mathbb{R};\mathbb{R}^n)$ and a wavespeed $c_0 \in \mathbb{R}$ so that the function

$$u(x,t) = \Phi_0(x - c_0 t) \tag{2.3}$$

satisfies the deterministic PDE

$$u_t = A_* u + f(u) \tag{2.4}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}$. In addition, there is a constant $K > 0$ together with exponents $\nu_\pm > 0$ so that the bound

$$|\Phi_0(\xi) - u_-| + |\Phi_0'(\xi)| \leq K e^{-\nu_- |\xi|} \tag{2.5}$$

holds for all $\xi \leq 0$, while the bound

$$|\Phi_0(\xi) - u_+| + |\Phi_0'(\xi)| \leq K e^{-\nu_+ |\xi|} \tag{2.6}$$

holds for all $\xi \geq 0$.

Throughout this paper, we will use the shorthands

$$L^2 = L^2(\mathbb{R};\mathbb{R}^n), \qquad H^1 = H^1(\mathbb{R};\mathbb{R}^n), \qquad H^2 = H^2(\mathbb{R};\mathbb{R}^n). \tag{2.7}$$

Linearizing the deterministic PDE (2.4) around the travelling wave (Φ_0, c_0), we obtain the linear operator

$$\mathcal{L}_{\mathrm{tw}} : H^2 \to L^2 \tag{2.8}$$

that acts as

$$[\mathcal{L}_{\mathrm{tw}} v](\xi) = c_0 v'(\xi) + [A_* v](\xi) + Df\big(\Phi_0(\xi)\big)v(\xi). \tag{2.9}$$

The formal adjoint

$$\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} : H^2 \to L^2 \tag{2.10}$$

of this operator acts as

$$[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w](\xi) = -c_0 w'(\xi) + [A_* w](\xi) + Df\big(\Phi_0(\xi)\big)w(\xi). \tag{2.11}$$

Indeed, one easily verifies that

$$\langle \mathcal{L}_{\mathrm{tw}} v, w\rangle_{L^2} = \langle v, \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} w\rangle_{L^2} \tag{2.12}$$

whenever $(v, w) \in H^2 \times H^2$. Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the standard inner-product on L^2.

We now impose a standard spectral stability condition on the wave. In particular, we require that the standard translational eigenvalue at zero is a simple eigenvalue. In addition, the remainder of the spectrum of $\mathcal{L}_{\mathrm{tw}}$ must be strictly bounded to the left of the imaginary axis.

(HS) There exists $\beta > 0$ so that the operator $\mathcal{L}_{\mathrm{tw}} - \lambda$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$ that have $\operatorname{Re}\lambda \geq -2\beta$, while $\mathcal{L}_{\mathrm{tw}}$ is a Fredholm operator with index zero. In addition, we have the identities

$$\mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}) = \mathrm{span}\{\Phi_0'\}, \qquad \mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}) = \mathrm{span}\{\psi_{\mathrm{tw}}\} \tag{2.13}$$

for some $\psi_{\mathrm{tw}} \in H^2$ that has

$$\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2} = 1. \tag{2.14}$$

We conclude by imposing a standard monotonicity condition on f, which ensures that the sPDE (2.1) fits into the variational framework of [27]. We remark here that we view this condition purely as a technical convenience, since it guarantees that solutions to (2.1) do not blow up. However, it does not a play a key role in the heart of our computations, where we restrict our attention to solutions that remain small in some sense.

(HVar) There exists $K_{\mathrm{var}} > 0$ so that the one-sided inequality

$$\langle f(u_A) - f(u_B), u_A - u_B\rangle_{\mathbb{R}^n} \leq K_{\mathrm{var}} \left|u_A - u_B\right|^2 \tag{2.15}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

2.2 Stochastic setup

Our first condition here states that the noise term in (2.1) is driven by a standard Brownian motion. Let us emphasize that we made this choice purely to enhance the readability of our arguments. Indeed, our results can easily be generalized to the situation where the noise is driven by Q-Wiener processes.

(Hβ) The process $(\beta_t)_{t\geq 0}$ is a Brownian-motion with respect to the complete filtered probability space

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big). \tag{2.16}$$

We require the function Dg to be globally Lipschitz and uniformly bounded. While the former condition is essential in our analysis to ensure that our cut-offs only depend on L^2-norms, the latter condition is only used to fit (2.1) into the framework of [27].

(Hg) We have $g \in C^2(\mathbb{R}^n; \mathbb{R}^n)$ with $g(u_-) = g(u_+) = 0$. In addition, there is $K_g > 0$ so that

$$|Dg(u)| \leq K_g \tag{2.17}$$

holds for all $u \in \mathbb{R}^n$, while

$$|g(u_A) - g(u_B)| + |Dg(u_A) - Dg(u_B)| \leq K_g \, |u_A - u_B| \tag{2.18}$$

holds for all pairs $(u_A, u_B) \in \mathbb{R}^n \times \mathbb{R}^n$.

We remark here that it is advantageous to view sPDEs as evolutions on Hilbert spaces, since powerful tools are available in this setting. However, in the case where $u_- \neq u_+$, the waveprofile Φ_0 does not lie in the natural statespace L^2. In order to circumvent this problem, we use Φ_0 as a reference function that connects u_- to u_+, allowing us to measure deviations from this function in the Hilbert spaces H^1 and L^2.

In order to highlight this dual role and prevent any confusion, we introduce the duplicate notation

$$\Phi_{\mathrm{ref}} = \Phi_0 \tag{2.19}$$

and emphasize the fact that Φ_{ref} remains fixed in the original frame, unlike the wave-solution (2.3). We also introduce the sets

$$\mathcal{U}_{L^2} = \Phi_{\mathrm{ref}} + L^2, \qquad \mathcal{U}_{H^1} = \Phi_{\mathrm{ref}} + H^1, \qquad \mathcal{U}_{H^2} = \Phi_{\mathrm{ref}} + H^2, \tag{2.20}$$

which we will use as the relevant state-spaces to capture the solutions U to (2.1).

We now set out to append a phase-tracking sODE to (2.1). As a preparation, we introduce the constant

$$K_{\mathrm{ip}} = \big[\, \|g(\Phi_0)\|_{L^2} + 2K_g \big] \, \|\psi_{\mathrm{tw}}\|_{L^2} \, . \tag{2.21}$$

In addition, we pick two C^∞-smooth non-decreasing cut-off functions

$$\chi_{\mathrm{low}} : \mathbb{R} \to [\frac{1}{4}, \infty), \qquad \chi_{\mathrm{high}} : \mathbb{R} \to [-K_{\mathrm{ip}} - 1, K_{\mathrm{ip}} + 1] \tag{2.22}$$

that satisfy the identities

$$\chi_{\mathrm{low}}(\vartheta) = \frac{1}{4} \text{ for } \vartheta \leq \frac{1}{4}, \qquad \chi_{\mathrm{low}}(\vartheta) = \vartheta \text{ for } \vartheta \geq \frac{1}{2}, \tag{2.23}$$

together with

$$\chi_{\rm high}(\vartheta) = \vartheta \text{ for } |\vartheta| \le K_{\rm ip}, \qquad \chi_{\rm high}(\vartheta) = \text{sign}(\vartheta)\big[K_{\rm ip}+1\big] \text{ for } |\vartheta| \ge K_{\rm ip}+1. \tag{2.24}$$

For any $u \in \mathcal{U}_{H^1}$ and $\psi \in H^1$, this allows us to introduce the functions

$$\begin{array}{lcl} b(u,\psi) & = & -\Big[\chi_{\rm low}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1}\chi_{\rm high}\big(\langle g(u),\psi\rangle_{L^2}\big), \\ \kappa_\sigma(u,\psi) & = & 1 + \frac{1}{2\rho}\sigma^2 b(u,\psi)^2. \end{array} \tag{2.25}$$

In addition, for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^1$ we define the expression

$$\mathcal{J}_\sigma(u,c,\psi) \quad = \quad \kappa_\sigma(u,\psi)^{-1}\Big[f(u) + cu' + \sigma^2 b(u,\psi)\partial_\xi[g(u)]\Big], \tag{2.26}$$

while for any $u \in \mathcal{U}_{H^1}$, $c \in \mathbb{R}$ and $\psi \in H^2$ we write

$$a_\sigma(u,c,\psi) \quad = \quad -\kappa_\sigma(u,\psi)\Big[\chi_{\rm low}\big(\langle \partial_\xi u,\psi\rangle_{L^2}\big)\Big]^{-1}\Big[\langle u, A_*\psi\rangle_{L^2} + \langle \mathcal{J}_\sigma(u,c,\psi),\psi\rangle_{L^2}\Big]. \tag{2.27}$$

Finally, we introduce the right-shift operators

$$[T_\gamma u](\xi) = u(\xi-\gamma) \tag{2.28}$$

that act on any function $u : \mathbb{R} \to \mathbb{R}^n$.

With these ingredients in hand, we are ready to introduce the main sPDE that we analyze in this paper. We formally write this sPDE as the coupled system

$$\begin{array}{lcl} dU & = & \big[A_* U + f(U)\big]dt + \sigma g(U) d\beta_t, \\ d\Gamma & = & \big[c + a_\sigma\big(U, c, T_\Gamma \psi_{\rm tw}\big)\big]dt + \sigma b\big(U, T_\Gamma\psi_{\rm tw}\big)\, d\beta_t, \end{array} \tag{2.29}$$

noting that we seek solutions with $\big(U(t),\Gamma(t)\big) \in \mathcal{U}_{H^1}\times\mathbb{R}$.

In order to make this precise, we introduce the spaces

$$\begin{array}{lcl} \mathcal{N}^2\big([0,T];(\mathcal{F}_t);\mathcal{H}\big) & = & \{X \in L^2\big([0,T]\times\Omega; dt\otimes\mathbb{P};\mathcal{H}\big) : \\ & & \qquad X \text{ has a } (\mathcal{F}_t)\text{-progressively measurable version}\}, \end{array} \tag{2.30}$$

where we allow $\mathcal{H} \in \{\mathbb{R}, L^2, H^1\}$. We note that we follow the convention of [32, 33] here by requiring progressive measurability instead of the usual stronger notion of predictability. Since we are exclusively dealing with Brownian motions, this choice suffices to construct stochastic integrals.

Our first result clarifies what we mean by a solution to (2.29). We note that (i) and (ii) in proposition 2.1 imply that (X,Γ) is an $L^2\times\mathbb{R}$-valued continuous $(\mathcal{F}_t)$-adapted process. We remark that in the integral equation (2.40) we interpret the diffusion operator A_* as an element of $\mathcal{L}(H^1;H^{-1})$, where H^{-1} is the dual of H^1 under the standard embeddings

$$H^1 \hookrightarrow L^2 \cong \big[L^2\big]^* \hookrightarrow H^{-1} = \big[H^1\big]^*. \tag{2.31}$$

We note that the set (H^1, L^2, H^{-1}) is commonly referred to as a Gelfand triple; see e.g. [13, §5.9] for a more detailed explanation. For $(v,w) \in H^{-1}\times H^1$ we write $\langle v,w\rangle_{H^{-1};H^1}$ to refer to the duality pairing between H^1 and H^{-1}. If in fact $v \in L^2$, then we have

$$\langle v,w\rangle_{H^{-1};H^1} = \langle v,w\rangle_{L^2}. \tag{2.32}$$

Proposition 2.1 (see §4). *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied and fix $T > 0$, $c \in \mathbb{R}$ and $0 \le \sigma \le 1$. In addition, pick an initial condition*

$$(X_0, \Gamma_0) \in L^2 \times \mathbb{R}. \tag{2.33}$$

Then there are maps

$$X : [0,T] \times \Omega \to L^2, \qquad \Gamma : [0,T] \times \Omega \to \mathbb{R} \tag{2.34}$$

that satisfy the following properties.

(i) For almost all $\omega \in \Omega$, the map

$$t \mapsto \big(X(t,\omega), \Gamma(t,\omega)\big) \tag{2.35}$$

is of class $C([0,T]; L^2 \times \mathbb{R})$.

(ii) For all $t \in [0,T]$, the map

$$\omega \mapsto \big(X(t,\omega), \Gamma(t,\omega)\big) \in L^2 \times \mathbb{R} \tag{2.36}$$

is $(\mathcal{F}_t)$-measurable.

(iii) We have the inclusion

$$X \in L^6\big(\Omega, \mathbb{P}; C([0,T]; L^2)\big), \tag{2.37}$$

together with

$$\begin{array}{lcl} X & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big), \\ \Gamma & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); \mathbb{R}\big) \end{array} \tag{2.38}$$

and

$$\begin{array}{lcl} g(X + \Phi_{\mathrm{ref}}) & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); L^2\big), \\ b\big(X + \Phi_{\mathrm{ref}}, T_{\Gamma}\psi_{\mathrm{tw}}\big) & \in & \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); \mathbb{R}\big). \end{array} \tag{2.39}$$

(iv) For almost all $\omega \in \Omega$, the identities

$$\begin{array}{lcl} X(t) & = & X_0 + \int_0^t A_*[X(s) + \Phi_{\mathrm{ref}}]\, ds + \int_0^t f\big(X(s) + \Phi_{\mathrm{ref}}\big)\, ds \\ & & \quad + \sigma \int_0^t g\big(X(s) + \Phi_{\mathrm{ref}}\big) d\beta_s \end{array} \tag{2.40}$$

and

$$\begin{array}{lcl} \Gamma(t) & = & \Gamma_0 + \int_0^t \big[c + a_\sigma\big(X(s) + \Phi_{\mathrm{ref}}, c, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big)\big]\, ds \\ & & \quad + \sigma \int_0^t b\big(X(s) + \Phi_{\mathrm{ref}}, T_{\Gamma(s)}\psi_{\mathrm{tw}}\big) d\beta_s \end{array} \tag{2.41}$$

hold for all $0 \le t \le T$.

(v) Suppose that the pair $(\tilde{X}, \tilde{\Gamma}) : [0,T] \times \Omega \to L^2 \times \mathbb{R}$ also satisfies (i)-(iv). Then for almost all $\omega \in \Omega$, we have

$$(\tilde{X}, \tilde{\Gamma})(t) = (X, \Gamma)(t) \qquad \text{for all } 0 \le t \le T. \tag{2.42}$$

2.3 Wave stability

By inserting the travelling wave Ansatz (2.3) into the deterministic PDE (2.4), we observe that

$$A_*\Phi_0 + \mathcal{J}_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0, \tag{2.43}$$

which means that $a_0(\Phi_0, c_0, \psi_{\mathrm{tw}}) = 0$. Our first result here shows that this can be extended into a branch of profiles and speeds for which

$$a_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0. \tag{2.44}$$

Roughly speaking, this means that the adjusted phase $\Gamma(t) - ct$ will (instantaneously) feel only stochastic forcing if one takes $c = c_\sigma$ and $U = T_{\Gamma(t)}\Phi_\sigma$ in (2.29).

Proposition 2.2 (see §7). *Suppose that* (HA), (Hf), (HTw), (HS) *and* (Hg) *are all satisfied and pick a sufficiently large constant* $K > 0$. *Then there exists* $\delta_\sigma > 0$ *so that for every* $0 \le \sigma \le \delta_\sigma$, *there is a unique pair*

$$(\Phi_\sigma, c_\sigma) \in \mathcal{U}_{H^2} \times \mathbb{R} \tag{2.45}$$

that satisfies the system

$$A_*\Phi_\sigma + \mathcal{J}_\sigma\big(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}\big) = 0 \tag{2.46}$$

and admits the bound

$$\|\Phi_\sigma - \Phi_0\|_{H^2} + |c_\sigma - c_0| \le K\sigma^2. \tag{2.47}$$

We are interested in solutions to (2.29) with an initial condition for U that is close to Φ_σ. We will use the remaining degree of freedom to pick the initial phase Γ in such a way that the orthogonality condition described in the following result is enforced.

Proposition 2.3 (see §7). *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied. Then there exist constants* $\delta_0 > 0$, $\delta_\sigma > 0$ *and* $K > 0$ *so that the following holds true. For every* $0 \le \sigma \le \delta_\sigma$ *and any* $u_0 \in \mathcal{U}_{L^2}$ *that has*

$$\|u_0 - \Phi_\sigma\|_{L^2} < \delta_0, \tag{2.48}$$

there exists $\gamma_0 \in \mathbb{R}$ *for which the function*

$$v_{\gamma_0} = T_{-\gamma_0}[u_0] - \Phi_\sigma \tag{2.49}$$

satisfies the identity

$$\langle v_{\gamma_0}, \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{2.50}$$

together with the bound

$$|\gamma_0| + \|v_{\gamma_0}\|_{L^2} \le K\,\|u_0 - \Phi_\sigma\|_{L^2}\,. \tag{2.51}$$

If in fact $u_0 \in \mathcal{U}_{H^1}$, *then we also have the estimate*

$$|\gamma_0| + \|v_{\gamma_0}\|_{H^1} \le K\,\|u_0 - \Phi_\sigma\|_{H^1}\,. \tag{2.52}$$

Let us now pick any $u_0 \in \mathcal{U}_{H^1}$ for which (2.48) holds. We write (X_{u_0}, Γ_{u_0}) for the process described in Proposition 2.1 with the initial condition

$$(X_0, \Gamma_0) = \big(u_0 - \Phi_{\mathrm{ref}}, \gamma_0\big), \tag{2.53}$$

in which γ_0 is the initial phase defined in Proposition 2.3. We then define the process

$$V_{u_0}(t) = T_{-\Gamma_{u_0}(t)}\big[X_{u_0}(t) + \Phi_{\mathrm{ref}}\big] - \Phi_\sigma, \tag{2.54}$$

which can be thought of as the deviation of the solution U of (2.29) from the stochastic wave Φ_σ shifted to the position $\Gamma_{u_0}(t)$.

In order to measure the size of the perturbation, we pick $\varepsilon > 0$ and introduce the scalar function

$$N_{\varepsilon;u_0}(t) = \|V_{u_0}(t)\|^2_{L^2} + \int_0^t e^{-\varepsilon(t-s)}\, \|V_{u_0}(s)\|^2_{H^1}\, ds. \tag{2.55}$$

For each $T > 0$ we now define a probability

$$p_\varepsilon(T, \eta, u_0) = P\Big(\sup_{0 \le t \le T} N_{\varepsilon;u_0}(t) > \eta\Big). \tag{2.56}$$

Our first main result shows that the probability that $N_{\varepsilon;u_0}$ remains small can be pushed arbitrarily close to one by restricting the strength of the noise and the size of the initial perturbation.

Theorem 2.4 (see §9). *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied and pick sufficiently small constants $\varepsilon > 0$, $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for every $0 \le \sigma \le \delta_\sigma$, any $u_0 \in \mathcal{U}_{H^1}$ that satisfies (2.48), any $0 < \eta \le \delta_\eta$ and any $T > 0$, we have the inequality*

$$p_\varepsilon(T, \eta, u_0) \le \eta^{-1} K\Big[\, \|u_0 - \Phi_\sigma\|^2_{H^1} + \sigma^2\Big]. \tag{2.57}$$

Our second main result concerns the special case where the noise pushes the stochastic wave Φ_σ in a rigid fashion. This is the case when

$$g(\Phi_0) = \vartheta_0 \Phi_0' \tag{2.58}$$

for some proportionality constant $\vartheta_0 \in \mathbb{R}$. It is easy to verify that (2.58) with $\vartheta_0 = -\sqrt{2}$ holds for (1.6).

Lemma 2.5. *Consider the setting of Proposition 2.2 and suppose that (2.58) holds. Then for all sufficiently small $0 \le \sigma \le \delta_\sigma$ we have the identities*

$$\begin{aligned} \Phi_\sigma(\xi) &= \Phi_0\Big(\big[1 + \tfrac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}\xi\Big), \\ c_\sigma &= \big[1 + \tfrac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2} c_0, \end{aligned} \tag{2.59}$$

together with

$$g(\Phi_\sigma) = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{-1/2}\vartheta_0\Phi_\sigma' = -b\big(\Phi_\sigma, \psi_{\mathrm{tw}}\big)\Phi_\sigma'. \tag{2.60}$$

Proof. For convenience, we introduce the notation

$$\alpha_\sigma = \big[1 + \frac{1}{2\rho}\sigma^2\vartheta_0^2\big]^{1/2}. \tag{2.61}$$

One easily verifies the identities

$$\Phi_\sigma'(\xi) = \alpha_\sigma \Phi_0'(\alpha_\sigma \xi), \qquad \Phi_\sigma''(\xi) = \alpha_\sigma^2 \Phi_0''(\alpha_\sigma \xi), \tag{2.62}$$

which yields

$$g\big(\Phi_\sigma(\xi)\big) = g\big(\Phi_0(\alpha_\sigma\xi)\big) = \vartheta_0 \Phi_0'\big(\alpha_\sigma \xi\big) = \vartheta_0 \alpha_\sigma^{-1} \Phi_\sigma'(\xi), \tag{2.63}$$

together with

$$f(\Phi_\sigma) + c_\sigma \Phi_\sigma' = -\alpha_\sigma^{-2} A_* \Phi_\sigma. \tag{2.64}$$

Since the cut-off functions in the definition of b act as the identity for small $\sigma \geq 0$, we obtain

$$\begin{array}{rcl} b(\Phi_\sigma, \psi_{\mathrm{tw}}) & = & -\vartheta_0 \alpha_\sigma^{-1}, \\ \kappa_\sigma(\Phi_\sigma, \psi_{\mathrm{tw}}) & = & 1 + \frac{1}{2\rho} \vartheta_0^2 \alpha_\sigma^{-2}, \end{array} \tag{2.65}$$

which implies

$$\begin{array}{rcl} \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{tw}) & = & [1 + \frac{1}{2\rho}\sigma^2 \vartheta_0^2 \alpha_\sigma^{-2}]^{-1} \big[f(\Phi_\sigma) + c_\sigma \Phi_\sigma' - \sigma^2 \vartheta_0^2 \alpha_\sigma^{-2} \Phi_\sigma'' \big] \\ & = & -[1 + \frac{1}{2\rho}\sigma^2 \vartheta_0^2 \alpha_\sigma^{-2}]^{-1} \big[\alpha_\sigma^{-2} A_* \Phi_\sigma + \frac{\sigma^2}{\rho} \vartheta_0^2 \alpha_\sigma^{-2} A_* \Phi_\sigma \big] \\ & = & -[\alpha_\sigma^2 + \frac{1}{2\rho}\sigma^2 \vartheta_0^2]^{-1} \big[1 + \frac{\sigma^2}{\rho} \vartheta_0^2 \big] A_* \Phi_\sigma \\ & = & -A_* \Phi_\sigma. \end{array} \tag{2.66}$$

The claims now follow from the uniqueness statement in Proposition 2.2. □

In this setting we expect the perturbation V to decay exponentially with large probability. In order to formalize this, we pick small constants $\varepsilon > 0$ and $\alpha > 0$ and introduce the scalar function

$$N_{\varepsilon,\alpha;u_0}(t) = e^{\alpha t} \left\| V_{u_0}(t) \right\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \left\| V_{u_0}(s) \right\|_{H^1}^2 \, ds, \tag{2.67}$$

together with the associated probabilities

$$p_{\varepsilon,\alpha}(T, \eta, u_0) = P\Big(\sup_{0 \leq t \leq T} N_{\varepsilon,\alpha;u_0}(t) > \eta \Big). \tag{2.68}$$

Theorem 2.6 (see §9). *Suppose that (HA), (Hf), $(HVar)$, (HTw), (HS), (Hg) and $(H\beta)$ are all satisfied . Suppose furthermore that (2.58) holds and pick sufficiently small constants $\varepsilon > 0$, $\alpha > 0$, $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for every $0 \leq \sigma \leq \delta_\sigma$, any $u_0 \in \mathcal{U}_{H^1}$ that satisfies (2.48) any $0 < \eta \leq \delta_\eta$ and any $T > 0$, we have the inequality*

$$p_{\varepsilon,\alpha}(T, \eta, u_0) \leq \eta^{-1} K \left\| u_0 - \Phi_\sigma \right\|_{H^1}^2. \tag{2.69}$$

3 Preliminary estimates

In this section we derive several preliminary estimates for the functions f, g, $\mathcal{J}_0$, b and κ_σ. We will write the arguments $(u, \overline{c}) \in \mathcal{U}_{H^1} \times \mathbb{R}$ as

$$u = \Phi + v, \qquad \overline{c} = c + d, \tag{3.1}$$

in which we take $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. We do not restrict ourselves to the case where $(\Phi, c) = (\Phi_0, c_0)$, but impose the following condition.

(hPar) The conditions (HTw) and (HS) hold and the pair $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ satisfies the bounds

$$\|\Phi - \Phi_0\|_{H^1} \leq \min\{1, [4\,\|\psi_{\mathrm{tw}}\|_{L^2}]^{-1}\}, \qquad |c - c_0| \leq 1. \tag{3.2}$$

In §3.1 we obtain global and Lipschitz bounds for the functions f and g. These bounds are subsequently used in §3.2 to analyze the auxilliary functions $\mathcal{J}_0$, b and κ_σ. Throughout this paper we use the convention that all numbered constants appearing in proofs are strictly positive and have the same dependencies as the constants appearing in the statement of the result.

3.1 Bounds on f and g

The conditions (Hf) and (Hg) allow us to obtain standard cubic bounds on f and globally Lipschitz bounds on g. We also consider expressions of the form $\partial_\xi g(u)$, which give rise to quadratic bounds.

Lemma 3.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{lcl} \|f(\Phi + v)\|_{L^2} & \leq & K\big[1 + \|v\|_{H^1}^2 \, \|v\|_{L^2}\big], \\ |\langle f(\Phi + v), \psi\rangle_{L^2}| & \leq & K\big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2\big] \, \|\psi\|_{H^1}\,, \end{array} \tag{3.3}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{lcl} \Delta_{AB} f & = & f(\Phi + v_A) - f(\Phi + v_B), \\ \Delta_{AB}\langle f, \cdot\rangle_{L^2} & = & \langle f(\Phi + v_A), \psi_A\rangle_{L^2} - \langle f(\Phi + v_B), \psi_B\rangle_{L^2} \end{array} \tag{3.4}$$

satisfy the estimates

$$\begin{array}{lcl} \|\Delta_{AB} f\|_{L^2} & \leq & K\,\|v_A - v_B\|_{L^2} \\ & & \quad +K\Big(\,\|v_A\|_{H^1}\,\|v_A\|_{L^2} + \|v_B\|_{H^1}\,\|v_B\|_{L^2}\,\Big)\,\|v_A - v_B\|_{H^1}\,, \\ |\Delta_{AB}\langle f, \cdot\rangle_{L^2}| & \leq & K\,\|v_A - v_B\|_{L^2}\,\|\psi_A\|_{H^1} \\ & & \quad +K\,\|v_A - v_B\|_{H^1}\,\big(\,\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\,\big)\,\|\psi_A\|_{H^1} \\ & & \quad +K\Big[1 + \|v_B\|_{H^1}\,\|v_B\|_{L^2}^2\Big]\,\|\psi_A - \psi_B\|_{H^1}\,. \end{array} \tag{3.5}$$

Proof. Exploiting (Hf) we obtain

$$\big|D^2 f(u)\big| \leq C_1[1 + |u|], \tag{3.6}$$

together with

$$|Df(u)| \leq C_1[1 + |u|^2] \tag{3.7}$$

for all $u \in \mathbb{R}^n$. In particular, (hPar) yields the pointwise Lipschitz bound

$$|f(\Phi + v_A) - f(\Phi + v_B)| \leq C_2[1 + |v_A|^2 + |v_B|^2]\,|v_A - v_B|\,. \tag{3.8}$$

Using the Sobolev embedding $\|\cdot\|_\infty \leq C_3\,\|\cdot\|_{H^1}$ this immediately implies the first estimate in (3.5). Applying this estimate with $v_A = 0$ and $v_B = \Phi_0 - \Phi$ we find

$$\begin{array}{lcl} \|f(\Phi)\|_{L^2} & \leq & \|f(\Phi_0)\|_{L^2} + \|f(\Phi) - f(\Phi_0)\|_{L^2} \\ & \leq & C_4. \end{array} \tag{3.9}$$

Exploiting

$$\|f(\Phi + v)\|_{L^2} \le \|f(\Phi)\|_{L^2} + \|f(\Phi + v) - f(\Phi)\|_{L^2}\,, \tag{3.10}$$

we hence obtain

$$\|f(\Phi + v)\|_{L^2} \quad \le \quad C_5\big[1 + \|v\|_{L^2} + \|v\|_{H^1}^2 \, \|v\|_{L^2}\,\big]. \tag{3.11}$$

The first estimate in (3.3) now follows by noting that $\|v\|_{L^2} \le \|v\|_{H^1}^2 \, \|v\|_{L^2}$ for $\|v\|_{L^2} \ge 1$.

Turning to the inner products, (3.8) allows us to compute

$$\begin{aligned} |\langle f(\Phi + v_A) - f(\Phi + v_B), \psi_A\rangle_{L^2}| \quad &\le \quad C_2 \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{L^2} \\ &\qquad + C_2\big[\, \|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2 \,\big] \, \|v_A - v_B\|_{H^1} \, \|\psi_A\|_{H^1}\,. \end{aligned} \tag{3.12}$$

Exploiting (3.9), the second estimate in (3.3) hence follows from the bound

$$|\langle f(\Phi + v), \psi\rangle_{L^2}| \quad \le \quad |\langle f(\Phi), \psi\rangle_{L^2}| + |\langle f(\Phi + v) - f(\Phi), \psi\rangle_{L^2}|\,, \tag{3.13}$$

using a similar observation as above to eliminate the $\|v\|_{L^2} \, \|\psi\|_{L^2}$ term. Finally, the second estimate in (3.5) can be obtained by applying (3.12) and (3.3) to the splitting

$$\begin{aligned} |\langle f(\Phi + v_A), \psi_A\rangle_{L^2} - \langle f(\Phi + v_B), \psi_B\rangle_{L^2}| \quad &\le \quad |\langle f(\Phi + v_A) - f(\Phi + v_B), \psi_A\rangle_{L^2}| \\ &\qquad + |f(\Phi + v_B), \psi_A - \psi_B\rangle_{L^2}|\,. \end{aligned} \tag{3.14}$$

□

Lemma 3.2. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ we have the bounds*

$$\begin{aligned} \|g(\Phi + v)\|_{L^2} \quad &\le \quad \|g(\Phi_0)\|_{L^2} + K_g(1 + \|v\|_{L^2}) \\ &\le \quad K[1 + \|v\|_{L^2}], \\ \|\partial_\xi g(\Phi + v)\|_{L^2} \quad &\le \quad K[1 + \|v\|_{H^1}], \end{aligned} \tag{3.15}$$

while for any pair $(v_A, v_B) \in H^1 \times H^1$ we have the estimates

$$\begin{aligned} \|g(\Phi + v_A) - g(\Phi + v_B)\|_{L^2} \quad &\le \quad K\,\|v_A - v_B\|_{L^2}\,, \\ \|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2} \quad &\le \quad K\big[1 + \|v_A\|_{H^1}\,\big]\,\|v_A - v_B\|_{H^1}\,. \end{aligned} \tag{3.16}$$

Proof. The Lipschitz estimate on g implies that

$$\|g(\Phi + v_A) - g(\Phi + v_B)\|_{L^2} \le K_g \, \|v_A - v_B\|_{L^2}\,. \tag{3.17}$$

Applying this inequality with $v_A = v$ and $v_B = \Phi_0 - \Phi$ we obtain

$$\|g(\Phi + v)\|_{L^2} \quad \le \quad \|g(\Phi_0)\|_{L^2} + K_g\big[\, \|\Phi - \Phi_0\|_{L^2} + \|v\|_{L^2}\,\big], \tag{3.18}$$

which in view of (hPar) yields the first line of (3.15).

The uniform bound

$$|Dg(\Phi + v)| \le K_g \tag{3.19}$$

together with the identity

$$\partial_\xi g(\Phi + v) = Dg(\Phi + v)\big(\Phi' + v'\big) \tag{3.20}$$

immediately imply the second estimate in (3.15). Finally, using

$$|Dg(\Phi + v) - Dg(\Phi + w)| \le K_g \, |v - w| \tag{3.21}$$

and the identity

$$\begin{array}{lcl} \partial_\xi \big[g(\Phi + v_A) - g(\Phi + v_B) \big] & = & \big[Dg(\Phi + v_A) - Dg(\Phi + v_B) \big] \big(\Phi' + v_A' \big) \\ & & + Dg(\Phi + v_B) \big(v_A' - v_B' \big), \end{array} \tag{3.22}$$

we obtain

$$\begin{array}{lcl} \big\| \partial_\xi \big[g(\Phi + v_A) - g(\Phi + v_B) \big] \big\|_{L^2} & \le & K_g \, \| v_A - v_B \|_\infty \, \big[\, \| \Phi' \|_{L^2} + \| v_A' \|_{L^2} \, \big] \\ & & + K_g \, \| v_A' - v_B' \|_{L^2} \, . \end{array} \tag{3.23}$$

The second estimate in (3.16) now follows easily. $\square$

Lemma 3.3. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{lcl} |\langle g(\Phi + v), \psi \rangle_{L^2}| & \le & K [1 + \|v\|_{L^2}] \, \|\psi\|_{L^2} \, , \\ |\langle \partial_\xi g(\Phi + v), \psi \rangle_{L^2}| & \le & K [1 + \|v\|_{L^2}] \, \|\psi\|_{H^1} \, , \end{array} \tag{3.24}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimates

$$\begin{array}{lcl} |\langle g(\Phi + v_A), \psi_A \rangle_{L^2} - \langle g(\Phi + v_B), \psi_B \rangle_{L^2}| & \le & K \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{L^2} \\ & & + K [1 + \|v_B\|_{L^2}] \, \|\psi_A - \psi_B\|_{L^2} \, , \\ |\langle \partial_\xi [g(\Phi + v_A)], \psi_A \rangle_{L^2} - \langle \partial_\xi [g(\Phi + v_B)], \psi_B \rangle_{L^2}| & \le & K \, \|v_A - v_B\|_{L^2} \, \|\psi_A\|_{H^1} \\ & & + K [1 + \|v_B\|_{L^2}] \, \|\psi_A - \psi_B\|_{H^1} \, . \end{array} \tag{3.25}$$

Proof. The estimates (3.24) follow immediately from the bound $\|g(\Phi + v)\|_{L^2} \le K \big[1 + \|v\|_{L^2} \big]$. The first bound in (3.25) can be obtained from Lemma 3.2 by noting that

$$\begin{array}{lcl} |\langle g(\Phi + v_A), \psi_A \rangle_{L^2} - \langle g(\Phi + v_B), \psi_B \rangle_{L^2}| & \le & |\langle g(\Phi + v_A) - g(\Phi + v_B), \psi_A \rangle_{L^2}| \\ & & + \, |\langle g(\Phi + v_B), \psi_A - \psi_B \rangle_{L^2}| \, . \end{array} \tag{3.26}$$

The final bound can be obtained by transferring the derivative to the functions ψ_A and ψ_B. $\square$

3.2 Bounds on $\mathcal{J}_0$, b and κ_σ

We are now ready to obtain global and Lipschitz bounds on the functions $\mathcal{J}_0$, b and κ_σ. In addition, we show that it suffices to impose an a-priori bound on $\|v\|_{L^2}$ in order to avoid hitting the cut-offs in the definition of b. This is crucial for the estimates in §9, where we have uniform control on $\|v\|_{L^2}$, but only an integrated form of control on $\|v\|_{H^1}$.

Lemma 3.4. *Suppose that (Hf) and (hPar) are satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $(v, d) \in H^1 \times \mathbb{R}$ and $\psi \in H^1$ we have the bounds*

$$\begin{array}{rcl} \|\mathcal{J}_0(\Phi + v, c + d)\|_{L^2} & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2 \|v\|_{L^2}\big], \\ |\langle \mathcal{J}_0(\Phi + v, c + d), \psi\rangle_{L^2}| & \leq & K(1 + |d|)\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \|\psi\|_{H^1}. \end{array} \tag{3.27}$$

In addition, for any set of pairs $(v_A, v_B) \in H^1 \times H^1$, $(d_A, d_B) \in \mathbb{R} \times \mathbb{R}$ and $(\psi_A, \psi_B) \in H^1 \times H^1$, the expressions

$$\begin{array}{rcl} \Delta_{AB}\mathcal{J}_0 & = & \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \\ \Delta_{AB}\langle \mathcal{J}_0, \cdot\rangle_{L^2} & = & \langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_A\rangle_{L^2} - \langle \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_B\rangle_{L^2} \end{array} \tag{3.28}$$

satisfy the estimates

$$\begin{array}{rcl} \|\Delta_{AB}\mathcal{J}_0\|_{L^2} & \leq & K[\|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2}] \|v_A - v_B\|_{H^1} \\ & & +[1 + \|v_A\|_{H^1}] |d_A - d_B| \\ & & +K(1 + |d_B|) \|v_A - v_B\|_{H^1}, \\ |\Delta_{AB}\langle \mathcal{J}_0, \cdot\rangle_{L^2}| & \leq & K[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2] \|v_A - v_B\|_{H^1} \|\psi_A\|_{H^1} \\ & & +[1 + \|v_A\|_{L^2}] |d_A - d_B| \|\psi_A\|_{H^1} \\ & & +K(1 + |d_B|) \|v_A - v_B\|_{L^2} \|\psi_A\|_{H^1} \\ & & +K(1 + |d_B|)\big[1 + \|v_B\|_{H^1} \|v_B\|_{L^2}^2\big] \|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{3.29}$$

Proof. We first note that the terms in (3.3)-(3.5) can be absorbed in (3.27)-(3.29), so it suffices to study the function

$$\mathcal{J}_{0;II}(u, \overline{c}) = \overline{c}u'. \tag{3.30}$$

Recalling that (hPar) implies

$$|c| + \|\Phi'\|_{L^2} \leq C_1, \tag{3.31}$$

we find

$$\|\mathcal{J}_{0;II}(\Phi + v, c + d)\|_{L^2} \leq C_2(1 + |d|)(1 + \|v\|_{H^1}), \tag{3.32}$$

together with

$$|\langle \mathcal{J}_{0;II}(\Phi + v, c + d), \psi\rangle_{L^2}| \leq C_2(1 + |d|)(1 + \|v\|_{L^2}) \|\psi\|_{H^1}, \tag{3.33}$$

which can be absorbed in (3.27).

In addition, writing

$$\begin{array}{rcl} \Delta_{AB}\mathcal{J}_{0;II} & = & \mathcal{J}_{0;II}(\Phi + v_A, c + d_A) - \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \\ \Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot\rangle_{L^2} & = & \langle \mathcal{J}_{0;II}(\Phi + v_A, c + d_A), \psi_A\rangle_{L^2} - \langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_B\rangle_{L^2}, \end{array} \tag{3.34}$$

we compute

$$\Delta_{AB}\mathcal{J}_{0;II} = (d_A - d_B)(\Phi' + v_A') + (c + d_B)(v_A' - v_B'). \tag{3.35}$$

This yields

$$\|\Delta_{AB}\mathcal{J}_{0;II}\|_{L^2} \le C_3\,|d_A - d_B|\,(1 + \|v_A\|_{H^1}) + C_3(1 + |d_B|)\,\|v_A - v_B\|_{H^1}\,, \tag{3.36}$$

which establishes the first estimate in (3.29).

In a similar fashion, we obtain

$$|\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi\rangle_{L^2}| \le C_3\,|d_A - d_B|\,(1 + \|v_A\|_{L^2})\,\|\psi\|_{H^1} + C_3(1 + |d_B|)\,\|v_A - v_B\|_{L^2}\,\|\psi\|_{H^1}\,. \tag{3.37}$$

The remaining estimate now follows from the inequality

$$\begin{array}{rcl} |\Delta_{AB}\langle \mathcal{J}_{0;II}, \cdot\rangle_{L^2}| & \le & |\langle \Delta_{AB}\mathcal{J}_{0;II}, \psi_A\rangle_{L^2}| \\ & & \quad + |\langle \mathcal{J}_{0;II}(\Phi + v_B, c + d_B), \psi_A - \psi_B\rangle_{L^2}|\,. \end{array} \tag{3.38}$$

□

Lemma 3.5. *Assume that (hPar) is satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|\langle \partial_\xi(\Phi + v), \psi\rangle_{L^2}| \le K\big[1 + \|v\|_{L^2}\big]\,\|\psi\|_{H^1}\,, \tag{3.39}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimate

$$\begin{array}{rcl} |\langle \partial_\xi[\Phi + v_A], \psi_A\rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B\rangle_{L^2}| & \le & K\,\|v_A - v_B\|_{L^2}\,\|\psi_A\|_{H^1} \\ & & \quad + K\big[1 + \|v_B\|_{L^2}\big]\,\|\psi_A - \psi_B\|_{H^1}\,. \end{array} \tag{3.40}$$

Proof. The desired bounds follow from the identity

$$|\langle \partial_\xi(\Phi + v), \psi\rangle_{L^2}| = |\langle \Phi + v, \partial_\xi\psi\rangle_{L^2}|\,, \tag{3.41}$$

together with the estimate

$$\begin{array}{rcl} |\langle \partial_\xi[\Phi + v_A], \psi_A\rangle_{L^2} - \langle \partial_\xi[\Phi + v_B], \psi_B\rangle_{L^2}| & \le & |\langle v_A - v_B, \partial_\xi\psi_A\rangle_{L^2}| + |\langle \partial_\xi\Phi, \psi_A - \psi_B\rangle_{L^2}| \\ & & \quad + |\langle v_B, \partial_\xi[\psi_A - \psi_B]\rangle_{L^2}|\,. \end{array} \tag{3.42}$$

□

Lemma 3.6. *Suppose that (hg) and (hPar) are satisfied. Then there exists constants $K_b > 0$ and $K > 0$, which do not depend on the pair (Φ, c), so that the following holds true. For any $v \in H^1$ and $\psi \in H^1$ we have the bound*

$$|b(\Phi + v, \psi)| \quad \le \quad K_b, \tag{3.43}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(\psi_A, \psi_B) \in H^1 \times H^1$ we have the estimate

$$\begin{array}{rcl} |b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)| & \le & K\,\|v_A - v_B\|_{L^2}\,\|\psi_A\|_{H^1} \\ & & \quad + K\big[1 + \|v_B\|_{L^2}\big]\,\|\psi_A - \psi_B\|_{H^1}\,. \end{array} \tag{3.44}$$

Proof. The uniform bound (3.43) follows directly from the properties of the cut-off functions. Upon introducing the function

$$\tilde{b}(x, y) = -\chi_{\text{low}}(x)^{-1}\chi_{\text{high}}(y), \tag{3.45}$$

the global Lipschitz smoothness of the cut-off functions implies that

$$\left|\tilde{b}(x_A, y_A) - \tilde{b}(x_B, y_B)\right| \le C_1 \big[\, |x_B - x_A| + |y_B - y_A| \,\big]. \tag{3.46}$$

Using the identity

$$b(u, \psi) = \tilde{b}\Big(\langle \partial_\xi u, \psi \rangle_{L^2}, \langle g(u), \psi \rangle_{L^2} \Big), \tag{3.47}$$

the desired bound (3.44) follows from Lemma's 3.3 and 3.5. □

Lemma 3.7. *Assume that (Hg) and (hPar) are satisfied. Then for any $v \in H^1$ that has*

$$\|v\|_{L^2} \le \min\{1, [4\,\|\psi_{\mathrm{tw}}\|_{H^1}]^{-1}\}, \tag{3.48}$$

we have the identity

$$b(\Phi + v, \psi_{\mathrm{tw}}) = -\big[\langle \partial_\xi [\Phi + v], \psi_{\mathrm{tw}} \rangle_{L^2} \big]^{-1} \langle g(\Phi + v), \psi_{\mathrm{tw}} \rangle_{L^2}. \tag{3.49}$$

Proof. Using (3.15) and recalling the definition (2.21), we find that

$$|\langle g(\Phi + v), \psi_{\mathrm{tw}} \rangle_{L^2}| \le \Big[\|g(\Phi_0)\|_{L^2} + 2K_g \Big] \|\psi_{\mathrm{tw}}\|_{L^2} = K_{\mathrm{ip}}. \tag{3.50}$$

In addition, we note that (hPar) and the normalization (2.14) imply that

$$\langle \partial_\xi \Phi, \psi_{\mathrm{tw}} \rangle_{L^2} = \langle \partial_\xi \Phi_0, \psi_{\mathrm{tw}} \rangle_{L^2} + \langle \partial_\xi [\Phi - \Phi_0], \psi_{\mathrm{tw}} \rangle_{L^2} \ge 1 - \frac{1}{4} = \frac{3}{4}. \tag{3.51}$$

This allows us to estimate

$$\langle \partial_\xi (\Phi + v), \psi_{\mathrm{tw}} \rangle_{L^2} \ge \frac{3}{4} - \langle v, \partial_\xi \psi_{\mathrm{tw}} \rangle_{L^2} \ge \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \tag{3.52}$$

which shows that the cut-off functions do not modify their arguments. □

Lemma 3.8. *Suppose that (Hg) and (hPar) are satisfied. Then there exists a constant $K_\kappa > 0$, which does not depend on the pair (Φ, c), so that for any $0 \le \sigma \le 1$, any $v \in H^1$ and any $\psi \in H^1$, we have the bound*

$$|\kappa_\sigma(\Phi + v, \psi)| + \left|\kappa_\sigma(\Phi + v, \psi)^{-1}\right| + \left|\kappa_\sigma(\Phi + v, \psi)^{-1/2}\right| \le K_\kappa. \tag{3.53}$$

Proof. This follows directly from the bound

$$1 \le \kappa_\sigma(\Phi + v, \psi) \le 1 + \frac{1}{2\rho} \sigma^2 K_b^2. \tag{3.54}$$

□

In order to state our final result, we introduce the functions

$$\begin{aligned} \nu_\sigma^{(1)}(u, \psi) &= \kappa_\sigma(u, \psi) - 1, \\ \nu_\sigma^{(-1)}(u, \psi) &= \kappa_\sigma(u, \psi)^{-1} - 1, \\ \nu_\sigma^{(-1/2)}(u, \psi) &= \kappa_\sigma(u, \psi)^{-1/2} - 1, \end{aligned} \tag{3.55}$$

which isolate the σ-dependence in κ_σ.

Lemma 3.9. *Suppose that (Hg) and (hPar) are satisfied and pick* $\vartheta \in \{-1, -\frac{1}{2}, 1\}$. *Then there exist constants* $K_\nu > 0$ *and* $K > 0$, *which do not depend on the pair* (Φ, c), *so that the following holds true. For any* $0 \le \sigma \le 1$, *any* $v \in H^1$ *and any* $\psi \in H^1$ *we have the bound*

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \le \sigma^2 K_\nu, \tag{3.56}$$

while for any $0 \le \sigma \le 1$ *and any set of pairs* $(v_A, v_B) \in H^1 \times H^1$ *and* $(\psi_A, \psi_B) \in H^1 \times H^1$, *we have the estimate*

$$\begin{aligned} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| \quad \le \quad & K\sigma^2 \left\|v_A - v_B\right\|_{L^2} \left\|\psi_A\right\|_{H^1} \\ & + K\sigma^2\big[1 + \left\|v_B\right\|_{L^2}\big] \left\|\psi_A - \psi_B\right\|_{H^1}. \end{aligned} \tag{3.57}$$

Proof. As a preparation, we observe that for any $x \ge 0$ and $y \ge 0$ we have the inequality

$$\left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \frac{|y-x|}{(1+x)(1+y)} \le |y - x|, \tag{3.58}$$

together with

$$\left|\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+y}}\right| = \frac{|y-x|}{\sqrt{(1+x)(1+y)}(\sqrt{1+x} + \sqrt{1+y})} \le \frac{1}{2}|y - x|. \tag{3.59}$$

Applying these bounds with $y = 0$, we obtain

$$\left|\nu_\sigma^{(\vartheta)}(\Phi + v, \psi)\right| \le \frac{1}{2\rho}\sigma^2 \left|b(\Phi + v, \psi)\right|^2 \le \frac{1}{2\rho}\sigma^2 K_b^2, \tag{3.60}$$

which yields (3.56). In addition, we may compute

$$\begin{aligned} \left|\nu_\sigma^{(\vartheta)}(\Phi + v_A, \psi_A) - \nu_\sigma^{(\vartheta)}(\Phi + v_B, \psi_B)\right| \quad & \le \quad \tfrac{1}{2\rho}\sigma^2 \left|b(\Phi + v_A, \psi_A)^2 - b(\Phi + v_B, \psi_B)^2\right| \\ & = \quad \tfrac{1}{2\rho}\sigma^2 \left|b(\Phi + v_A, \psi_A) + b(\Phi + v_B, \psi_B)\right| \\ & \qquad \times \left|b(\Phi + v_A, \psi_A) - b(\Phi + v_B, \psi_B)\right|. \end{aligned} \tag{3.61}$$

In particular, the bounds (3.57) follow from Lemma 3.6. ☐

4 Variational solution

In this section we set out to establish Proposition 2.1. Our strategy is to fit the first component of (2.29) into the framework of [27]. Indeed, the conditions (H1)-(H4) in this paper are explicitly verified in Lemma 4.1 below. The second line of (2.29) can subsequently be treated as an sODE for Γ with random coefficients. In Lemma 4.3 below we show that this sODE fits into the framework that was developed in [32, §3] to handle such equations.

Lemma 4.1. *Suppose that (HA), (Hf), (HTw), (HS), (HVar) and (Hg) are all satisfied. Then there exist constants* $K > 0$ *and* $\vartheta > 0$ *so that the following properties hold true.*

(i) For any triplet $(v_A, v_B, v) \in H^1 \times H^1 \times H^1$, *the map*

$$s \mapsto \langle A_*[v_A + s v_B], v\rangle_{H^{-1};H^1} + \langle f(\Phi_{\mathrm{ref}} + v_A + s v_B), v\rangle_{L^2} \tag{4.1}$$

is continuous.

(ii) For every pair $(v_A, v_B) \in H^1 \times H^1$, we have the inequality

$$\begin{aligned} K \|v_A - v_B\|_{L^2}^2 \quad \geq \quad & 2\langle A_*(v_A - v_B), v_A - v_B\rangle_{H^{-1};H^1} \\ & +2\langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2} \\ & + \|g(\Phi_{\mathrm{ref}} + v_A) - g(\Phi_{\mathrm{ref}} + v_B)\|_{L^2}^2 . \end{aligned} \tag{4.2}$$

(iii) For any $v \in H^1$ we have the inequality

$$2\langle A_* v, v\rangle_{H^{-1};H^1} + 2\langle f(\Phi_{\mathrm{ref}} + v), v\rangle_{L^2} + \|g(\Phi_{\mathrm{ref}} + v)\|_{L^2}^2 + \vartheta \|v\|_{H^1}^2 \leq K\big[1 + \|v\|_{L^2}^2\big]. \tag{4.3}$$

(iv) For any $v \in H^1$ we have the bound

$$\|A_* v\|_{H^{-1}}^2 + \|f(\Phi_{\mathrm{ref}} + v)\|_{H^{-1}}^2 \leq K\big[1 + \|v\|_{H^1}^2\big]\big[1 + \|v\|_{L^2}^4\big]. \tag{4.4}$$

Proof. Item (i) follows from the linearity of A_* and the Lipschitz bound (3.5). In addition, writing

$$\mathcal{I} = \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), v_A - v_B\rangle_{L^2}, \tag{4.5}$$

(HVar) implies the one-sided inequality

$$\begin{aligned} \mathcal{I} \quad = \quad & \langle f(\Phi_{\mathrm{ref}} + v_A) - f(\Phi_{\mathrm{ref}} + v_B), \Phi_{\mathrm{ref}} + v_A - (\Phi_{\mathrm{ref}} + v_B)\rangle_{L^2} \\ \leq \quad & C_1 \|v_A - v_B\|_{L^2}^2 . \end{aligned} \tag{4.6}$$

Item (ii) hence follows from the Lipschitz bound (3.16) together with the bound

$$\langle A_* v, v\rangle_{H^{-1},H^1} \leq -\rho \|v\|_{H^1}^2 . \tag{4.7}$$

A second consequence of (HVar) is that

$$\begin{aligned} \langle f(\Phi_{\mathrm{ref}} + v), v\rangle_{L^2} \quad = \quad & \langle f(\Phi_{\mathrm{ref}} + v) - f(\Phi_{\mathrm{ref}}), (\Phi_{\mathrm{ref}} + v) - \Phi_{\mathrm{ref}}\rangle_{L^2} \\ & +\langle f(\Phi_{\mathrm{ref}}), v\rangle_{L^2} \\ \leq \quad & C_1 \|v\|_{L^2}^2 + \|f(\Phi_{\mathrm{ref}})\|_{L^2} \|v\|_{L^2} \\ \leq \quad & C_2\big[1 + \|v\|_{L^2}^2\big]. \end{aligned} \tag{4.8}$$

In particular, we may obtain (iii) by combining (4.7) with (3.15).

Finally, for any $v \in H^1$ and $\psi \in H^1$ we may use (3.3) to compute

$$\begin{aligned} \langle f(\Phi_{\mathrm{ref}} + v), \psi\rangle_{H^{-1};H^1} \quad = \quad & \langle f(\Phi_{\mathrm{ref}} + v), \psi\rangle_{L^2} \\ \leq \quad & C_3\Big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\Big] \|\psi\|_{H^1} . \end{aligned} \tag{4.9}$$

In other words, we see that

$$\|f(\Phi_{\mathrm{ref}} + v)\|_{H^{-1}} \leq C_3\big[1 + \|v\|_{H^1} \|v\|_{L^2}^2\big] \leq C_3(1 + \|v\|_{H^1})(1 + \|v\|_{L^2}^2), \tag{4.10}$$

which yields (iv). □

Lemma 4.2. *Suppose that (HA), (Hf), (Hg) and (hPar) are all satisfied. Then there exists a constant $K > 0$, which does not depend on the pair (Φ, c), so that the following properties hold true for any $0 \leq \sigma \leq 1$.*

(i) For any $v \in H^1$ and any $\psi \in H^2$ with $\|\psi\|_{H^2} \le 2\|\psi_{\mathrm{tw}}\|_{H^2}$, we have the bound

$$|a_\sigma(\Phi + v, c, \psi)| \le K\Big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\Big]. \tag{4.11}$$

(ii) For any $v \in H^1$ and any pair $(\psi_A, \psi_B) \in H^2 \times H^2$ for which $\|\psi_A\|_{H^2} \le 2\|\psi_{\mathrm{tw}}\|_{H^2}$ and $\|\psi_B\|_{H^2} \le 2\|\psi_{\mathrm{tw}}\|_{H^2}$, the difference

$$\Delta_{AB} a_\sigma = a_\sigma(\Phi + v, c, \psi_A) - a_\sigma(\Phi + v, c, \psi_B) \tag{4.12}$$

satisfies the bound

$$|\Delta_{AB} a_\sigma| \le K\Big[1 + \|v\|_{H^1}(1 + \|v\|_{L^2}^3)\Big]\|\psi_A - \psi_B\|_{H^1}. \tag{4.13}$$

Proof. We first compute

$$\begin{array}{rcl} \kappa_\sigma(u,\psi)\langle \mathcal{J}_\sigma(u,c,\psi), \psi\rangle_{L^2} & = & \langle f(u) + cu' + \sigma^2 b(u,\psi)\partial_\xi[g(u)], \psi\rangle_{L^2} \\ & = & \langle \mathcal{J}_0(u,c), \psi\rangle_{L^2} + \sigma^2 b(u,\psi)\langle \partial_\xi[g(u)], \psi\rangle_{L^2}. \end{array} \tag{4.14}$$

Upon defining

$$\begin{array}{rcl} \mathcal{E}_I(u,c,\psi) & = & \langle \mathcal{J}_0(u,c), \psi\rangle_{L^2}, \\ \mathcal{E}_{II}(u,\psi) & = & \sigma^2 b(u,\psi)\langle \partial_\xi g(u), \psi\rangle_{L^2}, \\ \mathcal{E}_{III}(u,\psi) & = & \kappa_\sigma(u,\psi)\langle u, A_*\psi\rangle_{L^2}, \end{array} \tag{4.15}$$

we hence see that

$$a_\sigma(u,c,\psi) = -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi u, \psi\rangle_{L^2}\big)\Big]^{-1}\big[\mathcal{E}_I(u,c,\psi) + \mathcal{E}_{II}(u,\psi) + \mathcal{E}_{III}(u,\psi)\big]. \tag{4.16}$$

For $\# \in \{I, II, III\}$, we define

$$\Delta_{AB}\mathcal{E}_\# = \mathcal{E}_\#(\Phi + v, c, \psi_A) - \mathcal{E}_\#(\Phi + v, c, \psi_B). \tag{4.17}$$

We note that Lemma's 3.3, 3.4 and 3.6 yield the bounds

$$\begin{array}{rcl} |\mathcal{E}_I(\Phi + v, c, \psi)| & \le & C_1\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big], \\ |\mathcal{E}_{II}(\Phi + v, \psi)| & \le & C_1\big[1 + \|v\|_{L^2}\big], \end{array} \tag{4.18}$$

together with

$$\begin{array}{rcl} |\Delta_{AB}\mathcal{E}_I| & \le & C_1\big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\|\psi_A - \psi_B\|_{H^1}, \\ |\Delta_{AB}\mathcal{E}_{II}| & \le & C_1[1 + \|v\|_{L^2}]^2\|\psi_A - \psi_B\|_{H^1} \\ & & +C_1[1 + \|v\|_{L^2}]\|\psi_A - \psi_B\|_{H^1}. \end{array} \tag{4.19}$$

A direct estimate using the a-priori bound on $\|\psi\|_{H^2}$ and (3.53) yields

$$\begin{array}{rcl} |\mathcal{E}_{III}(\Phi + v, \psi)| & \le & K_\kappa\big[|\langle \Phi_{\mathrm{ref}}, A_*\psi\rangle_{L^2}| + |\langle v, A_*\psi\rangle_{L^2}|\big] \\ & \le & C_2\big[1 + \|v\|_{L^2}\big]. \end{array} \tag{4.20}$$

By transferring one of the derivatives in A_*, we also obtain

$$\begin{array}{rcl} |\Delta \mathcal{E}_{III}| & \leq & |\kappa_\sigma(\Phi + v, \psi_A) - \kappa_\sigma(\Phi + v, \psi_B)| \, |\langle \Phi + v, A_* \psi \rangle_{L^2}| \\ & & \quad + |\kappa_\sigma(\Phi + v, \psi_B)| \, |\langle \Phi + v, A_*[\psi_A - \psi_B] \rangle_{L^2}| \\ & \leq & C_3 (1 + \|v\|_{L^2})^2 \, \|\psi_A - \psi_B\|_{H^1} \\ & & \quad + C_3 \big[1 + \|v\|_{H^1} \big] \, \|\psi_A - \psi_B\|_{H^1} . \end{array} \tag{4.21}$$

Upon writing

$$\begin{array}{rcl} \mathcal{E}(u, c, \psi) & = & \mathcal{E}_I(u, c, \psi) + \mathcal{E}_{II}(u, \psi) + \mathcal{E}_{II}(u, \psi), \\ \Delta_{AB} \mathcal{E} & = & \Delta_{AB} \mathcal{E}_I + \Delta_{AB} \mathcal{E}_{II} + \Delta_{AB} \mathcal{E}_{III}, \end{array} \tag{4.22}$$

we hence conclude that

$$\begin{array}{rcl} |\mathcal{E}(\Phi + v, c, \psi)| & \leq & C_4 \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2 \big], \\ |\Delta_{AB} \mathcal{E}| & \leq & C_4 \big[1 + \|v\|_{H^1} \big] \big[1 + \|v\|_{L^2}^2 \big] \, \|\psi_A - \psi_B\|_{H^1} . \end{array} \tag{4.23}$$

Item (i) follows immediately from the first bound, since $\chi_{\mathrm{low}}(\cdot)^{-1}$ is globally bounded. To obtain (ii), we compute

$$\begin{array}{rcl} |\Delta_{AB} a_\sigma| & \leq & C_5 \, |\langle \partial_\xi (\Phi + v), \psi_A \rangle_{L^2} - \langle \partial_\xi (\Phi + v), \psi_B \rangle_{L^2}| \, |\mathcal{E}(\Phi + v, \psi_A)| \\ & & \quad + C_5 \, |\Delta_{AB} \mathcal{E}| \\ & \leq & C_6 [1 + \|v\|_{L^2}] \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^2 \big] \, \|\psi_A - \psi_B\|_{H^1} \\ & & \quad + C_6 \big[1 + \|v\|_{H^1} \big] \big[1 + \|v\|_{L^2}^2 \big] \, \|\psi_A - \psi_B\|_{H^1} \\ & \leq & C_7 \Big[1 + \|v\|_{H^1} \, (1 + \|v\|_{L^2}^3) \Big] \, \|\psi_A - \psi_B\|_{H^1} , \end{array} \tag{4.24}$$

in which we used several estimates of the form

$$\|v\|_{L^2} \leq C_8 \big[1 + \|v\|_{L^2}^4 \big] \leq C_8 \big[1 + \|v\|_{H^1} \, \|v\|_{L^2}^3 \big]. \tag{4.25}$$

□

Upon introducing the shorthands

$$\begin{array}{rcl} p(v, \gamma) & = & c + a_\sigma(\Phi_{\mathrm{ref}} + v, c, T_\gamma \psi_{\mathrm{tw}}), \\ q(v, \gamma) & = & b(\Phi_{\mathrm{ref}} + v, T_\gamma \psi_{\mathrm{tw}}), \end{array} \tag{4.26}$$

the second line of (2.29) can be written as

$$d\Gamma = p\big(X(t), \Gamma(t)\big) dt + \sigma q\big(X(t), \Gamma(t)\big) d\beta_t. \tag{4.27}$$

Taking the view-point that $X(t) = X(t, \omega)$ is known upon picking $\omega \in \Omega$, (4.27) can be viewed as an sODE with random coefficients. Our next result relates directly to the conditions of [32, Thm 3.1.1], which is specially tailored for equations of this type.

Lemma 4.3. *Suppose that (HA), (Hf), (HTw), (HS) and (Hg) are all satisfied and fix $c \in \mathbb{R}$ together with $0 \leq \sigma \leq 1$. Then there exists $K > 0$ so that the following properties are satisfied.*

(i) For any $v \in H^1$ and any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$, we have the inequality

$$\begin{array}{rcl} K \big[1 + \|v\|_{H^1} \, (1 + \|v\|_{L^2}^3) \big] \, |\gamma_A - \gamma_B|^2 & \geq & 2 \big[\gamma_A - \gamma_B \big] \big[p(v, \gamma_A) - p(v, \gamma_B) \big] \\ & & \quad + |q(v, \gamma_A) - q(v, \gamma_B)|^2 . \end{array} \tag{4.28}$$

(ii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the inequality

$$2\gamma p(v,\gamma) + |q(v,\gamma)|^2 \le K\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big]\big[1+\gamma^2\big]. \tag{4.29}$$

(iii) For any $v \in H^1$ and $\gamma \in \mathbb{R}$, we have the bound

$$|p(v,\gamma)| + |q(v,\gamma)|^2 \le K\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big]. \tag{4.30}$$

Proof. The exponential decay of ψ_{tw}' and ψ_{tw}'' implies that

$$\|T_{\gamma_A}\psi_{\mathrm{tw}} - T_{\gamma_B}\psi_{\mathrm{tw}}\|_{H^1} \le C_1\,|\gamma_A - \gamma_B|\,. \tag{4.31}$$

Using Lemma's 3.6 and 4.2, we hence find the bounds

$$\begin{array}{rcl} |p(v,\gamma)| & \le & C_2\big[1 + \|v\|_{H^1}\,\|v\|_{L^2}^2\,\big], \\ |q(v,\gamma)| & \le & K_b, \end{array} \tag{4.32}$$

together with

$$\begin{array}{rcl} |p(v,\gamma_A) - p(v,\gamma_B)| & \le & C_3\big[1 + \|v\|_{H^1}\,(1+\|v\|_{L^2}^3)\big]\,|\gamma_A - \gamma_B|\,, \\ |q(v,\psi_A) - q(v,\psi_B)| & \le & C_3[1 + \|v\|_{L^2}]\,|\gamma_A - \gamma_B|\,. \end{array} \tag{4.33}$$

Items (i), (ii) and (iii) can now be verified directly. □

Proof of Proposition 2.1. The existence of the $dt \otimes \mathbb{P}$ version of X that is $(\mathcal{F}_t)$-progressively measurable as a map into H^1, follows from [32, Ex. 4.2.3].

We remark that the conditions (H1) through (H4) appearing in [27] correspond directly with items (i)-(iv) of Lemma 4.1. In particular, we may apply the main result from this paper with $\alpha = 2$ and $\beta = 4$ to verify the remaining statements concerning X.

Finally, we note that items (i)-(iii) of Lemma 4.3 allow us to apply [32, Thm. 3.1.1], provided that the function

$$t \mapsto \big[1 + \|X(t)\|_{H^1}\,(1 + \|X(t)\|_{L^2}^3)\big] \tag{4.34}$$

is integrable on $[0,T]$ for almost all $\omega \in \Omega$. This however follows directly from the inclusions

$$X \in L^6\big(\Omega,\mathbb{P}; C([0,T];L^2)\big) \cap \mathcal{N}^2\big([0,T];(\mathcal{F}_t);H^1\big), \tag{4.35}$$

allowing us to verify the statements concerning Γ. The remaining inclusions (2.39) follow directly from the bounds in Lemma 3.2 and 3.6. □

5 The stochastic phase shift

In this section we consider the process (X,Γ) described in Proposition 2.1 and define the new process

$$V(t) = T_{-\Gamma(t)}[X(t) + \Phi_{\mathrm{ref}}] - \Phi \tag{5.1}$$

for some $\Phi \in \mathcal{U}_{H^1}$. In addition, we introduce the nonlinearity

$$\begin{array}{rcl} \mathcal{R}_{\sigma;\Phi,c}(v) & = & \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})A_*[\Phi + v] \\ & & \quad + f(\Phi + v) + \sigma^2 b(\Phi + v, \psi_{\mathrm{tw}})\partial_\xi[g(\Phi + v)] \\ & & \quad + \Big[c + a_\sigma\big(\Phi + v, c, \psi_{\mathrm{tw}}\big)\Big][\Phi' + v'] \\ & = & \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})\Big[A_*[\Phi + v] + \mathcal{J}_\sigma(\Phi + v, c, \psi_{\mathrm{tw}})\Big] + a_\sigma\big(\Phi + v, c, \psi_{\mathrm{tw}}\big)[\Phi' + v'], \end{array} \tag{5.2}$$

together with

$$\mathcal{S}_\Phi(v) \quad = \quad g(\Phi + v) + b(\Phi + v, \psi_{\mathrm{tw}})[\Phi' + v']. \tag{5.3}$$

Our main result states that the shifted process V can be interpreted as a weak solution to the sPDE

$$dV \quad = \quad \mathcal{R}_{\sigma;\Phi,c}(V)\,dt + \sigma \mathcal{S}_\Phi(V) d\beta_t. \tag{5.4}$$

Proposition 5.1. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map*

$$V : [0,T] \times \Omega \to L^2 \tag{5.5}$$

defined by (5.1) satisfies the following properties.

(i) For almost all $\omega \in \Omega$, the map $t \mapsto V(t,\omega)$ is of class $C([0,T];L^2)$.

(ii) For all $t \in [0,T]$, the map $\omega \mapsto V(t,\omega) \in L^2$ is $(\mathcal{F}_t)$-measurable.

(iii) We have the inclusion

$$V \quad \in \quad \mathcal{N}^2\big([0,T];(\mathcal{F}_t);H^1\big) \tag{5.6}$$

together with

$$\mathcal{S}_{\sigma;\Phi}(V) \quad \in \quad \mathcal{N}^2\big([0,T];(\mathcal{F}_t);L^2\big). \tag{5.7}$$

(iv) For almost all $\omega \in \Omega$, we have the inclusion

$$\mathcal{R}_{\sigma;\Phi,c}\big(V(\cdot,\omega)\big) \in L^1([0,T];H^{-1}). \tag{5.8}$$

(v) For almost all $\omega \in \Omega$, the identity

$$V(t) \quad = \quad V(0) + \int_0^t \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big)\,ds + \sigma \int_0^t \mathcal{S}_\Phi\big(V(s)\big) d\beta_s \tag{5.9}$$

holds for all $0 \le t \le T$.

Taking derivatives of translation operators typically requires extra regularity of the underlying function, which prevents us from applying an Ito formula directly to (5.1). In order to circumvent this technical issue, we pick a test function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$ and consider the two maps

$$\phi_{1;\zeta} : H^{-1} \times \mathbb{R} \to \mathbb{R}, \qquad\qquad \phi_{2;\zeta} : \mathbb{R} \to \mathbb{R} \tag{5.10}$$

that act as

$$\begin{aligned} \phi_{1;\zeta}\big(x,\gamma\big) \quad &= \quad \langle x, T_\gamma \zeta \rangle_{H^{-1};H^1}, \\ \phi_{2;\zeta}\big(\gamma\big) \quad &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{H^{-1};H^1} \\ &= \quad \langle T_{-\gamma}\Phi_{\mathrm{ref}} - \Phi, \zeta \rangle_{L^2}. \end{aligned} \tag{5.11}$$

These two maps do have sufficient smoothness for our purposes here.

Lemma 5.2. *Consider the setting of Proposition 2.1. Then for almost all* $\omega \in \Omega$ *the identity*

$$\begin{array}{lll}\phi_{1;\zeta}\big(X(t),\Gamma(t)\big) & = & \phi_{1;\zeta}\big(X(0),\Gamma(0)\big) \\ & & \quad + \int_0^t \langle A_*[X(s)+\Phi_{\rm ref}] + f(X(s)+\Phi_{\rm ref}), T_{\Gamma(s)}\zeta\rangle_{H^{-1};H^1}\, ds \\ & & \quad - \int_0^t \big[c + a_\sigma\big(X(s)+\Phi_{\rm ref}, c, T_{\Gamma(s)}\psi_{\rm tw}\big)\big]\langle X(s), T_{\Gamma(s)}\zeta'\rangle_{L^2}\, ds \\ & & \quad -\frac{1}{2}\sigma^2 \int_0^t 2b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)\langle g(X(s)+\Phi_{\rm ref}), T_{\Gamma(s)}\zeta'\rangle_{L^2}\, ds \\ & & \quad +\frac{1}{2}\sigma^2 \int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)^2 \langle X(s), T_{\Gamma(s)}\zeta''\rangle_{L^2}\, ds \\ & & \quad +\sigma \int_0^t \langle g(X(s)+\Phi_{\rm ref}), T_{\Gamma(s)}\zeta\rangle_{L^2}\, d\beta_s \\ & & \quad -\sigma \int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)\langle X(s), T_{\Gamma(s)}\zeta'\rangle_{L^2}\, d\beta_s \end{array} \tag{5.12}$$

holds for all $0 \le t \le T$.

Proof. We note that $\phi_{1;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{1;\zeta}(x,\gamma)[y,\beta] = \langle y, T_\gamma\zeta\rangle_{H^{-1};H^1} - \beta\langle x, T_\gamma\zeta'\rangle_{H^{-1};H^1}, \tag{5.13}$$

together with

$$D^2\phi_{1;\zeta}(x,\gamma)[y,\beta][y,\beta] = -2\beta\langle y, T_\gamma\zeta'\rangle_{H^{-1};H^1} + \beta^2\langle x, T_\gamma\zeta''\rangle_{H^{-1};H^1}. \tag{5.14}$$

Applying a standard Ito formula such as [11, Thm. 1] with $S = I$, the result readily follows. □

Lemma 5.3. *Consider the setting of Proposition 2.1. Then for almost all* $\omega \in \Omega$ *the identity*

$$\begin{array}{lll}\phi_{2;\zeta}\big(\Gamma(t)\big) & = & \phi_{2;\zeta}(\Gamma(0)) \\ & & \quad - \int_0^t \big[c + a_\sigma\big(X(s)+\Phi_{\rm ref}, c, T_{\Gamma(s)}\psi_{\rm tw}\big)\big]\langle \Phi_{\rm ref}, T_{\Gamma(s)}\zeta'\rangle_{L^2}\, ds \\ & & \quad +\frac{1}{2}\sigma^2 \int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)^2 \langle \Phi_{\rm ref}, T_{\Gamma(s)}\zeta''\rangle_{L^2}\, ds \\ & & \quad -\sigma \int_0^t b\big(X(s)+\Phi_{\rm ref}, T_{\Gamma(s)}\psi_{\rm tw}\big)\langle \Phi_{\rm ref}, T_{\Gamma(s)}\zeta'\rangle_{L^2}\, d\beta_s \end{array} \tag{5.15}$$

holds for all $0 \le t \le T$.

Proof. We note that $\phi_{2;\zeta}$ is C^2-smooth, with derivatives given by

$$D\phi_{2;\zeta}(\gamma)[\beta] = -\beta\langle \Phi_{\rm ref}, T_\gamma\zeta'\rangle_{L^2}, \tag{5.16}$$

together with

$$D^2\phi_{2;\zeta}(\gamma)[\beta][\beta] = \beta^2\langle \Phi_{\rm ref}, T_\gamma\zeta''\rangle_{L^2}. \tag{5.17}$$

The result again follows from the Ito formula. □

Corollary 5.4. *Consider the setting of Proposition 2.1, suppose that (hPar) is satisfied and pick a test-function* $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$. *Then for almost all* $\omega \in \Omega$, *the map* V *defined by (5.1) satisfies the identity*

$$\langle V(t),\zeta\rangle_{L^2} \quad = \quad \langle V(0),\zeta\rangle_{L^2} + \int_0^t \langle \mathcal{R}_{\sigma;\Phi,c}\big(V(s)\big),\zeta\rangle_{H^{-1};H^1}\, ds + \sigma\int_0^t \langle \mathcal{S}_\Phi\big(V(s)\big),\zeta\rangle_{L^2} d\beta_s \tag{5.18}$$

for all $0 \le t \le T$.

Proof. For any $\gamma \in \mathbb{R}$, we have the identities

$$\alpha_\sigma(u, c, T_\gamma \psi) = \alpha_\sigma(T_{-\gamma} u, c, \psi), \qquad b(u, T_\gamma \psi) = b(T_{-\gamma} u, \psi), \tag{5.19}$$

together with the commutation relations

$$T_\gamma f(u) = f(T_\gamma u), \qquad T_\gamma g(u) = g(T_\gamma u), \qquad T_\gamma A_* u = A_* T_\gamma u. \tag{5.20}$$

By construction, we also have

$$\langle V(t), \zeta \rangle_{L^2} = \phi_{1;\zeta}\big(X(t), \Gamma(t)\big) + \phi_{2;\zeta}\big(\Gamma(t)\big), \tag{5.21}$$

together with

$$T_{-\Gamma(s)}[X(s) + \Phi_{\mathrm{ref}}] = \Phi + V(s). \tag{5.22}$$

The derivatives in (5.12) and (5.15) can now be transferred from ζ to yield (5.18).

We emphasize that the identity

$$\frac{1}{2}\sigma^2 b(\Phi + V(s), \psi_{\mathrm{tw}})^2 [X'' + \Phi''_{\mathrm{ref}}] = \frac{1}{2\rho}\sigma^2 b(\Phi + V(s), \psi_{\mathrm{tw}})^2 A_* [X(s) + \Phi_{\mathrm{ref}}] \tag{5.23}$$

is a crucial ingredient in this computation. This is where we use the requirement in (HA) that all the diffusion coefficients in A_* are equal. □

Proof of Proposition 5.1. Items (i) and (ii) follow immediately from items (i) and (ii) of Proposition 2.1. Turning to (iii), notice first that we have the isometry

$$\|T_\gamma x\|_{H^1} = \|x\|_{H^1}. \tag{5.24}$$

Observe in addition that

$$\|T_\gamma \Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le \|T_\gamma \Phi_{\mathrm{ref}} - \Phi_{\mathrm{ref}}\|_{H^1} + \|\Phi_{\mathrm{ref}} - \Phi\|_{H^1} \le C_1\big[1 + |\gamma|\big], \tag{5.25}$$

since Φ'_{ref} and Φ''_{ref} decay exponentially. In particular, the inclusion (5.6) follows from the corresponding inclusions (2.38) for the pair (X, Γ). The second inclusion (5.7) now follows immediately from the bounds in Lemma's 3.2 and 3.6.

Using Lemma's 3.1, 3.2, 3.6 and 4.2, we obtain the bound

$$\begin{aligned} \|\mathcal{R}_{\sigma;\Phi,c}(v)\|_{H^{-1}} \le\ & C_2 K_\kappa\big[1 + \|v\|_{H^1}\big] \\ & + C_2\big[1 + \|v\|_{H^1}^2 \|v\|_{L^2}\big] \\ & + C_2 \sigma^2 K_b\big[1 + \|v\|_{H^1}\big] \\ & + \big[1 + \|v\|_{H^1}\|v\|_{L^2}^2\big]\big[1 + \|v\|_{H^1}\big]. \end{aligned} \tag{5.26}$$

Since items (i) and (iii) imply that

$$\sup_{0 \le t \le T} \|V(t,\omega)\|_{L^2} + \int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt < \infty \tag{5.27}$$

for almost all $\omega \in \Omega$, item (iv) follows from the standard bound

$$\int_0^T \|V(t,\omega)\|_{H^1} \, dt \le \sqrt{T}\Big[\int_0^T \|V(t,\omega)\|_{H^1}^2 \, dt\Big]^{1/2}. \tag{5.28}$$

Finally, we note that items (iii) and (iv) imply that the integrals in (5.9) are well-defined. In view of Corollary 5.4, we can apply a standard diagonalization argument involving the separability of L^2 and the density of test-functions to conclude that (v) holds. □

6 The stochastic time transform

We note that (5.9) can be interpreted as a quasi-linear equation due to the presence of the $\kappa_\sigma A_*$ term. In this section we transform our problem to a semi-linear form by rescaling time, using the fact that κ_σ is a scalar. In addition, we investigate the impact of this transformation on the probabilities (2.68).

Recalling the map V defined in Proposition 5.1, we write

$$\tau_\Phi(t,\omega) = \int_0^t \kappa_\sigma\big(\Phi + V(s,\omega), \psi_{\mathrm{tw}}\big)\, ds. \tag{6.1}$$

Using Lemma 3.8 we see that $t \mapsto \tau_\Phi(t)$ is a continuous strictly increasing $(\mathcal{F}_t)$-adapted process that satisfies

$$t \le \tau_\Phi(t) \le K_\kappa t \tag{6.2}$$

for $0 \le t \le T$. In particular, we can define a map

$$t_\Phi : [0,T] \times \Omega \to [0,T] \tag{6.3}$$

for which

$$\tau_\Phi(t_\Phi(\tau,\omega),\omega) = \tau. \tag{6.4}$$

We now introduce the time-transformed map

$$\overline{V} : [0,T] \times \Omega \to L^2 \tag{6.5}$$

that acts as

$$\overline{V}(\tau,\omega) \quad = \quad V\big(t_\Phi(\tau,\omega),\omega\big). \tag{6.6}$$

Before stating our main results, we first investigate the effects of this transformation on the terms appearing in (5.9).

Lemma 6.1. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map t_Φ defined in (6.3) satisfies the following properties.*

(i) For every $0 \le \tau \le T$, the random variable $\omega \mapsto t_\Phi(\tau,\omega)$ is an $(\mathcal{F}_t)$-stopping time.

(ii) The map $\tau \mapsto t_\Phi(\tau,\omega)$ is continuous and strictly increasing for all $\omega \in \Omega$.

(iii) For any $0 \le \tau \le T$ and $\omega \in \Omega$ we have the bounds

$$K_\kappa^{-1}\tau \le t_\Phi(\tau,\omega) \le \tau. \tag{6.7}$$

(iv) For every $0 \le t \le T$, the identity

$$t_\Phi(\tau_\Phi(t,\omega),\omega) = t \tag{6.8}$$

holds on the set $\{\omega : \tau_\Phi(t,\omega) \le T\}$.

Proof. On account of the identity

$$\{\omega : t_\Phi(\tau,\omega) \le t\} = \{\omega : \tau_\Phi(t,\omega) \ge \tau\} \tag{6.9}$$

and the fact that the latter set is in $\mathcal{F}_t$, we may conclude that $t_\Phi(\tau)$ is an $(\mathcal{F}_t)$-stopping time. The remaining properties follow directly from (6.2)-(6.4). □

Lemma 6.2. *Consider the setting of Proposition 2.1, recall the maps* $(t_\Phi, \overline{V})$ *defined by (6.3) and (6.6) and suppose that (hPar) is satisfied. Then there exists a filtration* $(\overline{\mathcal{F}}_\tau)_{\tau \geq 0}$ *together with a* $(\overline{\mathcal{F}}_\tau)$*-Brownian motion* $(\overline{\beta}_\tau)_{\tau\geq 0}$ *so that for any* $H \in \mathcal{N}^2([0,T];(\mathcal{F}_t);L^2)$*, the process*

$$\overline{H}(\tau,\omega) = H\big(t_\Phi(\tau,\omega),\omega\big) \tag{6.10}$$

satisfies the following properties.

(i) We have the inclusion

$$\overline{H} \in \mathcal{N}^2([0,T];(\overline{\mathcal{F}}_\tau);L^2), \tag{6.11}$$

together with the bound

$$E\int_0^T \big\|\overline{H}(\tau)\big\|_{L^2}^2 \, d\tau \leq K_\kappa E\int_0^T \|H(t)\|_{L^2}^2 \, dt. \tag{6.12}$$

(ii) For almost all $\omega \in \Omega$*, the identity*

$$\int_0^{t_\Phi(\tau)} H(s)\, d\beta_s = \int_0^\tau \overline{H}(\tau')\kappa_\sigma(\Phi + \overline{V}(\tau'),\psi_{\mathrm{tw}})^{-1/2}\, d\overline{\beta}_{\tau'} \tag{6.13}$$

holds for all $0 \leq \tau \leq T$*.*

Proof. Following [23, §1.2.3], we write

$$\overline{\mathcal{F}}_\tau = \{A \in \cup_{t\geq 0}\mathcal{F}_t : A \cap \{t_\Phi(\tau) \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}. \tag{6.14}$$

The fact that $\overline{H}$ is $(\overline{\mathcal{F}}_\tau)$-progressively measurable can be established following the proof of [22, Lemma 10.8(c)]. In addition, we note that for almost all $\omega \in \Omega$ the path

$$t \mapsto \|V(t,\omega)\|_{L^2}^2 \tag{6.15}$$

is in $L^1([0,T])$, which allows us to apply the deterministic substitution rule to obtain

$$\int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2 \, ds = \int_0^\tau \big\|V\big(t_\Phi(\tau')\big)\big\|_{L^2}^2 \,\partial_\tau t_\Phi(\tau')\, d\tau'. \tag{6.16}$$

We now note that

$$\begin{aligned}\partial_\tau t_\Phi(\tau') &= \big[\partial_t \tau_\Phi\big(t_\Phi(\tau')\big)\big]^{-1} \\ &= \kappa_\sigma\big(\Phi + V\big(t_\Phi(\tau')\big),\psi_{\mathrm{tw}}\big)^{-1} \\ &= \kappa_\sigma\big(\Phi + \overline{V}(\tau'),\psi_{\mathrm{tw}}\big)^{-1}.\end{aligned} \tag{6.17}$$

In particular, we see that

$$|\partial_\tau t_\Phi(\tau')| \geq K_\kappa^{-1} \tag{6.18}$$

and hence

$$\int_0^\tau \big\|V\big(t_\Phi(\tau')\big)\big\|_{L^2}^2 \, d\tau' \leq K_\kappa \int_0^{t_\Phi(\tau)} \|V(s)\|_{L^2}^2 \, ds. \tag{6.19}$$

The bound (6.12) now follows from $t_\Phi(T,\omega) \leq T$.

To obtain (ii), we introduce the Brownian-motion $(\overline{\beta}_\tau)_{\tau\geq 0}$ that is given by

$$\overline{\beta}_\tau = \int_0^\tau \frac{1}{\sqrt{\partial_\tau t_\Phi(\tau')}}\, d\beta_{t_\Phi(\tau')}. \tag{6.20}$$

For any test-function $\zeta \in C_c^\infty(\mathbb{R};\mathbb{R}^n)$ and $0 \leq t \leq T$, the proof of [23, Lem. 5.1.3.5] implies that for almost all $\omega \in \Omega$ the identity

$$\begin{aligned} \textstyle\int_0^{t_\Phi(\tau)} \langle H(s), \zeta\rangle_{L^2} d\beta_s &= \textstyle\int_0^\tau \langle H\big(t_\Phi(\tau')\big), \zeta\rangle_{L^2} \sqrt{\partial_\tau t_\Phi(\tau')}\, d\overline{\beta}_{\tau'} \\ &= \textstyle\int_0^\tau \langle \overline{H}(\tau'), \zeta\rangle_{L^2} \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1/2}\, d\overline{\beta}_{\tau'} \end{aligned} \tag{6.21}$$

holds for all $0 \leq \tau \leq T$. Since (i) and (ii) together imply that the right-hand side of (6.13) is well-defined as a stochastic-integral, a standard diagonalization argument involving the separability of L^2 shows that both sides must be equal for almost all $\omega \in \Omega$. □

In order to formulate the time-transformed sPDE, we introduce the nonlinearity

$$\begin{aligned} \overline{\mathcal{R}}_{\sigma;\Phi,c}(v) &= \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1}\mathcal{R}_{\sigma;\Phi,c}(v) - \mathcal{L}_{\mathrm{tw}} v \\ &= A_*[\Phi + v] + \mathcal{J}_\sigma(\Phi + v, c, \psi_{\mathrm{tw}}) + \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1} a(\Phi + v, c, \psi_{\mathrm{tw}})[\Phi' + v'] \\ &\quad -\mathcal{L}_{\mathrm{tw}} v, \end{aligned} \tag{6.22}$$

together with

$$\begin{aligned} \overline{\mathcal{S}}_{\sigma;\Phi}(v) &= \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1/2}\mathcal{S}_\Phi(v) \\ &= \kappa_\sigma(\Phi + v, \psi_{\mathrm{tw}})^{-1/2}\Big[g(\Phi + v) + b(\Phi + v, \psi_{\mathrm{tw}})[\Phi' + v']\Big]. \end{aligned} \tag{6.23}$$

Proposition 6.3. *Consider the setting of Proposition 2.1 and suppose that (hPar) is satisfied. Then the map*

$$\overline{V} : [0, T] \times \Omega \to L^2 \tag{6.24}$$

defined by the transformations (5.1) and (6.6) satisfies the following properties.

(i) For almost all $\omega \in \Omega$, the map $\tau \mapsto \overline{V}(\tau;\omega)$ is of class $C\big([0, T]; L^2\big)$.

(ii) For all $\tau \in [0, T]$, the map $\omega \mapsto \overline{V}(\tau, \omega)$ is $(\overline{\mathcal{F}}_\tau)$-measurable.

(iii) We have the inclusion

$$\overline{V} \in \mathcal{N}^2\big([0, T]; (\overline{\mathcal{F}})_\tau; H^1\big), \tag{6.25}$$

together with

$$\overline{\mathcal{S}}_{\sigma;\Phi}(\overline{V}) \in \mathcal{N}^2\big([0, T]; (\overline{\mathcal{F}})_\tau; L^2\big). \tag{6.26}$$

(iv) For almost all $\omega \in \Omega$, we have the inclusion

$$\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(V(\cdot, \omega)\big) \in L^1([0, T]; L^2). \tag{6.27}$$

(v) For almost all $\omega \in \Omega$, the identity

$$\begin{aligned} \overline{V}(\tau) &= \overline{V}(0) + \textstyle\int_0^\tau \Big[\mathcal{L}_{\mathrm{tw}}\overline{V}(\tau') + \overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\Big]\, d\tau' \\ &\quad + \sigma \textstyle\int_0^\tau \overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\, d\beta_{\tau'} \end{aligned} \tag{6.28}$$

holds for all $0 \leq t \leq T$.

(vi) For almost all $\omega \in \Omega$, the identity

$$\begin{array}{rcl}\overline{V}(\tau) & = & S(\tau)\overline{V}(0) + \int_0^\tau S(\tau-\tau')\overline{\mathcal{R}}_{\sigma;\Phi,c}\big(\overline{V}(\tau')\big)\,d\tau' \\ & & +\sigma\int_0^\tau S(\tau-\tau')\overline{\mathcal{S}}_{\sigma;\Phi}\big(\overline{V}(\tau')\big)\,d\beta_{\tau'}\end{array} \tag{6.29}$$

holds for all $\tau \in [0,T]$, in which

$$S : [0,\infty) \to \mathcal{L}(L^2;L^2) \tag{6.30}$$

denotes the analytic semigroup generated by $\mathcal{L}_{\mathrm{tw}}$.

Proof. Items (i)-(iii) follow by applying (i) of Lemma 6.2 to the maps V, $\partial_\xi V$ and using the definition (6.23). Item (iv) can be obtained from the computation (5.26), noting that the $A_* v$ contribution is no longer present.

Item (v) can be obtained by applying the stochastic time-transform (6.13) and the deterministic time-transform

$$\int_0^{t_\Phi(\tau)} \mathcal{R}_{\sigma;\Phi_c}\big(V(s)\big)\,ds \quad = \quad \int_0^\tau \mathcal{R}_{\sigma;\Phi_c}\big(\overline{V}(\tau')\big)\big[\kappa_\sigma\big(\Phi+\overline{V}(\tau'),\psi_{\mathrm{tw}}\big)\big]^{-1}\,d\tau' \tag{6.31}$$

to the integral equation (5.9).

Turning to (vi), we note that A_* generates a standard diagonal heat-semigroup, which is obviously analytic. Noting that

$$\mathcal{L}_{\mathrm{tw}} - A_* \in \mathcal{L}(H^1;L^2) \tag{6.32}$$

and recalling the interpolation estimate

$$\|v\|_{H^1} \le C_1\,\|v\|_{H^2}^{1/2}\,\|v\|_{L^2}^{1/2}, \tag{6.33}$$

we may apply [29, Prop 3.2.2(iii)] to conclude that also $\mathcal{L}_{\mathrm{tw}}$ generates an analytic semigroup. We may now apply [31, Prop 6.3] and the computation in the proof of [29, Prop 4.1.4] to conclude the integral identity (6.29). □

We now introduce the scalar functions

$$\begin{array}{rcl}N_{\varepsilon,\alpha}(t) & = & e^{\alpha t}\,\|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}e^{\alpha s}\,\|V(s)\|_{H^1}^2\,ds, \\ \overline{N}_{\varepsilon,\alpha}(\tau) & = & e^{\alpha\tau}\,\big\|\overline{V}(\tau)\big\|_{L^2}^2 + \int_0^\tau e^{-\varepsilon(\tau-\tau')}e^{\alpha\tau'}\,\big\|\overline{V}(\tau')\big\|_{H^1}^2\,d\tau',\end{array} \tag{6.34}$$

together with the associated probabilities

$$\begin{array}{rcl}p_{\varepsilon,\alpha}(T,\eta) & = & P\Big(\sup_{0\le t\le T} N_{\varepsilon,\alpha}(t) > \eta\Big), \\ \overline{p}_{\varepsilon,\alpha}(T,\eta) & = & P\Big(\sup_{0\le \tau\le T} \overline{N}_{\varepsilon,\alpha}(\tau) > \eta\Big).\end{array} \tag{6.35}$$

Our second main result shows that these two sets of probabilities can be effectively compared with each other.

Proposition 6.4. *Consider the setting of Proposition 2.1 and recall the maps V and $\overline{V}$ defined by (5.1) and (6.6). Then we have the bound*

$$p_{\varepsilon,\alpha}(T,\eta) \le \overline{p}_{K_\kappa^{-1}\varepsilon,\alpha}\big(K_\kappa T, K_\kappa^{-1}\eta\big). \tag{6.36}$$

Proof. We note that

$$e^{\alpha t} \left\| V(t) \right\|_{L^2} = e^{\alpha t} \left\| \overline{V}\big(\tau_\Phi(t)\big) \right\|_{L^2} \le e^{\alpha \tau_\Phi(t)} \left\| \overline{V}\big(\tau_\Phi(t)\big) \right\|_{L^2}, \tag{6.37}$$

which implies that

$$\sup_{0 \le t \le T} e^{\alpha t} \left\| V(t) \right\|_{L^2}^2 \le \sup_{0 \le \tau \le K_\kappa T} e^{\alpha \tau} \left\| \overline{V}(\tau) \right\|_{L^2}^2 . \tag{6.38}$$

In addition, we compute

$$\int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \left\| V(s) \right\|_{H^1}^2 \, ds = \int_0^{\tau_\Phi(t)} e^{-\varepsilon(t - t_\Phi(\tau'))} e^{\alpha t_\Phi(\tau')} \left\| \overline{V}(\tau') \right\|_{H^1}^2 \kappa_\sigma\big(\Phi + \overline{V}(\tau'), \psi_{\mathrm{tw}}\big)^{-1} \, d\tau'. \tag{6.39}$$

Using (6.18) we obtain the estimate

$$t - t_\Phi(\tau') = t_\Phi\big(\tau_\Phi(t)\big) - t_\Phi(\tau') = \int_{\tau'}^{\tau_\Phi(t)} \partial_\tau t_\Phi(\tau'') \, d\tau'' \ge K_\kappa^{-1} \left| \tau_\Phi(t) - \tau' \right|, \tag{6.40}$$

which yields

$$\int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \left\| V(s) \right\|_{H^1}^2 \, ds \le K_\kappa \int_0^{\tau_\Phi(t)} e^{-K_\kappa^{-1} \varepsilon (\tau_\Phi(t) - \tau')} e^{\alpha \tau'} \left\| \overline{V}(\tau') \right\|_{H^1}^2 \, d\tau'. \tag{6.41}$$

In particular, we conclude that

$$\sup_{0 \le t \le T} \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \left\| V(s) \right\|_{H^1}^2 \, ds \le \sup_{0 \le \tau \le K_\kappa T} K_\kappa \int_0^\tau e^{-K_\kappa^{-1} \varepsilon(\tau - \tau')} e^{\alpha \tau'} \left\| \overline{V}(\tau') \right\|_{H^1}^2 \, d\tau'. \tag{6.42}$$

This yields the implication

$$\sup_{0 \le \tau \le K_\kappa T} \overline{N}_{K_\kappa^{-1} \varepsilon, \alpha}(\tau) \le K_\kappa^{-1} \eta \Rightarrow \sup_{0 \le t \le T} N_{\varepsilon, \alpha}(t) \le \eta, \tag{6.43}$$

from which the desired inequality immediately follows. □

7 The stochastic wave

In this section we set out to construct the branch of modified waves (Φ_σ, c_σ) and analyze the phase condition

$$\langle T_{-\gamma_0}[u_0] - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{7.1}$$

for $u_0 \approx \Phi_\sigma$. In particular, we establish Propositions 2.2 and 2.3.

A key role in our analysis is reserved for the function

$$\begin{aligned} \mathcal{M}_{\sigma;\Phi,c}(v,d) \quad &= \quad \mathcal{J}_\sigma(\Phi + v, c + d, \psi_{\mathrm{tw}}) - \mathcal{J}_0(\Phi, c) \\ &\qquad -d\Phi_0' + [A_* - \mathcal{L}_{\mathrm{tw}}]v, \end{aligned} \tag{7.2}$$

defined for $(\Phi, c) \in \mathcal{U}_{H^1} \times \mathbb{R}$ and $(v, d) \in H^1 \times \mathbb{R}$. Indeed, we will construct a solution to

$$A_* \Phi_\sigma + \mathcal{J}_\sigma(\Phi_\sigma, c_\sigma, \psi_{\mathrm{tw}}) = 0 \tag{7.3}$$

by writing

$$\Phi_\sigma = \Phi_0 + v, \qquad c_\sigma = c_0 + d. \tag{7.4}$$

Using the fact that the pair (Φ_0, c_0) is a solution to (7.3) for $\sigma = 0$, one readily verifies that the pair $(v, d) \in H^2 \times \mathbb{R}$ must satisfy the system

$$d\Phi_0' + \mathcal{L}_{\mathrm{tw}} v = -\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{7.5}$$

In addition, the function $\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}$ will be used in §8 to obtain bounds on the nonlinearity $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$.

In §7.1 we obtain global and Lipschitz bounds on $\mathcal{M}_{\sigma;\Phi,c}$. These bounds are subsequently used in §7.2 to setup two fixed-point constructions that provide solutions to (7.1) and (7.3).

7.1 Bounds for $\mathcal{M}_\sigma$

In order to streamline our estimates, it is convenient to decompose the function $\mathcal{J}_\sigma$ as

$$\begin{aligned} \mathcal{J}_\sigma(u, \overline{c}, \psi_{\mathrm{tw}}) &= \kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1}\Big[f(u) + \overline{c}u' + \sigma^2 b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)]\Big] \\ &= \mathcal{J}_0(u, \overline{c}) + \mathcal{E}_{\sigma;I}(u, \overline{c}) + \mathcal{E}_{\sigma;II}(u). \end{aligned} \tag{7.6}$$

Here we have introduced the function

$$\begin{aligned} \mathcal{E}_{\sigma;I}(u, \overline{c}) &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})[f(u) + \overline{c}u'] \\ &= \nu_\sigma^{(-1)}(u, \psi_{\mathrm{tw}})\mathcal{J}_0(u, \overline{c}), \end{aligned} \tag{7.7}$$

together with

$$\mathcal{E}_{\sigma;II}(u) = \sigma^2 \kappa_\sigma(u, \psi_{\mathrm{tw}})^{-1} b(u, \psi_{\mathrm{tw}})\partial_\xi[g(u)] \tag{7.8}$$

where ν_σ^{-1} is as defined in (3.55).
This decomposition allows us to rewrite (7.2) in the intermediate form

$$\mathcal{M}_{\sigma;\Phi,c}(v, d) = \mathcal{M}_{0;\Phi,c}(v, d) + \mathcal{E}_{\sigma;I}(\Phi + v, c + d) + \mathcal{E}_{\sigma;II}(\Phi + v). \tag{7.9}$$

We now make a final splitting

$$\begin{aligned} \mathcal{M}_{0;\Phi,c}(v, d) &= \mathcal{J}_0(\Phi + v, c + d) - \mathcal{J}_0(\Phi, c) - Df(\Phi_0)v - c_0 v' - d\Phi_0' \\ &= \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v, d), \end{aligned} \tag{7.10}$$

in which we have introduced the function

$$\mathcal{N}_{I;f,\Phi}(v) = f(\Phi + v) - f(\Phi) - Df(\Phi)v, \tag{7.11}$$

together with

$$\mathcal{N}_{II;\Phi,c}(v, d) = dv' + \big[Df(\Phi) - Df(\Phi_0)\big]v + (c - c_0)v' + d[\Phi' - \Phi_0']. \tag{7.12}$$

We hence arrive at the convenient final expression

$$\mathcal{M}_{\sigma;\Phi,c}(v, d) = \mathcal{N}_{I;f,\Phi}(v) + \mathcal{N}_{II;\Phi,c}(v, d) + \mathcal{E}_{\sigma;I}(\Phi + v, c + d) + \mathcal{E}_{\sigma;II}(\Phi + v) \tag{7.13}$$

and set out to analyze each of these terms separately.

Lemma 7.1. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $v \in H^1$ we have the bound*

$$\|\mathcal{N}_{I;f,\Phi}(v)\|_{L^2} \quad \leq \quad K\big[1 + \|v\|_{H^1}\big]\, \|v\|_{H^1}\, \|v\|_{L^2}\,, \tag{7.14}$$

while for any pair $(v_A, v_B) \in H^1 \times H^1$ we have the estimates

$$\begin{array}{lcl} \|\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)\|_{L^2} & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\, \|v_A\|_{H^1} + \|v_B\|_{H^1}\big] \\ & & \times\, \|v_A - v_B\|_{L^2}\,, \\ |\langle \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B), \psi_{\rm tw}\rangle_{L^2}| & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\, \|v_A\|_{L^2} + \|v_B\|_{L^2}\big] \\ & & \times\, \|v_A - v_B\|_{L^2}\,. \end{array} \tag{7.15}$$

Proof. Using (3.6) and (hPar) we obtain the pointwise bound

$$|\mathcal{N}_{I;f,\Phi}(v)| \leq C_1[1 + |v|]\, |v|^2\,, \tag{7.16}$$

from which (7.14) easily follows. In addition, we may compute

$$\begin{array}{rcl} \mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B) & = & f(\Phi + v_A) - f(\Phi + v_B) - Df(\Phi + v_B)(v_A - v_B) \\ & & +\big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B) \\ & = & \mathcal{N}_{I;f,\Phi+v_B}(v_A - v_B) + \big(Df(\Phi + v_B) - Df(\Phi)\big)(v_A - v_B). \end{array} \tag{7.17}$$

Applying (3.6) and (hPar) a second time, we obtain the pointwise bound

$$\begin{array}{rcl} |\mathcal{N}_{I;f,\Phi}(v_A) - \mathcal{N}_{I;f,\Phi}(v_B)| & \leq & C_2[1 + |v_A| + |v_B|\,]\, |v_A - v_B|^2 \\ & & +C_2[1 + |v_B|]\, |v_B|\, |v_A - v_B| \\ & \leq & C_3[1 + |v_A| + |v_B|\,]\big[\, |v_A| + |v_B|\,\big]\, |v_A - v_B|\,, \end{array} \tag{7.18}$$

from which the estimates in (7.15) can be readily obtained. □

Lemma 7.2. *Suppose that (Hf) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $(v, d) \in H^1 \times \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{II;\Phi,c}(v, d)\|_{L^2} \quad \leq \quad K\big[\, |c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d|\,\big]\big[\, \|v\|_{H^1} + |d|\,\big], \tag{7.19}$$

while for any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ the expression

$$\Delta_{AB}\mathcal{N}_{II;\Phi,c} = \mathcal{N}_{II;\Phi,c}(v_A, d_A) - \mathcal{N}_{II;\Phi,c}(v_B, d_B) \tag{7.20}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{N}_{II;\Phi,c}\|_{L^2} & \leq & K\big[\, \|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0|\,\big] \\ & & \times\big[\, \|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big], \\ |\langle \Delta_{AB}\mathcal{N}_{II;\Phi,c}, \psi_{\rm tw}\rangle_{L^2}| & \leq & K\big[\, \|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\,\big] \\ & & \times\big[\, \|v_A - v_B\|_{L^2} + |d_A - d_B|\,\big]. \end{array} \tag{7.21}$$

Proof. In view of (hPar), we obtain the pointwise bound

$$|\mathcal{N}_{II;\Phi,c}(v, d)| \leq \big[\, |d| + |c - c_0|\,\big]\, |v'| + C_1\, |\Phi - \Phi_0|\, |v| + |\Phi' - \Phi_0'|\, |d|\,, \tag{7.22}$$

from which (7.19) follows. In addition, we obtain the pointwise bound

$$\begin{array}{rcl} |\Delta_{AB}\mathcal{N}_{II;\Phi,c}| & \leq & |d_A - d_B|\, |v_A'| + \big[\, |d_B| + |c - c_0|\,\big]\, |v_A' - v_B'| \\ & & +K\, |\Phi - \Phi_0|\,\big[\, |v_A - v_B|\,\big] + |\Phi' - \Phi_0'|\, |d_A - d_B| \end{array} \tag{7.23}$$

from which (7.21) follows. □

Lemma 7.3. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \le \sigma \le 1$ and $(v, d) \in H^1 \times \mathbb{R}$, we have the bound*

$$\|\mathcal{E}_{\sigma;I}(\Phi + v, c + d)\|_{L^2} \quad \le \quad K\sigma^2(1 + |d|)\big[1 + \|v\|_{H^1} + \|v\|_{H^1}^2 \|v\|_{L^2}\big], \tag{7.24}$$

while for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$, the expression

$$\Delta_{AB}\mathcal{E}_{\sigma;I} = \mathcal{E}_{\sigma;I}(\Phi + v_A, c + d_A) - \mathcal{E}_{\sigma;I}(\Phi + v_B, c + d_B) \tag{7.25}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} & \le & K\sigma^2(1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2 \|v_A\|_{L^2}\big] \|v_A - v_B\|_{L^2} \\ & & \quad +K\sigma^2\big[1 + |d_B| + \|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2}\big] \|v_A - v_B\|_{H^1} \\ & & \quad +K\sigma^2\big[1 + \|v_A\|_{H^1}\big] |d_A - d_B| , \\ |\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\mathrm{tw}} \rangle_{L^2}| & \le & K\sigma^2(1 + |d_A| + |d_B|)\big[1 + \|v_A\|_{H^1} \|v_A\|_{L^2}^2\big] \|v_A - v_B\|_{L^2} \\ & & \quad +K\sigma^2\big[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\big] \|v_A - v_B\|_{H^1} \\ & & \quad +K\sigma^2[1 + \|v_A\|_{L^2}] |d_A - d_B| . \end{array} \tag{7.26}$$

Proof. The bound (7.24) follows directly from Lemma's 3.4 and 3.9. In addition, these results allow us to compute

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;I}\|_{L^2} & \le & \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, \|\mathcal{J}_0(\Phi + v_A, c + d_A)\|_{L^2} \\ & & \quad + \Big|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, \|\mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B)\|_{L^2} \\ & \le & C_1\sigma^2 \|v_A - v_B\|_{L^2} (1 + |d_A|)\big[1 + \|v_A\|_{H^1} + \|v_A\|_{H^1}^2 \|v\|_{L^2}\big] \\ & & \quad +C_1\sigma^2\big[\|v_A\|_{H^1} \|v_A\|_{L^2} + \|v_B\|_{H^1} \|v_B\|_{L^2}\big] \|v_A - v_B\|_{H^1} \\ & & \quad +C_1\sigma^2[1 + \|v_A\|_{H^1}] |d_A - d_B| \\ & & \quad +C_1\sigma^2(1 + |d_B|) \|v_A - v_B\|_{H^1} , \end{array} \tag{7.27}$$

together with

$$\begin{array}{lcl} |\langle \Delta_{AB}\mathcal{E}_{\sigma;I}, \psi_{\mathrm{tw}} \rangle_{L^2}| & \le & \Big|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, |\langle \mathcal{J}_0(\Phi + v_A, c + d_A), \psi_{\mathrm{tw}} \rangle_{L^2}| \\ & & \quad + \Big|\nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\Big| \, |\langle \mathcal{J}_0(\Phi + v_A, c + d_A) - \mathcal{J}_0(\Phi + v_B, c + d_B), \psi_{\mathrm{tw}} \rangle_{L^2}| \\ & \le & C_2\sigma^2 \|v - w\|_{L^2} (1 + |d_A|)\big[1 + \|v_A\|_{H^1} \|v_A\|_{L^2}^2\big] \\ & & \quad +C_2\sigma^2[\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2] \|v_A - v_B\|_{H^1} \\ & & \quad +C_2\sigma^2[1 + \|v_A\|_{L^2}] |d_A - d_B| \\ & & \quad +C_2\sigma^2(1 + |d_B|) \|v_A - v_B\|_{L^2} . \end{array} \tag{7.28}$$

These terms can all be absorbed by the expressions in (7.26). □

Lemma 7.4. *Suppose that (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that for any $0 \le \sigma \le 1$ and $v \in H^1$ we have the bound*

$$\|\mathcal{E}_{\sigma;II}(\Phi + v)\|_{L^2} \quad \le \quad K\sigma^2\big[1 + \|v\|_{H^1}\big], \tag{7.29}$$

while for any $0 \le \sigma \le 1$ and any pair $(v_A, v_B) \in H^1 \times H^1$ the expression

$$\Delta_{AB}\mathcal{E}_{\sigma;II} = \mathcal{E}_{\sigma;II}(\Phi + v_A) - \mathcal{E}_{\sigma;II}(\Phi + v_B) \tag{7.30}$$

satisfies the estimates

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} & \le & K\sigma^2\big[1 + \|v_A\|_{H^1}\big]\, \|v_A - v_B\|_{H^1}\,, \\ |\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\mathrm{tw}}\rangle_{L^2}| & \le & K\sigma^2\big[1 + \|v_A\|_{L^2}\big]\, \|v_A - v_B\|_{L^2}\,. \end{array} \tag{7.31}$$

Proof. The bound (7.29) follows directly from Lemma's 3.2, 3.6 and 3.8. In addition, we may compute

$$\begin{array}{lcl} \|\Delta_{AB}\mathcal{E}_{\sigma;II}\|_{L^2} & \le & \sigma^2 \left|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\right| K_b \,\|\partial_\xi[g(\Phi + v)]\|_{L^2} \\ & & \quad + \sigma^2 K_\nu \,|b(\Phi + v_A, \psi_{\mathrm{tw}}) - b(\Phi + v_B, \psi_{\mathrm{tw}})|\, \|\partial_\xi[g(\Phi + v_A)]\|_{L^2} \\ & & \quad + \sigma^2 K_\kappa K_b \,\|\partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)]\|_{L^2} \\ & \le & C_1\sigma^2 \,\|v_A - v_B\|_{L^2}\, \big[1 + \|v_A\|_{H^1}\big] \\ & & \quad + C_1\sigma^2 \,\|v_A - v_B\|_{L^2}\, \big[1 + \|v_A\|_{H^1}\big] \\ & & \quad + C_1\sigma^2\big[1 + \|v_A\|_{H^1}\big]\, \|v_A - v_B\|_{H^1}\,, \end{array} \tag{7.32}$$

together with

$$\begin{array}{lcl} |\langle \Delta_{AB}\mathcal{E}_{\sigma;II}, \psi_{\mathrm{tw}}\rangle_{L^2}| & \le & \sigma^2 \left|\nu_\sigma^{(-1)}(\Phi + v_A, \psi_{\mathrm{tw}}) - \nu_\sigma^{(-1)}(\Phi + v_B, \psi_{\mathrm{tw}})\right| K_b \,|\langle \partial_\xi[g(\Phi + v)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\ & & \quad + \sigma^2 K_\nu \,|b(\Phi + v_A, \psi_{\mathrm{tw}}) - b(\Phi + v_B, \psi_{\mathrm{tw}})|\, |\langle \partial_\xi[g(\Phi + v)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\ & & \quad + \sigma^2 K_\nu K_b \,|\langle \partial_\xi[g(\Phi + v_A) - g(\Phi + v_B)], \psi_{\mathrm{tw}}\rangle_{L^2}| \\ & \le & C_2\sigma^2 \,\|v_A - v_B\|_{L^2}\, \big[1 + \|v_A\|_{L^2}\big] \\ & & \quad + C_2\sigma^2 \,\|v_A - v_B\|_{L^2}\, \big[1 + \|v_A\|_{L^2}\big] \\ & & \quad + C_2\sigma^2 \,\|v_A - v_B\|_{L^2}\,. \end{array} \tag{7.33}$$

These expressions can be absorbed into the bounds (7.31). □

Corollary 7.5. *Suppose that (Hf), (Hg) and (hPar) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \le \sigma \le 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $|d| \le 1$, we have the estimate*

$$\begin{array}{lcl} \|\mathcal{M}_{\sigma;\Phi,c}(v, d)\|_{L^2} & \le & K\big[1 + \|v\|_{H^1}\big]\, \|v\|_{H^1}\, \|v\|_{L^2} \\ & & \quad + K\big[\,|c - c_0| + \|\Phi - \Phi_0\|_{H^1} + |d|\,\big]\big[\,\|v\|_{H^1} + |d|\,\big] \\ & & \quad + K\sigma^2\big[1 + \|v\|_{H^1}\big]. \end{array} \tag{7.34}$$

In addition, for any $0 \le \sigma \le 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which $|d_A| \le 1$ and $|d_B| \le 1$, the expression

$$\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c} = \mathcal{M}_{\sigma;\Phi,c}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi,c}(v_B, d_B) \tag{7.35}$$

satisfies the estimates

$$\begin{array}{rcl} \|\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c}\|_{L^2} & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\,\|v_A - v_B\|_{L^2} \\ & & +K\big[\sigma^2 + \|v_A\|_{H^1} + |d_B| + \|\Phi - \Phi_0\|_{H^1} + |c - c_0|\,\big] \\ & & \quad\times\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big] \\ & & +K\sigma^2\,\|v_A\|_{H^1}^2\,\|v_A\|_{L^2}\,\|v_A - v_B\|_{L^2} \\ & & +K\sigma^2\Big[\,\|v_A\|_{H^1}\,\|v_A\|_{L^2} + \|v_B\|_{H^1}\,\|v_B\|_{L^2}\,\Big]\,\|v_A - v_B\|_{H^1}\,, \\ |\langle\Delta_{AB}\mathcal{M}_{\sigma;\Phi,c},\psi_{\mathrm{tw}}\rangle_{L^2}| & \leq & K\big[1 + \|v_A\|_{H^1} + \|v_B\|_{H^1}\big]\big[\,\|v_A\|_{L^2} + \|v_B\|_{L^2}\big]\,\|v_A - v_B\|_{L^2} \\ & & +K\big[\sigma^2 + \|v_A\|_{L^2} + |d_B| + \|\Phi - \Phi_0\|_{L^2} + |c - c_0|\,\big] \\ & & \quad\times\big[\,\|v_A - v_B\|_{L^2} + |d_A - d_B|\,\big] \\ & & +K\sigma^2\,\|v_A\|_{H^1}\,\|v_A\|_{L^2}^2\,\|v_A - v_B\|_{L^2} \\ & & +K\sigma^2\Big[\,\|v_A\|_{L^2}^2 + \|v_B\|_{L^2}^2\,\Big]\,\|v_A - v_B\|_{H^1}\,. \end{array} \tag{7.36}$$

Proof. In view of the identity (7.13) it suffices to note that the terms (7.14), (7.19), (7.24) and (7.29) can be absorbed in (7.34), while the expressions (7.15), (7.21), (7.26) and (7.31) can be absorbed in (7.36). □

Corollary 7.6. *Suppose that (Hf) and (Hg) are satisfied. Then there exists $K > 0$ so that the following holds true. For any $0 \leq \sigma \leq 1$ and any $(v, d) \in H^1 \times \mathbb{R}$ that has $\|v\|_{H^1} \leq 1$ together with $|d| \leq 1$, we have the estimate*

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v,d)\|_{L^2} \quad \leq \quad K\big[\,\|v\|_{L^2} + |d|\,\big]\big[\,\|v\|_{H^1} + |d|\,\big] + K\sigma^2. \tag{7.37}$$

In addition, for any $0 \leq \sigma \leq 1$ and any set of pairs $(v_A, v_B) \in H^1 \times H^1$ and $(d_A, d_B) \in \mathbb{R}^2$ for which the bounds

$$\|v_A\|_{H^1} \leq 1, \qquad |d_A| \leq 1, \qquad \|v_B\|_{H^1} \leq 1, \qquad |d_B| \leq 1 \tag{7.38}$$

hold, the expression

$$\Delta_{AB}\mathcal{M}_{\sigma;\Phi_0,c_0} = \mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B) \tag{7.39}$$

satisfies the estimate

$$\|\Delta_{AB}\mathcal{M}_{\sigma;\Phi_0,c_0}\|_{L^2} \quad \leq \quad K\big[\sigma^2 + \|v_A\|_{H^1} + \|v_B\|_{H^1} + |d_B|\,\big]\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big]. \tag{7.40}$$

Proof. These bounds can easily be obtained by simplifying the corresponding expressions from Corollary 7.5. □

7.2 Fixed-point constructions

As a final preparation before setting up our fixed-point problems, we need to control the higher order effects that arise when translating the adjoint eigenfunction ψ_{tw}. In particular, for any $\gamma \in \mathbb{R}$ we introduce the function

$$\mathcal{N}_{\mathrm{tw}}(\gamma) = T_\gamma\psi_{\mathrm{tw}} - \psi_{\mathrm{tw}} + \gamma\psi'_{\mathrm{tw}} \tag{7.41}$$

and obtain the following bounds.

Lemma 7.7. *Suppose that (HTw) and (HS) hold. Then there exists $K > 0$ so that for any $\gamma \in \mathbb{R}$ we have the bound*

$$\|\mathcal{N}_{\rm tw}(\gamma)\|_{L^2} \le K\gamma^2, \tag{7.42}$$

while for any pair $(\gamma_A, \gamma_B) \in \mathbb{R}^2$ we have the estimate

$$\|\mathcal{N}_{\rm tw}(\gamma_A) - \mathcal{N}_{\rm tw}(\gamma_B)\|_{L^2} \le K\big[\,|\gamma_A| + |\gamma_B|\,\big]\,|\gamma_A - \gamma_B|\,. \tag{7.43}$$

Proof. In view of (4.31), we have the a-priori bound

$$\|\mathcal{N}_{\rm tw}(\gamma)\|_{L^2} \le C_1\big[1 + |\gamma|\,\big], \tag{7.44}$$

together with

$$\|\mathcal{N}_{\rm tw}(\gamma_A) - \mathcal{N}_{\rm tw}(\gamma_B)\|_{L^2} \le C_1\,|\gamma_A - \gamma_B|\,. \tag{7.45}$$

In particular, we can restrict our attention to the situation where $|\gamma| \le 1$ and $|\gamma_A| + |\gamma_B| \le 1$. In this case we obtain the pointwise bounds

$$|\mathcal{N}_{\rm tw}(\gamma)(\xi)| \le \frac{1}{2}\gamma^2 \sup_{\xi - 1 \le \xi' \le \xi + 1} |\psi''_{\rm tw}(\xi')| \tag{7.46}$$

together with

$$|\mathcal{N}_{\rm tw}(\gamma_A)(\xi) - \mathcal{N}_{\rm tw}(\gamma_B)(\xi)| \le \big[\sup_{\xi-1\le\xi'\le\xi+1} |\psi''_{\rm tw}(\xi')|\,\big]\,\big[\frac{1}{2}(\gamma_A - \gamma_B)^2 + |\gamma_B|\,|\gamma_A - \gamma_B|\,\big]. \tag{7.47}$$

The desired bounds now follow from the exponential decay of $\psi''_{\rm tw}$. □

Proof of Proposition 2.2. As a consequence of (HS), there exists a bounded linear map

$$\mathcal{L}_{\rm inv} : L^2 \to H^2 \times \mathbb{R} \tag{7.48}$$

so that for any $h \in L^2$, the pair $(v, d) = \mathcal{L}_{\rm inv} h$ is the unique solution in $H^2 \times \mathbb{R}$ to the problem

$$\mathcal{L}_{\rm tw} v = h - \Phi_0' d. \tag{7.49}$$

Indeed, we take $d = \langle h, \psi_{\rm tw}\rangle_{L^2}$, which in view of the normalization (2.14) ensures that the right-hand side of (7.49) is in the range of $\mathcal{L}_{\rm tw}$.

It now suffices to find a solution to the fixed-point problem

$$(v, d) = -\mathcal{L}_{\rm inv}\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d). \tag{7.50}$$

Upon introducing the set

$$\mathcal{Z}_\Theta = \big\{(v, d) \in H^2 \times \mathbb{R} : \|v\|_{H^2} + |d| \le \min\{1, \Theta\sigma^2\}\big\} \subset H^2 \times \mathbb{R} \tag{7.51}$$

and applying Corollary 7.6, we see that for any $(v, d) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v, d)\|_{L^2} \le K\big(\Theta^4\sigma^4 + \sigma^2\big) = K\sigma^2\big(\Theta^2\sigma^2 + 1\big), \tag{7.52}$$

while for any two pairs $(v_A, d_A) \in \mathcal{Z}_\Theta$ and $(v_B, d_B) \in \mathcal{Z}_\Theta$ we have

$$\|\mathcal{M}_{\sigma;\Phi_0,c_0}(v_A, d_A) - \mathcal{M}_{\sigma;\Phi_0,c_0}(v_B, d_B)\|_{L^2} \le K\sigma^2\big[1 + 2\Theta\big]\big[\,\|v_A - v_B\|_{H^1} + |d_A - d_B|\,\big]. \tag{7.53}$$

In particular, choosing Θ to be sufficiently large and $\delta_\sigma > 0$ to be sufficiently small, we see that the map $-\mathcal{L}_{\rm inv}\mathcal{M}_{\sigma;\Phi_0,c_0}$ is a contraction on $\mathcal{Z}_\Theta$ for all $0 \le \sigma \le \delta_\sigma$. □

Proof of Proposition 2.3. We first recall that

$$\langle \Phi_{\mathrm{ref}}, \psi_{\mathrm{tw}}' \rangle_{L^2} = -\langle \Phi_{\mathrm{ref}}', \psi_{\mathrm{tw}} \rangle_{L^2} = -\langle \Phi_0', \psi_{\mathrm{tw}} \rangle_{L^2} = -1. \tag{7.54}$$

Writing $u_0 = x_0 + \Phi_{\mathrm{ref}}$, this allows us to compute

$$\begin{array}{lcl} \langle v_\gamma, \psi_{\mathrm{tw}} \rangle_{L^2} & = & \langle x_0 + \Phi_{\mathrm{ref}}, T_\gamma \psi_{\mathrm{tw}} \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle x_0 + \Phi_{\mathrm{ref}}, \psi_{\mathrm{tw}} - \gamma \psi_{\mathrm{tw}}' + \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2} - \langle \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \gamma + \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \mathcal{E}_\sigma(x_0, \gamma), \end{array} \tag{7.55}$$

in which we have introduced the expression

$$\mathcal{E}_\sigma(x_0, \gamma) = -\gamma \langle x_0, \psi_{\mathrm{tw}}' \rangle_{L^2} + \langle x_0 + \Phi_{\mathrm{ref}}, \mathcal{N}_{\mathrm{tw}}(\gamma) \rangle_{L^2}. \tag{7.56}$$

Using Lemma 7.7, we obtain the estimate

$$\| \mathcal{E}_\sigma(x_0, \gamma) \|_{L^2} \le C_1 \, \|x_0\|_{L^2} \, |\gamma| + C_1 \big[1 + \|x_0\|_{L^2} \big] \gamma^2, \tag{7.57}$$

together with the Lipschitz bound

$$\begin{array}{lcl} \| \mathcal{E}_\sigma(x_0, \gamma_A) - \mathcal{E}_\sigma(x_0, \gamma_B) \|_{L^2} & \le & C_2 \, \|x_0\|_{L^2} \, |\gamma_A - \gamma_B| \\ & & \quad + C_2 \big[1 + \|x_0\|_{L^2} \big] \big[\, |\gamma_A| + |\gamma_B| \, \big] \, |\gamma_A - \gamma_B| \, . \end{array} \tag{7.58}$$

In particular, upon choosing $\delta_{\mathrm{fix}} > 0$ to be sufficiently small and imposing the restriction

$$\|x_0\|_{L^2} + \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} < \delta_{\mathrm{fix}}, \tag{7.59}$$

we can define γ_0 as the unique solution to the fixed-point problem

$$-\gamma_0 = \langle x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \mathcal{E}_\sigma(x_0, \gamma) \tag{7.60}$$

on the set

$$\Sigma_{x_0} = \{ \gamma_0 : |\gamma_0| \le 2 \, \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} \, \|\psi_{\mathrm{tw}}\|_{L^2} \}. \tag{7.61}$$

By choosing $\delta_\sigma > 0$ and $\delta_0 > 0$ to be sufficiently small, the bound (2.47) allows us to conclude that (7.59) is satisfied whenever (2.48) holds.

For any $\gamma \in \mathbb{R}$ we can compute

$$\begin{array}{lcl} \|T_{-\gamma} \Phi_\sigma - \Phi_\sigma\|_{L^2}^2 & = & \int \big(\Phi_\sigma(\xi + \gamma) - \Phi_\sigma(\xi) \big)^2 \, d\xi \\ & = & \int \big[\int_0^\gamma \Phi_\sigma'(\xi + s) \, ds \big]^2 \, d\xi \\ & \le & \int |\gamma| \int_0^\gamma \Phi_\sigma'(\xi + s)^2 \, ds \, d\xi \\ & = & |\gamma|^2 \int \Phi_\sigma'(\xi)^2 \, d\xi \\ & = & |\gamma|^2 \, \|\Phi_\sigma'\|_{L^2}^2 \, . \end{array} \tag{7.62}$$

In particular, we obtain the bound

$$\begin{array}{lcl} \|v_{\gamma_0}\|_{L^2} & = & \|x_0 + \Phi_{\mathrm{ref}} - T_{\gamma_0} \Phi_\sigma\|_{L^2} \\ & \le & \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + \|T_{\gamma_0} \Phi_\sigma - \Phi_\sigma\|_{L^2} \\ & \le & \|x_0 + \Phi_{\mathrm{ref}} - \Phi_\sigma\|_{L^2} + C_3 \, |\gamma_0| \, . \end{array} \tag{7.63}$$

The desired estimate (2.51) hence follows from $\gamma_0 \in \Sigma_{x_0}$. The final estimate (2.52) follows in a similar fashion, exploiting $\Phi_\sigma'' \in L^2$. □

8 Bounds on mild nonlinearities

In this section we set out to obtain bounds on the nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ and $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ defined in (6.22)-(6.23). In addition, we show that our choices (2.25) and (2.27) for a_σ and b prevent these nonlinearities from having a component in the subspace of L^2 on which the semigroup $S(t)$ does not decay, provided the cut-offs are not hit.

Our main result below shows that the construction of Φ_σ has eliminated all $O(1)$-terms from the deterministic nonlinearity $\overline{\mathcal{R}}$, leaving only a small linear contribution together with the expected higher order terms. It is important to note here that these higher order terms depend at most quadratically on $\|v\|_{H^1}$, besides powers of $\|v\|_{L^2}$.

In general, the stochastic nonlinearity $\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}$ will have an $O(1)$-term, but we have an explicit expression for this contribution. In this case, the higher order terms depend at most linearly on $\|v\|_{H^1}$.

Proposition 8.1. *Consider the setting of Proposition 2.2 and recall the definitions (6.22) and (6.23). Then there exists $K > 0$ so that for any $0 \le \sigma \le \delta_\sigma$ and any $v \in H^1$, the following properties hold true.*

(i) We have the bound

$$\left\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\right\|_{L^2} \quad \le \quad K\sigma^2 \|v\|_{H^1} + K \|v\|_{H^1}^2 \left[1 + \|v\|_{L^2}^2 + \sigma^2 \|v\|_{L^2}^3\right]. \tag{8.1}$$

(ii) We have the estimate

$$\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\right\|_{L^2} \le K\left[1 + \|v\|_{H^1}\right]. \tag{8.2}$$

(iii) If the inequality

$$\|v\|_{L^2} \le \min\{1, [4\|\psi_{\mathrm{tw}}\|_{H^1}]^{-1}\} \tag{8.3}$$

holds, then we have the identities

$$\langle \overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = \langle \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v), \psi_{\mathrm{tw}}\rangle_{L^2} = 0. \tag{8.4}$$

(iv) If the identity

$$g(\Phi_\sigma) = -b(\Phi_\sigma, \psi_{\mathrm{tw}})\Phi_\sigma' \tag{8.5}$$

holds, then we have the bound

$$\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\right\|_{L^2} \le K\|v\|_{H^1}. \tag{8.6}$$

In order to derive a compact expression for $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$, it is convenient to recall the definition (7.2) and introduce the function

$$\overline{\mathcal{R}}_{\sigma;I}(v) = \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0). \tag{8.7}$$

We note that the bounds in Corollary 7.5 are directly applicable to this function.

Lemma 8.2. *Consider the setting of Proposition 2.2. Then for any $0 \le \sigma \le \delta_\sigma$ and $v \in H^1$, we have the identity*

$$\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v) = \overline{\mathcal{R}}_{\sigma;I}(v) - \Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi[\Phi_\sigma + v], \psi_{\mathrm{tw}}\rangle_{L^2}\big)\Big]^{-1} \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}}\rangle_{L^2}[\Phi_\sigma' + v']. \tag{8.8}$$

Proof. Inspecting (7.2) and using the defining property (2.46) for (Φ_σ, c_σ), we see that

$$-\mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) = A_* \Phi_\sigma + \mathcal{J}_0(\Phi_\sigma, c_\sigma). \tag{8.9}$$

Applying (7.2) once more, we hence find

$$\begin{array}{lcl} \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}) & = & \mathcal{J}_0(\Phi_\sigma, c_\sigma) + [\mathcal{L}_{\mathrm{tw}} - A_*]v + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) \\ & = & [\mathcal{L}_{\mathrm{tw}} - A_*]v - A_*\Phi_\sigma + \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(v,0) - \mathcal{M}_{\sigma;\Phi_\sigma,c_\sigma}(0,0) \\ & = & [\mathcal{L}_{\mathrm{tw}} - A_*]v - A_*\Phi_\sigma + \overline{\mathcal{R}}_{\sigma;I}(v). \end{array} \tag{8.10}$$

Writing

$$\mathcal{I}_\sigma(v) \quad = \quad \langle \Phi_\sigma + v, A_* \psi_{\mathrm{tw}} \rangle_{L^2} + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \tag{8.11}$$

and using $\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} \psi_{\mathrm{tw}} = 0$, we may compute

$$\begin{array}{lcl} \mathcal{I}_\sigma(v) & = & \langle \Phi_\sigma, A_* \psi_{\mathrm{tw}} \rangle_{L^2} + \langle v, [A_* - \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}] \psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle A_* \Phi_\sigma, \psi_{\mathrm{tw}} \rangle_{L^2} + \langle [A_* - \mathcal{L}_{\mathrm{tw}}] v, \psi_{\mathrm{tw}} \rangle_{L^2} \\ & & \quad + \langle \mathcal{J}_\sigma(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}), \psi_{\mathrm{tw}} \rangle_{L^2} \\ & = & \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}} \rangle_{L^2}. \end{array} \tag{8.12}$$

In view of the definition (2.27) for a_σ, we now obtain

$$\begin{array}{lcl} \kappa_\sigma(\Phi_\sigma + v, \psi_{\mathrm{tw}})^{-1} a_\sigma\big(\Phi_\sigma + v, c_\sigma, \psi_{\mathrm{tw}}\big) & = & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi [\Phi_\sigma + v], \psi_{\mathrm{tw}} \rangle_{L^2}\big)\Big]^{-1} \mathcal{I}_\sigma(v) \\ & = & -\Big[\chi_{\mathrm{low}}\big(\langle \partial_\xi [\Phi_\sigma + v], \psi_{\mathrm{tw}} \rangle_{L^2}\big)\Big]^{-1} \langle \overline{\mathcal{R}}_{\sigma;I}(v), \psi_{\mathrm{tw}} \rangle_{L^2}. \end{array} \tag{8.13}$$

In particular, the desired identity (8.8) follows directly from the definition (6.22). □

Lemma 8.3. *Consider the setting of Proposition 2.2. Then there exists $K > 0$ so that for any $v \in H^1$ and $0 \le \sigma \le \delta_\sigma$ we have the bound*

$$\big\| \overline{\mathcal{R}}_{\sigma;I}(v) \big\|_{L^2} \quad \le \quad K\sigma^2 \|v\|_{H^1} + K \|v\|_{H^1}^2 \big[1 + \|v\|_{L^2} + \sigma^2 \|v\|_{L^2}^2\big], \tag{8.14}$$

together with

$$\big| \langle \overline{\mathcal{R}}_\sigma(v), \psi_{\mathrm{tw}} \rangle_{L^2} \big| \quad \le \quad K \|v\|_{L^2} \big[\sigma^2 + \|v\|_{L^2}\big] + K \|v\|_{H^1} \big[\|v\|_{L^2}^2 + \sigma^2 \|v\|_{L^2}^3\big]. \tag{8.15}$$

Proof. Applying Corollary 7.5, we find

$$\begin{array}{lcl} \big\| \overline{\mathcal{R}}_{\sigma;I}(v) \big\|_{L^2} & \le & C_1\big[1 + \|v\|_{H^1}\big] \|v\|_{H^1} \|v\|_{L^2} \\ & & + C_1\big[\sigma^2 + \|v\|_{H^1}\big] \|v\|_{H^1} \\ & & + C_1 \sigma^2 \|v\|_{H^1}^2 \|v\|_{L^2} \|v\|_{L^2} \\ & & + C_1 \sigma^2 \|v\|_{H^1} \|v\|_{L^2} \|v\|_{H^1}, \end{array} \tag{8.16}$$

together with

$$
\begin{aligned}
\left|\langle\overline{\mathcal{R}}_{\sigma;I}(v),\psi_{\rm tw}\rangle_{L^2}\right| \quad \leq \quad & C_2\big[1+\|v\|_{H^1}\big]\,\|v\|_{L^2}\,\|v\|_{L^2} \\
& +C_2\big[\sigma^2+\|v\|_{L^2}\big]\,\|v\|_{L^2} \\
& +C_2\sigma^2\,\|v\|_{H^1}\,\|v\|_{L^2}^2\,\|v\|_{L^2} \\
& +C_2\sigma^2\,\|v\|_{L^2}^2\,\|v\|_{H^1}\,.
\end{aligned} \tag{8.17}
$$

These expressions can be absorbed into (8.14) and (8.15). □

Lemma 8.4. *Consider the setting of Proposition 2.2. Then there exists $K>0$ so that for any $0\leq\sigma\leq\delta_\sigma$ and any $v\in H^1$ we have the bound*

$$
\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2}\leq K\,\|v\|_{H^1}\,. \tag{8.18}
$$

Proof. Writing

$$
\begin{aligned}
\mathcal{I} \quad &= \quad \overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) \\
&= \quad \kappa_\sigma(\Phi_\sigma+v,\psi_{\rm tw})^{-1/2}\Big[g(\Phi_\sigma+v)+b(\Phi_\sigma+v,\psi_{\rm tw})\partial_\xi[\Phi_\sigma+v]\Big] \\
&\qquad -\kappa_\sigma(\Phi_\sigma,\psi_{\rm tw})^{-1/2}\Big[g(\Phi_\sigma)+b(\Phi_\sigma,\psi_{\rm tw})\partial_\xi[\Phi_\sigma]\Big]
\end{aligned} \tag{8.19}
$$

and using Lemma's 3.2, 3.6, 3.8 and 3.9, we compute

$$
\begin{aligned}
\|\mathcal{I}\|_{L^2} \quad \leq \quad & \Big|\nu_\sigma^{(-1/2)}(\Phi_\sigma+v,\psi_{\rm tw})-\nu_\sigma^{(-1/2)}(\Phi_\sigma,\psi_{\rm tw})\Big|\,\big[\,\|g(\Phi_\sigma)\|_{L^2}+K_b\,\|\Phi_\sigma'\|_{L^2}\,\big] \\
& +K_\kappa\,\|g(\Phi_\sigma+v)-g(\Phi_\sigma)\|_{L^2} \\
& +K_\kappa\,|b(\Phi_\sigma+v,\psi_{\rm tw})-b(\Phi_\sigma,\psi_{\rm tw})|\,\|\Phi_\sigma'\|_{L^2} \\
& +K_\kappa K_b\,\|v'\|_{L^2}\,.
\end{aligned} \tag{8.20}
$$

Applying these results once more, we find

$$
\begin{aligned}
\|\mathcal{I}\|_{L^2} \quad &\leq \quad C_1\sigma^2\,\|v\|_{L^2}+C_1\,\|v\|_{L^2}+C_1\,\|v\|_{L^2}+C_1\,\|v\|_{H^1} \\
&\leq \quad C_2\,\|v\|_{H^1}\,,
\end{aligned} \tag{8.21}
$$

as desired. □

Proof of Proposition 8.1. To obtain (i), we use (8.8) together with Lemma 8.3 to compute

$$
\begin{aligned}
\left\|\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}(v)\right\|_{L^2} \quad \leq \quad & \left\|\overline{\mathcal{R}}_{\sigma;I}(v)\right\|_{L^2}+C_1\left|\langle\overline{\mathcal{R}}_{\sigma;I}(v),\psi_{\rm tw}\rangle_{L^2}\right|\,\big[1+\|v\|_{H^1}\big] \\
\leq \quad & C_2\sigma^2\,\|v\|_{H^1}+C_2\,\|v\|_{H^1}^2\,\big[1+\|v\|_{L^2}+\sigma^2\,\|v\|_{L^2}^2\big] \\
& +C_2\,\|v\|_{L^2}\,\big[\sigma^2+\|v\|_{L^2}\big]\big[1+\|v\|_{H^1}\big] \\
& +C_2\,\|v\|_{H^1}\,\big[\,\|v\|_{L^2}^2+\sigma^2\,\|v\|_{L^2}^3\big]\big[1+\|v\|_{H^1}\big].
\end{aligned} \tag{8.22}
$$

These terms can all be absorbed into (8.1).

The bound (ii) follows directly from Lemma 8.4, using the estimate

$$
\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)\right\|_{L^2}\leq\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2}+\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(v)-\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2} \tag{8.23}
$$

and the a-priori bound

$$
\left\|\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0)\right\|_{L^2}\leq C_3. \tag{8.24}
$$

The bound (iv) follows in the same fashion, since the condition (8.5) implies that

$$\overline{\mathcal{S}}_{\sigma;\Phi_\sigma}(0) = 0. \tag{8.25}$$

Finally, (iii) follows from the identities (8.8) and (3.49), using the proof of Lemma 3.7 to show that the cut-off function χ_{low} in (8.8) acts as the identity. □

9 Nonlinear stability of mild solutions

In this section we prove Theorems 2.4 and 2.6, providing an orbital and an exponential stability result for the stochastic waves (Φ_σ, c_σ). Recalling the function (6.34), our key statement is that $E\overline{N}_{\varepsilon,\alpha}$ can be bounded in terms of itself, the noise-strength σ and the initial condition $\|\overline{V}(0)\|_{H^1}^2$. This requires a number of technical regularity estimates, which we obtain in §9.1.

In order to prevent cumbersome notation and to highlight the broad applicability of our techniques here, we do not refer to the specific functions $\overline{V}$ and the specific nonlinearities $\overline{\mathcal{R}}_{\sigma;\Phi_\sigma,c_\sigma}$ here. Instead, we assume the following general condition concerning the form of our nonlinearities.

(hFB) We have $B_{\mathrm{cn}} \in L^2$ and the maps

$$F_{\mathrm{lin}} : H^1 \to L^2, \qquad F_{\mathrm{nl}} : H^1 \to L^2, \qquad B_{\mathrm{lin}} : H^1 \to L^2 \tag{9.1}$$

satisfy the bounds

$$\begin{array}{rcl} \|F_{\mathrm{lin}}(v)\|_{L^2} & \leq & K_{F;\mathrm{lin}} \|v\|_{H^1}, \\ \|F_{\mathrm{nl}}(v)\|_{L^2} & \leq & K_{F;\mathrm{nl}} \|v\|_{H^1}^2 (1 + \|v\|_{L^2}^m), \\ \|B_{\mathrm{lin}}(v)\|_{L^2} & \leq & K_{B;\mathrm{lin}} \|v\|_{H^1} \end{array} \tag{9.2}$$

for some $m > 0$. In addition, there exists $\eta_0 > 0$ so that

$$\langle \sigma^2 F_{\mathrm{lin}}(v) + F_{\mathrm{nl}}(v), \psi_{\mathrm{tw}} \rangle_{L^2} = 0, \qquad \langle B_{\mathrm{cn}} + B_{\mathrm{lin}}(v), \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{9.3}$$

whenever $\|v\|_{L^2} \leq \eta_0$.

Using the nonlinearities above, we can discuss the mild formulation of the sPDE that we are interested in. At present, we simply assume that a solution is a-priori available, but one can also set out to construct such a solution directly.

(hSol) For any $T > 0$, there exists a continuous $(\mathcal{F}_t)$-adapted process $V : \Omega \times [0, T] \to L^2$ for which we have the inclusions

$$V \in \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); H^1\big), \qquad B_{\mathrm{lin}}(V) \in \mathcal{N}^2\big([0,T]; (\mathcal{F}_t); L^2\big). \tag{9.4}$$

In addition, for almost all $\omega \in \Omega$ we have the inclusions

$$F_{\mathrm{lin}}\big(V(\cdot,\omega)\big) \in L^1([0,T]; L^2), \qquad F_{\mathrm{nl}}\big(V(\cdot,\omega)\big) \in L^1([0,T]; L^2) \tag{9.5}$$

together with the identity

$$\begin{array}{rcl} V(t) & = & S(t)V(0) + \sigma^2 \int_0^t S(t-s) F_{\mathrm{lin}}\big(V(s)\big)\, ds + \int_0^t S(t-s) F_{\mathrm{nl}}\big(V(s)\big)\, ds \\ & & +\sigma \int_0^t S(t-s) B_{\mathrm{cn}}\, d\beta_s + \sigma \int_0^t S(t-s) B_{\mathrm{lin}}\big(V(s)\big)\, d\beta_s, \end{array} \tag{9.6}$$

which holds for all $t \in [0, T]$. Finally, we have $\langle V(0), \psi_{\mathrm{tw}} \rangle_{L^2} = 0$.

For any $\varepsilon > 0$ and $\alpha \geq 0$, we recall the notation

$$N_{\varepsilon,\alpha}(t) = e^{\alpha t} \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds. \tag{9.7}$$

For any $T > 0$ and $\eta > 0$, we introduce the $(\mathcal{F}_t)$-stopping time

$$\tau_{\varepsilon,\alpha}(T, \eta) = \inf \Big\{ 0 \leq t < T : N_{\alpha,\varepsilon}(t) > \eta \Big\}, \tag{9.8}$$

writing $\tau_{\varepsilon,\alpha}(T, \eta) = T$ if the set is empty. Our two main results here, which we establish in §9.1, provide T-independent bounds on the expectation of $N_{\varepsilon,\alpha}$.

Proposition 9.1. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied. Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma$ we have the bound*

$$E N_{\varepsilon,0}\big(T \wedge \tau_{\varepsilon,0}(T, \eta)\big) \quad \leq \quad K \Big[\|V(0)\|_{H^1}^2 + \sigma^2 \Big]. \tag{9.9}$$

Proposition 9.2. *Assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) are satisfied and that $B_{\mathrm{cn}} = 0$. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma$ we have the bound*

$$E N_{\varepsilon,\alpha}\big(T \wedge \tau_{\varepsilon,\alpha}(T, \eta)\big) \quad \leq \quad K \|V(0)\|_{H^1}^2 . \tag{9.10}$$

Exploiting the technique used in Stannat [36], these bounds can be turned into estimates concerning the probabilities

$$p_{\varepsilon,\alpha}(T, \eta) = P\Big(\sup_{0 \leq t \leq T} \big[N_{\varepsilon,\alpha}(t) \big] > \eta \Big). \tag{9.11}$$

This allows our main stability theorems to be established.

Corollary 9.3. *Consider the setting of Proposition 9.1. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma$, we have the bound*

$$p_{\varepsilon,0}(T, \eta) \leq \eta^{-1} K \big[\|V(0)\|_{H^1}^2 + \sigma^2 \big]. \tag{9.12}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,0}(T, \eta) &= \eta P\big(\tau_{\varepsilon,0}(T, \eta) < T\big) \\ &= E\Big[\mathbf{1}_{\tau_{\varepsilon,0}(T,\eta) < T} N_{\varepsilon,0}\big(\tau_{\varepsilon,0}(T, \eta)\big) \Big] \\ &\leq E N_{\varepsilon,0}\big(T \wedge \tau_{\varepsilon,0}(T, \eta)\big), \end{aligned} \tag{9.13}$$

the result follows from (9.9). □

Corollary 9.4. *Consider the setting of Proposition 9.2. Then there exists a constant $K > 0$ so that for any $T > 0$, any $0 < \eta \leq \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma$ we have the bound*

$$p_{\varepsilon,\alpha}(T, \eta) \leq \eta^{-1} K \|V(0)\|_{H^1}^2 . \tag{9.14}$$

Proof. Upon computing

$$\begin{aligned} \eta p_{\varepsilon,\alpha}(T, \eta) &= \eta P\big(\tau_{\varepsilon,\alpha}(T, \eta) < T\big) \\ &= E\Big[\mathbf{1}_{\tau_{\varepsilon,\alpha}(T,\eta) < T} N_{\varepsilon,\alpha}\big(\tau_{\varepsilon,\alpha}(T, \eta)\big) \Big] \\ &\leq E N_{\varepsilon,\alpha}\big(T \wedge \tau_{\varepsilon,\alpha}(T, \eta)\big), \end{aligned} \tag{9.15}$$

the result follows from (9.10). □

Proof of Theorems 2.4 and 2.6. On account of Propositions 2.3 and 6.3, the map $\overline{V}$ defined in (6.6) satisfies the conditions of (hSol) with $\big(\overline{\beta}_\tau, \overline{\mathcal{F}}_\tau\big)_{\tau\geq 0}$ as the relevant Brownian motion. In addition, Proposition 8.1 guarantees that (hFB) is satisfied. The desired estimates now follow from Corollaries 9.3 and 9.4, using Proposition 6.4 to reverse the time-transform. □

9.1 Regularity estimates

In this subsection we establish Propositions 9.1-9.2 by estimating each of the terms featuring in the identity (9.6). The fact that the semigroup $S(t)$ is analytic is crucial for our purposes here. In fact, introducing the map $Q : L^2 \to L^2$ that acts as

$$Qv = v - \langle v, \psi_{\mathrm{tw}} \rangle_{L^2} \Phi_0', \tag{9.16}$$

all the bounds on S that we require are collected in our first result.

Lemma 9.5 (see [29]). *Assume that (HTw) and (HS) hold and consider the analytic semigroup $S(t)$ generated by $\mathcal{L}_{\mathrm{tw}}$. Then there is a constant $M \geq 1$ for which we have the bounds*

$$\begin{array}{lcll} \|S(t)Q\|_{\mathcal{L}(H^1,L^2)} & \leq & Me^{-\beta t}, & 0 < t < \infty, \\ \|S(t)Q\|_{\mathcal{L}(L^2,H^1)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \|S(t)Q\|_{\mathcal{L}(L^2,H^1)} & \leq & Me^{-\beta t}, & t \geq 1, \\ \|[\mathcal{L}_{\mathrm{tw}} - A_*]S(t)Q\|_{\mathcal{L}(L^2,L^2)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2, \\ \left\|[\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} - A_*]S(t)Q\right\|_{\mathcal{L}(L^2,L^2)} & \leq & Mt^{-\frac{1}{2}}, & 0 < t \leq 2. \end{array} \tag{9.17}$$

Writing $\nu = \alpha + \varepsilon$, we introduce the splitting

$$\begin{array}{rcl} N_{\varepsilon,\alpha;I}(t) & = & e^{\alpha t} \|V(t)\|_{L^2}^2 , \\ N_{\varepsilon,\alpha;II}(t) & = & \int_0^t e^{-\varepsilon(t-s)} e^{\alpha s} \|V(s)\|_{H^1}^2 \, ds \\ & = & e^{\alpha t} \int_0^t e^{-\nu(t-s)} \|V(s)\|_{H^1}^2 \, ds. \end{array} \tag{9.18}$$

In order to understand $N_{\varepsilon,\alpha;I}$, we introduce the expression

$$\mathcal{E}_0(t) \quad = \quad S(t)QV(0), \tag{9.19}$$

together with the long-term and short-term integrals

$$\begin{array}{rclrcl} \mathcal{E}^{\mathrm{lt}}_{F;\mathrm{lin}}(t) & = & \int_0^{t-1} S(t-s)QF_{\mathrm{lin}}\big(V(s)\big)ds, & \mathcal{E}^{\mathrm{sh}}_{F;\mathrm{lin}}(t) & = & \int_{t-1}^{t} S(t-s)QF_{\mathrm{lin}}\big(V(s)\big)ds, \\ \mathcal{E}^{\mathrm{lt}}_{F;\mathrm{nl}}(t) & = & \int_0^{t-1} S(t-s)QF_{\mathrm{nl}}\big(V(s)\big)ds, & \mathcal{E}^{\mathrm{sh}}_{F;\mathrm{nl}}(t) & = & \int_{t-1}^{t} S(t-s)QF_{\mathrm{nl}}\big(V(s)\big)ds, \\ \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{lin}}(t) & = & \int_0^{t-1} S(t-s)QB_{\mathrm{lin}}\big(V(s)\big)d\beta_s, & \mathcal{E}^{\mathrm{sh}}_{B;\mathrm{lin}}(t) & = & \int_{t-1}^{t} S(t-s)QB_{\mathrm{lin}}\big(V(s)\big)d\beta_s, \\ \mathcal{E}^{\mathrm{lt}}_{B;\mathrm{cn}}(t) & = & \int_0^{t-1} S(t-s)QB_{\mathrm{cn}}\big(V(s)\big)d\beta_s, & \mathcal{E}^{\mathrm{sh}}_{B;\mathrm{cn}}(t) & = & \int_{t-1}^{t} S(t-s)QB_{\mathrm{cn}}\big(V(s)\big)d\beta_s. \end{array} \tag{9.20}$$

Here we use the convention that integrands are set to zero for $s < 0$. For convenience, we also write

$$\mathcal{E}_{F;\#}(t) = \mathcal{E}^{\mathrm{lt}}_{F;\#}(t) + \mathcal{E}^{\mathrm{sh}}_{F;\#}(t) \tag{9.21}$$

for $\# \in \{\mathrm{lin}, \mathrm{nl}\}$ and

$$\mathcal{E}_{B;\#}(t) = \mathcal{E}^{\mathrm{lt}}_{B;\#}(t) + \mathcal{E}^{\mathrm{sh}}_{B;\#}(t) \tag{9.22}$$

for $\# \in \{\mathrm{lin}, \mathrm{cn}\}$.

Turning to the terms in (9.6) that are relevant for evaluating $N_{\varepsilon,\alpha;II}$, we introduce the expression

$$\mathcal{I}_{\nu,\delta;0}(t) \quad = \quad \int_0^t e^{-\nu(t-s)} \left\| S(\delta)\mathcal{E}_0(s) \right\|_{H^1}^2 \, ds, \tag{9.23}$$

together with

$$\begin{array}{rcl} \mathcal{I}^{\#}_{\nu,\delta;F;\mathrm{lin}}(t) & = & \int_0^t e^{-\nu(t-s)} \left\| S(\delta)\mathcal{E}^{\#}_{F;\mathrm{lin}}(s) \right\|_{H^1}^2 \, ds, \\ \mathcal{I}^{\#}_{\nu,\delta;F;nl}(t) & = & \int_0^t e^{-\nu(t-s)} \left\| S(\delta)\mathcal{E}^{\#}_{F;\mathrm{nl}}(s) \right\|_{H^1}^2 \, ds, \\ \mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{lin}}(t) & = & \int_0^t e^{-\nu(t-s)} \left\| S(\delta)\mathcal{E}^{\#}_{B;\mathrm{lin}}(s) \right\|_{H^1}^2 \, ds, \\ \mathcal{I}^{\#}_{\nu,\delta;B;\mathrm{cn}}(t) & = & \int_0^t e^{-\nu(t-s)} \left\| S(\delta)\mathcal{E}^{\#}_{B;\mathrm{cn}}(s) \right\|_{H^1}^2 \, ds \end{array} \tag{9.24}$$

for $\# \in \{\mathrm{lt}, \mathrm{sh}\}$. The extra $S(\delta)$ factor will be used to ensure that all the integrals we encounter are well-defined. We emphasize that all our estimates are uniform in $0 < \delta < 1$, allowing us to take $\delta \downarrow 0$. The estimates concerning $\mathcal{I}^{\mathrm{st}}_{\nu,\delta;F;\mathrm{nl}}$ and $\mathcal{I}^{\mathrm{st}}_{\nu,\delta;B;\mathrm{lin}}$ in Lemma's 9.10 and 9.13 are particularly delicate in this respect, as a direct application of the bounds in Lemma 9.5 would result in expressions that diverge as $\delta \downarrow 0$.

Lemma 9.6. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $0 < \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$e^{\alpha t} \left\| \mathcal{E}_0(t) \right\|_{L^2}^2 \quad \leq \quad M^2 \left\| V(0) \right\|_{L^2}^2, \tag{9.25}$$

together with

$$e^{\alpha t} \mathcal{I}_{\nu,\delta;0}(t) \quad \leq \quad \tfrac{M^2}{2\beta - \nu} \left\| V(0) \right\|_{H^1}^2, \tag{9.26}$$

Proof. We compute

$$\begin{array}{rcl} e^{\alpha t} \left\| \mathcal{E}_0(t) \right\|_{L^2}^2 & \leq & M^2 e^{\alpha t} e^{-2\beta t} \left\| V(0) \right\|_{L^2}^2 \\ & \leq & M^2 \left\| V(0) \right\|_{L^2}^2, \end{array} \tag{9.27}$$

together with

$$\begin{array}{rcl} e^{\alpha t} \mathcal{I}_{\nu,\delta;0}(t) & \leq & M^2 e^{\alpha t} \int_0^t e^{-\nu(t-s)} e^{-2\beta(s+\delta)} \left\| V(0) \right\|_{H^1}^2 \, ds \\ & \leq & \tfrac{M^2}{2\beta - \nu} \left\| V(0) \right\|_{H^1}^2. \end{array} \tag{9.28}$$

□

Lemma 9.7. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $\eta > 0$, any $0 < \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$e^{\alpha t} \left\| \mathcal{E}_{F;\mathrm{lin}}(t) \right\|_{L^2}^2 \quad \leq \quad K_{F;\mathrm{lin}}^2 \tfrac{M^2}{2\beta - \nu} N_{\varepsilon,\alpha;II}(t), \tag{9.29}$$

together with

$$e^{\alpha t} \mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(t) \quad \leq \quad K_{F;\mathrm{lin}}^2 \tfrac{M^2}{2(\beta + \frac{\alpha}{2} - \nu)\varepsilon} N_{\varepsilon,\alpha;II}(t). \tag{9.30}$$

Proof. We first observe that

$$\begin{array}{lcl} \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^t e^{-\beta(t-s)} \|V(s)\|_{H^1} \, ds \right)^2, \\ \left\| S(\delta)\mathcal{E}_{F;\mathrm{lin}}^{\mathrm{lt}}(t) \right\|_{H^1}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2 \left(\int_0^t e^{-\beta(t-s)} \|V(s)\|_{H^1} \, ds \right)^2. \end{array} \tag{9.31}$$

This allows us to compute

$$\begin{array}{lcl} e^{\alpha t} \|\mathcal{E}_{F;\mathrm{lin}}(t)\|_{L^2}^2 & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{\alpha t} \left(\int_0^t e^{-(\beta - \frac{\nu}{2})(t-s)} e^{-\frac{\nu}{2}(t-s)} \|V(s)\|_{H^1} \, ds \right)^2 \\ & \leq & K_{F;\mathrm{lin}} L^2 \frac{M^2}{2\beta - \nu} e^{\alpha t} \int_0^t e^{-\nu(t-s)} \|V(s)\|_{H^1}^2 \, ds \\ & = & K_{F;\mathrm{lin}}^2 \frac{M^2}{2\beta-\nu} N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.32}$$

Exploiting the inequality $2\beta - \nu > \varepsilon$, we write

$$\gamma_2 = \frac{\varepsilon + \nu}{2\beta} < 1 \tag{9.33}$$

and observe that

$$2\gamma_2\beta - \nu = \varepsilon. \tag{9.34}$$

Upon fixing $\gamma_1 = 1 - \gamma_2$, we readily see that

$$2\gamma_1\beta = 2\beta - \varepsilon - \nu = 2(\beta + \frac{\alpha}{2} - \nu). \tag{9.35}$$

This allows us to compute

$$\begin{array}{lcl} e^{\alpha t}\mathcal{I}_{\nu,\delta;F;\mathrm{lin}}^{\mathrm{lt}}(t) & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{\alpha t} \int_0^t e^{-\nu(t-s)} \left(\int_0^s e^{-\beta(s-s')} \|V(s')\|_{H^1} \, ds' \right)^2 ds \\ & \leq & K_{F;\mathrm{lin}}^2 M^2 e^{\alpha t} \int_0^t e^{-\nu(t-s)} \left(\int_0^s e^{-2\gamma_1\beta(s-s')} \, ds' \right) \left(\int_0^s e^{-2\gamma_2\beta(s-s')} \|V(s')\|_{H^1}^2 \, ds' \right) ds \\ & \leq & K_{F;\mathrm{lin}}^2 \frac{M^2}{2\gamma_1\beta} e^{\alpha t} \int_0^t e^{-\nu(t-s)} \int_0^s e^{-2\gamma_2\beta(s-s')} \|V(s')\|_{H^1}^2 \, ds' \, ds \\ & = & K_{F;\mathrm{lin}}^2 \frac{M^2}{2\gamma_1\beta} e^{\alpha t} \int_0^t \int_{s'}^t e^{-\nu(t-s)} e^{-2\gamma_2\beta(s-s')} \|V(s')\|_{H^1}^2 \, ds \, ds' \\ & = & K_{F;\mathrm{lin}}^2 \frac{M^2}{2\gamma_1\beta} e^{\alpha t} \int_0^t e^{-\nu t} \Big[\int_{s'}^t e^{-(2\gamma_2\beta - \nu)s} \, ds \Big] e^{2\gamma_2\beta s'} \|V(s')\|_{H^1}^2 \, ds' \\ & \leq & K_{F;\mathrm{lin}}^2 \frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)} e^{\alpha t} \int_0^t e^{-\nu t} e^{-(2\gamma_2\beta-\nu)s'} e^{2\gamma_2\beta s'} \|V(s')\|_{H^1}^2 \, ds' \\ & = & K_{F;\mathrm{lin}}^2 \frac{M^2}{(2\gamma_1\beta)(2\gamma_2\beta-\nu)} e^{\alpha t} \int_0^t e^{-\nu(t-s')} \|V(s')\|_{H^1}^2 \, ds' \\ & = & K_{F;\mathrm{lin}}^2 \frac{M^2}{2(\beta + \frac{\alpha}{2} - \nu)\varepsilon} N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.36}$$

□

Lemma 9.8. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ and write $\nu = \alpha + \varepsilon$. Then for any $0 < \delta < 1$ and any $0 \leq t \leq T$, we have the bound*

$$e^{\alpha t}\mathcal{I}_{\nu,\delta;F;\mathrm{lin}}^{\mathrm{sh}}(t) \quad \leq \quad 4e^{\nu} M^2 K_{F;\mathrm{lin}}^2 N_{\varepsilon,\alpha;II}(t). \tag{9.37}$$

Proof. Using Cauchy-Schwarz, we compute

$$\begin{array}{rcl} e^{\alpha t}\mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{lin}}(t) & \leq & M^2 K^2_{F;\mathrm{lin}} e^{\alpha t}\int_0^t e^{-\nu(t-s)}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\left\|V(s')\right\|_{H^1} ds'\right)^2 ds \\ & \leq & M^2 K^2_{F;\mathrm{lin}} e^{\alpha t}\int_0^t e^{-\nu(t-s)}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}} ds'\right)\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\left\|V(s')\right\|^2_{H^1} ds'\right) ds \\ & \leq & 2M^2 K^2_{F;\mathrm{lin}} e^{\alpha t}\int_0^t e^{-\nu(t-s)}\left(\int_{s-1}^s \frac{1}{\sqrt{s+\delta-s'}}\left\|V(s')\right\|^2_{H^1} ds'\right) ds \\ & = & 2M^2 K^2_{F;\mathrm{lin}} e^{\alpha t}\int_0^t e^{-\nu t}\Big[\int_{s'}^{\min\{t,s'+1\}} e^{\nu s}\frac{1}{\sqrt{s+\delta-s'}} ds\Big]\left\|V(s')\right\|^2_{H^1} ds' \\ & \leq & 4e^{\nu} M^2 K^2_{F;\mathrm{lin}} e^{\alpha t}\int_0^t e^{-\nu(t-s')}\left\|V(s')\right\|^2_{H^1} ds' \\ & = & 4e^{\nu} M^2 K^2_{F;\mathrm{lin}} N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.38}$$

□

Lemma 9.9. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon > 0$, $\alpha \geq 0$ for which $\varepsilon + \frac{\alpha}{2} < \beta$ and write $\nu = \alpha + \varepsilon$. Then for any $\eta > 0$, any $0 < \delta < 1$ and any $0 \leq t \leq \tau_{\varepsilon,\alpha}(T,\eta)$, we have the bound*

$$e^{\alpha t}\left\|\mathcal{E}_{F;\mathrm{nl}}(t)\right\|^2_{L^2} \quad \leq \quad \eta K^2_{F;\mathrm{nl}} M^2 (1+\eta^m)^2 N_{\varepsilon,\alpha;II}(t), \tag{9.39}$$

together with

$$e^{\alpha t}\mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{nl}}(t) \quad \leq \quad \eta K^2_{F;\mathrm{nl}}(1+\eta^m)^2 \frac{M^2}{\beta+\frac{\alpha}{2}-\nu} N_{\varepsilon,\alpha;II}(t). \tag{9.40}$$

Proof. We first notice that

$$\begin{array}{rcl} \left\|\mathcal{E}_{F;\mathrm{nl}}(t)\right\|^2_{L^2} + \left\|S(\delta)\mathcal{E}^{\mathrm{lt}}_{F;\mathrm{nl}}(t)\right\|^2_{H^1} & \leq & K_{F;\mathrm{nl}}(1+\eta^m)^2 M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1} ds\right)^2, \\ \left\|S(\delta)\mathcal{E}^{\mathrm{lt}}_{F;\mathrm{nl}}(t)\right\|^2_{H^1} & \leq & K_{F;\mathrm{nl}}(1+\eta^m)^2 M^2\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1} ds\right)^2. \end{array} \tag{9.41}$$

Using $\beta > \nu - \frac{1}{2}\alpha = \frac{1}{2}\alpha + \varepsilon$, we compute

$$\begin{array}{rcl} \int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1} ds & = & e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}t}\left\|V(s)\right\|^2_{H^1} ds \\ & \leq & e^{\frac{\alpha}{2}t}\int_0^t e^{-\beta(t-s)}e^{-\frac{\alpha}{2}(t-s)}\left\|V(s)\right\|^2_{H^1} ds \\ & \leq & e^{\frac{\alpha}{2}t}\int_0^t e^{-\nu(t-s)}\left\|V(s)\right\|^2_{H^1} ds. \end{array} \tag{9.42}$$

This yields the desired bound

$$\begin{array}{rcl} e^{\alpha t}\left\|\mathcal{E}_{F;\mathrm{nl}}(t)\right\|^2_{L^2} & \leq & K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 e^{\alpha t}\left(\int_0^t e^{-\beta(t-s)}\left\|V(s)\right\|^2_{H^1} ds\right)^2 \\ & \leq & K^2_{F;\mathrm{nl}}(1+\eta^2)^2 M^2 e^{2\alpha t}\left(\int_0^t e^{-\nu(t-s)}\left\|V(s)\right\|^2_{H^1} ds\right)^2 \\ & \leq & K^2_{F;\mathrm{nl}}(1+\eta^m)^2 M^2 \eta N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.43}$$

In a similar spirit, we compute

$$
\begin{array}{rcl}
e^{\alpha t}\mathcal{I}^{\rm lt}_{\nu,\delta;F;{\rm nl}}(t) & \leq & K_{F;{\rm nl}}^2(1+\eta^m)^2 M^2 e^{\alpha t}\int_0^t e^{-\nu(t-s)}\Big(\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|_{H^1}^2\,ds'\Big)^2\,ds \\
& \leq & K_{F;{\rm nl}}^2(1+\eta^m)^2 M^2 e^{\alpha t}\int_0^t e^{-\nu(t-s)}e^{\frac{1}{2}\alpha s}\Big(\int_0^s e^{-\nu(s-s')}\left\|V(s')\right\|_{H^1}^2\,ds'\Big) \\
& & \qquad\qquad \times\Big(\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|_{H^1}^2\,ds'\Big)\,ds \\
& \leq & \eta K_{F;{\rm nl}}^2(1+\eta^m)^2 M^2 e^{\alpha t}\int_0^t e^{-\nu(t-s)}e^{-\frac{\alpha}{2}s}\int_0^s e^{-\beta(s-s')}\left\|V(s')\right\|_{H^1}^2\,ds'\,ds \\
& = & \eta K_{F;{\rm nl}}^2(1+\eta^m)^2 M^2 e^{\alpha t}\int_0^t e^{-\nu t}\Big[\int_{s'}^t e^{-(\frac{\alpha}{2}-\nu+\beta)s}\,ds\Big]e^{\beta s'}\left\|V(s')\right\|_{H^1}^2\,ds' \\
& \leq & \eta K_{F;{\rm nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}e^{\alpha t}\int_0^t e^{-\nu t}e^{-(\frac{\alpha}{2}-\nu+\beta)s'}e^{\beta s'}\left\|V(s')\right\|_{H^1}^2\,ds' \\
& = & \eta K_{F;{\rm nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}e^{\alpha t}\int_0^t e^{-\nu(t-s')}e^{-\frac{\alpha}{2}s'}\left\|V(s')\right\|_{H^1}^2\,ds' \\
& \leq & \eta K_{F;{\rm nl}}^2(1+\eta^m)^2\frac{M^2}{\beta+\frac{\alpha}{2}-\nu}N_{\varepsilon,\alpha;II}(t).
\end{array}
\tag{9.44}
$$

□

Lemma 9.10. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon>0$, $\alpha\geq 0$ and write $\nu=\alpha+\varepsilon$. Then for any $\eta>0$, any $0<\delta<1$ and any $0\leq t\leq\tau_{\varepsilon,\alpha}(T,\eta)$, we have the bound*

$$
e^{\alpha t}\mathcal{I}^{\rm sh}_{\nu,\delta;F,nl}(t) \quad\leq\quad \eta M^2 K_{F;{\rm nl}}^2(1+\eta^m)^2(1+\rho^{-1})e^{3\nu}(4+\nu)N_{\varepsilon,\alpha;II}(t). \tag{9.45}
$$

Proof. We start by observing that

$$
\left\|v\right\|_{H^1}^2=\left\|v\right\|_{L^2}^2+\rho^{-1}\left\|A_*^{1/2}v\right\|_{L^2}^2. \tag{9.46}
$$

In addition, for any $w\in L^2$, $\vartheta>0$, $\vartheta_A\geq 0$ and $\vartheta_B\geq 0$ we have

$$
\begin{array}{rcl}
\frac{d}{d\vartheta}\langle S(\vartheta+\vartheta_A)w,S(\vartheta+\vartheta_B)w\rangle_{L^2} & = & \langle\mathcal{L}_{\rm tw}S(\vartheta+\vartheta_A)w,S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad+\langle S(\vartheta+\vartheta_A)w,\mathcal{L}_{\rm tw}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& = & \langle S(\vartheta+\vartheta_A)w,\mathcal{L}_{\rm tw}^{\rm adj}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad+\langle S(\vartheta+\vartheta_A)w,\mathcal{L}_{\rm tw}S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& = & \langle S(\vartheta+\vartheta_A)w,\big[\mathcal{L}_{\rm tw}^{\rm adj}-A_*\big]S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad+\langle S(\vartheta+\vartheta_A)w,\ \big[\mathcal{L}_{\rm tw}-A_*\big]S(\vartheta+\vartheta_B)w\rangle_{L^2} \\
& & \quad-2\langle A_*^{1/2}S(\vartheta+\vartheta_A)w,A_*^{1/2}S(\vartheta+\vartheta_B)w\rangle_{L^2}.
\end{array}
\tag{9.47}
$$

For convenience, we introduce the expression

$$
\mathcal{E}_{s,s',s'';\mathcal{H}}=\langle S(s+\delta-s')QF_{\rm nl}\big(V(s')\big),S(s+\delta-s'')QF_{\rm nl}\big(V(s'')\big)\rangle_{\mathcal{H}}, \tag{9.48}
$$

where we allow $\mathcal{H}\in\{L^2,H^1\}$. Exploiting (9.47) and the fact that $\delta>0$, we obtain the bound

$$
\begin{array}{rcl}
\mathcal{E}_{s;s's'';H^1} & \leq & M^2K_{F;{\rm nl}}^2(1+\eta^m)^2\left\|V(s')\right\|_{H^1}^2\left\|V(s'')\right\|_{H^1}^2 \\
& & \quad+M^2K_{F;{\rm nl}}^2(1+\eta^m)^2\rho^{-1}\frac{1}{\sqrt{s+\delta-s''}}\left\|V(s')\right\|_{H^1}^2\left\|V(s'')\right\|_{H^1}^2 \\
& & \quad-\rho^{-1}\frac{1}{2}\frac{d}{ds}\mathcal{E}_{s,s',s'';L^2}
\end{array}
\tag{9.49}
$$

for the values of (s, s', s'') that are relevant below. Upon introducing the integrals

$$\begin{array}{lcl} \mathcal{I}_I & = & e^{\alpha t} \int_0^t e^{-\nu(t-s)} \int_{s-1}^s \int_{s-1}^s \big[1 + \frac{1}{\sqrt{s+\delta-s''}}\big] \, \|V(s')\|_{H^1}^2 \, \|V(s'')\|_{H^1}^2 \, ds' \, ds'' \, ds, \\ \mathcal{I}_{II} & = & e^{\alpha t} \int_0^t e^{-\nu(t-s)} \int_{s-1}^s \int_{s-1}^s \frac{d}{ds} \mathcal{E}_{s,s',s'';L^2} ds' \, ds'' \, ds, \end{array} \tag{9.50}$$

we hence readily obtain the estimate

$$e^{\alpha t} \mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}(t) \quad \le \quad (1 + \rho^{-1}) M^2 K_{F;\rm nl}^2 (1 + \eta^m)^2 \mathcal{I}_I - \tfrac{1}{2} \rho^{-1} \mathcal{I}_{II}. \tag{9.51}$$

Changing the order of the integrals, we find

$$\begin{array}{lcl} \mathcal{I}_I & = & e^{\alpha t} \int_0^t e^{-\nu t} \int_{s'-1}^{\min\{t, s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s} \big[1 + \frac{1}{\sqrt{s+\delta-s''}}\big] \, ds \Big] \, \|V(s')\|_{H^1}^2 \, \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \le & 3 e^{\alpha t} \int_0^t e^{-\nu(t-s')} e^{\nu} \, \|V(s')\|_{H^1}^2 \int_{s'-1}^{\min\{t,s'+1\}} \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \le & 3 e^{\alpha t} \int_0^t e^{-\nu(t-s')} e^{\nu} \, \|V(s')\|_{H^1}^2 \, e^{2\nu} \int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\} - s'')} \, \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \le & 3 \eta e^{3\nu} e^{\alpha t} \int_0^t e^{-\nu(t-s')} \, \|V(s')\|_{H^1}^2 \, e^{-\alpha \min\{t,s'+1\}} \, ds' \\ & \le & 3 \eta e^{3 \nu} N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.52}$$

In a similar fashion, we may use an integration by parts to write

$$\begin{array}{lcl} \mathcal{I}_{II} & = & e^{\alpha t} \int_0^t e^{-\nu t} \int_{s'-1}^{\min\{t,s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} e^{\nu s} \frac{d}{ds} \mathcal{E}_{s,s',s'';L^2} \, ds \Big] \, ds'' \, ds' \\ & = & \mathcal{I}_{II;A} + \mathcal{I}_{II;B} + \mathcal{I}_{II;C}, \end{array} \tag{9.53}$$

in which we have introduced

$$\begin{array}{lcl} \mathcal{I}_{II;A} & = & e^{\alpha t} \int_0^t e^{-\nu t} \int_{s'-1}^{\min\{t,s'+1\}} e^{\nu \min\{t, s'+1, s''+1\}} \mathcal{E}_{\min\{t,s'+1,s''+1\},s',s'';L^2} \, ds'' \, ds', \\ \mathcal{I}_{II;B} & = & -e^{\alpha t} \int_0^t e^{-\nu t} \int_{s'-1}^{\min\{t,s'+1\}} e^{\nu \max\{s',s''\}} \mathcal{E}_{\max\{s',s''\},s',s'';L^2} \, ds'' \, ds', \\ \mathcal{I}_{II;C} & = & -e^{\alpha t} \int_0^t e^{-\nu t} \int_{s'-1}^{\min\{t,s'+1\}} \Big[\int_{\max\{s',s''\}}^{\min\{t,s'+1,s''+1\}} \nu e^{\nu s} \mathcal{E}_{s,s',s'';L^2} \, ds \Big] \, ds'' \, ds'. \end{array} \tag{9.54}$$

A direct inspection of these terms yields the bound

$$\begin{array}{lcl} |\mathcal{I}_{II}| & \le & e^{\nu} (2 + \nu) M^2 K_{F;\rm nl}^2 (1 + \eta^m)^2 e^{\alpha t} \int_0^t e^{-\nu(t-s')} \, \|V(s')\|_{H^1}^2 \int_{s'-1}^{\min\{t,s'+1\}} \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \le & e^{\nu} (2 + \nu) M^2 K_{F;\rm nl}^2 (1 + \eta^m)^2 e^{\alpha t} \int_0^t e^{-\nu(t-s')} \, \|V(s')\|_{H^1}^2 \, e^{2\nu} \\ & & \qquad \times \int_{s'-1}^{\min\{t,s'+1\}} e^{-\nu(\min\{t,s'+1\} - s'')} \, \|V(s'')\|_{H^1}^2 \, ds'' \, ds' \\ & \le & \eta e^{3\nu} (2 + \nu) M^2 K_{F;\rm nl}^2 (1 + \eta^m)^2 e^{\alpha t} \int_0^t e^{-\nu(t-s')} \, \|V(s')\|_{H^1}^2 \, e^{-\alpha \min\{t,s'+1\}} \, ds' \\ & \le & \eta e^{3\nu} (2 + \nu) M^2 K_{F;\rm nl}^2 (1 + \eta^m)^2 N_{\varepsilon,\alpha;II}(t). \end{array} \tag{9.55}$$

□

Lemma 9.11. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Then for any $(\mathcal{F}_t)$-stopping time τ and any $0 \le t \le T$, we have the identities*

$$\begin{array}{lcl} E e^{\alpha(t \wedge \tau)} \, \|\mathcal{E}_{B;\rm lin}(t \wedge \tau)\|_{L^2}^2 & = & E e^{\alpha(t\wedge\tau)} \int_0^{t\wedge\tau} \big\| S(t \wedge \tau - s) Q B_{\rm lin}\big(V(s)\big) \big\|_{L^2}^2 \, ds, \\ E e^{\alpha(t \wedge \tau)} \, \|\mathcal{E}_{B;\rm cn}(t \wedge \tau)\|_{L^2}^2 & = & E e^{\alpha(t\wedge\tau)} \int_0^{t\wedge\tau} \| S(t \wedge \tau - s) Q B_{\rm cn} \|_{L^2}^2 \, ds, \end{array} \tag{9.56}$$

together with

$$\begin{array}{rcl} Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm lt}_{\nu,\delta;B;\rm lin}(t\wedge\tau) & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\int_0^{s-1}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|_{L^2}^2\,ds'\,ds,\\ Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm lt}_{\nu,\delta;B;\rm cn}(t\wedge\tau) & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\int_0^{s-1}\left\|S(s+\delta-s')QB_{\rm cn}\right\|_{L^2}^2\,ds'\,ds \end{array} \tag{9.57}$$

and their short-time counterparts

$$\begin{array}{rcl} Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}(t\wedge\tau) & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\int_{s-1}^{s}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|_{L^2}^2\,ds'\,ds,\\ Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm cn}(t\wedge\tau) & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\int_{s-1}^{s}\left\|S(s+\delta-s')QB_{\rm cn}\right\|_{L^2}^2\,ds'\,ds. \end{array} \tag{9.58}$$

Proof. Using the Ito-isometry, we readily compute

$$\begin{array}{rcl} Ee^{\alpha(t\wedge\tau)}\left\|\mathcal{E}_{B;\rm lin}(t\wedge\tau)\right\|_{L^2}^2 & = & E\left\|\int_0^{t\wedge\tau} e^{\frac12\alpha(t\wedge\tau)}S(t\wedge\tau-s)QB_{\rm lin}\big(V(s)\big)d\beta_s\right\|_{L^2}^2\\ & = & E\left\|\int_0^{t}\mathbf{1}_{s<\tau} e^{\frac12\alpha(t\wedge\tau)}S(t\wedge\tau-s)QB_{\rm lin}\big(V(s)\big)d\beta_s\right\|_{L^2}^2\\ & = & E\int_0^{t}\left\|\mathbf{1}_{s<\tau} e^{\frac12\alpha(t\wedge\tau)}S(t\wedge\tau-s)QB_{\rm lin}\big(V(s)\big)\right\|_{L^2}^2 ds\\ & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}\left\|S(t\wedge\tau-s)QB_{\rm lin}\big(V(s)\big)\right\|_{L^2}^2 ds. \end{array} \tag{9.59}$$

In addition, the linearity of the expectation operator, the Ito-isometry and the integrability of the integrands imply that

$$\begin{array}{rcl} Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm lt}_{B;\rm lin}(t\wedge\tau) & = & E\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\left\|\int_0^{s-1} e^{\frac12\alpha(t\wedge\tau)}S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)d\beta_{s'}\right\|_{H_1}^2 ds\\ & = & E\int_0^{t}\left\|\int_0^{s-1}\mathbf{1}_{s<\tau}e^{\frac12\alpha(t\wedge\tau)}e^{-\frac12\nu(t\wedge\tau-s)}S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)d\beta_{s'}\right\|_{L^2}^2 ds\\ & = & E\int_0^{t}\int_0^{s-1}\left\|\mathbf{1}_{s<\tau}e^{\frac12\alpha(t\wedge\tau)}e^{-\frac12\nu(t\wedge\tau-s)}S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|_{L^2}^2 ds'\,ds\\ & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\int_0^{s-1}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|_{L^2}^2 ds'\,ds. \end{array} \tag{9.60}$$

The remaining expressions follow in a similar fashion. □

Lemma 9.12. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick constants $\varepsilon>0$, $\alpha\geq 0$ for which $\varepsilon+\frac{\alpha}{2}<\beta$ and write $\nu=\alpha+\varepsilon$. Then for any $0<\delta<1$, any $(\mathcal{F}_t)$-stopping time τ and any $0\leq t\leq T$, we have the bound*

$$Ee^{\alpha(t\wedge\tau)}\left\|\mathcal{E}_{B;\rm lin}(t\wedge\tau)\right\|_{L^2}^2 \quad\leq\quad M^2K^2_{B;\rm lin}EN_{\varepsilon,\alpha;II}(t\wedge\tau), \tag{9.61}$$

together with

$$Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm lt}_{\nu,\delta;B;\rm lin}(t\wedge\tau) \quad\leq\quad \tfrac{M^2}{2\beta-\nu}K^2_{B;\rm lin}EN_{\varepsilon;\alpha;II}(t\wedge\tau). \tag{9.62}$$

Proof. Using (9.56) together with $2\beta>\nu+\varepsilon>\nu$, we compute

$$\begin{array}{rcl} Ee^{\alpha(t\wedge\tau)}\left\|\mathcal{E}_{B;\rm lin}(t\wedge\tau)\right\|_{L^2}^2 & \leq & M^2K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-2\beta(t\wedge\tau-s)}\left\|V(s)\right\|_{H^1}^2 ds\\ & \leq & M^2K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s)}\left\|V(s)\right\|_{H^1}^2 ds\\ & = & M^2K^2_{B;\rm lin}EN_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{9.63}$$

In addition, using (9.57) and switching the integration order, we obtain

$$\begin{array}{lll}Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm lt}_{\nu,\delta;B;\rm lin}(t\wedge\tau) & \leq & M^2K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau-s)}\int_0^s e^{-2\beta(s-s')}\left\|V(s')\right\|^2_{H^1}\,ds'\,ds\\ & = & M^2K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}\Big[\int_{s'}^{t\wedge\tau}e^{-(2\beta-\nu)s}\,ds\Big]e^{2\beta s'}\left\|V(s')\right\|^2_{H^1}\,ds'\\ & \leq & \frac{M^2}{2\beta-\nu}K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}e^{-(2\beta-\nu)s'}e^{2\beta s'}\left\|V(s')\right\|^2_{H^1}\,ds'\\ & = & \frac{M^2}{2\beta-\nu}K^2_{B;\rm lin}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau-s)}\left\|V(s')\right\|^2_{H^1}\,ds'\\ & = & \frac{M^2}{2\beta-\nu}K^2_{B;\rm lin}EN_{\varepsilon,\alpha;II}(t\wedge\tau).\end{array}\tag{9.64}$$

□

Lemma 9.13. *Fix $T>0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick two constants $\varepsilon>0$, $\alpha\geq 0$ and write $\nu=\alpha+\varepsilon$. Then for any $0<\delta<1$, any $(\mathcal{F}_t)$-stopping time τ and any $0\leq t\leq T$, we have the bound*

$$Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}(t\wedge\tau)\quad\leq\quad K^2_{B;\rm lin}M^2(1+\rho^{-1})e^{\nu}(4+\nu)EN_{\varepsilon,\alpha;II}(t\wedge\tau).\tag{9.65}$$

Proof. Applying the identity (9.47) with $\vartheta_A=\vartheta_B$, we obtain the bound

$$\begin{array}{lll}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{H^1} & \leq & M^2K^2_{B;\rm lin}\left\|V(s')\right\|^2_{H^1}\\ & & +M^2K^2_{B;\rm lin}\rho^{-1}\frac{1}{\sqrt{s+\delta-s'}}\left\|V(s')\right\|^2_{H^1}\\ & & -\rho^{-1}\frac{1}{2}\frac{d}{ds}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}\end{array}\tag{9.66}$$

for the values of (s,s') that are relevant below. Upon writing

$$\begin{array}{lll}\mathcal{I}_I & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau-s)}\int_{s-1}^s\big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]\left\|V(s')\right\|^2_{H^1}\,ds'\,ds,\\ \mathcal{I}_{II} & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau-s)}\int_{s-1}^s\frac{d}{ds}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}ds'\,ds,\end{array}\tag{9.67}$$

we obtain the estimate

$$Ee^{\alpha(t\wedge\tau)}\mathcal{I}^{\rm sh}_{\nu,\delta;B;\rm lin}(t\wedge\tau)\quad\leq\quad(1+\rho^{-1})M^2K^2_{B;\rm lin}\mathcal{I}_I-\tfrac{1}{2}\rho^{-1}\mathcal{I}_{II}.\tag{9.68}$$

Changing the integration order, we obtain

$$\begin{array}{lll}\mathcal{I}_I & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}\int_{s'}^{\min\{t\wedge\tau,s'+1\}}e^{\nu s}\big[1+\frac{1}{\sqrt{s+\delta-s'}}\big]\,ds\Big]\left\|V(s')\right\|^2_{H^1}\,ds'\\ & \leq & 3e^{\nu}Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau-s')}\left\|V(s')\right\|^2_{H^1}\,ds'\\ & = & 3e^{\nu}EN_{\varepsilon,\alpha;II}(t\wedge\tau).\end{array}\tag{9.69}$$

Integrating by parts, we arrive at the identity

$$\begin{array}{lll}\mathcal{I}_{II} & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}\Big[\int_{s'}^{\min\{t\wedge\tau,s'+1\}}e^{\nu s}\frac{d}{ds}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}\,ds\Big]\,ds'\\ & = & \mathcal{I}_{II;A}+\mathcal{I}_{II;B}+\mathcal{I}_{II;C},\end{array}\tag{9.70}$$

in which we have introduced the expressions

$$\begin{array}{lll}\mathcal{I}_{II;A} & = & Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}e^{\nu\min\{t\wedge\tau,s'+1\}}\left\|S(\min\{t\wedge\tau,s'+1\}+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}\,ds',\\ \mathcal{I}_{II;B} & = & -Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}e^{\nu s'}\left\|S(\delta)QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}\,ds',\\ \mathcal{I}_{II;C} & = & -Ee^{\alpha(t\wedge\tau)}\int_0^{t\wedge\tau}e^{-\nu(t\wedge\tau)}\Big[\int_{s'}^{\min\{t\wedge\tau,s'+1\}}\nu e^{\nu s}\left\|S(s+\delta-s')QB_{\rm lin}\big(V(s')\big)\right\|^2_{L^2}\,ds\Big]\,ds'.\end{array}\tag{9.71}$$

Inspecting these expressions, we readily obtain the bound

$$\begin{array}{rcl} |\mathcal{I}_{II}| & \leq & e^{\nu}(2+\nu)M^2 K_{B;\mathrm{lin}}^2 E e^{\alpha(t\wedge\tau)} \int_0^{t\wedge\tau} e^{-\nu(t\wedge\tau-s')} \left\|V(s')\right\|_{H_1}^2 \, ds' \\ & = & e^{\nu}(2+\nu)M^2 K_{B;\mathrm{lin}}^2 E N_{\varepsilon,\alpha;II}(t\wedge\tau). \end{array} \tag{9.72}$$

□

Lemma 9.14. *Fix $T > 0$ and assume that (HA), (HTw), (HS), (Hβ), (hSol) and (hFB) all hold. Pick a constant $0 < \varepsilon < \beta$. Then for any $0 < \delta < 1$, any $(\mathcal{F}_t)$-stopping time τ and any $0 \leq t \leq T$, we have the bound*

$$E \left\|\mathcal{E}_{B;\mathrm{cn}}(t\wedge\tau)\right\|_{L^2}^2 \quad \leq \quad M^2 K_{B;\mathrm{cn}}^2 \tfrac{1}{2\varepsilon}, \tag{9.73}$$

together with

$$\begin{array}{rcl} E\mathcal{I}^{\mathrm{lt}}_{\varepsilon,\delta;B;\mathrm{cn}}(t\wedge\tau) & \leq & \frac{M^2}{(2\beta-\varepsilon)\varepsilon} K_{B;\mathrm{cn}}^2, \\ E\mathcal{I}^{\mathrm{sh}}_{\varepsilon,\delta;B;\mathrm{cn}}(t\wedge\tau) & \leq & K_{B;\mathrm{cn}}^2 \frac{M^2}{\varepsilon}(1+\rho^{-1})e^{\varepsilon}(4+\varepsilon). \end{array} \tag{9.74}$$

Proof. Since

$$\int_0^{t\wedge\tau} e^{-\varepsilon(t\wedge\tau-s)} \, ds \leq \frac{1}{\varepsilon}, \tag{9.75}$$

we can obtain these bounds from Lemma's 9.12 and 9.13 by picking $\alpha = 0$ and making the substitutions

$$K_{B;\mathrm{lin}} \mapsto K_{B;\mathrm{cn}}, \qquad\qquad E N_{\varepsilon,0;II}(t\wedge\tau) \mapsto \frac{1}{\varepsilon}. \tag{9.76}$$

□

Proof of Proposition 9.1. Pick $T > 0$ and $0 < \eta < \eta_0$ and write $\tau = \tau_{\varepsilon,\alpha}(T,\eta)$. Since the identities (9.3) with $v = V(t\wedge\tau)$ hold for all $0 \leq t \leq T$, we may compute

$$\begin{array}{rcl} E N_{\varepsilon,0;I}(T\wedge\tau) & \leq & 5E\Big[\left\|\mathcal{E}_0(T\wedge\tau)\right\|_{L^2}^2 + \sigma^4 \left\|\mathcal{E}_{F;\mathrm{lin}}(T\wedge\tau)\right\|_{L^2}^2 + \left\|\mathcal{E}_{F;\mathrm{nl}}(T\wedge\tau)\right\|_{L^2}^2 \\ & & \quad + \sigma^2 \left\|\mathcal{E}_{B;\mathrm{lin}}(T\wedge\tau)\right\|_{L^2}^2 + \sigma^2 \left\|\mathcal{E}_{B;\mathrm{cn}}(T\wedge\tau)\right\|_{L^2}^2 \Big] \end{array} \tag{9.77}$$

by applying Young's inequality. The inequalities in Lemma's 9.6-9.14 now imply that

$$E N_{\varepsilon,0;I}(T\wedge\tau) \quad \leq \quad C_1 \big[\left\|V(0)\right\|_{H^1}^2 + \eta N_{\varepsilon,0;II}(T\wedge\tau) + (\sigma^2+\sigma^4) N_{\varepsilon,0;II}(T\wedge\tau) + \sigma^2 \big]. \tag{9.78}$$

In addition, picking $0 < \delta < 1$ and writing

$$N_{\varepsilon,0,\delta;II}(t) = \int_0^t e^{-\nu(t-s)} \left\|S(\delta)V(s)\right\|_{H^1}^2 \, ds, \tag{9.79}$$

we note that

$$\begin{array}{rcl} E N_{\varepsilon,0,\delta;II}(T\wedge\tau) & \leq & 9E\Big[\mathcal{I}_{\nu,\delta;0}(T\wedge\tau) + \sigma^4 \mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{lin}}(T\wedge\tau) + \sigma^4 \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{lin}}(T\wedge\tau) \\ & & \quad + \mathcal{I}^{\mathrm{lt}}_{\nu,\delta;F;\mathrm{nl}}(T\wedge\tau) + \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;F;\mathrm{nl}}(T\wedge\tau) \\ & & \quad + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{lin}}(T\wedge\tau) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{lin}}(T\wedge\tau) \\ & & \quad + \sigma^2 \mathcal{I}^{\mathrm{lt}}_{\nu,\delta;B;\mathrm{cn}}(T\wedge\tau) + \sigma^2 \mathcal{I}^{\mathrm{sh}}_{\nu,\delta;B;\mathrm{cn}}(T\wedge\tau) \Big]. \end{array} \tag{9.80}$$

The inequalities in Lemma's 9.6-9.14 now imply that

$$EN_{\varepsilon,0,\delta;II}(T\wedge\tau) \quad \leq \quad C_2\big[\,\|V(0)\|_{H^1}^2 + \eta N_{\varepsilon,0;II}(T\wedge\tau) + (\sigma^2+\sigma^4)N_{\varepsilon,0;II}(T\wedge\tau) + \sigma^2\big], \quad (9.81)$$

where we emphasize that C_2 does not depend on $\delta > 0$. This allows us to apply Fatou's lemma to conclude

$$\begin{array}{rcl} EN_{\varepsilon,0;II}(T\wedge\tau) & = & E\int_0^{T\wedge\tau} e^{\nu(T\wedge\tau-s)}(\lim_{\delta\to 0}\|S(\delta)V(s)\|_{H^1})^2\,ds \\ & \leq & \liminf_{\delta\to 0} EN_{\varepsilon,0,\delta;II}(T\wedge\tau). \end{array} \quad (9.82)$$

In particular, we see that

$$EN_{\varepsilon,0}(T\wedge\tau) \leq C_3\big[\,\|V(0)\|_{H^1}^2 + (\eta+\sigma^2+\sigma^4)EN_{\varepsilon,0}(T\wedge\tau) + \sigma^2\big]. \quad (9.83)$$

The desired bound hence follows by appropriately restricting the size of $\eta+\sigma^2+\sigma^4$. □

Proof of Proposition 9.2. Ignoring the contributions arising from $B_{\rm cn}$, we can follow the proof of Proposition 9.2 to obtain the bound

$$EN_{\varepsilon,\alpha}(T\wedge\tau) \leq C_4\big[\,\|V(0)\|_{H^1}^2 + (\eta+\sigma^2+\sigma^4)EN_{\varepsilon,\alpha}(T\wedge\tau)\big]. \quad (9.84)$$

The desired estimate hence follows by appropriately restricting the size of $\eta+\sigma^2+\sigma^4$. □

References

[1] M. Beck, H. J. Hupkes, B. Sandstede and K. Zumbrun (2010), Nonlinear Stability of Semidiscrete Shocks for Two-Sided Schemes. *SIAM J. Math. Anal.* **42**, 857–903.

[2] M. Beck, B. Sandstede and K. Zumbrun (2010), Nonlinear stability of time-periodic viscous shocks. *Archive for rational mechanics and analysis* **196**(3), 1011–1076.

[3] H. Berestycki, F. Hamel and H. Matano (2009), Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.* **62**(6), 729–788.

[4] S. Brassesco, A. De Masi and E. Presutti (1995), Brownian fluctuations of the interface in the D= 1 Ginzburg-Landau equation with noise. *Ann. Inst. H. Poincaré Probab. Statist* **31**(1), 81–118.

[5] P. C. Bressloff and Z. P. Kilpatrick (2015), Nonlinear Langevin equations for wandering patterns in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **14**(1), 305–334.

[6] P. C. Bressloff and M. A. Webber (2012), Front propagation in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **11**(2), 708–740.

[7] C.-N. Chen and Y. Choi (2015), Traveling pulse solutions to FitzHugh–Nagumo equations. *Calculus of Variations and Partial Differential Equations* **54**(1), 1–45.

[8] P.-L. Chow (2014), *Stochastic partial differential equations.* CRC Press.

[9] P. Cornwell (2017), Opening the Maslov Box for Traveling Waves in Skew-Gradient Systems. *arXiv preprint arXiv:1709.01908.*

[10] P. Cornwell and C. K. Jones (2017), On the Existence and Stability of Fast Traveling Waves in a Doubly-Diffusive FitzHugh-Nagumo System. *arXiv preprint arXiv:1709.09132.*

[11] G. Da Prato, A. Jentzen and M. Röckner (2010), A mild Itô formula for SPDEs. *arXiv preprint arXiv:1009.3526.*

[12] G. di Nunno and B. O. (editors) (2011), *Advanced Mathematical Methods for Finance.* Springer.

[13] L. Evans (1998), *Partial differential equations.* American Mathematical Society, Providence, R.I.

[14] P. C. Fife and J. B. McLeod (1977), The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65**(4), 335–361.

[15] C. L. E. Franzke, T. J. O'Kane, J. Berner, P. D. Williams and V. Lucarini (2015), Stochastic climate theory and modeling. *Wiley Interdisciplinary Reviews: Climate Change* **6**(1), 63–78.

[16] T. Funaki (1995), The scaling limit for a stochastic PDE and the separation of phases. *Probability Theory and Related Fields* **102**(2), 221–288.

[17] L. Gawarecki and V. Mandrekar (2010), *Stochastic differential equations in infinite dimensions: with applications to stochastic partial differential equations.* Springer Science & Business Media.

[18] K. Gowda and C. Kuehn (2015), Early-warning signs for pattern-formation in stochastic partial differential equations. *Communications in Nonlinear Science and Numerical Simulation* **22**(1), 55–69.

[19] A. Hoffman, H. J. Hupkes and E. S. Van Vleck, Entire Solutions for Bistable Lattice Differential Equations with Obstacles. *Memoirs of the AMS, to appear.*

[20] A. Hoffman, H. J. Hupkes and E. S. Van Vleck, Multi-Dimensional Stability of Waves Travelling through Rectangular Lattices in Rational Directions. *Transactions of the AMS, to appear.*

[21] J. Inglis and J. MacLaurin (2016), A general framework for stochastic traveling waves and patterns, with application to neural field equations. *SIAM Journal on Applied Dynamical Systems* **15**(1), 195–234.

[22] J. Jacod (2006), *Calcul stochastique et problemes de martingales*, Vol. 714. Springer.

[23] M. Jeanblanc, M. Yor and M. Chesney (2009), *Mathematical methods for financial markets.* Springer Science & Business Media.

[24] T. Kapitula (1997), Multidimensional Stability of Planar Travelling Waves. *Trans. Amer. Math. Soc.* **349**, 257–269.

[25] R. Kuske, C. Lee and V. Rottschäfer (2017), Patterns and coherence resonance in the stochastic Swift-Hohenberg equation with Pyragas control: The Turing bifurcation case. *Physica D: Nonlinear Phenomena* pp. –.

[26] E. Lang (2016), A multiscale analysis of traveling waves in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **15**(3), 1581–1614.

[27] W. Liu and M. Röckner (2010), SPDE in Hilbert space with locally monotone coefficients. *Journal of Functional Analysis* **259**(11), 2902–2922.

[28] G. Lord and V. Thümmler (2012), Computing stochastic traveling waves. *SIAM Journal on Scientific Computing* **34**(1), B24–B43.

[29] L. Lorenzi, A. Lunardi, G. Metafune and D. Pallara (2004), Analytic semigroups and reaction-diffusion problems. In: *Internet Seminar*, Vol. 2005. p. 127.

[30] C. Mascia and K. Zumbrun (2002), Pointwise Green's function bounds and stability of relaxation shocks. *Indiana Univ. Math. J.* **51**(4), 773–904.

[31] G. Prato and J. Zabczyk (1992), *Stochastic equations in infinite dimensions.* Cambridge University Press, Cambridge New York.

[32] C. Prévôt and M. Röckner (2007), *A concise course on stochastic partial differential equations*, Vol. 1905. Springer.

[33] D. Revuz and M. Yor (2013), *Continuous martingales and Brownian motion*, Vol. 293. Springer Science & Business Media.

[34] D. H. Sattinger (1976), On the stability of waves of nonlinear parabolic systems. *Advances in Mathematics* **22**(3), 312–355.

[35] T. Shardlow (2005), Numerical simulation of stochastic PDEs for excitable media. *Journal of computational and applied mathematics* **175**(2), 429–446.

[36] W. Stannat (2013), Stability of travelling waves in stochastic Nagumo equations. *arXiv preprint arXiv:1301.6378.*

[37] W. Stannat (2014), Stability of travelling waves in stochastic bistable reaction-diffusion equations. *arXiv preprint arXiv:1404.3853.*

[38] J. Viñals, E. Hernández-García, M. San Miguel and R. Toral (1991), Numerical study of the dynamical aspects of pattern selection in the stochastic Swift-Hohenberg equation in one dimension. *Physical Review A* **44**(2), 1123.

[39] J. Zhang, A. Holden, O. Monfredi, M. Boyett and H. Zhang (2009), Stochastic vagal modulation of cardiac pacemaking may lead to erroneous identification of cardiac chaos. *Chaos: An Interdisciplinary Journal of Nonlinear Science* **19**(2), 028509.

[40] K. Zumbrun (2011), Instantaneous Shock Location and One-Dimensional Nonlinear Stability of Viscous Shock Waves. *Quarterly of applied mathematics* **69**(1), 177–202.