Well-Posedness of Initial Value Problems for Vector-Valued Functional Differential Equations of Mixed Type

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Abstract

We study initial value problems for functional differential (algebraic) equations of mixed type posed on Hilbert spaces. We develop key tools such as exponential dichotomies and Wiener-Hopf factorizations for such systems that allow us to characterize in what sense such problems are well-posed. The key mathematical issue is that the natural technical state space for such systems is bigger than the appropriate space containing the initial conditions.

We illustrate our techniques by studying an optimal control problem with time delays posed on an integer lattice, which can be used to weigh the costs and benefits of utilizing polluting chemicals to enhance crop yields. The conditions defining Nash equilibria can be explicitly analyzed in our framework, allowing us to give conditions under which such optimal strategies exist.

Key words: functional differential equations, exponential dichotomies, advanced and retarded arguments, indeterminacy, initial value problems, spatial lattices.

1 Introduction

In this paper we consider a class of initial value problems that includes the prototypes

$$\begin{array}{rcll} \mathcal{I}x'(\xi) & = & \mathcal{A}x(\xi) + \mathcal{B}\int_{-1}^{1} x(\xi+\sigma)\,d\sigma + f\big(x(\xi)\big) & \text{for all } \xi \geq 0, \\ x(\xi) & \to & 0 & \text{as } \xi \to \infty, \end{array} \tag{1.1}$$

at times coupled with an initial condition

$$x(\vartheta) \;=\; \phi(\vartheta) \qquad \text{for all } -1 \leq \vartheta \leq 0. \tag{1.2}$$

Here x takes values in a Hilbert space $\mathcal{H}$, while $\mathcal{A}, \mathcal{B} \in \mathcal{L}(\mathcal{H};\mathcal{H})$ are bounded linear maps and the nonlinearity f is assumed to have $f(0) = Df(0) = 0$. We require the solution x to be continuous for

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all $\xi > 0$, but we will allow it to have a single discontinuity at $\xi = 0$. More precisely, the right-hand limit $x(0^+) = \lim_{\xi \downarrow 0} x(\xi)$ must exist but may satisfy $x(0^+) \neq \phi(0)$.

For now, the operator $\mathcal{I}$ should be thought of as being either the identity on $\mathcal{H}$ or the zero operator. In the former case, we refer to (1.1) as a functional differential equation of mixed type (MFDE), in which the term mixed refers to the fact that the system features both delayed and advanced arguments. In the latter case, we impose the additional constraint that the system under consideration can be turned into a differential equation after a finite number of differentiations. For example, if $f = 0$ and $\mathcal{A}$ is invertible this is true for (1.1), in which case we refer to this system as a functional differential-algebraic equation of mixed type (MFDAE).

In a broad sense, our goal is to classify which initial conditions ϕ and jumps $x(0^+) - \phi(0)$ lead to solutions of (1.1)-(1.2) and whether such solutions are unique. This paper hence continues the program in [4, 10, 11], where it was always the case that $\mathcal{H} = \mathbb{R}^n$. The extra ingredient in this paper is that $\mathcal{H}$ is allowed to be a Hilbert space. Indeed, we are especially interested in situations where $\mathcal{H} = \ell^2(\mathbb{R}; \mathbb{Z}^n)$ and the operators $\mathcal{A}$ and $\mathcal{B}$ are convolution operators, allowing us to make effective use of Fourier transforms.

Our earlier results for $\mathcal{H} = \mathbb{R}^n$ relied heavily upon the framework developed in [20] by Mallet-Paret and Verduyn Lunel. These authors exploited the classic Ascoli-Arzela theorem to obtain exponential dichotomies and Wiener-Hopf factorizations for MFDEs. In addition, certain important restriction operators were shown to be Fredholm. As we will explain in §1.2-§1.3, one cannot expect these properties to hold in infinite-dimensional settings such as $\mathcal{H} = \ell^2(\mathbb{R}; \mathbb{Z}^n)$. We refer the casual reader to the results stated in §3.4 and the worked out example in §4 to gain an appreciation of what can still be expected in this situation.

1.1 Application to optimization problems

It is well-known that the solutions to optimal control problems often satisfy ill-posed equations, even if the dynamics of the state and control variables themselves are well-posed. For example, minimizing a convex cost functional over the path of a controlled heat equation leads to a backward heat equation for the adjoint variable [26] that is coupled to the original system. In a similar spirit, a result due to Hughes [8] shows that MFDEs arise as the Euler-Lagrange conditions when studying optimal control problems that involve time delays [9, 25]. In both cases, the resulting equations are ill-posed in the sense that initial conditions cannot always be continuated either in forward or backward time. Research in this area has traditionally focussed on approximation techniques and abstract existence theorems for optimal paths, whereas our approach here directly tackles the *structure* of the ill-posed system.

Pollution and crop yields For concreteness, let us consider a toy model that describes the interaction between a grid of farmers that each use fertilizers to enhance their crop yields, but who all suffer from the environmental damage caused by the polluting chemicals spreading through the ground water. Indexing the farms by $j \in \mathbb{Z}$, we use $p_j(t)$ to indicate the amount of pollutant used at farm j at time t and write

$$p(t) = \{p_j(t)\}_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{R}). \tag{1.3}$$

The goal of each farmer individually is to maximize the functional

$$\mathcal{J}_j(p) = \int_0^\infty \big(2\sqrt{p_j(t)} - \frac{1}{2} c_j[p](t)^2\big) e^{-\rho t}\, dt \tag{1.4}$$

by appropriately choosing his own fertilizer use

$$p_j(t) \in L^\infty([0, \infty); \mathbb{R}) \cap C([0, \infty), \mathbb{R}). \tag{1.5}$$

The parameter $\rho > 0$ is a temporal discount factor and the function $2\sqrt{p_j(t)}$ encodes the private economic benefits of pollutant use. The concavity of this function models the decreasing marginal benefits associated to increased fertilizer usage. On the other hand, the function $c_j[p](t)$ is given by

$$c_j[p](t) = \sum_{k \in \mathbb{Z}} h_{j-k} \int_{t-1}^{t} p_k(\sigma)\, d\sigma \tag{1.6}$$

and reflects the volume of chemicals accumulated at site j, crudely taking into account leakages from neighbouring farms and time delays caused by the finite spreading speed. This is a highly simplified version of a so-caled lumped parameter model [21–23], originally developed to describe the spread of tracers through ground water. The square in (1.4) reflects the increasing marginal costs caused by the extra pollution and the associated cleanup costs.

As a consequence of the delay in (1.6), we need to fix an initial condition

$$p_j(\vartheta) = \phi_j(\vartheta) \tag{1.7}$$

for all $-1 \le \vartheta \le 0$ and $j \in \mathbb{Z}$, with

$$\phi = \{\phi_j\}_{j \in \mathbb{Z}} \in C\big([-1, 0]; \ell^\infty(\mathbb{Z}; \mathbb{R})\big). \tag{1.8}$$

This initial condition can be viewed in some sense as the current environmental state, based upon which each farmer needs to make decisions in order to attain his maximal pollution-adjusted welfare.

Naturally, the optimal choice for farmer j depends on ϕ and the functions $\{p_k\}_{k \neq j}$ associated to the other farmers. In §4 we search for a so-called Nash equilibrium [24], which is a choice for p that ensures that all the individual functionals (1.4) cannot be increased by modifying only p_j. We show that such a simultaneous optimum must satisfy

$$\frac{1}{\sqrt{p_j(t)}} = h_0 \sum_{k \in \mathbb{Z}} h_{j-k} \int_0^1 \int_{-1}^{0} p_k(t + \sigma + \sigma') e^{-\rho\sigma}\, d\sigma'\, d\sigma, \tag{1.9}$$

which is an algebraic equation covered by the theory in this paper. Under some technical conditions on h, it is not hard to see that there exists a spatially and temporally homogeneous simultaneous optimum

$$\phi_j(\vartheta) = p_*, \qquad p_j(t) = p_*, \qquad j \in \mathbb{Z} \tag{1.10}$$

for $\vartheta \in [-1, 0]$ and $t \ge 0$.

The main question for models such as these is whether it is still possible to find a bounded Nash equilibrium for initial conditions ϕ that are sufficiently close to p_*. Using the theory developed in this paper this question can be answered affirmatively, provided one interprets closeness in an ℓ^2-sense, i.e.

$$\sup_{-1 \le \vartheta \le 0} \|\phi(\vartheta) - p_*\|_{\ell^2(\mathbb{Z}; \mathbb{R})} \ll 1 \tag{1.11}$$

and also allows for discontinuities

$$p_j(0^+) \neq p_j(0^-) = \phi(0). \tag{1.12}$$

Our main results can be split into three main themes, which we each briefly discuss below.

1.2 Exponential dichotomies

Linearizing (1.1) around the equilibrium $x = 0$, we arrive at the autonomous MFDE

$$x'(\xi) = \mathcal{A}x(\xi) + \mathcal{B}\int_{-1}^{1} x(\xi + \sigma)\, d\sigma. \tag{1.13}$$

A typical analysis of (1.13) hinges upon an understanding of the characteristic function, which is given by

$$\Delta(z) = z - \mathcal{A} - \mathcal{B} \int_{-1}^{1} e^{z\sigma} \, d\sigma. \tag{1.14}$$

Notice that any solution pair $(z, v) \in \mathbb{C} \times \mathcal{H} \setminus \{0\}$ to $\Delta(z)v = 0$ yields a non-trivial solution $x(\xi) = e^{z\xi} v$ to (1.13).

In the finite dimensional case $\mathcal{H} = \mathbb{R}^n$, one can hence study the characteristic equation

$$\det \Delta(z) = 0, \qquad z \in \mathbb{C} \tag{1.15}$$

which typically has an infinite number of roots on both sides of the imaginary axis. This corresponds to the fact that (1.13) is typically ill-posed on its mathematical state space $C([-1, 1]; \mathbb{R}^n)$. In particular, one cannot expect it to be possible to extend arbitrary initial conditions $\phi \in C([-1, 1]; \mathbb{R}^n)$ to solutions of (1.13) on the half-lines $[0, \infty)$ or $(-\infty, 0]$.

When studying such ill-posed problems, exponential dichotomies become the methods of choice. For the finite dimensional case $\mathcal{H} = \mathbb{R}^n$, it was established in [6, 20] that if the hyperbolicity condition

$$0 \notin \det \Delta(i\mathbb{R}) \tag{1.16}$$

holds, one has an exponential splitting

$$C([-1, 1]; \mathbb{R}^n) = P \oplus Q. \tag{1.17}$$

Here Q contains all initial conditions $\phi \in C([-1, 1]; \mathcal{H})$ for which a bounded continuous function $x = x[\phi] : [-1, \infty) \to \mathcal{H}$ exists that satisfies (1.13) for $\xi \geq 0$ and has $x(\vartheta) = \phi(\vartheta)$ for $-1 \leq \vartheta \leq 1$. Such solutions satisfy the estimate

$$\|x[\phi](\xi)\|_{\mathcal{H}} \leq C e^{-\epsilon\xi} \|\phi\|_{C([-1,1];\mathcal{H})} \tag{1.18}$$

for some $C > 0$ and $\epsilon > 0$. Similarly, P contains all initial conditions that can be extended to bounded solutions on $(-\infty, 0]$.

These exponential dichotomies, together with their generalizations for non-autonomous systems, have played a critical role in the analysis of MFDEs during the past decade. For example, they have been used to construct travelling pulses for discrete FitzHugh-Nagumo equations [12], to analyze the nonlinear stability of these pulses [13], to investigate Lax shocks in discrete conservation laws [2] and to study the scattering of wave-fronts from obstacles in discrete planar systems [7].

In this paper we generalize the splitting (1.17) of the mathematical state space to settings where $\mathcal{H}$ is an infinite-dimensional Hilbert space. The main obstacle that needs to be overcome is that $\mathcal{H}$ is no longer locally compact, which prevents the use of the Ascoli-Arzela theorem that plays a crucial role in the abstract existence results of [20]. Our novel ingredient is that we explicitly characterize the spaces P and Q as the solution of two fixed point problems involving integral expressions related to the inverse Laplace transform. This explicit approach allows us obtain detailed estimates of the form

$$\|x[\phi](\xi)\|_{\mathcal{H}} \leq C e^{-\epsilon\xi} \big[\|\phi(0)\|_{\mathcal{H}} + \|\phi\|_{L^2([-1,1];\mathcal{H})} \big]. \tag{1.19}$$

This should be contrasted to estimates of the form (1.18) that can be obtained from a more abstract approach. It also allows us, in some settings, to replace $\mathcal{H}$ by a Banach space. Both these ingredients are crucial to allow us to perform the Fourier decompositions discussed in the sequel in a transparent fashion.

We remark that our use of inverse Laplace transforms is the sole reason that we restrict our attention to the Hilbert space setting. In particular, we are confident that our results are also valid for intermediate UMD-spaces [15], but we do not pursue this here.

1.3 Restriction operators

Here we discuss the linear homogeneous initial value problem

$$\begin{array}{rclr} x'(\xi) & = & \mathcal{A}x(\xi) + \mathcal{B}\int_{-1}^{1} x(\xi+\sigma)\,d\sigma & \text{for all } \xi \geq 0, \\ x(\xi) & \to & 0 & \text{as } \xi \to \infty, \end{array} \tag{1.20}$$

coupled with an initial condition

$$x(\vartheta) \;=\; \phi(\vartheta) \qquad \text{for all } -1 \leq \vartheta \leq 0, \tag{1.21}$$

for some $\phi \in C([-1,0];\mathcal{H})$. This prototype system differs from the traditional initial value problems that one typically encounters when studying ODEs or delay equations. Indeed, the initial condition ϕ does not provide sufficient information to calculate $x'(0)$. As discussed in §1.2, the natural mathematical state space for (1.20) is given by $C([-1,1];\mathcal{H})$, which of course differs from the space $C([-1,0];\mathcal{H})$ that ϕ belongs to. One of the interesting consequences of this discrepancy is that even after fixing ϕ, the problem (1.20)-(1.21) can still have multiple solutions rather than just a unique solution or none at all.

Indeterminacy The potential for (1.20)-(1.21) to have multiple solutions ties directly into a well-known problem in the area of macro-economic modelling. In particular, it is known that societies with seemingly similar economic structures and initial conditions can nevertheless experience remarkably distinct growth trajectories. For example, the expectations of market participants often play a major role in the evolution of markets, allowing several different sequences of self-fulfilling expectations to exist simultaneously [18]. The term *indeterminacy* is widely used to refer to models that reproduce this uncertainty in some fashion; see [3] for an informative survey.

State space(s) The key mathematical question is how the exponential dichotomy that splits $C([-1,1];\mathcal{H})$ projects down onto the modelling space $C([-1,0];\mathcal{H})$. More precisely, let us define the restriction operator

$$\pi^- : C([-1,1];\mathcal{H}) \to C([-1,0];\mathcal{H}), \qquad \psi \mapsto \psi_{|[-1,0]} \tag{1.22}$$

and recall the space Q introduced in §1.2. Our goal now is to understand the space

$$\pi^-(Q) \subset C([-1,0];\mathcal{H}) \tag{1.23}$$

containing all modelling initial conditions for which (1.20)-(1.21) has a solution, together with the space

$$\mathrm{Ker}\big(\pi^-_{|Q}\big) \subset C([-1,1];\mathcal{H}), \tag{1.24}$$

which characterizes the uniqueness of such extensions.

In the scalar case $\mathcal{H} = \mathbb{R}$, the key tool [20] to understand this restriction operator $\pi^-_{|Q}$ is the existence of a Wiener-Hopf factorization

$$z\Delta(z) = \Delta_{\mathrm{del}}(z)\Delta_{\mathrm{adv}}(z), \tag{1.25}$$

in which Δ_{del} is the characteristic equation of a (typically unknown) delay differential equation with state space $C([-1,0];\mathbb{C})$ and Δ_{adv} corresponds similarly to a (typically unknown) advanced differential equation posed on $C([0,1];\mathbb{C})$. Once such a splitting is obtained, one can compute an integer $n^\sharp$ by counting the roots of the characteristic equations $\Delta_{\mathrm{del}}(z) = 0$ and $\Delta_{\mathrm{adv}}(z) = 0$ that lie on the 'wrong' side of the imaginary axis. With this integer $n^\sharp$ in hand, one can easily determine the codimension of $\pi^-(Q)$ and the dimension of $\mathrm{Ker}\big(\pi^-_{|Q}\big)$, which are both finite. In particular, the restriction operator $\pi^-_{|Q}$ is Fredholm.

Unfortunately, explicit factorizations (1.25) are usually extremely hard to find. Motivated by this complication, we developed continuation techniques in [4, 10] that allow us to track the integer $n^\sharp$ through carefully constructed homotopies that lead to systems for which factorizations are explicitly available. In addition, we generalized the well-posedness results above to include certain differential-algebraic systems and to allow for the possibility of a single jump at $\xi = 0$. A small part of our contribution here centers on streamlining these results by introducing an invariant similar to $n^\sharp$ for differential-algebraic systems.

In the non-scalar but finite dimensional setting $\mathcal{H} = \mathbb{R}^n$ with $n > 1$, it is still the case that the restriction operator $\pi^-_{|Q}$ is Fredholm and that Wiener-Hopf factorizations are available, but the codimension of $\pi^-(Q)$ and the dimension of $\mathrm{Ker}\big(\pi^-_{|Q}\big)$ can at present [1] only be given in ranges involving $n^\sharp$. Indeed, possible linear dependencies between eigenvectors that are hard to track through continuations prevent the use of simple counting arguments.

Our main contribution in this regime is to give a description of the range $\pi^-(Q)$ and kernel $\mathrm{Ker}\big(\pi^-_{|Q}\big)$ of the restriction operator in terms of the Hale inner product, which naturally couples the linear system (1.13) with its formal adjoint. In the future this description might help to develop a practical tool for understanding the well-posedness of (1.20)-(1.21). In the present paper however, we need this description to show how the range and kernel of $\pi^-_{|Q}$ vary after parameters (such as Fourier frequencies) are introduced to the linear system (1.13).

1.4 Fourier decompositions

We here fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for $n \geq 1$ and assume that the operators $\mathcal{A}$ and $\mathcal{B}$ are convolution operators. In particular, we pick $a \in \ell^1(\mathbb{Z};\mathbb{R}^{n\times n})$ and $b \in \ell^1(\mathbb{Z};\mathbb{R}^{n\times n})$ and study the system

$$\begin{array}{lcl} x_j'(\xi) & = & \sum_{k\in\mathbb{Z}} a_{j-k} x_k(\xi) + \sum_{k\in\mathbb{Z}} b_{j-k} \int_{-1}^1 x_k(\xi+\sigma)\, d\sigma, \\ \|x(\xi)\|_{\ell^2(\mathbb{Z};\mathbb{R}^n)} & \to & 0 \text{ as } \xi \to \infty, \end{array} \tag{1.26}$$

together with the initial condition

$$x_j(\vartheta) \quad = \quad \phi_j(\vartheta) \qquad \text{for all } -1 \leq \vartheta \leq 0 \text{ and } j \in \mathbb{Z}, \tag{1.27}$$

for some $\phi \in C\big([-1,0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$. Formally taking Fourier transforms

$$x(\omega,\xi) = \sum_{j\in\mathbb{Z}} e^{-ij\omega} x_j(\xi), \qquad a(\omega) = \sum_{j\in\mathbb{Z}} e^{-ij\omega} a_j, \qquad b(\omega) = \sum_{j\in\mathbb{Z}} e^{-ij\omega} b_j, \tag{1.28}$$

the first line of (1.26) decouples as

$$\partial_\xi x(\omega,\xi) = a(\omega)x(\omega,\xi) + b(\omega)\int_{-1}^1 x(\omega,\xi+\sigma)\, d\sigma. \tag{1.29}$$

For each $\omega \in [-\pi,\pi]$ this is hence a system of the form (1.13) posed on $\mathcal{H} = \mathbb{R}^n$. Assuming that the characteristic functions

$$\Delta_\omega(z) = z - a(\omega) - b(\omega)\int_{-1}^1 e^{z\sigma}\, d\sigma \tag{1.30}$$

satisfy $0 \notin \det \Delta_\omega(i\mathbb{R})$ for all $\omega \in [-\pi,\pi]$, the discussion in §1.2 shows that we have decompositions

$$C([-1,1];\mathbb{R}^n) = P_\omega \oplus Q_\omega \tag{1.31}$$

together with

$$C\big([-1,1];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) = P \oplus Q. \tag{1.32}$$

In the final part of this paper we show how information concerning the set of restriction operators $\pi^-_{|Q_\omega}$ associated to the Fourier frequencies can be used to study the properties of the full restriction operator $\pi^-_{|Q}$. Special care here is required because in contrast to the setting of §1.3, the latter restriction operator is not Fredholm when not invertible. Indeed, the dimension of the kernel $\mathrm{Ker}\big(\pi^-_{|Q}\big)$ and the codimension of the range $\pi^-(Q)$ will either be zero or infinite dimensional. This follows directly from the translation invariance of (1.26), as kernel elements or functions missing from the range can be arbitrarily shifted in the j-direction.

A second factor that requires delicate attention is the interplay between the Fourier transform and the spaces of continuous functions on which the restriction operators are defined. Indeed, for $x(\xi) \in \ell^2(\mathbb{Z};\mathbb{R}^n)$, the Fourier components (1.28) need not be defined for all $\omega \in [-\pi,\pi]$. This can be alleviated by demanding $x(\xi) \in \ell^1(\mathbb{Z};\mathbb{R}^n)$, which is why part of the efforts described above in §1.2 focus on Banach spaces. In addition, to exploit the power of the Plancherel theorem we need to focus on L^2-based norms instead of supremum norms, which accounts for the discussion in §1.2 above concerning the detailed estimates (1.19) for the exponential splittings.

1.5 Organization

This paper is organized as follows. In §2 and §3 we state our main results, focussing on exponential dichotomies in §2 and on the restriction operators in §3. We illustrate our results in §4 by analyzing the model discussed above in §1.1. The remainder of the paper is devoted to the proof of the main results. We set up two fixed point problems in §5 in order to establish the existence of exponential splittings. We move on in §6 to discuss restriction operators for the finite dimensional case $\mathcal{H} = \mathbb{R}^n$. Differential-algebraic problems are analyzed in §7 and we conclude in §8 by studying Fourier decompositions.

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2 Exponential Splittings

Fix a Hilbert space $\mathcal{H}$. In this section we are interested in bounded linear operators

$$L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}, \tag{2.1}$$

in which we include the special case $L \in \mathcal{L}(\mathcal{H};\mathcal{H})$ by imposing the notation

$$C([0,0];\mathcal{H}) = \mathcal{H} \tag{2.2}$$

throughout the entire paper. Most of our results will require the following form condition to be satisfied.

$(\mathrm{HF})_L$ We have $r_{\min} \le 0 \le r_{\max}$. There exists an integer $N \ge 0$ together with real numbers

$$r_{\min} = r_0 < r_1 < \ldots < r_N = r_{\max}, \qquad r_{\min} \le s_j^- \le s_j^+ \le r_{\max}, \tag{2.3}$$

operators

$$A_j \in \mathcal{L}(\mathcal{H};\mathcal{H}), \qquad 0 \le j \le N \tag{2.4}$$

and functions

$$B_j \in C\big([s_j^-, s_j^+]; \mathcal{L}(\mathcal{H};\mathcal{H})\big), \qquad 0 \le j \le N, \tag{2.5}$$

so that

$$L\phi = \sum_{j=0}^{N} \Big[A_j \phi(r_j) + \int_{s_j^-}^{s_j^+} B_j(\sigma)\phi(\sigma)\, d\sigma \Big]. \tag{2.6}$$

We remark that the integrals in (2.6) are well-defined both as Riemann and Bochner integrals, since the integrands are continuous functions on compact intervals.

Fix $\xi \in \mathbb{R}$. For any $\mathcal{H}$-valued function x that is continuous on the interval $[\xi + r_{\min}, \xi + r_{\max}]$, we introduce the notation

$$\mathrm{ev}_\xi\, x \in C([r_{\min}, r_{\max}]; \mathcal{H}) \tag{2.7}$$

to refer to the function that has

$$[\mathrm{ev}_\xi x](\sigma) = x(\xi + \sigma) \qquad \text{for all } r_{\min} \leq \sigma \leq r_{\max}. \tag{2.8}$$

Our goal here is to characterize which functions $\phi \in C([r_{\min}, r_{\max}]; \mathcal{H})$ can be extended to solutions to the homogeneous problem

$$x'(\xi) = L\, \mathrm{ev}_\xi\, x \tag{2.9}$$

that are defined on half-lines and bounded by prescribed exponentials. For any $\eta \in \mathbb{R}$ and any interval $\mathcal{I} \subset \mathbb{R}$, we therefore introduce the function space

$$BC_\eta(\mathcal{I}; \mathcal{H}) \;\; = \;\; \{x \in C(\mathcal{I}; \mathcal{H}) : \|x\|_\eta := \sup_{\xi \in \mathcal{I}} e^{-\eta\xi} \, \|x(\xi)\|_{\mathcal{H}} < \infty\}. \tag{2.10}$$

This allows us to define the two families

$$\begin{aligned} BC_\eta^\ominus(\mathcal{H}) &= BC_\eta\big((-\infty, r_{\max}]; \mathcal{H}\big), \\ BC_\eta^\oplus(\mathcal{H}) &= BC_\eta\big([r_{\min}, \infty); \mathcal{H}\big), \end{aligned} \tag{2.11}$$

together with the solution spaces

$$\begin{aligned} \mathfrak{P}_L(\eta) &= \big\{x \in BC_\eta^\ominus(\mathcal{H}) : x'(\xi) = L\, \mathrm{ev}_\xi\, x \text{ for all } \xi \leq 0\big\}, \\ \mathfrak{Q}_L(\eta) &= \big\{y \in BC_\eta^\oplus(\mathcal{H}) : y'(\xi) = L\, \mathrm{ev}_\xi\, y \text{ for all } \xi \geq 0\big\}. \end{aligned} \tag{2.12}$$

The initial segments of these solutions are contained in the spaces

$$\begin{aligned} P_L(\eta) &= \big\{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) : \phi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{P}_L(\eta)\big\}, \\ Q_L(\eta) &= \big\{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) : \phi = \mathrm{ev}_0\, y \text{ for some } y \in \mathfrak{Q}_L(\eta)\big\}. \end{aligned} \tag{2.13}$$

As a final preparation, we introduce the characteristic function

$$\Delta_L : \mathbb{C} \mapsto \mathcal{L}\big(\mathcal{H}; \mathcal{H}\big) \tag{2.14}$$

defined by

$$\Delta_L(z) = z - Le^{z\cdot} \tag{2.15}$$

for any $z \in \mathbb{C}$. Here we are implicitly assuming that $\mathcal{H}$ has been complexified if necessary. If $(\mathrm{HF})_L$ is satisfied, then we may write

$$\Delta_L(z) = z - \sum_{j=0}^{N} \Big[A_j e^{z r_j} + \int_{s_j^-}^{s_j^+} B_j(\sigma) e^{z\sigma}\, d\sigma \Big]. \tag{2.16}$$

Our first main result states that (2.9) admits exponential dichotomies, hence generalizing [20, Thm. 3.2] to the current infinite dimensional setting.

Theorem 2.1 (see §5). *Fix a Hilbert space $\mathcal{H}$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. If the linear operators $\Delta_L(z) \in \mathcal{L}(\mathcal{H}; \mathcal{H})$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$, then the spaces $P_L(\eta)$ and $Q_L(\eta)$ are both closed and we have the splitting*

$$C\big([r_{\min}, r_{\max}]; \mathcal{H}\big) = P_L(\eta) \oplus Q_L(\eta). \tag{2.17}$$

In order to allow solutions of (2.9) to have a jump discontinuity at $\xi = 0$, we introduce the shorthands

$$\begin{array}{lcl} BC^-_\eta(\mathcal{H}) & = & BC_\eta((-\infty, 0]; \mathcal{H}), \\ BC^+_\eta(\mathcal{H}) & = & BC_\eta([0, \infty); \mathcal{H}), \end{array} \tag{2.18}$$

together with the two families of function spaces

$$\begin{array}{lcl} \widehat{BC}^\oplus_\eta(\mathcal{H}) & = & C([r_{\min}, 0]; \mathcal{H}) \times BC^+_\eta(\mathcal{H}), \\ \widehat{BC}^\ominus_\eta(\mathcal{H}) & = & BC^-_\eta(\mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}), \end{array} \tag{2.19}$$

all parametrized by $\eta \in \mathbb{R}$.

For $\widehat{y} \in \widehat{BC}^\oplus_\eta(\mathcal{H})$, we will simply write $\widehat{y}(\xi)$ to refer to the approriate function value, using the notation $\widehat{y}(0^+)$ and $\widehat{y}(0^-)$ to resolve the ambiguity at $\xi = 0$ whenever necessary. For any $0 \le \xi \le -r_{\min}$ and $\widehat{y} \in BC^\oplus_\eta$, we introduce the notation

$$\widehat{\mathrm{ev}}_\xi\, \widehat{y} = (\phi^l, \phi^r) \in C\big([r_{\min}, -\xi]; \mathcal{H}\big) \times C\big([-\xi, r_{\max}]; \mathcal{H}\big) \tag{2.20}$$

to refer to the pair of functions that have

$$\begin{array}{lcl} \phi^l(\sigma) & = & \begin{cases} \widehat{y}(\xi + \sigma) & r_{\min} \le \sigma < -\xi, \\ \widehat{y}(0^-) & \sigma = -\xi, \end{cases} \\ \phi^r(\sigma) & = & \begin{cases} \widehat{y}(\xi + \sigma) & -\xi < \sigma \le r_{\max}, \\ \widehat{y}(0^+) & \sigma = -\xi. \end{cases} \end{array} \tag{2.21}$$

We then write

$$\begin{array}{lcl} \widehat{L}_+\, \widehat{\mathrm{ev}}_\xi\, \widehat{y} & = & \sum_{r_j = -\xi} A_j \phi^r(r_j) + \sum_{r_j > -\xi} A_j \phi^r(r_j) + \sum_{r_j < -\xi} A_j \phi^l(r_j) \\ & & \quad + \sum_{j=0}^N \Big[\int_{\min\{s_j^-, -\xi\}}^{\min\{-\xi, s_j^+\}} B_j(\sigma) \phi^l(\sigma)\, d\sigma + \int_{\max\{-\xi, s_j^-\}}^{\max\{-\xi, s_j^+\}} B_j(\sigma) \phi^r(\sigma)\, d\sigma \Big] \\ & = & \sum_{j=0}^N \Big[A_j \widehat{y}\big((\xi + r_j)^+\big) + \int_{s_j^-}^{s_j^+} B_j(\sigma) \widehat{y}(\xi + \sigma)\, d\sigma \Big]. \end{array} \tag{2.22}$$

The plus sign hence stands for the fact that every reference to $\widehat{y}(0)$ is interpreted as $\widehat{y}(0^+)$. For $\xi > -r_{\min}$ we simply write

$$[\widehat{\mathrm{ev}}_\xi\, \widehat{y}](\sigma) = \widehat{y}(\xi + \sigma), \qquad r_{\min} \le \sigma \le r_{\max} \tag{2.23}$$

as there is no cause for confusion.

For $\widehat{x} \in \widehat{BC}^\ominus_\eta(\mathcal{H})$ and $-r_{\max} \le \xi \le 0$, we again write $\widehat{\mathrm{ev}}_\xi\, \widehat{x} = (\phi^l, \phi^r)$ with (ϕ^l, ϕ^r) defined as in (2.21) with $\widehat{y}$ replaced by $\widehat{x}$. We then write

$$\begin{array}{lcl} \widehat{L}_-\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} & = & \sum_{r_j = -\xi} A_j \phi^l(r_j) + \sum_{r_j > -\xi} A_j \phi^r(r_j) + \sum_{r_j < -\xi} A_j \phi^l(r_j) \\ & & \quad + \sum_{j=0}^N \Big[\int_{\min\{s_j^-, -\xi\}}^{\min\{-\xi, s_j^+\}} B_j(\sigma) \phi^l(\sigma)\, d\sigma + \int_{\max\{-\xi, s_j^-\}}^{\max\{-\xi, s_j^+\}} B_j(\sigma) \phi^r(\sigma)\, d\sigma \Big] \\ & = & \sum_{j=0}^N \Big[A_j \widehat{x}\big((\xi + r_j)^-\big) + \int_{s_j^-}^{s_j^+} B_j(\sigma) \widehat{x}(\xi + \sigma)\, d\sigma \Big]. \end{array} \tag{2.24}$$

For convenience, we introduce the set

$$\mathcal{R} = \{0\} \cup \{-r_j\}_{j=0}^{N}. \tag{2.25}$$

We note that any discontinuities in the functions and $\xi \mapsto \widehat{L}_+ \,\widehat{\mathrm{ev}}_\xi\, \widehat{y}$ and $\xi \mapsto \widehat{L}_- \,\widehat{\mathrm{ev}}_\xi\, \widehat{x}$ will arise for values of $\xi \in \mathcal{R}$. In fact, for $\xi \notin \mathcal{R}$ there is no ambiguity between the two definitions (2.22) and (2.24) and we simply use the notation $\widehat{L}$ in this case.

Proposition 2.2 (see §5). *Fix a Hilbert space $\mathcal{H}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. If the linear operators $\Delta_L(z) \in \mathcal{L}(\mathcal{H};\mathcal{H})$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$, then there exists a function*

$$\widehat{G}_L(\eta) = \big(G_L^-(\eta), G_L^+(\eta)\big) \in C\big((-\infty, 0]; \mathcal{L}(\mathcal{H};\mathcal{H})\big) \times C\big([0,\infty); \mathcal{L}(\mathcal{H};\mathcal{H})\big) \tag{2.26}$$

that satisfies the following properties.

(i) For every $\xi \in \mathbb{R} \setminus \mathcal{R}$, the function $\widehat{G}_L(\eta)$ satisfies the differential equation

$$[\widehat{G}_L(\eta)]'(\xi) = \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,[\widehat{G}_L](\eta). \tag{2.27}$$

(ii) There exist constants $K > 0$ and $\kappa > 0$ such that

$$\Big| e^{-\eta\xi}[\widehat{G}_L(\eta)](\xi) \Big| \le K e^{-\kappa|\xi|}, \qquad \xi \in \mathbb{R}. \tag{2.28}$$

(iii) Writing I for the identity on $\mathcal{H}$, we have

$$[G_L^+(\eta)](0) - [G_L^-(\eta)](0) = I. \tag{2.29}$$

We are now ready to introduce the solution spaces

$$\begin{aligned} \widehat{\mathfrak{P}}_L(\eta) &= \big\{\widehat{x} \in \widehat{BC}_\eta^{\ominus}(\mathcal{H}) : \widehat{x}'(\xi) = \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,\widehat{x} \text{ for all } \xi \in (-\infty, 0) \setminus \mathcal{R}\big\}, \\ \widehat{\mathfrak{Q}}_L(\eta) &= \big\{\widehat{y} \in \widehat{BC}_\eta^{\oplus}(\mathcal{H}) : \widehat{y}'(\xi) = \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,\widehat{y} \text{ for all } \xi \in (0, \infty) \setminus \mathcal{R}\big\}, \end{aligned} \tag{2.30}$$

together with the associated initial segment spaces

$$\begin{aligned} \widehat{P}_L(\eta) &= \big\{\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) : \widehat{\phi} = \widehat{\mathrm{ev}}_0\,\widehat{x} \text{ for some } \widehat{x} \in \widehat{\mathfrak{P}}_L(\eta)\big\}, \\ \widehat{Q}_L(\eta) &= \big\{\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) : \widehat{\phi} = \widehat{\mathrm{ev}}_0\,\widehat{y} \text{ for some } \widehat{y} \in \widehat{\mathfrak{Q}}_L(\eta)\big\}. \end{aligned} \tag{2.31}$$

Our second main result shows that the Green's function described in Proposition 2.2 acts as a bridge between solution spaces that do and do not permit jumps at $\xi = 0$.

Theorem 2.3 (see §5). *Fix a Hilbert space $\mathcal{H}$ and consider a linear map $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. If the linear operators $\Delta_L(z) \in \mathcal{L}(\mathcal{H};\mathcal{H})$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$, then we have the relations*

$$\widehat{P}_L(\eta) = P_L(\eta) \oplus \mathrm{span}_{\mathcal{H}}\{\widehat{\mathrm{ev}}_0 \widehat{G}_L(\eta)\}, \qquad \widehat{Q}_L(\eta) = Q_L(\eta) \oplus \mathrm{span}_{\mathcal{H}}\{\widehat{\mathrm{ev}}_0 \widehat{G}_L(\eta)\}, \tag{2.32}$$

in which we have introduced the notation

$$\begin{aligned} \mathrm{span}_{\mathcal{H}}\{\widehat{\mathrm{ev}}_0 \widehat{G}_L(\eta)\} &= \{\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) : \\ &\qquad \widehat{\phi} = \widehat{\mathrm{ev}}_0[\widehat{G}_L(\eta) v] \textit{ for some } v \in \mathcal{H}\}. \end{aligned} \tag{2.33}$$

We now turn our attention to differential-algebraic equations. In particular, we fix a bounded linear operator

$$M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H} \tag{2.34}$$

and consider equations of the form

$$\mathcal{I}x'(\xi) = M \operatorname{ev}_\xi x \tag{2.35}$$

that satisfy the following structural condition.

(HS) We have $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ or $\mathcal{H} = \mathbb{R}^n$ for some $n \geq 1$ and the $n \times n$-matrix $\mathcal{I}$ is diagonal with $\mathcal{I}^2 = \mathcal{I}$. If $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$, then the matrix multiplication in (2.35) should be interpreted in a pointwise fashion, e.g.

$$[\mathcal{I}v]_j = \mathcal{I}v_j \text{ for any } v \in \ell^2(\mathbb{Z}; \mathbb{R}^n). \tag{2.36}$$

We are interested in systems that can be closely related to a differential system of the form (2.9). In order to clarify this relationship, we introduce the characteristic operator

$$\delta_{\mathcal{I},M}(z) = \mathcal{I}z - Me^{z\cdot} \tag{2.37}$$

that is associated to (2.35). The restriction on the differential-algebraic structure of this system that we need to impose can now be captured by the following condition on the characteristic function.

$(\text{HAlg})_{\mathcal{I},M}$ There exists a linear operator

$$L \in \mathcal{L}\big(C([r_{\min}, r_{\max}]; \mathcal{H}); \mathcal{H}\big) \tag{2.38}$$

together with a constant $\alpha \in \mathbb{C}$ and a set of non-negative integers $(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n_{\geq 0}$ such that

$$\mathcal{J}_\alpha(z)\delta_{\mathcal{I},M}(z) = \Delta_L(z), \tag{2.39}$$

where $\mathcal{J}_\alpha : \mathbb{C} \to \mathbb{C}^{n \times n}$ denotes the diagonal matrix function

$$\mathcal{J}_\alpha(z) = \operatorname{diag}\big((z-\alpha)^{\ell_1}, \ldots, (z-\alpha)^{\ell_n}\big). \tag{2.40}$$

If $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$, then the matrix multiplication in (2.39) should be interpreted in the pointwise fashion described in (HS).

This condition roughly states that one arrives at a pure differential equation by differentiating the i-th component of (2.35) ℓ_i times. We note that we do not require L above to satisfy $(\text{HF})_L$. In addition, the demand $\mathcal{I}^2 = \mathcal{I}$ means that all entries of the diagonal matrix $\mathcal{I}$ are either zero or one.

Please note that the purely algebraic components of the system (2.35) are unaffected if the corresponding components of M are multiplied by a non-zero factor. In particular, the corresponding rows of $\delta_{\mathcal{I},M}$ can be rescaled without affecting the dynamics of (2.35). Adjusting M in such a manner will typically be necessary in order to show that all the terms $(z-\alpha)^\ell$ appearing in (2.40) have coefficient one. Furthermore, we remark that a simple matching of asymptotics along the imaginary axis shows that

$$\mathcal{J}_\alpha(\alpha) = \mathcal{I}, \tag{2.41}$$

or alternatively, that $\ell_i = 0$ if and only if $\mathcal{I}_{ii} = 1$.

We will be interested in the solution spaces

$$\begin{array}{rcl} \mathfrak{p}_{\mathcal{I},M}(\eta) & = & \big\{x \in BC^{\ominus}_\eta : \mathcal{I}x'(\xi) = M \operatorname{ev}_\xi x \text{ for all } \xi \leq 0\big\}, \\ \mathfrak{q}_{\mathcal{I},M}(\eta) & = & \big\{y \in BC^{\oplus}_\eta : \mathcal{I}y'(\xi) = M \operatorname{ev}_\xi y \text{ for all } \xi \geq 0\big\}, \end{array} \tag{2.42}$$

together with their initial segments

$$\begin{aligned} p_{\mathcal{I},M}(\eta) &= \big\{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) \mid \phi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{p}_{\mathcal{I},M}(\eta)\big\}, \\ q_{\mathcal{I},M}(\eta) &= \big\{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) \mid \phi = \mathrm{ev}_0\, y \text{ for some } y \in \mathfrak{q}_{\mathcal{I},M}(\eta)\big\}, \end{aligned} \tag{2.43}$$

which all describe solutions to (2.35) that do not admit a discontinuity at $\xi = 0$.

However, when considering functions that are allowed to be multi-valued at $\xi = 0$, care must be taken to ensure that (2.35) is well-posed. The following result is important in this respect, as it shows that the part of the right-hand side of (2.35) that corresponds to the purely algebraic equations is continuous.

Lemma 2.4 (see §7). *Consider the differential-algebraic system (2.35) and suppose that* $(\mathrm{HF})_M$*,* (HS) *and* $(\mathrm{HAlg})_{\mathcal{I},M}$ *are all satisfied. Then for any* $\widehat{y} \in \widehat{BC}_\eta^{\oplus}$*, the function*

$$\xi \mapsto (I - \mathcal{I})\widehat{M}_+\, \widehat{\mathrm{ev}}_\xi\, \widehat{y} \tag{2.44}$$

is continuous on $[0, \infty)$*, while for any* $\widehat{x} \in \widehat{BC}_\eta^{\ominus}$*, the function*

$$\xi \mapsto (I - \mathcal{I})\widehat{M}_-\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} \tag{2.45}$$

is continuous on $(-\infty, 0]$*.*

Recalling the set $\mathcal{R}$ defined in (2.25), this result motivates the introduction of the solution spaces

$$\begin{aligned} \widehat{\mathfrak{p}}_{\mathcal{I},M}(\eta) &= \big\{\widehat{x} \in \widehat{BC}_\eta^{\oplus} : \mathcal{I}\,\widehat{x}'(\xi) = \mathcal{I}\,\widehat{M}_-\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} \text{ for all } \xi \in (-\infty, 0) \setminus \mathcal{R} \\ &\qquad \text{and } 0 = (I - \mathcal{I})\,\widehat{M}_-\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} \text{ for all } \xi \le 0\big\}. \\ \widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta) &= \big\{\widehat{y} \in \widehat{BC}_\eta^{\oplus} : \mathcal{I}\,\widehat{y}'(\xi) = \mathcal{I}\,\widehat{M}_+\, \widehat{\mathrm{ev}}_\xi\, \widehat{y} \text{ for all } \xi \in (0, \infty) \setminus \mathcal{R} \\ &\qquad \text{and } 0 = (I - \mathcal{I})\,\widehat{M}_+\, \widehat{\mathrm{ev}}_\xi\, \widehat{y} \text{ for all } \xi \ge 0\big\}, \end{aligned} \tag{2.46}$$

together with their initial segments

$$\begin{aligned} \widehat{p}_{\mathcal{I},M}(\eta) &= \big\{\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) : \widehat{\phi} = \widehat{\mathrm{ev}}_0\, \widehat{x} \text{ for some } \widehat{x} \in \mathfrak{p}_{\mathcal{I},M}(\eta)\big\}, \\ \widehat{q}_{\mathcal{I},M}(\eta) &= \big\{\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) : \widehat{\phi} = \widehat{\mathrm{ev}}_0\, \widehat{y} \text{ for some } \widehat{y} \in \mathfrak{q}_{\mathcal{I},M}(\eta)\big\}. \end{aligned} \tag{2.47}$$

Our final result in this section relates these spaces to their counterparts that were defined for the differential equation (2.9). In particular, initial value problems for the differential-algebraic system (2.35) can be studied by techniques similar to those that we will develop for (2.9).

Theorem 2.5 (see §7). *Consider the differential-algebraic equation (2.35) and suppose that* $(\mathrm{HF})_M$*,* (HS) *and* $(\mathrm{HAlg})_{\mathcal{I},M}$ *are all satisfied. Choose any* $\eta_* \in \mathbb{R}$ *for which the characteristic operator* $\delta_{\mathcal{I},M}(z)$ *is invertible for all* $\mathrm{Re}\, z = \eta_*$*. Then there exists a bounded linear map* $L' : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ *that satisfies* $(\mathrm{HF})_{L'}$ *and for which*

$$\mathcal{J}_{\eta_*}(z)\delta_{\mathcal{I},M}(z) = \Delta_{L'}(z) \tag{2.48}$$

holds for all $z \in \mathbb{C}$*.*

In addition, for every sufficiently small $\epsilon > 0$ *we have*

$$\mathfrak{q}_{\mathcal{I},M}(\eta_*) = \mathfrak{Q}_{L'}(\eta_* - \epsilon), \qquad \widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta_*) = \widehat{\mathfrak{Q}}_{L'}(\eta_* - \epsilon), \tag{2.49}$$

together with

$$\mathfrak{p}_{\mathcal{I},M}(\eta_*) = \mathfrak{P}_{L'}(\eta_* + \epsilon), \qquad \widehat{\mathfrak{p}}_{\mathcal{I},M}(\eta_*) = \widehat{\mathfrak{P}}_{L'}(\eta_* + \epsilon). \tag{2.50}$$

Alternatively, for every $\eta < \eta_*$ *we have*

$$\mathfrak{q}_{\mathcal{I},M}(\eta) = \mathfrak{Q}_{L'}(\eta), \qquad \widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta) = \widehat{\mathfrak{Q}}_{L'}(\eta), \tag{2.51}$$

while for every $\eta > \eta_*$ *we have*

$$\mathfrak{p}_{\mathcal{I},M}(\eta) = \mathfrak{P}_{L'}(\eta), \qquad \widehat{\mathfrak{p}}_{\mathcal{I},M}(\eta) = \widehat{\mathfrak{P}}_{L'}(\eta). \tag{2.52}$$

3 State Space Restrictions

In this section we state our main results concerning two pairs of restriction operators. The first pair

$$\begin{array}{rcl} \pi^- : C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) & \to & C([r_{\min}, 0]; \mathcal{H}), \\ \pi^+ : C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) & \to & C([0, r_{\max}]; \mathcal{H}) \end{array} \tag{3.1}$$

acts as

$$\pi^-(\phi^-, \phi^+) = \phi^-, \qquad \pi^+(\phi^-, \phi^+) = \phi^+, \tag{3.2}$$

while the second augmented pair

$$\begin{array}{rcl} \widehat{\pi}^- : C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) & \to & C([r_{\min}, 0]; \mathcal{H}) \times \mathcal{H}, \\ \widehat{\pi}^+ : C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H}) & \to & \mathcal{H} \times C([0, r_{\max}]; \mathcal{H}) \end{array} \tag{3.3}$$

acts as

$$\widehat{\pi}^-(\phi^-, \phi^+) = \big(\phi^-, \phi^+(0)\big), \qquad \widehat{\pi}^+(\phi^-, \phi^+) = \big(\phi^-(0), \phi^+\big). \tag{3.4}$$

We are specifically interested in the action of these base restriction operators on the initial segment spaces related to the exponential splittings that were introduced in §2.

In particular, for any $\eta \in \mathbb{R}$ we introduce the shorthands

$$\begin{array}{rclcrcl} \pi^+_{P_L(\eta)} & = & [\pi^+]_{|P_L(\eta)}, & \qquad & \pi^-_{Q_L(\eta)} & = & [\pi^-]_{|Q_L(\eta)}, \\ \pi^+_{\widehat{P}_L(\eta)} & = & [\pi^+]_{|\widehat{P}_L(\eta)}, & & \pi^-_{\widehat{Q}_L(\eta)} & = & [\pi^-]_{|\widehat{Q}_L(\eta)}, \\ \widehat{\pi}^+_{\widehat{P}_L(\eta)} & = & [\widehat{\pi}^+]_{|\widehat{P}_L(\eta)}, & & \widehat{\pi}^-_{\widehat{Q}_L(\eta)} & = & [\widehat{\pi}^-]_{|\widehat{Q}_L(\eta)}, \end{array} \tag{3.5}$$

which are all associated to the differential system (2.9). In addition, we introduce the spaces

$$\begin{array}{rclcrcl} K^+_{P_L(\eta)} & = & \mathrm{Ker}\, \pi^+_{P_L(\eta)}, & \qquad & R^+_{P_L(\eta)} & = & \mathrm{Range}\, \pi^+_{P_L(\eta)}, \\ K^+_{\widehat{P}_L(\eta)} & = & \mathrm{Ker}\, \pi^+_{\widehat{P}_L(\eta)}, & & R^+_{\widehat{P}_L(\eta)} & = & \mathrm{Range}\, \pi^+_{\widehat{P}_L(\eta)}, \\ \widehat{K}^+_{\widehat{P}_L(\eta)} & = & \mathrm{Ker}\, \widehat{\pi}^+_{\widehat{P}_L(\eta)}, & & \widehat{R}^+_{\widehat{P}_L(\eta)} & = & \mathrm{Range}\, \widehat{\pi}^+_{\widehat{P}_L(\eta)}, \end{array} \tag{3.6}$$

together with

$$\begin{array}{rclcrcl} K^-_{Q_L(\eta)} & = & \mathrm{Ker}\, \pi^-_{Q_L(\eta)}, & \qquad & R^-_{Q_L(\eta)} & = & \mathrm{Range}\, \pi^-_{Q_L(\eta)}, \\ K^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Ker}\, \pi^-_{\widehat{Q}_L(\eta)}, & & R^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Range}\, \pi^-_{\widehat{Q}_L(\eta)}, \\ \widehat{K}^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Ker}\, \widehat{\pi}^-_{\widehat{Q}_L(\eta)}, & & \widehat{R}^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Range}\, \widehat{\pi}^-_{\widehat{Q}_L(\eta)}. \end{array} \tag{3.7}$$

In order to conveniently formulate our results, we also introduce the collection of triplets

$$\begin{array}{rcl}\Theta_L(\eta) & = & \Big\{ \quad \big(\pi^+_{P_L(\eta)}, P_L(\eta), C([0, r_{\max}];\mathcal{H})\big), \qquad \big(\pi^-_{Q_L(\eta)}, Q_L(\eta), C([r_{\min}, 0];\mathcal{H})\big), \\ & & \big(\pi^+_{\widehat{P}_L(\eta)}, \widehat{P}_L(\eta), C([0, r_{\max}];\mathcal{H})\big), \qquad \big(\pi^-_{\widehat{Q}_L(\eta)}, \widehat{Q}_L(\eta), C([r_{\min}, 0];\mathcal{H})\big), \\ & & \big(\widehat{\pi}^+_{\widehat{P}_L(\eta)}, \widehat{P}_L(\eta), \mathcal{H} \times C([0, r_{\max}];\mathcal{H})\big), \quad \big(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}, \widehat{Q}_L(\eta), C([r_{\min}, 0];\mathcal{H}) \times \mathcal{H}\big)\Big\}.\end{array} \tag{3.8}$$

For any triplet $\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_L(\eta)$, we will be interested in determining the properties of the restriction operator

$$\pi_\theta : \mathcal{S}_\theta \to \mathcal{T}_\theta. \tag{3.9}$$

To accompany the differential-algebraic system (2.35), we write

$$\begin{array}{rclrcl}\pi^+_{p_{\mathcal{I},M}(\eta)} & = & [\pi^+]_{|p_{\mathcal{I},M}(\eta)}, & \pi^-_{q_{\mathcal{I},M}(\eta)} & = & [\pi^-]_{|q_{\mathcal{I},M}(\eta)}, \\ \pi^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & [\pi^+]_{|\widehat{p}_{\mathcal{I},M}(\eta)}, & \pi^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & [\pi^-]_{|\widehat{q}_{\mathcal{I},M}(\eta)}, \\ \widehat{\pi}^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & [\widehat{\pi}^+]_{|\widehat{p}_{\mathcal{I},M}(\eta)}, & \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & [\widehat{\pi}^-]_{|\widehat{q}_{\mathcal{I},M}(\eta)}.\end{array} \tag{3.10}$$

As above, we also introduce the spaces

$$\begin{array}{rclrcl}K^+_{p_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \pi^+_{p_{\mathcal{I},M}(\eta)}, & R^+_{p_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \pi^+_{p_{\mathcal{I},M}(\eta)}, \\ K^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \pi^+_{\widehat{p}_{\mathcal{I},M}(\eta)}, & R^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \pi^+_{\widehat{p}_{\mathcal{I},M}(\eta)}, \\ \widehat{K}^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \widehat{\pi}^+_{\widehat{p}_{\mathcal{I},M}(\eta)}, & \widehat{R}^+_{\widehat{p}_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \widehat{\pi}^+_{\widehat{p}_{\mathcal{I},M}(\eta)}\end{array} \tag{3.11}$$

and

$$\begin{array}{rclrcl}K^-_{q_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \pi^-_{q_{\mathcal{I},M}(\eta)}, & R^-_{q_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \pi^-_{q_{\mathcal{I},M}(\eta)}, \\ K^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(\eta)}, & R^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(\eta)}, \\ \widehat{K}^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & \mathrm{Ker}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(\eta)}, & \widehat{R}^-_{\widehat{q}_{\mathcal{I},M}(\eta)} & = & \mathrm{Range}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(\eta)},\end{array} \tag{3.12}$$

together with the collection of triplets

$$\begin{array}{rl}\Theta_{I,M}(\eta) = \Big\{ & \big(\pi^+_{p_{\mathcal{I},M}(\eta)}, p_{\mathcal{I},M}(\eta), C([0, r_{\max}];\mathcal{H})\big), \qquad \big(\pi^-_{q_{\mathcal{I},M}(\eta)}, q_{\mathcal{I},M}(\eta), C([r_{\min}, 0];\mathcal{H})\big), \\ & \big(\pi^+_{\widehat{p}_{\mathcal{I},M}(\eta)}, \widehat{p}_{\mathcal{I},M}(\eta), C([0, r_{\max}];\mathcal{H})\big), \qquad \big(\pi^-_{\widehat{q}_{\mathcal{I},M}(\eta)}, \widehat{q}_{\mathcal{I},M}(\eta), C([r_{\min}, 0];\mathcal{H})\big), \\ & \big(\widehat{\pi}^+_{\widehat{p}_{\mathcal{I},M}(\eta)}, \widehat{p}_{\mathcal{I},M}(\eta), \mathcal{H} \times C([0, r_{\max}];\mathcal{H})\big), \quad \big(\widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(\eta)}, \widehat{q}_{\mathcal{I},M}(\eta), C([r_{\min}, 0];\mathcal{H}) \times \mathcal{H}\big)\Big\}.\end{array} \tag{3.13}$$

For any triplet $\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_{I,M}(\eta)$, we will again be interested in understanding the restriction

$$\pi_\theta : \mathcal{S}_\theta \to \mathcal{T}_\theta. \tag{3.14}$$

In §3.1 we present our results for the general finite dimensional case $\mathcal{H} = \mathbb{R}^n$. In the scalar setting $\mathcal{H} = \mathbb{R}$ more detailed characterizations are possible, which we formulate for differential systems in §3.2 and for differential-algebraic systems in §3.3. Finally, in §3.4 we discuss the infinite dimensional case $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$.

3.1 Finite dimensional systems

Our interest here is in the finite dimensional case $\mathcal{H} = \mathbb{R}^n$. Our first results state that the restriction operators (3.5) and (3.10) are Fredholm provided that the relevant vertical line in the complex plane is free of eigenvalues. This property can already be found in [20] for $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$, together with the index formula (3.16). Our extensions are relatively minor, but we include them here for completeness. In fact, we provide an alternative proof for these facts that is based upon the fixed point setup developed in §5.

Proposition 3.1 (see §6). *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then for every $\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_L(\eta)$, the restriction operator*

$$\pi_\theta : \mathcal{S}_\theta \to \mathcal{T}_\theta \tag{3.15}$$

is a Fredholm operator. In addition, we have the index formula

$$\mathrm{ind}(\pi^+_{P_L(\eta)}) + \mathrm{ind}(\pi^-_{Q_L(\eta)}) \quad = \quad -n, \tag{3.16}$$

together with the variants

$$\begin{array}{rcccl} \mathrm{ind}(\pi^+_{P_L(\eta)}) + \mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) & = & \mathrm{ind}(\pi^+_{\widehat{P}_L(\eta)}) + \mathrm{ind}(\pi^-_{Q_L(\eta)}) & = & 0, \\ \mathrm{ind}(\pi^+_{\widehat{P}_L(\eta)}) + \mathrm{ind}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}) & = & \mathrm{ind}(\widehat{\pi}^+_{\widehat{P}_L(\eta)}) + \mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) & = & 0. \end{array} \tag{3.17}$$

Corollary 3.2. *Consider the differential-algebraic equation (2.35) with $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and suppose that $(\mathrm{HF})_M$ and $(\mathrm{HAlg})_{\mathcal{I},M}$ are satisfied. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \delta_{\mathcal{I},M}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then for every $\xi = (\pi_\xi, \mathcal{S}_\xi, \mathcal{T}_\xi) \in \Xi_{\mathcal{I},M}(\eta)$, the restriction operator*

$$\pi_\xi : \mathcal{S}_\xi \to \mathcal{T}_\xi \tag{3.18}$$

is a Fredholm operator.

Proof. This follows directly from Theorem 2.5 and Proposition 3.1. □

Assuming the form condition $(\mathrm{HF})_L$ is satisfied for the system (2.9), we define the formal adjoint

$$L_* : C\big([-r_{\max}, -r_{\min}]; \mathbb{R}^n\big) \to \mathbb{R}^n \tag{3.19}$$

that acts as[1]

$$\begin{array}{rcl} L_*\psi & = & -\sum_{j=0}^N \big[A_j^* \psi(-r_j) + \int_{s_j^-}^{s_j^+} B_j(\sigma)^* \psi(-\sigma)\, d\sigma\big] \\ & = & -\sum_{j=0}^N \big[A_j^* \psi(-r_j) + \int_{-s_j^+}^{-s_j^-} B_j(-\sigma)^* \psi(\sigma)\, d\sigma\big]. \end{array} \tag{3.20}$$

The coupling between L and L_* is provided through the Hale inner product

$$\langle \cdot, \cdot \rangle_L : C([-r_{\max}, -r_{\min}]; \mathbb{R}^n) \times C([r_{\min}, r_{\max}]; \mathbb{R}^n) \to \mathbb{R}^n, \tag{3.21}$$

which acts as

$$\langle \psi, \phi \rangle_L = \psi(0)^* \phi(0) - \sum_{j=0}^N \int_0^{r_j} \psi(\tau - r_j)^* A_j \phi(\tau)\, d\tau - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} \int_0^\sigma \psi(\tau - \sigma)^* B_j(\sigma) \phi(\tau)\, d\tau\, d\sigma. \tag{3.22}$$

[1] For later use, we are deliberately using complex notation here, even though all terms are real-valued at present.

Introducing the two bounded linear operators

$$\begin{array}{rcll} L_{>0} & : & C([0, r_{\max}]; \mathbb{R}^n) & \to & \mathbb{R}^n, \\ L_{<0} & : & C([r_{\min}, 0]; \mathbb{R}^n) & \to & \mathbb{R}^n \end{array} \tag{3.23}$$

that act as

$$\begin{array}{rcl} L_{>0}\phi & = & \sum_{r_j > 0} A_j \phi(r_j) + \sum_{s_j^+ > 0} \int_{\max\{0, s_j^-\}}^{s_j^+} B_j(\sigma)\phi(\sigma)\, d\sigma, \\ L_{<0}\phi & = & \sum_{r_j < 0} A_j \phi(r_j) + \sum_{s_j^- < 0} \int_{s_j^-}^{\min\{0, s_j^+\}} B_j(\sigma)\phi(\sigma)\, d\sigma, \end{array} \tag{3.24}$$

we note that the corresponding Hale inner products

$$\begin{array}{rcll} \langle \cdot, \cdot \rangle_{L_{>0}} & : & C([-r_{\max}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) & \to & \mathbb{R}^n, \\ \langle \cdot, \cdot \rangle_{L_{<0}} & : & C([0, -r_{\min}]; \mathbb{R}^n) \times C([r_{\min}, 0]; \mathbb{R}^n) & \to & \mathbb{R}^n \end{array} \tag{3.25}$$

are given by

$$\begin{array}{rcl} \langle \psi, \phi \rangle_{L_{>0}} & = & \psi(0)^* \phi(0) - \sum_{r_j > 0} \int_0^{r_j} \psi(\tau - r_j)^* A_j \phi(\tau)\, d\tau \\ & & \quad - \sum_{s_j^+ > 0} \int_{\max\{0, s_j^-\}}^{s_j^+} \int_0^{\sigma} \psi(\tau - \sigma)^* B_j(\sigma) \phi(\tau)\, d\tau\, d\sigma, \end{array} \tag{3.26}$$

together with

$$\begin{array}{rcl} \langle \psi, \phi \rangle_{L_{<0}} & = & \psi(0)^* \phi(0) - \sum_{r_j < 0} \int_0^{r_j} \psi(\tau - r_j)^* A_j \phi(\tau)\, d\tau \\ & & \quad - \sum_{s_j^- < 0} \int_{s_j^-}^{\min\{0, s_j^+\}} \int_0^{\sigma} \psi(\tau - \sigma)^* B_j(\sigma) \phi(\tau)\, d\tau\, d\sigma. \end{array} \tag{3.27}$$

By construction, we have the identity

$$\langle \psi, \phi \rangle_L = \langle \pi^- \psi, \pi^+ \phi \rangle_{L_{>0}} + \langle \pi^+ \psi, \pi^- \phi \rangle_{L_{<0}} - \psi(0)^* \phi(0) \tag{3.28}$$

for all $\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n)$ and $\psi \in C([-r_{\max}, -r_{\min}]; \mathbb{R}^n)$.

The remainder of our results in this subsection require the Hale inner product to be complete. To ensure that this is the case, we need to impose the following non-degeneracy condition on the linear operator L. It roughly states that for every non-zero $u \in \mathbb{R}^n$, computation of the expression $\langle u, L\phi \rangle_{\mathbb{R}^n}$ requires knowledge of ϕ on the entire interval $[r_{\min}, r_{\max}]$.

$(\mathrm{HRnk})_L$ If $r_{\max} > 0$, there exist $s_+ \geq 0$ and $J_+ \in \mathbb{R}^{n \times n}$ with $\det J_+ \neq 0$ so that

$$\Delta_L(z) = z^{-s_+} e^{z r_{\max}} \big(J_+ + o(1) \big) \text{ as } z \to \infty. \tag{3.29}$$

In addition, if $r_{\min} < 0$, there exist $s_- \geq 0$ and $J_- \in \mathbb{R}^{n \times n}$ with $\det J_- \neq 0$ so that

$$\Delta_L(z) = z^{-s_-} e^{z r_{\min}} \big(J_- + o(1) \big) \text{ as } z \to -\infty. \tag{3.30}$$

Proposition 3.3 (see §6). *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HRnk})_L$. Then the Hale inner products $\langle \cdot, \cdot \rangle_L$, $\langle \cdot, \cdot \rangle_{L_{>0}}$ and $\langle \cdot, \cdot \rangle_{L_{<0}}$ are all non-degenerate in the sense that the following properties hold.*

(i) Any $\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n)$ for which

$$\langle \psi, \phi \rangle_L = 0 \tag{3.31}$$

holds for all $\psi \in C([-r_{\max}, -r_{\min}]; \mathbb{R}^n)$ must satisfy $\phi = 0$.

(ii) Any $\phi \in C([r_{\min}, 0]; \mathbb{R}^n)$ for which

$$\langle \psi, \phi \rangle_{L_{<0}} = 0 \tag{3.32}$$

holds for all $\psi \in C([0, -r_{\min}]; \mathbb{R}^n)$ must satisfy $\phi = 0$.

(iii) Any $\phi \in C([0, r_{\max}]; \mathbb{R}^n)$ for which

$$\langle \psi, \phi \rangle_{L_{>0}} = 0 \tag{3.33}$$

holds for all $\psi \in C([-r_{\max}, 0]; \mathbb{R}^n)$ must satisfy $\phi = 0$.

To prepare for the next results, we need to extend the Hale inner product to functions with discontinuities. To this end, fix any $r_{\min} \leq \alpha \leq r_{\max}$ and consider a pair of functions

$$\widehat{\psi} \in C([-r_{\max}, -\alpha]; \mathbb{R}^n) \times C([-\alpha, -r_{\min}]; \mathbb{R}^n), \qquad \widehat{\phi} \in C([r_{\min}, \alpha]; \mathbb{R}^n) \times C([\alpha, r_{\max}]; \mathbb{R}^n) \tag{3.34}$$

together with a pair $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$. We now introduce the notation

$$\begin{array}{lcl} \big\langle (\widehat{\psi}, w), (\widehat{\phi}, v) \big\rangle_L & = & w^* v - \sum_{j=0}^N \int_0^{r_j} \widehat{\psi}(\tau - r_j)^* A_j \widehat{\phi}(\tau) \, d\tau \\ & & \quad - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} \int_0^{\sigma} \widehat{\psi}(\tau - \sigma)^* B_j(\sigma) \widehat{\phi}(\tau) \, d\tau \, d\sigma, \end{array} \tag{3.35}$$

where the integrals are now taken over piecewise continuous functions. This notation isolates the possible ambiguity corresponding to the evaluations at zero. We also use the analogous expressions for the operators $L_{<0}$ and $L_{>0}$.

The Hale inner product can be used to characterize the restriction operators (3.5) and the initial segment spaces (2.13) and (2.31) in terms of their counterparts defined for the adjoint operator L_*.

Proposition 3.4 (see §6). *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HRnk})_L$. Pick $\eta \in \mathbb{R}$ in such a way that $\det \Delta_L(z) = 0$ has no roots with $\operatorname{Re} z = \eta$. Then we have the representations*

$$\begin{array}{lcl} P_L(\eta) & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \langle \psi, \phi \rangle_L = 0 \text{ for all } \psi \in P_{L_*}(-\eta)\} \\ & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^-)\big), \big(\phi, \phi(0)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{P}_{L_*}(-\eta)\}, \\ \widehat{P}_L(\eta) & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \qquad \Big\langle \big(\psi, \psi(0)\big), \big(\widehat{\phi}, \widehat{\phi}(0^-)\big) \Big\rangle_L = 0 \text{ for all } \psi \in P_{L_*}(-\eta)\} \\ & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \qquad \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^-)\big), \big(\widehat{\phi}, \widehat{\phi}(0^-)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{P}_{L_*}(-\eta)\}, \end{array} \tag{3.36}$$

together with

$$\begin{array}{lcl} Q_L(\eta) & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \langle \psi, \phi \rangle_L = 0 \text{ for all } \psi \in Q_{L_*}(-\eta)\} \\ & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\phi, \phi(0)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{Q}_{L_*}(-\eta)\}, \\ \widehat{Q}_L(\eta) & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \qquad \Big\langle \big(\psi, \psi(0)\big), \big(\widehat{\phi}, \widehat{\phi}(0^+)\big) \Big\rangle_L = 0 \text{ for all } \psi \in Q_{L_*}(-\eta)\} \\ & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \qquad \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\widehat{\phi}, \widehat{\phi}(0^+)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{Q}_{L_*}(-\eta)\}. \end{array} \tag{3.37}$$

Theorem 3.5 (see §6). *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HRnk})_L$. Pick $\eta \in \mathbb{R}$ in such a way that $\det \Delta_L(z) = 0$ has no roots with $\mathrm{Re}\, z = \eta$. Then we have the representations*

$$\begin{array}{rcl}
\mathrm{Range}\, \pi^+_{P_L(\eta)} & = & \{\phi \in C([0, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\pi^- \widehat{\psi}, \widehat{\psi}(0^-)\big), \big(\phi, \phi(0)\big) \Big\rangle_{L_{>0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \pi^+_{\widehat{P}_{L_*}(-\eta)}\}, \\
\mathrm{Range}\, \pi^+_{\widehat{P}_L(\eta)} & = & \{\phi \in C([0, r_{\max}]; \mathbb{R}^n) : \langle \psi, \phi \rangle_{L_{>0}} = 0 \textit{ for all } \psi \in \mathrm{Ker}\, \pi^+_{P_{L_*}(-\eta)}\} \\
 & = & \{\phi \in C([0, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\pi^- \widehat{\psi}, 0\big), \big(\phi, \phi(0)\big) \Big\rangle_{L_{>0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \widehat{\pi}^+_{\widehat{P}_{L_*}(-\eta)}\}, \\
\mathrm{Range}\, \widehat{\pi}^+_{\widehat{P}_L(\eta)} & = & \{(v, \phi) \in \mathbb{R}^n \times C([0, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\widehat{\psi}^-, \widehat{\psi}(0^-)\big), \big(\phi, v\big) \Big\rangle_{L_{>0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \pi^+_{\widehat{P}_{L_*}(-\eta)}\},
\end{array} \tag{3.38}$$

together with

$$\begin{array}{rcl}
\mathrm{Range}\, \pi^-_{Q_L(\eta)} & = & \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \Big\langle \big(\pi^+ \widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\phi, \phi(0)\big) \Big\rangle_{L_{<0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \pi^-_{\widehat{Q}_{L_*}(-\eta)}\}, \\
\mathrm{Range}\, \pi^-_{\widehat{Q}_L(\eta)} & = & \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \langle \psi, \phi \rangle_{L_{<0}} = 0 \textit{ for all } \psi \in \mathrm{Ker}\, \pi^-_{Q_{L_*}(-\eta)}\} \\
 & = & \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \Big\langle \big(\pi^+ \widehat{\psi}, 0\big), \big(\phi, \phi(0)\big) \Big\rangle_{L_{<0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \widehat{\pi}^-_{\widehat{Q}_{L_*}(-\eta)}\}, \\
\mathrm{Range}\, \widehat{\pi}^-_{\widehat{Q}_L(\eta)} & = & \{(\phi, v) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n : \Big\langle \big(\pi^+ \widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\phi, v\big) \Big\rangle_{L_{<0}} = 0 \\
 & & \quad \textit{for all } \widehat{\psi} \in \mathrm{Ker}\, \pi^-_{\widehat{Q}_{L_*}(-\eta)}\}.
\end{array} \tag{3.39}$$

3.2 Scalar differential equations

We now turn to the differential equation (2.9) in the well-studied scalar case $\mathcal{H} = \mathbb{R}$. We recall a number of results from [10] and [20] which together allow for a detailed understanding of the restriction operators (3.5). We emphasize that none of these results require the form condition $(\mathrm{HF})_L$ to hold.

Proposition 3.6 (see [20, Thm. 5.2] and [10, Prop. 2.2]). *Fix $r_{\min} \leq 0 \leq r_{\max}$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ that satisfies $(\mathrm{HRnk})_L$. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then for any $\alpha \in \mathbb{C}$, there exist linear operators*

$$L_- \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{C}); \mathbb{C}\big), \qquad L_+ \in \mathcal{L}\big(C([0, r_{\max}]; \mathbb{C}); \mathbb{C}\big), \tag{3.40}$$

with associated characteristic matrices

$$\Delta_{L_\pm}(z) = zI - L_\pm e^{z \cdot} I \tag{3.41}$$

for which the splitting

$$(z - \alpha)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z) \tag{3.42}$$

holds for all $z \in \mathbb{C}$.

The splitting (3.42) is referred to as a Wiener-Hopf factorization for the symbol Δ_L and we will call any such triplet (α, L_-, L_+) a Wiener-Hopf triplet for L. In general, such triplets need not be unique, but it is possible to extract a quantity that does not depend on the chosen splitting (3.42). Indeed, for any Wiener-Hopf triplet (α, L_-, L_+) for L and any $\eta \in \mathbb{R} \setminus \{\mathrm{Re}\,\alpha\}$, we introduce the integer

$$n_L^\sharp(\eta) = n_{L_+}^-(\eta) - n_{L-}^+(\eta) + n_\alpha^+(\eta) \tag{3.43}$$

that is defined by

$$\begin{aligned} n_{L_-}^+(\eta) &= \#\{z \in \mathbb{C} \mid \Delta_{L_-}(z) = 0 \text{ and } \mathrm{Re}\, z > \eta\}, \\ n_{L_+}^-(\eta) &= \#\{z \in \mathbb{C} \mid \Delta_{L_+}(z) = 0 \text{ and } \mathrm{Re}\, z < \eta\}, \\ n_\alpha^+(\eta) &= \{1 \text{ if } \mathrm{Re}\,\alpha > \eta \text{ and } 0 \text{ otherwise}\}. \end{aligned} \tag{3.44}$$

Here all roots are counted with their multiplicity. This quantity $n_L^\sharp(\eta)$ is invariant in the following sense.

Proposition 3.7 (see [20, Thm. 5.2] and [10, Prop. 2.3]). *Fix $r_{\min} \le 0 \le r_{\max}$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ that satisfies $(\mathrm{HRnk})_L$. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the quantity $n_L^\sharp(\eta)$ is invariant across all Wiener-Hopf triplets (α, L_-, L_+) for L that have $\mathrm{Re}\,\alpha \neq \eta$.*

Unfortunately, it is often intractable to find Wiener-Hopf triplets for a prescribed operator L. The following result can often be used to calculate $n_L^\sharp(\eta)$ in such settings. The only requirement is that a Wiener-Hopf triplet is available for some reference system that can be continuously transformed into the original system without violating the completeness condition (HRnk). Please note however that the exponents $s_\pm$ appearing in this condition need not remain constant during this transformation.

Proposition 3.8 (see [10, Thm. 2.5]). *Fix $r_{\min} \le 0 \le r_{\max}$, consider a continuous path*

$$\Gamma : [0, 1] \to \mathcal{L}\big(C([r_{\min}, r_{\max}]; \mathbb{C}); \mathbb{C}\big) \tag{3.45}$$

and suppose that $(\mathrm{HRnk})_{\Gamma(\mu)}$ is satisfied for each $0 \le \mu \le 1$. Fix any $\eta \in \mathbb{R}$ and suppose that the characteristic equation $\Delta_{\Gamma(\mu)}(z) = 0$ admits roots with $\mathrm{Re}\, z = \eta$ for only finitely many values of $\mu \in [0, 1]$ and that $\mu \in (0, 1)$ for all such μ. Then we have the identity

$$n_{\Gamma(1)}^\sharp(\eta) - n_{\Gamma(0)}^\sharp(\eta) = -\mathrm{cross}(\Gamma, \eta), \tag{3.46}$$

in which the crossing number $\mathrm{cross}(\Gamma, \eta)$ denotes the net number of roots of the characteristic equation $\Delta_{\Gamma(\mu)}(z) = 0$, counted with multiplicity, that cross the line $\mathrm{Re}\, z = \eta$ from left to right as μ increases from 0 to 1.

We conclude this short review by showing how the quantities $n_L^\sharp(\eta)$ can be used to characterize the kernels and ranges of the Fredholm operators (3.5).

Proposition 3.9 (see [20, Thms. 6.1-6.2], [10, Prop. 2.4] and [4, Thm. 3.10]). *Fix $r_{\min} \le 0 \le r_{\max}$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathbb{R}) \to \mathbb{R}$ that satisfies $(\mathrm{HRnk})_L$. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then we have the identities*

$$\begin{aligned} \dim K_{P_L(\eta)}^+ &= \max\{-n_L^\sharp(\eta), 0\}, & \mathrm{codim}\, R_{P_L(\eta)}^+ &= \max\{n_L^\sharp(\eta), 0\}, \\ \dim K_{\widehat{P}_L(\eta)}^+ &= \max\{1 - n_L^\sharp(\eta), 0\}, & \mathrm{codim}\, R_{\widehat{P}_L(\eta)}^+ &= \max\{n_L^\sharp(\eta) - 1, 0\}, \\ \dim \widehat{K}_{\widehat{P}_L(\eta)}^+ &= \max\{-n_L^\sharp(\eta), 0\}, & \mathrm{codim}\, \widehat{R}_{\widehat{P}_L(\eta)}^+ &= \max\{n_L^\sharp(\eta), 0\}, \end{aligned} \tag{3.47}$$

together with

$$\begin{array}{rclrcl} \dim K^-_{Q_L(\eta)} & = & \max\{n^\sharp_L(\eta)-1,0\}, & \operatorname{codim} R^-_{Q_L(\eta)} & = & \max\{1-n^\sharp_L(\eta),0\}. \\ \dim K^-_{\widehat{Q}_L(\eta)} & = & \max\{n^\sharp_L(\eta),0\}, & \operatorname{codim} R^-_{\widehat{Q}_L(\eta)} & = & \max\{-n^\sharp_L(\eta),0\}, \\ \dim \widehat{K}^-_{\widehat{Q}_L(\eta)} & = & \max\{n^\sharp_L(\eta)-1,0\}, & \operatorname{codim} \widehat{R}^-_{\widehat{Q}_L(\eta)} & = & \max\{1-n^\sharp_L(\eta),0\}. \end{array} \tag{3.48}$$

3.3 Scalar algebraic equations

Our goal here is to show how the explicit techniques outlined in §3.2 can be transferred to the differential-algebraic setting of (2.35). In particular, we pick a bounded linear operator

$$M : C([r_{\min}, r_{\max}];\mathbb{R}) \to \mathbb{R} \tag{3.49}$$

and study the system

$$0 = M\mathrm{ev}_\xi x. \tag{3.50}$$

The conditions (HS) and $(\mathrm{HAlg})_{\mathcal{I},M}$ can then be restated as the following assumption.

$(\mathrm{HAlgSc})_M$ There is a linear operator

$$L \in \mathcal{L}\big(C([r_{\min}, r_{\max}];\mathbb{C});\mathbb{C}\big) \tag{3.51}$$

together with a constant $\alpha \in \mathbb{C}$ and integer $\ell \geq 1$ so that

$$(z-\alpha)^\ell \delta_{0,M}(z) = \Delta_L(z) \tag{3.52}$$

for all $z \in \mathbb{C}$.

Our first task is to generalize the notation of a Wiener-Hopf factorization to the symbol

$$\delta_{0,M}(z) = -Me^{z\cdot}. \tag{3.53}$$

Proposition 3.10 (see §7). *Fix* $r_{\min} \leq 0 \leq r_{\max}$ *and consider a bounded linear operator* $M : C([r_{\min}, r_{\max}];\mathbb{C}) \to \mathbb{C}$ *that satisfies both* $(\mathrm{HRnk})_M$ *and* $(\mathrm{HAlgSc})_M$*. Then there exist linear operators*

$$M_- \in \mathcal{L}\big(C([r_{\min}, 0];\mathbb{C});\mathbb{C}\big), \qquad M_+ \in \mathcal{L}\big(C([0, r_{\max}];\mathbb{C});\mathbb{C}\big), \tag{3.54}$$

that satisfy $(\mathrm{HAlgSc})_{M_\pm}$ *and for which the splitting*

$$\delta_{0,M}(z) = \delta_{0,M_-}(z)\delta_{0,M_+}(z) \tag{3.55}$$

holds for all $z \in \mathbb{C}$.

Writing $\ell_\pm \geq 1$ for the integers appearing in the conditions $(\mathrm{HAlgSc})_{M_\pm}$, we refer to any set $(M_-, \ell_-, M_+, \ell_+)$ that satisfies the statements in Proposition 3.10 as a Wiener-hopf set for M. We note that any such set automatically satisfies the relation

$$\ell_- + \ell_+ = 1 + \ell, \tag{3.56}$$

which can be seen by taking $\operatorname{Im} z \to \infty$ in (3.55).

For any $\eta \in \mathbb{R}$, we now define the integer

$$m^\sharp_M(\eta) = m^-_{M_+}(\eta) - m^+_{M_-}(\eta) + \frac{1}{2}(\ell_+ - \ell_-) + \frac{1}{2}, \tag{3.57}$$

in which we have

$$m^{+}_{M_-}(\eta) = \#\{z \in \mathbb{C} : \delta_{M_-}(z) = 0 \text{ and } \operatorname{Re} z > \eta\} \tag{3.58}$$

together with

$$m^{-}_{M_+}(\eta) = \#\{z \in \mathbb{C} : \delta_{M_+}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}. \tag{3.59}$$

As usual, roots are counted according to their multiplicity in these definitions.

Proposition 3.11 (see §7). *Fix $r_{\min} \le 0 \le r_{\max}$ and consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ that satisfies both $(\mathrm{HRnk})_M$ and $(\mathrm{HAlgSc})_M$. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\delta_{0,M}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the quantity $m^{\sharp}_M(\eta)$ is invariant across all Wiener-Hopf sets $(M_-, \ell_-, M_+, \ell_+)$ for M.*

Our next step is to formulate a convenient tool to track the invariant $m^{\sharp}_M(\eta)$ through homotopies. To assist us, we define the quantity

$$\operatorname{cross}(\alpha_0, \alpha_1; \eta) = \begin{cases} 1 & \alpha_0 < \eta < \alpha_1, \\ -1 & \alpha_1 < \eta < \alpha_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.60}$$

Proposition 3.12 (see §7). *Fix $r_{\min} \le 0 \le r_{\max}$ and consider two bounded linear operators*

$$M_0 : C([r_{\min}, r_{max}]; \mathbb{C}) \to \mathbb{C}, \qquad M_1 : C([r_{\min}, r_{max}]; \mathbb{C}) \to \mathbb{C}. \tag{3.61}$$

Consider a continuous path

$$\Gamma : [0, 1] \to \mathcal{L}(C([r_{\min}, r_{\max}]; \mathbb{C}), \mathbb{C}) \tag{3.62}$$

for which we have the factorizations

$$\delta_{0,M_0}(z) = (z - \alpha_0)^{-\ell} \Delta_{\Gamma(0)}(z), \qquad \delta_{0,M_1}(z) = (z - \alpha_1)^{-\ell} \Delta_{\Gamma(1)}(z) \tag{3.63}$$

and for which $(\mathrm{HRnk})_{\Gamma(\mu)}$ is satisfied for each $0 \le \mu \le 1$. Pick $\eta \in \mathbb{R}$ in such a way that the two characteristic equations

$$\delta_{0,M_0}(z) = 0, \qquad \delta_{0,M_1}(z) = 0 \tag{3.64}$$

both have no roots with $\operatorname{Re} z = \eta$. Then the following statements all hold.[2]

(i) If $(\alpha_0 - \eta)(\alpha_1 - \eta) \neq 0$, we have

$$m^{\sharp}_{M_1}(\eta) - m^{\sharp}_{M_0}(\eta) = -\operatorname{cross}(\Gamma; \eta) + \ell \operatorname{cross}(\alpha_0, \alpha_1; \eta). \tag{3.65}$$

(ii) If $\min\{\alpha_1, \alpha_2\} \ge \eta$, then for all sufficiently small $\epsilon > 0$ we have

$$m^{\sharp}_{M_1}(\eta) - m^{\sharp}_{M_0}(\eta) = -\operatorname{cross}(\Gamma; \eta - \epsilon). \tag{3.66}$$

(iii) If $\max\{\alpha_1, \alpha_2\} \le \eta$, then for all sufficiently small $\epsilon > 0$ we have

$$m^{\sharp}_{M_1}(\eta) - m^{\sharp}_{M_0}(\eta) = -\operatorname{cross}(\Gamma; \eta + \epsilon). \tag{3.67}$$

[2] Please note that these conditions are not mutually exclusive.

Here again the crossing number $\mathrm{cross}(\Gamma, \eta)$ *denotes the net number of roots of the characteristic equation* $\Delta_{\Gamma(\mu)}(z) = 0$*, counted with multiplicity, that cross the line* $\mathrm{Re}\, z = \eta$ *from left to right as* μ *increases from* 0 *to* 1*.*

In a fashion analogous to Proposition 3.9, the invariant $m^\sharp_M(\eta)$ yields valuable information concerning the restriction operators (3.10).

Proposition 3.13 (see §7). *Write* $\mathcal{H} = \mathbb{R}$ *and consider a linear operator* $M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ *that satisfies* $(\mathrm{HF})_M$*,* $(\mathrm{HAlgSc})_M$ *and* $(\mathrm{HRnk})_M$*. Fix any* $\eta \in \mathbb{R}$ *for which the characteristic equation* $\delta_{0,M}(z) = 0$ *admits no roots with* $\mathrm{Re}\, z = \eta$*. Then we have the identities*

$$\begin{array}{rclrcl} \dim K^+_{p_{0,M}(\eta)} & = & \max\{-\frac{1}{2}\ell - m^\sharp_M(\eta), 0\}, & \mathrm{codim}\, R^+_{p_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell + m^\sharp_M(\eta), 0\}, \\ \dim K^+_{\widehat{p}_{0,M}(\eta)} & = & \max\{1 - \frac{1}{2}\ell - m^\sharp_M(\eta), 0\}, & \mathrm{codim}\, R^+_{\widehat{p}_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell + m^\sharp_M(\eta) - 1, 0\}, \\ \dim \widehat{K}^+_{\widehat{p}_{0,M}(\eta)} & = & \max\{-\frac{1}{2}\ell - m^\sharp_M(\eta), 0\}, & \mathrm{codim}\, \widehat{R}^+_{\widehat{p}_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell + m^\sharp_M(\eta), 0\}, \end{array} \tag{3.68}$$

together with

$$\begin{array}{rclrcl} \dim K^-_{q_{0,M}(\eta)} & = & \max\{-\frac{1}{2}\ell + m^\sharp_M(\eta) - 1, 0\}, & \mathrm{codim}\, R^-_{q_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell - m^\sharp_M(\eta) + 1, 0\}. \\ \dim K^-_{\widehat{q}_{0,M}(\eta)} & = & \max\{-\frac{1}{2}\ell + m^\sharp_M(\eta), 0\}, & \mathrm{codim}\, R^-_{\widehat{q}_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell - m^\sharp_M(\eta), 0\}, \\ \dim \widehat{K}^-_{\widehat{q}_{0,M}(\eta)} & = & \max\{-\frac{1}{2}\ell + m^\sharp_M(\eta) - 1, 0\}, & \mathrm{codim}\, \widehat{R}^-_{\widehat{q}_{0,M}(\eta)} & = & \max\{\frac{1}{2}\ell - m^\sharp_M(\eta) + 1, 0\}. \end{array} \tag{3.69}$$

3.4 Fourier decompositions

We now address the infinite-dimensional situation where $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ and the operator

$$L : C\big([r_{\min}, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big) \to \ell^2(\mathbb{Z}; \mathbb{R}^n) \tag{3.70}$$

is of convolution type. More precisely, we assume that $(\mathrm{HF})_L$ is satisfied and impose the following extra condition.

$(\mathrm{HFrr})_L$ We have $\mathcal{H} = \ell^2(\mathbb{R}^n)$ for some integer $n \geq 1$. Recalling the terminology of $(\mathrm{HF})_L$, there exist sequences $a_j \in \ell^1\big(\mathbb{Z}; \mathbb{R}^{n\times n}\big)$ together with functions

$$b_j \in C\Big([s_j^-, s_j^+]; \ell^1\big(\mathbb{Z}; \mathbb{R}^{n\times n}\big)\Big), \tag{3.71}$$

both defined for all $0 \leq j \leq N$, so that for all such j we have

$$[A_j v]_k = \sum_{m\in\mathbb{Z}} [a_j]_{k-m} v_m \tag{3.72}$$

for all $v \in \ell^2(\mathbb{Z}; \mathbb{R}^n)$, together with

$$[B_j(\sigma) v]_k = \sum_{m\in\mathbb{Z}} [b_j(\sigma)]_{k-m} v_m \tag{3.73}$$

for all $\sigma \in [s_j^-, s_j^+]$ and $v \in \ell^2(\mathbb{Z}; \mathbb{R}^n)$.

For any $v \in \ell^1(\mathbb{Z}; \mathbb{R}^{m\times n})$, we define the Fourier transform $\mathcal{F}v \in C([-\pi, \pi]; \mathbb{R}^{m\times n})$ by

$$[\mathcal{F}v](\omega) = \sum_{l\in\mathbb{Z}} v_l e^{-il\omega}. \tag{3.74}$$

In a standard fashion, this map can be extended to a map from $\ell^2(\mathbb{Z};\mathbb{R}^{m\times n})$ into $L^2([-\pi,\pi];\mathbb{R}^{m\times n})$ that is bounded and invertible via

$$v = \mathcal{F}_{\mathrm{inv}}[\mathcal{F}v], \tag{3.75}$$

in which we have defined

$$[\mathcal{F}_{\mathrm{inv}}\, w]_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{il\omega} w(\omega)\, d\omega \tag{3.76}$$

for $w \in L^2([\pi,\pi];\mathbb{R}^{m\times n})$.

In particular, for all $0 \le j \le N$, all $\omega \in [-\pi,\pi]$ and $\sigma \in [s_j^-, s_j^+]$, we have the estimates

$$\big|[\mathcal{F}(a_j)](\omega)\big|_{\mathbb{R}^{n\times n}} \le \|a_j\|_{\ell^1(\mathbb{Z};\mathbb{R}^{n\times n})}, \qquad \big|[\mathcal{F}\big(b_j(\sigma)\big)](\omega)\big|_{\mathbb{R}^{n\times n}} \le \|b_j(\sigma)\|_{\ell^1(\mathbb{Z};\mathbb{R}^{n\times n})}. \tag{3.77}$$

In addition, the function

$$(\sigma,\omega) \mapsto [\mathcal{F}\big(b_j(\sigma)\big)](\omega) \tag{3.78}$$

is continuous as a map from $[s_j^-, s_j^+] \times [-\pi,\pi]$ into $\mathbb{R}^{n\times n}$.

For any $\omega \in [-\pi,\pi]$, these observations allow us to define a linear operator $L(\omega) : C([r_{\min}, r_{\max}];\mathbb{R}^n) \to \mathbb{R}^n$ that acts as

$$L(\omega)\phi \;=\; \textstyle\sum_{j=0}^{N}[\mathcal{F}(a_j)](\omega)\phi(r_j) + \sum_{j=0}^{N}\int_{s_j^-}^{s_j^+}[\mathcal{F}\big(b_j(\sigma)\big)](\omega)\phi(\sigma)\, d\sigma \tag{3.79}$$

and satisfies $(\mathrm{HF})_{L(\omega)}$.

Pick any $\eta \in \mathbb{R}$ and recall the collection $\Theta_L(\eta)$ defined in (3.8). For any triplet

$$\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_L(\eta), \tag{3.80}$$

we now introduce the notation

$$\theta(\omega) = (\pi_{\theta(\omega)}, \mathcal{S}_{\theta(\omega)}, \mathcal{T}_{\theta(\omega)}) \in \Theta_{L(\omega)}(\eta) \tag{3.81}$$

to refer to the same projections and spaces but with L replaced by $L(\omega)$ and $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ replaced by $\mathcal{H}(\omega) = \mathbb{R}^n$. For example, if

$$\theta = \Big(\pi^-_{Q_L(\eta)}, Q_L(\eta), C\big([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big)\Big), \tag{3.82}$$

then we have

$$\theta(\omega) = \big(\pi^-_{Q_{L(\omega)}(\eta)}, Q_{L(\omega)}(\eta), C([r_{\min},0];\mathbb{R}^n)\big). \tag{3.83}$$

Our main result gives conditions under which the kernel and range of the restriction operators (3.5) are both closed, with a closed complement for the kernel. As explained in §1.4, this is the closest one can hope to get to the Fredholm properties discussed in (3.1).

Theorem 3.14. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \ge 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{HRnk})_{L(\omega)}$ is satisfied for all $\omega \in [-\pi,\pi]$.*

Fix $\eta \in \mathbb{R}$ in such a way that

$$\det \Delta_{L(\omega)}(\eta + i\nu) \ne 0 \tag{3.84}$$

for all $\nu \in \mathbb{R}$ and $\omega \in [-\pi, \pi]$ and choose a triplet

$$\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_L(\eta). \tag{3.85}$$

Then $\mathcal{S}_\theta$ is a closed subspace of $C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H})$. In addition, if there exists an integer $J \geq 0$ so that

$$\dim \operatorname{Ker} \pi_{\theta(\omega)} = J \tag{3.86}$$

for all $\omega \in [-\pi, \pi]$, then the following properties hold true.

(i) The two spaces

$$\begin{aligned} K_\theta &= \operatorname{Ker} \pi_\theta \subset \mathcal{S}_\theta, \\ R_\theta &= \operatorname{Range} \pi_\theta \subset \mathcal{T}_\theta \end{aligned} \tag{3.87}$$

are both closed.

(ii) There exists a closed subspace $K_\theta^\perp \subset \mathcal{S}_\theta$ so that we have the decomposition

$$\mathcal{S}_\theta = K_\theta \oplus K_\theta^\perp. \tag{3.88}$$

(iii) If for all $\omega \in [-\pi, \pi]$ we have

$$\operatorname{Range} \pi_{\theta(\omega)} = \mathcal{T}_{\theta(\omega)}, \tag{3.89}$$

then in fact $R_\theta = \mathcal{T}_\theta$.

(iv) If for all $\omega \in [-\pi, \pi]$ we have

$$\operatorname{Ker} \pi_{\theta(\omega)} = \{0\} \subset \mathcal{S}_{\theta(\omega)}, \tag{3.90}$$

then in fact

$$K_\theta = \{0\} \subset \mathcal{S}_\theta, \qquad K_\theta^\perp = \mathcal{S}_\theta. \tag{3.91}$$

We note that items (i) and (ii) above imply that the restriction $\pi_\theta : K_\theta^\perp \to R_\theta$ is an invertible bounded linear operator, which hence has a bounded inverse.

Corollary 3.15. *Consider the setting of Theorem 3.14 with $n = 1$. Then the condition (3.86) is automatically satisfied. In addition, if $n^\sharp_{L(\omega)}(\eta) = 0$ for all $\omega \in [-\pi, \pi]$, then the operators*

$$\begin{aligned} \pi^+_{P_L(\eta)} &: P_L(\eta) \to C\big([0, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R})\big), \\ \widehat{\pi}^+_{\widehat{P}_L(\eta)} &: \widehat{P}_L(\eta) \to \ell^2(\mathbb{Z}; \mathbb{R}) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R})\big), \\ \pi^-_{\widehat{Q}_L(\eta)} &: \widehat{Q}_L(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R})\big) \end{aligned} \tag{3.92}$$

are all invertible. On the other hand, if $n^\sharp_{L(\omega)}(\eta) = 1$ for all $\omega \in [-\pi, \pi]$, then the operators

$$\begin{aligned} \pi^+_{\widehat{P}_L(\eta)} &: \widehat{P}_L(\eta) \to C\big([0, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R})\big), \\ \pi^-_{Q_L(\eta)} &: Q_L(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R})\big), \\ \widehat{\pi}^-_{\widehat{Q}_L(\eta)} &: \widehat{Q}_L(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R})\big) \times \ell^2(\mathbb{Z}; \mathbb{R}) \end{aligned} \tag{3.93}$$

are all invertible.

Our final results concern the differential-algebraic system (2.35). If M satisfies $(\mathrm{HF})_M$ and $(\mathrm{HFrr})_M$, then the Fourier components

$$M(\omega) : C([r_{\min}, r_{\max}]; \mathbb{R}^n) \to \mathbb{R}^n \tag{3.94}$$

can be defined in exactly the same fashion as $L(\omega)$ in (3.79).

Theorem 3.16. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ and consider the differential-algebraic system (2.35). Suppose that $(\mathrm{HF})_M$, (HS), $(\mathrm{HAlg})_{\mathcal{I},M}$ and $(\mathrm{HFrr})_M$ are all satisfied and that $(\mathrm{HRnk})_{M(\omega)}$ holds for all $\omega \in [-\pi, \pi]$.*

Fix $\eta \in \mathbb{R}$ in such a way that

$$\det \delta_{\mathcal{I},M(\omega)}(\eta + i\nu) \neq 0 \tag{3.95}$$

for all $\nu \in \mathbb{R}$ and $\omega \in [-\pi, \pi]$ and choose a triplet

$$\theta = (\pi_\theta, \mathcal{S}_\theta, \mathcal{T}_\theta) \in \Theta_{\mathcal{I},M}(\eta). \tag{3.96}$$

Then $\mathcal{S}_\theta$ is a closed subspace of $C([r_{\min}, 0]; \mathcal{H}) \times C([0, r_{\max}]; \mathcal{H})$. In addition, if there exists an integer $J \geq 0$ so that

$$\dim \operatorname{Ker} \pi_{\theta(\omega)} = J \tag{3.97}$$

for all $\omega \in [-\pi, \pi]$, then the following properties hold true.

(i) The two spaces

$$\begin{aligned} K_\theta &= \operatorname{Ker} \pi_\theta \subset \mathcal{S}_\theta, \\ R_\theta &= \operatorname{Range} \pi_\theta \subset \mathcal{T}_\theta, \end{aligned} \tag{3.98}$$

are both closed.

(ii) There exists a closed subspace $K_\theta^\perp \subset \mathcal{S}_\theta$ so that we have the decomposition

$$\mathcal{S}_\theta = K_\theta \oplus K_\theta^\perp. \tag{3.99}$$

(iii) If for all $\omega \in [-\pi, \pi]$ we have

$$\operatorname{Range} \pi_{\theta(\omega)} = \mathcal{T}_{\theta(\omega)}, \tag{3.100}$$

then in fact $R_\theta = \mathcal{T}_\theta$

(iv) If for all $\omega \in [-\pi, \pi]$ we have

$$\operatorname{Ker} \pi_{\theta(\omega)} = \{0\} \subset \mathcal{S}_{\theta(\omega)}, \tag{3.101}$$

then in fact

$$K_\theta = \{0\} \subset \mathcal{S}_\theta, \qquad K_\theta^\perp = \mathcal{S}_\theta. \tag{3.102}$$

Corollary 3.17. *Consider the setting of Theorem 3.16 with $n = 1$ and $\mathcal{I} = 0$. Then the condition (3.97) is automatically satisfied. In addition, writing $\ell = \ell_1$ for the integer defined in $(\mathrm{HAlg})_{\mathcal{I},M}$, the following statements hold true.*

(i) If $m^{\sharp}_{0,M(\omega)}(\eta) = -\frac{1}{2}\ell$ *for all* $\omega \in [-\pi, \pi]$*, then the operators*

$$\begin{array}{rcl} \pi^{+}_{p_{0,M}(\eta)} & : & p_{0,M}(\eta) \to C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R})\big), \\ \widehat{\pi}^{+}_{\widehat{p}_M(\eta)} & : & \widehat{p}_{0,M}(\eta) \to \ell^2(\mathbb{Z};\mathbb{R}) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R})\big) \end{array} \tag{3.103}$$

are both invertible.

(ii) If $m^{\sharp}_{0,M(\omega)} = 1 - \frac{1}{2}\ell$ *for all* $\omega \in [-\pi, \pi]$ *the operator*

$$\pi^{+}_{\widehat{p}_{0,M}(\eta)} : \widehat{p}_{0,M}(\eta) \to C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R})\big) \tag{3.104}$$

is invertible.

(iii) If $m^{\sharp}_{0,M(\omega)} = \frac{1}{2}\ell$ *for all* $\omega \in [-\pi, \pi]$*, the operator*

$$\pi^{-}_{\widehat{q}_{0,M}(\eta)} : \widehat{q}_{0,M}(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R})\big) \tag{3.105}$$

is invertible.

(iv) If $m^{\sharp}_{0,M(\omega)} = 1 + \frac{1}{2}\ell$ *for all* $\omega \in [-\pi, \pi]$*, the operators*

$$\begin{array}{rcl} \pi^{-}_{q_{0,M}(\eta)} & : & q_{0,M}(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R})\big), \\ \widehat{\pi}^{-}_{\widehat{q}_{0,M}(\eta)} & : & \widehat{q}_{0,M}(\eta) \to C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R})\big) \times \ell^2(\mathbb{Z};\mathbb{R}) \end{array} \tag{3.106}$$

are both invertible.

4 The Model

In this section we analyze the toy pollution model that was introduced in §1.1. In particular, we recall the functionals

$$\mathcal{J}_j(p) = \int_0^\infty \big(2\sqrt{p_j(t)} - \frac{1}{2}c_j[p](t)^2\big)e^{-\rho t}\,dt \tag{4.1}$$

together with the cost functions

$$c_j[p](t) = \sum_{k\in\mathbb{Z}} h_{j-k} \int_{t-1}^{t} p_k(\sigma)\,d\sigma, \tag{4.2}$$

both defined for $j \in \mathbb{Z}$ and

$$p \in C\big([-1, 0]; \ell^\infty(\mathbb{Z};\mathbb{R})\big) \times BC_0\big([0, \infty); \ell^\infty(\mathbb{Z};\mathbb{R})\big). \tag{4.3}$$

Our modelling assumptions are summarized in the following condition.

(hMod) The discount rate satisfies $\rho > 0$ and the spatial kernel h satisfies $h \in \ell^1(\mathbb{Z};\mathbb{R})$ with

$$h_0 > 0, \qquad \sum_{j\in\mathbb{Z}} h_j > 0. \tag{4.4}$$

In addition, the symmetry condition

$$h_{-j} = h_j, \qquad j \in \mathbb{Z} \tag{4.5}$$

holds, which implies that the Fourier transform of h is real-valued. In fact, the inequality

$$[\mathcal{F}h](\omega) > 0 \tag{4.6}$$

holds for all $\omega \in [-\pi, \pi]$.

We now fix an integer $j \in \mathbb{Z}$ and attempt to find p_j in such a way that the functional (4.1) is maximized, under the initial condition

$$p_j(\vartheta) = \phi_j(\vartheta) \tag{4.7}$$

for $-1 \le \vartheta \le 0$. We do allow for p_j to have a jump at zero, i.e. $p_j(0^+) \neq p_j(0^-)$.

To separate out the functions that need to be considered as fixed, we write

$$q(t) = \{q_k(t)\}_{k \neq j} = \{p_k(t)\}_{k \neq j} \tag{4.8}$$

for all $t \ge -1$. In addition, we write

$$d_j[q](t) = \sum_{k \in \mathbb{Z} \setminus \{j\}} \int_{t-1}^{t} h_{j-k} q_k(u) du \tag{4.9}$$

for all $t \ge 0$. Our goal is hence to find, for fixed q, the function p_j that maximizes the expression

$$\mathcal{J}_j(p_j, q) = \int_0^\infty \Big(2\sqrt{p_j(t)} - \frac{1}{2}\big[\int_{t-1}^{t} h_0 p_j(u)\, du + d_j[q](t) \big]^2 \Big) e^{-\rho t} dt. \tag{4.10}$$

In order to apply the method of variations, we now write

$$p_j = \overline{p}_j + \epsilon p_{\mathrm{var}} \tag{4.11}$$

and assume that

$$p_{\mathrm{var}}(\vartheta) = 0, \qquad -1 \le \vartheta \le 0 \tag{4.12}$$

to account for the initial condition (4.7). We also write

$$\overline{c}_j[\overline{p}_j, q](t) = \int_{t-1}^{t} h_0 \overline{p}_j(u)\, du + d_j[q](t). \tag{4.13}$$

We may now formally write

$$\mathcal{J}_j(p_j, q) \quad = \quad \mathcal{J}_j(\overline{p}_j, q) + \epsilon \mathcal{Q}_j(\overline{p}_j, q)[p_{\mathrm{var}}] + O(\epsilon^2) \tag{4.14}$$

in which

$$\mathcal{Q}(\overline{p}_j, q)[p_{\mathrm{var}}] = \int_0^\infty \Big(\frac{1}{\sqrt{\overline{p}_j(t)}} p_{\mathrm{var}}(t) - \overline{c}_i[\overline{p}_j, q](t) \int_{t-1}^{t} h_0 p_{\mathrm{var}}(u)\, du \Big) e^{-\rho t} dt. \tag{4.15}$$

A short computation shows that

$$\begin{aligned} \mathcal{Q}(\overline{p}_j, q)[p_{\mathrm{var}}] \quad = \quad & \int_0^\infty \frac{1}{\sqrt{\overline{p}_j(t)}} p_{\mathrm{var}}(t) e^{-\rho t}\, dt \\ & - h_0 \int_{u=-1}^{0} \int_{t=0}^{u+1} \overline{c}_j[\overline{p}_j, q](t) e^{-\rho t} p_{\mathrm{var}}(u)\, dt\, du \\ & - h_0 \int_{u=0}^{\infty} \int_{t=u}^{u+1} \overline{c}_j[\overline{p}_j, q](t) e^{-\rho t} p_{\mathrm{var}}(u)\, dt\, du. \end{aligned} \tag{4.16}$$

Exploiting (4.12), this can be rephrased as

$$\begin{aligned} \mathcal{Q}(\overline{p}_j, q)[p_{\mathrm{var}}] \quad = \quad & \int_0^\infty \frac{1}{\sqrt{\overline{p}_j(t)}} p_{\mathrm{var}}(t) e^{-\rho t}\, dt \\ & - h_0 \int_{u=0}^{\infty} \int_{t=u}^{u+1} \overline{c}_j[\overline{p}_j, q](t) e^{-\rho t} p_{\mathrm{var}}(u)\, dt\, du \\ = \quad & \int_0^\infty \frac{1}{\sqrt{\overline{p}_j(u)}} p_{\mathrm{var}}(u) e^{-\rho u}\, du \\ & - h_0 \int_{u=0}^{\infty} \int_{\sigma=0}^{1} \overline{c}_j[\overline{p}_j, q](u+\sigma) e^{-\rho\sigma} e^{-\rho u} p_{\mathrm{var}}(u)\, d\sigma\, du, \end{aligned} \tag{4.17}$$

which gives the optimality condition

$$\frac{1}{\sqrt{\overline{p}_j(u)}} = h_0 \int_{\sigma=0}^{1} \overline{c}_j[\overline{p}_j, q](u+\sigma)e^{-\rho\sigma}d\sigma \tag{4.18}$$

for all $u \geq 0$.

We now look for a so-called simultaneous optimum $\overline{p} = \{\overline{p}_j\}_{j\in\mathbb{Z}}$, for which we must have

$$\frac{1}{\sqrt{\overline{p}_j(u)}} = h_0 \int_{\sigma=0}^{1} \overline{c}_j[\overline{p}_j, \overline{p}](u+\sigma)e^{-\rho\sigma}d\sigma \tag{4.19}$$

for all $j \in \mathbb{Z}$. Substituting (4.13), we find that such an optimum must satisfy

$$\frac{1}{\sqrt{\overline{p}_j(u)}} = h_0 \sum_{k\in\mathbb{Z}} h_{j-k} \int_{\sigma=0}^{1} \int_{\sigma'=-1}^{0} \overline{p}_k(u+\sigma+\sigma')d\sigma' e^{-\rho\sigma}d\sigma \tag{4.20}$$

for all $j \in \mathbb{Z}$ and $u \geq 0$. Upon introducing the expression

$$\nu(\sigma) = \frac{1}{2}e^{\frac{\rho}{2}\sigma} \begin{cases} \int_{-\sigma}^{2+\sigma} e^{-\frac{\rho}{2}(\sigma+\sigma')}d\sigma' & \text{for } \sigma \leq 0, \\ \int_{\sigma}^{2-\sigma} e^{-\frac{\rho}{2}(\sigma+\sigma')}d\sigma' & \text{for } \sigma > 0, \end{cases} \tag{4.21}$$

the simultaneous optimum condition (4.20) can be rewritten in the convenient form

$$\frac{1}{\sqrt{\overline{p}_j(u)}} = h_0 \sum_{k\in\mathbb{Z}} h_{j-k} \int_{-1}^{1} \nu(\sigma)e^{-\frac{\rho}{2}\sigma}\overline{p}_k(u+\sigma)\, d\sigma, \tag{4.22}$$

which must hold for all $u \geq 0$.

We note that we have included the extra exponential factor $e^{-\frac{\rho}{2}\sigma}$ in (4.22) for symmetry purposes. Indeed, with this choice the function ν satisfies the following properties.

Lemma 4.1. *Suppose that* (hMod) *is satisfied. Then we have* $\nu(\sigma) = \nu(-\sigma)$ *for* $\sigma \in [-1, 1]$. *In addition, we have*

$$\nu \in C([-1, 1]; \mathbb{R}) \cap C^1([-1, 0]; \mathbb{R}) \cap C^1([0, 1]; \mathbb{R}), \tag{4.23}$$

with $\nu(-1) = \nu(1) = 0$ *and* $\nu(\sigma) > 0$ *for* $\sigma \in (-1, 1)$. *For any* $z \in \mathbb{C}$ *we have the identity*

$$\int_{-1}^{1} e^{z\sigma}\nu(\sigma)\, d\sigma = \int_{0}^{1} e^{z\sigma}e^{-\frac{\rho}{2}\sigma}\, d\sigma \int_{-1}^{0} e^{z\sigma}e^{\frac{\rho}{2}\sigma}\, d\sigma, \tag{4.24}$$

which for $z \notin \{-\frac{\rho}{2}, \frac{\rho}{2}\}$ *can be evaluated as*

$$\int_{-1}^{1} e^{z\sigma}\nu(\sigma)\, d\sigma = e^{-\frac{\rho}{2}} \frac{1}{z-\frac{\rho}{2}} \frac{1}{z+\frac{\rho}{2}} \big(e^{z} + e^{-z} - e^{\frac{\rho}{2}} - e^{-\frac{\rho}{2}}\big). \tag{4.25}$$

Finally, we have the identities

$$\begin{aligned} \int_{-1}^{1} e^{\pm\frac{\rho}{2}\sigma}\nu(\sigma)\, d\sigma &= \rho^{-1}(1-e^{-\rho}), \\ \int_{-1}^{1} \nu(\sigma)\, d\sigma &= \frac{4}{\rho^2}(1-e^{-\frac{\rho}{2}})^2. \end{aligned} \tag{4.26}$$

Proof. These statements follow directly from the explicit expressions

$$\nu(\sigma) = \begin{cases} \rho^{-1}e^{-\frac{\rho}{2}\sigma}[e^{\rho\sigma} - e^{-\rho}] & \text{for } \sigma \leq 0, \\ \rho^{-1}e^{\frac{\rho}{2}\sigma}[e^{-\rho\sigma} - e^{-\rho}] & \text{for } \sigma > 0. \end{cases} \tag{4.27}$$

□

For convenience, we now introduce the constant

$$\kappa = 2\Big[\big[\sum_{j\in\mathbb{Z}} h_j\big]\rho^{-1}(1-e^{-\rho})\Big]^{-1} > 0. \qquad (4.28)$$

Any spatially and temporally homogeneous solution

$$\overline{p}_j(u) = p_* \qquad (4.29)$$

to (4.22) must satisfy

$$\frac{1}{\sqrt{p_*}} = h_0\big[\sum_{j\in\mathbb{Z}} h_j\big]\big[\int_{\sigma=-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu(\sigma)d\sigma\big]p_* = 2h_0\kappa^{-1}p_*, \qquad (4.30)$$

which immediately gives

$$p_* = [2h_0]^{-2/3}\kappa^{2/3}. \qquad (4.31)$$

Our main goal in this section is to study solutions to (4.22) that are close to this homogeneous state p_*. In particular, substituting the Ansatz

$$\overline{p}_j = p_* + w_j \qquad (4.32)$$

into (4.22), we find

$$-\tfrac{1}{2p_*^{3/2}}w_j(u) \quad = \quad h_0\int_{\sigma=-1}^{1}\sum_{k\in\mathbb{Z}} h_{j-k}w_k(u+\sigma)e^{-\frac{\rho}{2}\sigma}\nu(\sigma)d\sigma - \mathcal{N}_1\big(w_j(u)\big), \qquad (4.33)$$

in which we have introduced the nonlinear expression

$$\mathcal{N}_1(w) \quad = \quad \big[\tfrac{1}{\sqrt{p_*+w}} - \tfrac{1}{\sqrt{p_*}} + \tfrac{1}{2p_*^{3/2}}w\big] = O(w^2). \qquad (4.34)$$

Substituting (4.31), this can be written as

$$w_j(u) \quad = \quad -\kappa\sum_{k\in\mathbb{Z}} h_{j-k}\int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu(\sigma)w_k(u+\sigma)d\sigma + \kappa h_0^{-1}\mathcal{N}_1\big(w_j(u)\big). \qquad (4.35)$$

Introducing the bounded linear operator

$$M_* : C\big([-1,1];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \to \ell^2(\mathbb{Z};\mathbb{R}^n) \qquad (4.36)$$

that acts as

$$[M_*\psi]_j = -\psi_j(0) - \kappa\sum_{k\in\mathbb{Z}} h_{j-k}\int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu(\sigma)\psi_k(\sigma)d\sigma, \qquad (4.37)$$

the system (4.35) can be written as

$$-M_*\mathrm{ev}_u w = \kappa h_0^{-1}\mathcal{N}_1\big(w(u)\big), \qquad (4.38)$$

which must hold for all $u \geq 0$. Here $\mathcal{N}_1$ acts in a pointwise fashion.

In the remainder of this section we use the theory outlined in §2 and §3 to analyze the linear part of (4.38). In particular, we establish the following well-posedness result.

Proposition 4.2 (see §4.1). *Suppose that* (hMod) *holds. Then we have*

$$\mathrm{Range}\,\pi^-_{\widehat{q}_{0,M_*(0)}} = C\big([-1,0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \qquad (4.39)$$

together with

$$\mathrm{Ker}\,\pi^-_{\widehat{q}_{0,M_*(0)}} = \{0\}. \qquad (4.40)$$

Using the nonlinear techniques of [4, §3.4], this result shows that for every $\phi \in C\big([-1,0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$ with

$$\sup_{-1\le\vartheta\le 0} |\phi(\vartheta) - p_*|_{\ell^2(\mathbb{Z};\mathbb{R}^n)} \ll 1, \tag{4.41}$$

there exists a unique $\overline{p}$ with

$$\overline{p} - p_* \in C\big([-1,0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times BC_0\big([0,\infty);\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \tag{4.42}$$

that solves (4.22) with the initial condition

$$\overline{p}_j(\vartheta) = \phi_j(\vartheta) \qquad \text{for } -1 \le \vartheta \le 0. \tag{4.43}$$

In particular, the jump $\overline{p}(0^+) - \overline{p}(0^-)$ is determined uniquely by ϕ.

4.1 Preparations

In this subsection we set up the machinery that we will need to establish Proposition 4.2. In particular, we illustrate the steps which typically are necessary to apply the theory developed in this paper to explicit models.

Lemma 4.3. *Assume that* (hMod) *is satisfied. Then the linear operator M_* defined in (4.37) satisfies* $(\mathrm{HAlg})_{0,M_*}$.

Proof. The result follows immediately by applying Proposition 7.5 with $J = 1$, exploiting the piecewise differentiability of ν. Alternatively and more directly, one can introduce the bounded linear operator

$$L_* : C\big([-1,1];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \to \ell^2(\mathbb{Z};\mathbb{R}^n) \tag{4.44}$$

that acts as

$$[L_*\psi]_j = \frac{\rho}{2}\psi_j(0) + \kappa \sum_{k\in\mathbb{Z}} h_{j-k} \int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu'(\sigma)\psi_k(\sigma)\,d\sigma. \tag{4.45}$$

Exploiting the identities $\nu(0^-) = \nu(0^+)$ and $\nu(-1) = \nu(+1) = 0$, we may then compute

$$\begin{array}{rcl} [\Delta_{L_*}(z)v]_j & = & \big[[z - Le^{z\cdot}]v\big]_j \\ & = & zv_j - \frac{\rho}{2}v_j - \kappa\sum_{k\in\mathbb{Z}} h_{j-k}v_k \int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu'(\sigma)e^{z\sigma}\,d\sigma \\ & = & (z-\frac{\rho}{2})v_j + \kappa\sum_{k\in\mathbb{Z}} h_{j-k}v_k(z-\frac{\rho}{2})\int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu(\sigma)e^{z\sigma}\,d\sigma \end{array} \tag{4.46}$$

for all $v \in \ell^2(\mathbb{Z};\mathbb{R})$ and $j \in \mathbb{Z}$. In addition, we can compute

$$[-M_*e^{z\cdot}v]_j = v_j + \kappa\sum_{k\in\mathbb{Z}} h_{j-k}v_k \int_{-1}^{1} e^{-\frac{\rho}{2}\sigma}\nu(\sigma)e^{z\sigma}\,d\sigma, \tag{4.47}$$

which implies

$$(z - \frac{\rho}{2})\delta_{0,M_*}(z) = \Delta_{L_*}(z). \tag{4.48}$$

□

In order to efficiently exploit the symmetry properties in (4.1), we will study the exponentially shifted operator

$$M : C\big([-1,1];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \to \ell^2(\mathbb{Z};\mathbb{R}^n) \tag{4.49}$$

that acts as

$$M\psi \quad = \quad M_* e^{\frac{\rho}{2}\cdot}\psi(\cdot). \tag{4.50}$$

In particular, we have

$$[M\psi]_j \quad = \quad -\psi_j(0) - \kappa \textstyle\sum_{k\in\mathbb{Z}} h_{j-k} \int_{-1}^{1} \nu(\sigma)\psi_k(\sigma)\, d\sigma. \tag{4.51}$$

The Fourier components

$$M(\omega) : C([-1,1];\mathbb{R}) \to \mathbb{R} \tag{4.52}$$

act as

$$[M(\omega)\psi](u) = -\psi(0) - \kappa[\mathcal{F}h](\omega) \int_{-1}^{1} \nu(\sigma)\psi(\sigma)\, d\sigma. \tag{4.53}$$

For convenience, we write

$$\beta_\omega = \sqrt{\kappa[\mathcal{F}h](\omega)} > 0, \tag{4.54}$$

which is well-defined on account of (hMod). We note that

$$\delta_{0,M(\omega)}(z) = 1 + \beta_\omega^2 \int_{-1}^{1} \nu(\sigma) e^{z\sigma}\, d\sigma, \tag{4.55}$$

which in view of Lemma 4.1 implies that

$$\delta_{0,M_\omega}(z) = \delta_{0,M_\omega}(-z). \tag{4.56}$$

In addition, upon introducing the functions

$$S_\omega(z) = (z - \frac{\rho}{2})(z + \frac{\rho}{2})\delta_{0,M(\omega)}(z), \tag{4.57}$$

Lemma 4.1 implies

$$S_\omega(z) = z^2 - \frac{\rho^2}{4} + \beta_\omega^2 e^{-\frac{\rho}{2}}\big(e^z + e^{-z} - e^{\frac{\rho}{2}} - e^{-\frac{\rho}{2}}\big). \tag{4.58}$$

We now show that $\delta_{0,M_\omega}(z) = 0$ has no roots in the strip $|\mathrm{Re}\, z| \le \frac{\rho}{2}$. This will allow us to conclude that $\widehat{\mathfrak{q}}_{0,M_*}(0) = \widehat{\mathfrak{q}}_{0,M}(0)$.

Lemma 4.4. *Suppose that* (hMod) *holds. Then for every $\omega \in [-\pi,\pi]$, the characteristic equation*

$$\delta_{0,M_\omega}(z) = 0 \tag{4.59}$$

has no roots with $|\mathrm{Re}\, z| \le \frac{\rho}{2}$.

Proof. For $p \in \mathbb{R}$ and $q \in \mathbb{R}$ we may compute

$$\mathrm{Re}\, S_\omega(p+iq) = p^2 - q^2 - \frac{\rho^2}{4} + \beta_\omega^2 e^{-\frac{\rho}{2}}\big(2\cosh(p)\cos(q) - 2\cosh(\frac{\rho}{2})\big). \tag{4.60}$$

In particular, for $|p| \le \frac{\rho}{2}$ we may estimate

$$\mathrm{Re}\, S_\omega(p+iq) \le p^2 - \frac{\rho^2}{4} - q^2 \le -q^2, \tag{4.61}$$

while for $p \in \mathbb{R}$ we have $\delta_{0,M_\omega}(p) \ge 1$. □

In §4.2 below we establish the following result concerning the invariants $m^{\sharp}_{0,M(\omega)}(0)$, which in fact turn out to be independent of ω. This allows us to apply one of our main results and establish Proposition 4.2.

Proposition 4.5 (see §4.2). *Suppose that* (hMod) *holds. Then for every* $\omega \in [-\pi, \pi]$ *we have*

$$m^{\sharp}_{0,M(\omega)}(0) = \frac{1}{2}. \tag{4.62}$$

Proof of Proposition 4.2. Corollary 3.17 and Proposition 4.5 imply that

$$\begin{array}{lcl} \mathrm{Range}\, \pi^{-}_{\widehat{q}_{0,M_*}(0)} & = & C\big([-1,0];\ell^2(\mathbb{Z};\mathbb{R})\big), \\ \mathrm{Ker}\, \pi^{-}_{\widehat{q}_{0,M_*}(0)} & = & \{0\}, \end{array} \tag{4.63}$$

while Lemma 4.4 yields the identities

$$\begin{array}{lcl} \mathrm{Range}\, \pi^{-}_{\widehat{q}_{0,M}(0)} & = & \mathrm{Range}\, \pi^{-}_{\widehat{q}_{0,M_*}(0)}, \\ \mathrm{Ker}\, \pi^{-}_{\widehat{q}_{0,M}(0)} & = & \mathrm{Ker}\, \pi^{-}_{\widehat{q}_{0,M_*}(0)}. \end{array} \tag{4.64}$$

□

4.2 Wiener-Hopf factorizations

Our goal here is to establish Proposition 4.5 by constructing a path of operators that connects $M(\omega)$ to a reference operator $M^{\mathrm{fct}}_{\omega}$ for which a Wiener-Hopf factorization is available. Inspired by the factorization (4.24) for ν, we define

$$\begin{array}{lcl} M^{-}_{\omega}\phi & = & -\phi(0) - \beta_\omega \int_{-1}^{0} e^{\frac{\rho}{2}\sigma}\phi(\sigma)\, d\sigma, \\ M^{+}_{\omega}\phi & = & -\phi(0) - \beta_\omega \int_{0}^{1} e^{-\frac{\rho}{2}\sigma}\phi(\sigma)\, d\sigma, \end{array} \tag{4.65}$$

together with

$$M^{\mathrm{fct}}_{\omega}\phi = 2\phi(0) + M(\omega)\phi + M^{-}_{\omega}\phi + M^{+}_{\omega}\phi. \tag{4.66}$$

The characteristic functions are given by

$$\begin{array}{lcl} \delta_{0,M^{-}_{\omega}}(z) & = & -M^{-}_{\omega} e^{z\cdot} \\ & = & 1 + \beta_\omega \int_{-1}^{0} e^{\frac{\rho}{2}\sigma} e^{z\sigma}\, d\sigma, \\ \delta_{0,M^{+}_{\omega}}(z) & = & -M^{+}_{\omega} e^{z\cdot} \\ & = & 1 + \beta_\omega \int_{0}^{1} e^{-\frac{\rho}{2}\sigma} e^{z\sigma}\, d\sigma, \end{array} \tag{4.67}$$

together with

$$\begin{array}{lcl} \delta_{0,M^{\mathrm{fct}}_{\omega}}(z) & = & -2 + \delta_{0,M(\omega)}(z) + \delta_{0,M^{-}_{\omega}}(z) + \delta_{0,M^{+}_{\omega}}(z) \\ & = & \delta_{0,M^{-}_{\omega}}(z)\delta_{0,M^{+}_{\omega}}(z). \end{array} \tag{4.68}$$

Lemma 4.6. *Suppose that* (hMod) *holds. Then for every* $\omega \in [-\pi, \pi]$*, we have the identity*

$$m^{\sharp}_{0,M^{\mathrm{fct}}_{\omega}}(0) = \frac{1}{2}. \tag{4.69}$$

Proof. We first verify that the conditions $(\mathrm{HAlgSc})_{M^{\pm}_{\omega}}$ and $(\mathrm{HAlgSc})_{M^{\mathrm{fct}}_{\omega}}$ are satisfied. To see this, one can either apply Proposition 7.5 with $J = 1$ or directly introduce the operators

$$L_\omega : C([-1,1];\mathbb{R}) \to \mathbb{R}, \qquad L^-_\omega : C([-1,0];\mathbb{R}) \to \mathbb{R}, \qquad L^+_\omega : C([0,1];\mathbb{R}) \to \mathbb{R} \tag{4.70}$$

that act as

$$\begin{array}{lcl} L_\omega \psi & = & \beta_\omega^2 \int_{-1}^1 \nu'(\sigma)\psi(\sigma)\,d\sigma, \\ L^-_\omega \phi & = & -\beta_\omega\big[\phi(0) - e^{-\frac{\rho}{2}}\phi(-1) - \frac{\rho}{2}\int_{-1}^0 e^{\frac{\rho}{2}\sigma}\phi(\sigma)\,d\sigma\big], \\ L^+_\omega \phi & = & -\beta_\omega\big[e^{-\frac{\rho}{2}}\phi(1) - \phi(0) - \frac{\rho}{2}\int_0^1 e^{-\frac{\rho}{2}\sigma}\phi(\sigma)\,d\sigma\big], \end{array} \tag{4.71}$$

together with

$$L^{\mathrm{fct}}_\omega = L_\omega + L^-_\omega + L^+_\omega. \tag{4.72}$$

We can then compute

$$\begin{array}{lcl} \delta_{0,L_\omega}(z) & = & -\beta_\omega^2 \int_{-1}^1 \nu'(\sigma) e^{z\sigma}\,d\sigma \\ & = & z\beta_\omega^2 \int_{-1}^1 \nu(\sigma) e^{z\sigma}\,d\sigma \\ & = & z(\delta_{0,M(\omega)}(z) - 1), \end{array} \tag{4.73}$$

together with

$$\begin{array}{lcl} \delta_{0,L^-_\omega}(z) & = & \beta_\omega\big[1 - e^{-\frac{\rho}{2}}e^{-z} - \frac{\rho}{2}\int_{-1}^0 e^{\frac{\rho}{2}\sigma}e^{z\sigma}\,d\sigma\big] \\ & = & z(\delta_{0,M^-_\omega}(z) - 1), \\ \delta_{0,L^+_\omega}(z) & = & \beta_\omega\big[e^{-\frac{\rho}{2}}e^{z} - 1 - \frac{\rho}{2}\int_0^1 e^{-\frac{\rho}{2}\sigma}e^{z\sigma}\,d\sigma\big] \\ & = & z(\delta_{0,M^+_\omega}(z) - 1), \end{array} \tag{4.74}$$

showing that $(\mathrm{HAlgSc})_{M^{\pm}_{\omega}}$ are satisfied. In addition, this allows us to write

$$\begin{array}{lcl} \Delta_{L^{\mathrm{fct}}_\omega}(z) & = & z + \delta_{0,L_\omega}(z) + \delta_{0,L^-_\omega}(z) + \delta_{0,L^+_\omega}(z) \\ & = & z\delta_{0,M_\omega}(z) + z(\delta_{0,M^-_\omega}(z) - 1) + z(\delta_{0,M^+_\omega}(z) - 1) \\ & = & z\delta_{0,M^{\mathrm{fct}}_\omega}(z), \end{array} \tag{4.75}$$

which establishes $(\mathrm{HAlgSc})_{M^{\mathrm{fct}}_{\omega}}$.

To compute $m^\sharp_{0,M^{\mathrm{fct}}_\omega}(0)$, we observe that

$$\begin{array}{lcl} \delta_{0,M^+_\omega(-z)} & = & 1 + \beta_\omega \int_0^1 e^{-\frac{\rho}{2}\sigma}e^{-z\sigma}\,d\sigma \\ & = & 1 + \beta_\omega \int_{-1}^0 e^{\frac{\rho}{2}\sigma}e^{z\sigma}\,d\sigma \\ & = & \delta_{0,M^-_\omega}(z). \end{array} \tag{4.76}$$

This implies that

$$m^+_{M^-_\omega}(0) = m^-_{M^+_\omega}(0) \tag{4.77}$$

and hence

$$m^\sharp_{M^{\mathrm{fct}}_\omega}(0) = m^-_{M^+_\omega}(0) - m^+_{M^-_\omega}(0) + \frac{1}{2}(\ell_+ - \ell_-) + \frac{1}{2} = \frac{1}{2}, \tag{4.78}$$

exploiting the identities $\ell_\pm = 1$. □

We now define, for $0 \le \mu \le 1$, the bounded linear operators

$$\gamma_\omega(\mu) : C([-1,1];\mathbb{R}) \to \mathbb{R}, \tag{4.79}$$

that act as

$$\gamma_\omega(\mu) \quad = \quad M(\omega)\phi + (1-\mu)\big[M_\omega^- \phi + M_\omega^+ \phi + 2\phi(0)\big]. \tag{4.80}$$

By construction, we have

$$\gamma_\omega(0) = M_\omega^{\mathrm{fct}}, \qquad \gamma_\omega(1) = M(\omega). \tag{4.81}$$

Lemma 4.7. *Suppose that* (hMod) *holds. Then for every* $\omega \in [-\pi,\pi]$ *and every* $0 \le \mu \le 1$, *we have*

$$\delta_{0,\gamma_\omega(\mu)}(iq) \neq 0 \tag{4.82}$$

for all $q \in \mathbb{R}$.

Proof. Observe that

$$\begin{aligned} \delta_{0,\gamma_\omega(\mu)}(iq) &= \delta_{0,M_\omega}(iq) + (1-\mu)\big[\delta_{0,M_\omega^-}(iq) + \delta_{0,M_\omega^+}(iq) - 2\big] \\ &= \delta_{0,M_\omega}(iq) + (1-\mu)\beta_\omega\big[\int_{-1}^0 e^{(iq+\frac{\rho}{2})\sigma}\,d\sigma + \int_0^1 e^{(iq-\frac{\rho}{2})\sigma}\,d\sigma\big] \\ &= \delta_{0,M_\omega}(iq) + (1-\mu)\beta_\omega\big[\int_{-1}^0 \cos(q\sigma)e^{\frac{\rho}{2}\sigma}\,d\sigma + \int_0^1 \cos(q\sigma)e^{-\frac{\rho}{2}\sigma}\,d\sigma\big]. \end{aligned} \tag{4.83}$$

In particular, writing

$$T_{\omega,\mu}(q) = (q^2 + \frac{\rho^2}{4})\delta_{0,\gamma_\omega(\mu)}(iq), \tag{4.84}$$

we find that

$$\begin{aligned} T_{\omega,\mu}(q) &= -S_\omega(iq) + (1-\mu)\beta_\omega\big[\rho - e^{-\frac{\rho}{2}}\rho\cos(q) + 2e^{-\frac{\rho}{2}}q\sin(q)\big] \\ &= q^2 + \tfrac{\rho^2}{4} - \beta_\omega^2 e^{-\frac{\rho}{2}}\big(2\cos(q) - 2\cosh(\tfrac{\rho}{2})\big) \\ &\quad +(1-\mu)\beta_\omega\big[\rho - e^{-\frac{\rho}{2}}\rho\cos(q) + 2e^{-\frac{\rho}{2}}q\sin(q)\big] \\ &\ge q^2 + \tfrac{\rho^2}{4} + 2e^{-\frac{\rho}{2}}(1-\mu)\beta_\omega q\sin(q). \end{aligned} \tag{4.85}$$

Exploiting (4.28), we now observe that

$$\beta_\omega^2 = \kappa[\mathcal{F}h](\omega) \le 2\frac{\rho}{1-e^{-\rho}} \le 2(1+\rho). \tag{4.86}$$

This in turn implies that

$$2(1-\mu)e^{-\frac{\rho}{2}}\beta_\omega \le 2\sqrt{2}e^{-\frac{\rho}{2}}\sqrt{1+\rho} \le 3. \tag{4.87}$$

Noting that

$$q^2 \ge -\alpha q\sin(q) \tag{4.88}$$

holds for all $0 \le \alpha \le 3$ and $q \in \mathbb{R}$, we hence see that $T_{\omega,\mu}(q) > 0$. □

Proof of Proposition 4.5. In order to apply Proposition 3.12, we construct a continuous map

$$[0,1] \ni \mu \mapsto \Gamma_\omega(\mu) \in \mathcal{L}\big(C([-1,1];\mathbb{R});\mathbb{R}\big) \tag{4.89}$$

for which

$$\Delta_{\Gamma_\omega(\mu)}(z) = z\delta_{0,\gamma_\omega(\mu)}(z) \tag{4.90}$$

holds for all $\mu \in [0,1]$. The identity (4.62) then follows from Proposition 3.12 and Lemma 4.6, since Lemma 4.7 implies that one can pick $0 < \epsilon \ll 1$ in such a way that no roots of the characteristic equation $\Delta_{\Gamma_\omega(\mu)}(z) = 0$ cross the lines $\mathrm{Re}\, z = \pm\epsilon$ as μ is increased from $\mu = 0$ to $\mu = 1$.

To construct the branch (4.89), one can either invoke Proposition 7.5 with $J = 1$ or directly write

$$\Gamma_\omega(\mu) \quad = \quad L_\omega + (1-\mu)(L_\omega^- + L_\omega^+), \tag{4.91}$$

in which we have reused the operators defined in the proof of Lemma 4.6. Indeed, a short computation shows that

$$\begin{aligned} \Delta_{\Gamma_\omega(\mu)}(z) &= z + \delta_{0,L_\omega}(z) + (1-\mu)\big(\delta_{0,L_\omega^-}(z) + \delta_{0,L_\omega^+}(z)\big) \\ &= z\delta_{0,M_\omega}(z) + z(1-\mu)(\delta_{0,M_\omega^-}(z) - 1) + z(1-\mu)(\delta_{0,M_\omega^+}(z) - 1) \\ &= z\delta_{0,\gamma_\omega(\mu)}(z). \end{aligned} \tag{4.92}$$

□

5 Exponential splittings via fixed point problems

In this section we set out to prove the existence of exponential dichotomies by characterizing the spaces $P_L(\eta)$ and $Q_L(\eta)$ as solution spaces to two fixed point problems. In particular, we establish Theorems 2.1 and 2.3 together with Proposition 2.2.

Most of our results here concern MFDEs posed on Banach-spaces. We therefore fix a Banach space $\mathcal{B}$ and consider a bounded linear operator

$$L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B} \tag{5.1}$$

on which we impose the following condition.

$(\mathrm{hF})_{L;\mathcal{B}}$ The form condition $(\mathrm{HF})_L$ holds with $\mathcal{H}$ replaced by $\mathcal{B}$.

In addition, we introduce the spaces

$$\begin{aligned} \widehat{\mathfrak{P}}_{L;\mathcal{B}}(\eta) &= \big\{\widehat{x} \in \widehat{BC}_\eta^\ominus(\mathcal{B}) : \widehat{x}'(\xi) = \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,\widehat{x} \text{ for all } \xi \in (-\infty, 0) \setminus \mathcal{R}\big\}, \\ \widehat{\mathfrak{Q}}_{L;\mathcal{B}}(\eta) &= \big\{\widehat{y} \in \widehat{BC}_\eta^\oplus(\mathcal{B}) : \widehat{y}'(\xi) = \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,\widehat{y} \text{ for all } \xi \in (0, \infty) \setminus \mathcal{R}\big\}, \end{aligned} \tag{5.2}$$

which simply generalize the definitions (2.30) by replacing $\mathcal{H}$ with $\mathcal{B}$.

Fix $\eta \in \mathbb{R}$. For any function $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$, any element $v \in \mathcal{B}$ and any $\xi \in \mathbb{R}$, we formally introduce the expression

$$\mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) = \frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \Delta_L(z)^{-1} \big\langle (e^{-z^*\cdot}, 1), (\widehat{\phi}, v)\big\rangle_L \, dz. \tag{5.3}$$

This expression is related to the inverse Laplace transform of (2.9) and hence plays a fundamental role in this section. We first show in what sense $\mathcal{T}_{L;\eta}$ is well-defined.

Lemma 5.1 (see §5.1). *Fix a Banach space $\mathcal{B}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ that satisfies $(\mathrm{hF})_{L;\mathcal{B}}$. Pick $\eta \in \mathbb{R}$ in such a way that the linear operators $\Delta_L(z)$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$.*

Then for any $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$ and any $v \in \mathcal{B}$, the symbol $\mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi)$ introduced in (5.3) is well-defined for $\xi \neq 0$. In addition, the functions

$$\begin{aligned} f_-(\xi) &= \mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) \qquad \text{for } \xi < 0, \\ f_+(\xi) &= \mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) \qquad \text{for } \xi > 0 \end{aligned} \tag{5.4}$$

can be extended at $\xi = 0$ to satisfy

$$f_- \in BC^-_\eta(\mathcal{B}), \qquad f_+ \in BC^+_\eta(\mathcal{B}). \tag{5.5}$$

In particular, the one-sided limits $f_-(0^-)$ and $f_+(0^+)$ both exist and the jump at zero is explicitly given by

$$f_+(0^+) - f_-(0^-) = v. \tag{5.6}$$

Finally, there exists a constant $K \geq 1$ that does not depend on $\widehat{\phi}$ and v such that

$$\left\| \mathcal{T}_{L;\eta}[\widehat{\phi}, v] \right\|_{BC^-_\eta(\mathcal{B})} + \left\| \mathcal{T}_{L;\eta}[\widehat{\phi}, v] \right\|_{BC^+_\eta(\mathcal{B})} \leq K \Big[\left\| \widehat{\phi} \right\|_{C([r_{\min},0];\mathcal{B}) \times C([0,r_{\max}];\mathcal{B})} + \|v\|_{\mathcal{B}} \Big]. \tag{5.7}$$

When $\mathcal{B}$ is finite dimensional, the map $\mathcal{T}_{L;\eta}$ admits certain compactness properties that we describe below. In §6 this result will allow us to establish Fredholm properties for the restriction operators (3.5).

Lemma 5.2 (see §5.1). *Fix a finite dimensional Banach space $\mathcal{B}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ that satisfies $(\mathrm{hF})_{L;\mathcal{B}}$. Pick $\eta \in \mathbb{R}$ in such a way that the linear operators $\Delta_L(z)$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$. Then the linear operator*

$$\widehat{\mathrm{ev}}_0 \mathcal{T}_\eta : C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B}) \times \mathcal{B} \to C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B}) \tag{5.8}$$

is compact.

In the Hilbert space setting $\mathcal{B} = \mathcal{H}$, the map $\mathcal{T}_{L;\eta}$ can be used to relate functions in $\widehat{\mathfrak{P}}_L(\eta)$ and $\widehat{\mathfrak{Q}}_L(\eta)$ back to their initial segments. The almost-everywhere pointwise convergence of the inverse Laplace transform lies at the basis of this result, which is why the restriction $\mathcal{B} = \mathcal{H}$ is necessary.

Proposition 5.3 (see §5.2). *Fix a Hilbert space $\mathcal{H}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Pick $\eta \in \mathbb{R}$ in such a way that the linear operators $\Delta_L(z)$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$.*

Then for any $\widehat{q} \in \widehat{\mathfrak{Q}}_L(\eta)$ we have

$$\widehat{q}(\xi) = \mathcal{T}_{L;\eta}[\widehat{\mathrm{ev}}_0\, \widehat{q}, \widehat{q}(0^+)](\xi) \tag{5.9}$$

for all $\xi > 0$, while for any $\widehat{p} \in \widehat{\mathfrak{P}}_L(\eta)$ we have

$$\widehat{p}(\xi) = -\mathcal{T}_{L;\eta}[\widehat{\mathrm{ev}}_0\, \widehat{p}, \widehat{p}(0^-)](\xi) \tag{5.10}$$

for all $\xi < 0$.

Pick $\eta \in \mathbb{R}$ in such a way that the linear operators $\Delta_L(z) \in \mathcal{L}(\mathcal{B}; \mathcal{B})$ are invertible for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$. Inspired by the identities (5.9) and (5.10), we now introduce the two operators

$$\begin{aligned} E_{\widehat{Q}_L(\eta)} &: C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B}) \to \widehat{BC}^{\oplus}_\eta(\mathcal{B}), \\ E_{\widehat{P}_L(\eta)} &: C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B}) \to \widehat{BC}^{\ominus}_\eta(\mathcal{B}) \end{aligned} \tag{5.11}$$

that act as

$$[E_{\widehat{Q}_L(\eta)}\widehat{\phi}](\xi) = \begin{cases} \mathcal{T}_{L;\eta}[\widehat{\phi}, \widehat{\phi}(0^+)](\xi), & \xi > 0 \text{ and } \xi = 0^+, \\ \widehat{\phi}(\xi) + \mathcal{T}_{L;\eta}[\widehat{\phi}, \widehat{\phi}(0^+)](\xi), & r_{\min} \le \xi < 0 \text{ and } \xi = 0^-, \end{cases} \tag{5.12}$$

together with

$$[E_{\widehat{P}_L(\eta)}\widehat{\phi}](\xi) = \begin{cases} -\mathcal{T}_{L;\eta}[\widehat{\phi}, \widehat{\phi}(0^-)](\xi), & \xi < 0 \text{ and } \xi = 0^-, \\ \widehat{\phi}(\xi) - \mathcal{T}_{L;\eta}[\widehat{\phi}, \widehat{\phi}(0^-)](\xi), & 0 < \xi \le r_{\max} \text{ and } \xi = 0^+. \end{cases} \tag{5.13}$$

A direct consequence of Proposition 5.3 is that in the Hilbert space setting $\mathcal{B} = \mathcal{H}$, all elements of $\widehat{P}_L(\eta)$ and $\widehat{\mathfrak{Q}}_L(\eta)$ can be interpreted as solutions to a fixed point problem.

Corollary 5.4. *Consider the setting of Proposition 5.3. Then for any $\widehat{\phi} \in \widehat{P}_L(\eta)$ we have*

$$\widehat{\phi} = \widehat{\mathrm{ev}}_0 E_{\widehat{P}_L(\eta)}\widehat{\phi}, \tag{5.14}$$

while for any $\widehat{\phi} \in \widehat{Q}_L(\eta)$ we have

$$\widehat{\phi} = \widehat{\mathrm{ev}}_0 E_{\widehat{Q}_L(\eta)}\widehat{\phi}. \tag{5.15}$$

The main technical result of this section can be interpreted as a converse to Corollary 5.4 that also works in a Banach space setting. In addition, it provides detailed bounds for the operators $E_{\widehat{P}_L(\eta)}$ and $E_{\widehat{Q}_L(\eta)}$ that reference the L^2-norm instead of the usual supremum norm.

Proposition 5.5 (see §5.3). *Fix a Banach space $\mathcal{B}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ that satisfies $(\mathrm{hF})_{L;\mathcal{B}}$. Pick $\eta \in \mathbb{R}$ in such a way that the linear operators $\Delta_L(z)$ are invertible in $\mathcal{L}(\mathcal{B}; \mathcal{B})$ for all $z \in \mathbb{C}$ that have $\mathrm{Re}\, z = \eta$.*

Then for any $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$, we have the inclusions

$$\begin{array}{rcl} E_{\widehat{Q}_L(\eta)}\widehat{\phi} & \in & \widehat{\mathfrak{Q}}_{L;\mathcal{B}}(\eta), \\ E_{\widehat{P}_L(\eta)}\widehat{\phi} & \in & \widehat{\mathfrak{P}}_{L;\mathcal{B}}(\eta). \end{array} \tag{5.16}$$

In addition, there exist constants $K \ge 1$ and $\epsilon > 0$ that do not depend on $\widehat{\phi}$ so that the estimates

$$\left\| [E_{\widehat{Q}_L(\eta)}\widehat{\phi}](\xi) \right\|_{\mathcal{B}} \le K e^{(\eta - \epsilon)\xi} \big[\left\| \widehat{\phi}(0^+) \right\|_{\mathcal{B}} + \left\| \widehat{\phi} \right\|_{L^2([r_{\min}, r_{\max}]; \mathcal{B})} \big] \tag{5.17}$$

hold for all $\xi > 0$, while the estimates

$$\left\| [E_{\widehat{P}_L(\eta)}\widehat{\phi}](\xi) \right\|_{\mathcal{B}} \le K e^{(\eta + \epsilon)\xi} \big[\left\| \widehat{\phi}(0^-) \right\|_{\mathcal{B}} + \left\| \widehat{\phi} \right\|_{L^2([r_{\min}, r_{\max}]; \mathcal{B})} \big] \tag{5.18}$$

hold for all $\xi < 0$.

As a consequence of (5.6), we note that we have the restrictions

$$E_{\widehat{P}_L(\eta)}\big(C([r_{\min}, r_{\max}]; \mathcal{B})\big) \subset BC_\eta^{\ominus}(\mathcal{B}), \qquad E_{\widehat{Q}_L(\eta)}\big(C([r_{\min}, r_{\max}]; \mathcal{B})\big) \subset BC_\eta^{\oplus}(\mathcal{B}). \tag{5.19}$$

This observation lies at the heart of our final result, which is more explicit than Theorem 2.1. In particular, it shows how the operators $E_{\widehat{P}_L(\eta)}$ and $E_{\widehat{Q}_L(\eta)}$ can be interpreted as the projections associated to the desired exponential splitting of the state space $C([r_{\min}, r_{\max}]; \mathcal{H})$.

Proposition 5.6 (see §5.3). *Fix a Hilbert space $\mathcal{H}$ and consider a linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. If the linear operators $\Delta_L(z) \in \mathcal{L}(\mathcal{H}; \mathcal{H})$ are invertible for all $z \in \mathbb{C}$ that have $\operatorname{Re} z = \eta$, then the spaces $P_L(\eta)$ and $Q_L(\eta)$ are both closed and we have the splitting*

$$C\big([r_{\min}, r_{\max}]; \mathcal{H}\big) = P_L(\eta) \oplus Q_L(\eta), \tag{5.20}$$

which can be made explicit by writing

$$\phi = \mathrm{ev}_0 E_{\widehat{P}_L(\eta)} \phi \oplus \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)} \phi \tag{5.21}$$

for any $\phi \in C([r_{\min}, r_{\max}]; \mathcal{H})$.

5.1 Decomposition of $\mathcal{T}_{L;\eta}$

In this subsection we set out to gain a detailed understanding of the formal expressions

$$\begin{array}{rcl} \mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) & = & \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \Delta_L(z)^{-1} \big\langle (e^{-z^* \cdot}, 1), (\widehat{\phi}, v) \big\rangle_L, \\ \frac{d}{d\xi} \mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) & = & \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} z \Delta_L(z)^{-1} \big\langle (e^{-z^* \cdot}, 1), (\widehat{\phi}, v) \big\rangle_L. \end{array} \tag{5.22}$$

In particular, we will obtain an explicit description of the components in the integral representations above that are not integrable with respect to z on the vertical line $\operatorname{Re} z = \eta$.

Let us therefore pick a Banach space $\mathcal{B}$ and a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$. In order to isolate the slowest decaying portion of $\Delta_L(z)^{-1}$, we formally introduce the expression

$$R_{L;\alpha}(z) \quad = \quad \Delta_L(z)^{-1} - \frac{1}{z - \alpha} I - \frac{L e^{z \cdot} - \alpha}{(z - \alpha)^2}. \tag{5.23}$$

We note that $R_{L;\alpha}(z) \in \mathcal{L}(\mathcal{B}; \mathcal{B})$ whenever $\Delta_L(z) \in \mathcal{L}(\mathcal{B}; \mathcal{B})$ is invertible and $z \neq \alpha$.

Lemma 5.7. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ and suppose that $\Delta_L(z) \in \mathcal{L}(\mathcal{B}; \mathcal{B})$ is invertible for all $z \in \mathbb{C}$ with $\operatorname{Re} z = \eta$. Pick $\alpha \in \mathbb{R} \setminus \{\eta\}$. Then there exist constants $\kappa > 0$ and $K \geq 1$ so that*

$$\|R_{L;\alpha}(z)\|_{\mathcal{L}(\mathcal{B};\mathcal{B})} \leq \frac{K}{1 + |z|^3} \tag{5.24}$$

for all $z \in \mathbb{C}$ with $|\operatorname{Re} z - \eta| \leq \kappa$.

Proof. For large $|\operatorname{Im} z|$, the desired behaviour follows from the expansion

$$\begin{array}{rcl} \Delta_L(z)^{-1} & = & \big[z - \alpha + (\alpha - L e^{z \cdot})\big]^{-1} \\ & = & (z - \alpha)^{-1} \big[I + (z - \alpha)^{-1} (\alpha - L e^{z \cdot})\big]^{-1} \\ & = & (z - \alpha)^{-1} \big[I - (z - \alpha)^{-1} (\alpha - L e^{z \cdot}) + O\big((z - \alpha)^{-2}\big)\big], \end{array} \tag{5.25}$$

exploiting the fact that $z \mapsto L e^{z \cdot}$ is bounded in vertical strips of the complex plane. ◻

To exploit the decomposition (5.23), we introduce, for any $\alpha \in \mathbb{R} \setminus \{\eta\}$, any $v \in \mathcal{B}$ and any $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$, the expressions

$$\begin{array}{rcl} \mathcal{M}^1_\alpha[v] & = & \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \Big[\frac{1}{z - \alpha} + \frac{L e^{z \cdot} - \alpha}{(z - \alpha)^2} \Big] v, \\ \mathcal{R}^1_\alpha[v](\xi) & = & \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} e^{\xi z} R_{L;\alpha}(z) v \, dz, \end{array} \tag{5.26}$$

together with

$$\begin{array}{lcl} \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) & = & \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \Big[\frac{1}{z-\alpha} + \frac{L e^{z \cdot} - \alpha}{(z-\alpha)^2} \Big] \big\langle (e^{-z^*}, 0), (\widehat{\phi}, 0) \big\rangle_L \, dz, \\ \mathcal{R}^2_\alpha[\widehat{\phi}](\xi) & = & \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} e^{\xi z} R_{L;\alpha}(z) \big\langle (e^{-z^*}, 0), (\widehat{\phi}, 0) \big\rangle_L \, dz. \end{array} \tag{5.27}$$

Observe that we have dropped the limit with respect to Ω in the expressions for $\mathcal{R}^1_\alpha$ and $\mathcal{R}^2_\alpha$, which is possible because Lemma 5.7 guarantees that the integrands are in L^1. The expressions (5.26)-(5.27) allow us to write

$$\mathcal{T}_{L;\eta}[\widehat{\phi}, v](\xi) = \mathcal{M}^1_\alpha[v](\xi) + \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) + \mathcal{R}^1_\alpha[v](\xi) + \mathcal{R}^2_\alpha[\widehat{\phi}](\xi). \tag{5.28}$$

We now set out to derive explicit expressions for $\mathcal{M}^1_\alpha[v]$ and $\mathcal{M}^2_\alpha[\widehat{\phi}]$. For convenience, we define the function

$$H(\xi) = \begin{cases} 1 & \xi > 0, \\ \frac{1}{2} & \xi = 0, \\ 0 & \xi < 0 \end{cases} \tag{5.29}$$

and evaluate several key integrals.

Lemma 5.8. *For any $\alpha > \eta$ and $\xi \in \mathbb{R}$ we have*

$$\frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \frac{1}{z-\alpha} \, dz \quad = \quad -e^{\alpha \xi} H(-\xi), \tag{5.30}$$

together with

$$\frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi z} \frac{1}{(z-\alpha)^2} \, dz \quad = \quad -\xi e^{\alpha \xi} H(-\xi). \tag{5.31}$$

Proof. The expressions follow from standard computations using Jordan's lemma and the residue theorem. □

For $\alpha > \eta$, we hence have the explicit representation

$$\mathcal{M}^1_\alpha[v](\xi) \quad = \quad -\Big[e^{\alpha \xi} H(-\xi) + (L - \alpha) \mathrm{ev}_\xi [\xi' \mapsto \xi' e^{\alpha \xi'} H(-\xi')] \Big] v, \tag{5.32}$$

in which we have introduced the shorthand $(L - \alpha)\psi = L\psi - \alpha\psi(0)$ for $\psi \in C([r_{\min}, r_{\max}]; \mathcal{B})$. The following result summarizes some facts that can be read off directly from this representation.

Lemma 5.9. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ that satisfies* $(\mathrm{hF})_{L;\mathcal{B}}$ *and pick $\alpha > \eta$. Then for any sufficiently small $\epsilon > 0$, the map*[3]

$$\mathcal{B} \ni v \mapsto \mathcal{M}^1_\alpha[v] \in BC^-_{\eta + \epsilon}(\mathcal{B}) \cap BC^+_{\eta - \epsilon}(\mathcal{B}) \tag{5.33}$$

is well-defined and bounded. Upon fixing $v \in \mathcal{B}$, we have the jump discontinuity

$$\mathcal{M}^1_\alpha[v](0^+) - \mathcal{M}^1_\alpha[v](0^-) = v. \tag{5.34}$$

In addition, $\mathcal{M}^1_\alpha[v]$ is continuously differentiable on $\mathbb{R} \setminus \mathcal{R}$ with

$$\mathcal{M}^1_\alpha[v]'(\xi) \quad = \quad \alpha \mathcal{M}^1_\alpha[v](\xi) - (\widehat{L} - \alpha) \widehat{\mathrm{ev}}_\xi [\xi' \mapsto e^{\alpha \xi'} H(-\xi')] v \tag{5.35}$$

[3]Actually, the map (5.33) should be interpreted as two separate maps, one into $BC^-_{\eta+\epsilon}(\mathcal{B})$ and one into $BC^+_{\eta-\epsilon}(\mathcal{B})$. Throughout this section we will slightly abuse the symbol $\cap$ in this fashion.

for $\xi \in \mathbb{R} \setminus \mathcal{R}$. In particular, for all $0 \le j \le N$ we have the jump

$$\mathcal{M}^1_\alpha[v]'(-r_j^+) - \mathcal{M}^1_\alpha[v]'(-r_j^-) = A_j v. \tag{5.36}$$

Finally, upon writing

$$\mathcal{H}_{\mathcal{M}^1_\alpha}[v](\xi) = \mathcal{M}^1_\alpha[v]'(\xi) - \widehat{L}\,\widehat{\mathrm{ev}}_\xi \mathcal{M}^1_\alpha[v] \tag{5.37}$$

for all $\xi \in \mathbb{R} \setminus \mathcal{R}$, we have

$$\mathcal{H}_{\mathcal{M}^1_\alpha}[v](\xi) \quad = \quad (L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto \xi'' e^{\alpha\xi''} H(-\xi'')v]\big] \tag{5.38}$$

for all such ξ.

We remark that the right-hand side of (5.38) is in fact a member of $BC_\eta(\mathbb{R};\mathcal{B})$. Formally however one can say that the functions $\mathcal{M}^1_\alpha[v]'$ and $\mathcal{H}_{\mathcal{M}^1_\alpha}[v]$ both have a $\delta(\xi)v$ component at $\xi = 0$.

For any $\alpha \in \mathbb{R} \setminus \{\eta\}$, any $\widehat{\phi} \in L^2([r_{\min}, r_{\max}];\mathcal{B})$ and any $\vartheta \in [r_{\min}, r_{\max}]$, we now introduce the two expressions

$$\begin{array}{lcl} \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) & = & \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi z}\frac{1}{z-\alpha}e^{z\vartheta}\int_\vartheta^0 e^{-z\sigma}\widehat{\phi}(\sigma)\,d\sigma\,dz, \\ \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) & = & \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi z}\frac{1}{(z-\alpha)^2}e^{z\vartheta}\int_\vartheta^0 e^{-z\sigma}\widehat{\phi}(\sigma)\,d\sigma\,dz. \end{array} \tag{5.39}$$

We remark that both expressions are identically zero if $\vartheta = 0$. The other cases are studied in the following result, in which we use the notation

$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases} \tag{5.40}$$

Lemma 5.10. *Suppose that $r_{\min} \le 0 \le r_{\max}$ and pick $\alpha > \eta$ and $\vartheta \in [r_{\min}, r_{\max}] \setminus \{0\}$. Then for all sufficiently small $\epsilon > 0$, the maps*

$$\begin{array}{lcl} L^2([r_{\min}, r_{\max}];\mathcal{B}) \ni \widehat{\phi} & \mapsto & \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}] \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}), \\ L^2([r_{\min}, r_{\max}];\mathcal{B}) \ni \widehat{\phi} & \mapsto & \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}] \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}), \\ L^2([r_{\min}, r_{\max}];\mathcal{B}) \ni \widehat{\phi} & \mapsto & \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]' \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}) \end{array} \tag{5.41}$$

are all well-defined and bounded. Upon fixing $\widehat{\phi} \in C([r_{\min}, 0];\mathcal{B}) \times C([0, r_{\max}];\mathcal{B})$, we have the explicit identities

$$\begin{array}{lcl} \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) & = & -e^{\alpha(\xi+\vartheta)}\int_\vartheta^0 H(\sigma-\xi-\vartheta)e^{-\alpha\sigma}\widehat{\phi}(\sigma)\,d\sigma \\ & = & -e^{\alpha(\xi+\vartheta)}\int_{\max\{\vartheta,\xi+\vartheta\}}^{\max\{0,\xi+\vartheta\}} e^{-\alpha\sigma}\widehat{\phi}(\sigma)\,d\sigma \end{array} \tag{5.42}$$

together with

$$\begin{array}{lcl} \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) & = & -e^{\alpha(\xi+\vartheta)}\int_\vartheta^0 (\xi+\vartheta-\sigma)H(\sigma-\xi-\vartheta)e^{-\alpha\sigma}\widehat{\phi}(\sigma)\,d\sigma \\ & = & -e^{\alpha(\xi+\vartheta)}\int_{\max\{\vartheta,\xi+\vartheta\}}^{\max\{0,\xi+\vartheta\}} (\xi+\vartheta-\sigma)e^{-\alpha\sigma}\widehat{\phi}(\sigma)\,d\sigma, \end{array} \tag{5.43}$$

which both hold for any $\xi \in \mathbb{R}$.

In addition, for any $\xi \notin \{0, -\vartheta\}$ we have

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) - \mathrm{sign}(\vartheta)\widehat{\phi}(\xi+\vartheta)\mathbf{1}_{\min\{-\vartheta,0\}<\xi<\max\{-\vartheta,0\}}, \tag{5.44}$$

while for any $\xi \in \mathbb{R}$ *we have*

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) + \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi). \tag{5.45}$$

In particular, if $\vartheta < 0$*, then we have*

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]' \in BC_\eta\big((-\infty, 0]; \mathcal{B}\big) \cap C([0, -\vartheta]; \mathcal{B}) \cap BC_\eta([-\vartheta, \infty); \mathcal{B}) \tag{5.46}$$

with jumps

$$\begin{array}{lcl} \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(0^+) - \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(0^-) & = & \widehat{\phi}(\vartheta), \\ \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(-\vartheta^+) - \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(-\vartheta^-) & = & -\widehat{\phi}(0^-). \end{array} \tag{5.47}$$

On the other hand, if $\vartheta > 0$ *then we have*

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]' \in BC_\eta\big((-\infty, -\vartheta]; \mathcal{B}\big) \cap C([-\vartheta, 0]; \mathcal{B}) \cap BC_\eta([0, \infty); \mathcal{B}) \tag{5.48}$$

with jumps

$$\begin{array}{lcl} \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(-\vartheta^+) - \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(-\vartheta^-) & = & -\widehat{\phi}(0^+), \\ \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(0^+) - \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(0^-) & = & \widehat{\phi}(\vartheta). \end{array} \tag{5.49}$$

Proof. Exploiting Fubini, we write

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad \tfrac{1}{2\pi i} \lim_{\Omega \to \infty} \int_\vartheta^0 \int_{\eta - i\Omega}^{\eta + i\Omega} e^{z(\xi + \vartheta - \sigma)} \tfrac{1}{z - \alpha} \widehat{\phi}(\sigma)\, dz\, d\sigma. \tag{5.50}$$

Applying the limits in Lemma 5.8, which hold uniformly for compact sets of ξ, we find

$$\begin{array}{lcl} \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) & = & \tfrac{1}{2\pi i} \int_\vartheta^0 \lim_{\Omega \to \infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{z(\xi + \vartheta - \sigma)} \tfrac{1}{z - \alpha} \widehat{\phi}(\sigma)\, dz\, d\sigma \\ & = & -\int_\vartheta^0 H(\sigma - \xi - \vartheta) e^{\alpha(\xi + \vartheta - \sigma)} \widehat{\phi}(\sigma)\, d\sigma \\ & = & -e^{\alpha(\xi + \vartheta)} \int_\vartheta^0 H(\sigma - \xi - \vartheta) e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma. \end{array} \tag{5.51}$$

We first fix $\vartheta < 0$. For $\xi + \vartheta \le 0$, we have

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad -e^{\alpha(\xi + \vartheta)} \int_{\max\{\vartheta, \xi + \vartheta\}}^0 e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma, \tag{5.52}$$

while $\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) = 0$ for $\xi + \vartheta > 0$. In both cases this matches the desired identity (5.42). In addition, for $0 < \xi < -\vartheta$ we have

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) + \widehat{\phi}(\xi + \vartheta), \tag{5.53}$$

while

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) = \alpha \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \tag{5.54}$$

for $\xi < 0$ and $\xi > -\vartheta$. Both identities agree with (5.44).

We now fix $\vartheta > 0$. For $\xi \le 0$ we may compute

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad -e^{\alpha(\xi + \vartheta)} \int_\vartheta^{\max\{0, \xi + \vartheta\}} e^{-\alpha\sigma} \widehat{\phi}(\sigma) d\sigma, \tag{5.55}$$

while for $\xi \ge 0$ we have $\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) = 0$, again matching (5.42). For $-\vartheta < \xi < 0$ we have

$$\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) - \widehat{\phi}(\xi + \vartheta), \tag{5.56}$$

which matches (5.44).

In a similar fashion as above we may compute

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad -e^{\alpha(\xi+\vartheta)} \int_{\vartheta}^{0} (\xi+\vartheta-\sigma) H(\sigma-\xi-\vartheta) e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma. \tag{5.57}$$

We again fix $\vartheta < 0$. For $\xi + \vartheta \leq 0$, we now see

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad -e^{\alpha(\xi+\vartheta)} \int_{\max\{\vartheta,\xi+\vartheta\}}^{0} (\xi+\vartheta-\sigma) e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma, \tag{5.58}$$

while $\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) = 0$ for $\xi + \vartheta > 0$. Both expression agree with (5.43). In addition, for $\xi + \vartheta < 0$ we may compute

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) - e^{\alpha(\xi+\vartheta)} \int_{\max\{\vartheta,\xi+\vartheta\}}^{0} e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma, \tag{5.59}$$

while $\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) = 0$ for $\xi + \vartheta > 0$. Both expressions are equal at $\xi + \vartheta = 0$, allowing us to conclude that $\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]$ is continuously differentiable on $\mathbb{R}$.

Finally, fix $\vartheta > 0$. For $\xi \leq 0$ we have

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) \quad = \quad -e^{\alpha(\xi+\vartheta)} \int_{\vartheta}^{\max\{0,\xi+\vartheta\}} (\xi+\vartheta-\sigma) e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma, \tag{5.60}$$

while for $\xi \geq 0$ we have $\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) = 0$, which both agree with (5.43). For $\xi < 0$ we may compute

$$\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) \quad = \quad \alpha \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi) - e^{\alpha(\xi+\vartheta)} \int_{\vartheta}^{\max\{0,\xi+\vartheta\}} e^{-\alpha\sigma} \widehat{\phi}(\sigma)\, d\sigma, \tag{5.61}$$

while $\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}]'(\xi) = 0$ for $\xi > 0$. As before, both expressions are equal at $\xi = 0$.

The remaining statements follow easily from inspection of the explicit expressions (5.42)-(5.45). □

For any $\alpha > \eta$, we can hence explicitly evaluate (5.27) as

$$\begin{aligned} \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) \quad = \quad & \sum_{j=0}^N \left[A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\widehat{\phi}](\xi) + \int_{s_j^-}^{s_j^+} B_j(\vartheta) \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi)\, d\vartheta \right] \\ & +(L-\alpha)\mathrm{ev}_\xi \Big[\xi' \mapsto \sum_{j=0}^N \left[A_j \mathcal{J}^{(2)}_{\alpha,r_j}[\widehat{\phi}](\xi') + \int_{s_j^-}^{s_j^+} B_j(\vartheta) \mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi')\, d\vartheta \right] \Big]. \end{aligned} \tag{5.62}$$

In addition, a direct computation shows that

$$\begin{aligned} \mathcal{M}^2_\alpha[\widehat{\phi}]'(\xi) \quad = \quad & \alpha \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) - \sum_{j=0}^N A_j \mathrm{sign}(r_j) \widehat{\phi}(\xi + r_j) \mathbf{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}} \\ & - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} B_j(\vartheta) \mathrm{sign}(\vartheta) \widehat{\phi}(\xi+\vartheta) \mathbf{1}_{\min\{-\vartheta,0\}<\xi<\max\{-\vartheta,0\}}\, d\vartheta \\ & +(L-\alpha)\mathrm{ev}_\xi \Big[\xi' \mapsto \sum_{j=0}^N \left[A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\widehat{\phi}](\xi') + \int_{s_j^-}^{s_j^+} B_j(\vartheta) \mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi')\, d\vartheta \right] \Big] \end{aligned} \tag{5.63}$$

for $\xi \in \mathbb{R} \setminus \mathcal{R}$. Inspection of these identities readily yields the following result.

Lemma 5.11. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ that satisfies* $(\mathrm{hF})_{L;\mathcal{B}}$ *and pick $\alpha > \eta$. Then for any sufficiently small $\epsilon > 0$, the map*

$$L^2([r_{\min}, r_{\max}]; \mathcal{B}) \ni \widehat{\phi} \mapsto \mathcal{M}^2_\alpha[\widehat{\phi}] \in BC_\eta(\mathbb{R}; \mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}) \tag{5.64}$$

is well-defined and bounded.

Upon fixing $\widehat{\phi} \in C([r_{\min},0];\mathcal{B}) \times C([0,r_{\max}];\mathcal{B})$, the function $\mathcal{M}^2_\alpha[\widehat{\phi}]$ is continuously differentiable on $\mathbb{R} \setminus \mathcal{R}$. For $\xi \in (-\infty,0) \setminus \mathcal{R}$, we have

$$\begin{array}{rcl}\mathcal{M}^2_\alpha[\widehat{\phi}]'(\xi) & = & \alpha \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) - \sum_{r_j>0} A_j \widehat{\phi}(\xi + r_j)\mathbf{1}_{-r_j<\xi<0} \\ & & - \sum_{s_j^+>0} \int_{\max\{s_j^-,-\xi\}}^{\max\{s_j^+,-\xi\}} B_j(\vartheta)\widehat{\phi}(\xi+\vartheta)\,d\vartheta \\ & & +(L-\alpha)\mathrm{ev}_\xi\Big[\xi' \mapsto \sum_{j=0}^N \big[A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\widehat{\phi}](\xi') + \int_{s_j^-}^{s_j^+} B_j(\vartheta)\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi')\,d\vartheta\big]\Big],\end{array} \tag{5.65}$$

while for $\xi \in (0,\infty) \setminus \mathcal{R}$ we have

$$\begin{array}{rcl}\mathcal{M}^2_\alpha[\widehat{\phi}]'(\xi) & = & \alpha \mathcal{M}^2_\alpha[\widehat{\phi}](\xi) + \sum_{r_j<0} A_j \widehat{\phi}(\xi + r_j)\mathbf{1}_{0<\xi<-r_j} \\ & & + \sum_{s_j^-<0} \int_{\min\{s_j^-,-\xi\}}^{\min\{s_j^+,-\xi\}} B_j(\vartheta)\widehat{\phi}(\xi+\vartheta)\,d\vartheta \\ & & +(L-\alpha)\mathrm{ev}_\xi\Big[\xi' \mapsto \sum_{j=0}^N \big[A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\widehat{\phi}](\xi') + \int_{s_j^-}^{s_j^+} B_j(\vartheta)\mathcal{J}^{(1)}_{\alpha;\vartheta}[\widehat{\phi}](\xi')\,d\vartheta\big]\Big].\end{array} \tag{5.66}$$

In particular, for $r_j < 0$ we have the jump

$$[\mathcal{M}^2_{\alpha;\phi}]'(-r_j^+) - [\mathcal{M}^2_{\alpha;\phi}]'(-r_j^-) = -A_j \widehat{\phi}(0^-), \tag{5.67}$$

while for $r_j > 0$ we have

$$[\mathcal{M}^2_{\alpha;\phi}]'(-r_j^+) - [\mathcal{M}^2_{\alpha;\phi}]'(-r_j^-) = -A_j \widehat{\phi}(0^+). \tag{5.68}$$

In addition, the discontinuity at $\xi = 0$ is given by

$$[\mathcal{M}^2_{\alpha;\phi}]'(0^+) - [\mathcal{M}^2_{\alpha;\phi}]'(0^-) = \sum_{r_j<0} A_j \widehat{\phi}(r_j) + \sum_{r_j>0} A_j \widehat{\phi}(r_j) + \sum_{j=0}^N \int_{s_j^-}^{s_j^+} B_j(\vartheta)\widehat{\phi}(\vartheta)\,d\vartheta. \tag{5.69}$$

Finally, upon introducing the expression

$$\mathcal{H}_{\mathcal{M}^2_\alpha}[\widehat{\phi}](\xi) = \mathcal{M}^2_\alpha[\widehat{\phi}]'(\xi) - L\,\mathrm{ev}_\xi\,\mathcal{M}^2_\alpha[\widehat{\phi}] \tag{5.70}$$

for $\xi \in \mathbb{R} \setminus \mathcal{R}$, we have

$$\begin{array}{rcl}\mathcal{H}_{\mathcal{M}^2_\alpha}[\widehat{\phi}](\xi) & = & -\sum_{j=0}^N (L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\widehat{\phi}](\xi'')]\big] \\ & & - \sum_{j=0}^N (L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto \int_{s_j^-}^{s_j^+} B_j(\vartheta)\mathcal{J}^{(2)}_{\alpha;r_j}[\widehat{\phi}](\xi'')]\big] \\ & & - \sum_{j=0}^N A_j \mathrm{sign}(r_j)\widehat{\phi}(\xi + r_j)\mathbf{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}} \\ & & - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} B_j(\vartheta)\mathrm{sign}(\vartheta)\widehat{\phi}(\xi+\vartheta)\mathbf{1}_{\min\{-\vartheta,0\}<\xi<\max\{-\vartheta,0\}}\,d\vartheta\end{array} \tag{5.71}$$

for all such ξ.

Observe that the right-hand side of (5.71) admits the same jump discontinuities as $\mathcal{M}^2_\alpha[\widehat{\phi}]'$, which are described in (5.67)-(5.69).

Lemma 5.12. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{B}) \to \mathcal{B}$ that satisfies* $(\mathrm{hF})_{L;\mathcal{B}}$, *suppose that $\Delta_L(z) \in \mathcal{L}(\mathcal{B};\mathcal{B})$ is invertible for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z = \eta$ and pick $\alpha > \eta$. Then for any sufficiently small $\epsilon > 0$, the maps*

$$\begin{array}{rcl}\mathcal{B} \ni v & \mapsto & \mathcal{R}^1_\alpha[v] \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}), \\ \mathcal{B} \ni v & \mapsto & \mathcal{R}^1_\alpha[v]' \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B})\end{array} \tag{5.72}$$

together with

$$
\begin{array}{rcl}
L^2([r_{\min}, r_{\max}]; \mathcal{B}) \ni \widehat{\phi} & \mapsto & \mathcal{R}^2_\alpha[\widehat{\phi}] \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B}), \\
L^2([r_{\min}, r_{\max}]; \mathcal{B}) \ni \widehat{\phi} & \mapsto & \mathcal{R}^2_\alpha[\widehat{\phi}]' \in BC_\eta(\mathbb{R};\mathcal{B}) \cap BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B})
\end{array}
\tag{5.73}
$$

are all well-defined and bounded.

Upon fixing $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$ *together with* $v \in \mathcal{B}$ *and writing*

$$
\begin{array}{rcl}
\mathcal{H}_{\mathcal{R}^1_\alpha}[v](\xi) & = & \mathcal{R}^1_\alpha[v]'(\xi) - L\,\mathrm{ev}_\xi\, \mathcal{R}^1_\alpha[v], \\
\mathcal{H}_{\mathcal{R}^2_\alpha}[\widehat{\phi}](\xi) & = & \mathcal{R}^2_\alpha[\widehat{\phi}]'(\xi) - L\,\mathrm{ev}_\xi\, \mathcal{R}^2_\alpha[\widehat{\phi}]
\end{array}
\tag{5.74}
$$

for any $\xi \in \mathbb{R}$*, we have the identity*

$$
\mathcal{H}_{\mathcal{R}^1_\alpha}[v](\xi) \quad = \quad -(L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto \xi'' e^{\alpha\xi''} H(-\xi'')v]\big]
\tag{5.75}
$$

together with

$$
\begin{array}{rcl}
\mathcal{H}_{\mathcal{R}^2_\alpha}[\widehat{\phi}](\xi) & = & \sum_{j=0}^N (L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\widehat{\phi}](\xi'')]\big] \\
& & + \sum_{j=0}^N (L-\alpha)\mathrm{ev}_\xi\big[\xi' \mapsto (L-\alpha)\mathrm{ev}_{\xi'}[\xi'' \mapsto \int_{s_j^-}^{s_j^+} B_j(\vartheta)\mathcal{J}^{(2)}_{\alpha;\vartheta}[\widehat{\phi}](\xi'')\, d\vartheta]\big]
\end{array}
\tag{5.76}
$$

for all $\xi \in \mathbb{R}$*.*

Proof. As a consequence of Cauchy-Schwartz, we can find $\epsilon > 0$ and $K \geq 1$ so that

$$
\left\| \langle (e^{-z^*}, 0), (\widehat{\phi}, 0) \rangle_L \right\|_{\mathcal{B}} \leq K \left\| \widehat{\phi} \right\|_{L^2([r_{\min}, r_{\max}];\mathcal{B})}
\tag{5.77}
$$

holds for all $z \in \mathbb{C}$ with $|\mathrm{Re}\, z - \eta| < \epsilon$. On account of Lemma 5.7, we see that the function

$$
z \quad \mapsto \quad (1 + |z|) \, \| R_{L;\alpha}(z) \|_{\mathcal{L}(\mathcal{B};\mathcal{B})}
\tag{5.78}
$$

is in both $L^1(\eta + i\mathbb{R})$ and $L^1(\eta \pm \epsilon + i\mathbb{R})$. The estimates in Lemma 5.7 also allow us to shift the integration path from $\mathrm{Re}\, z = \eta$ to $\mathrm{Re}\, z = \eta \pm \epsilon$, which guarantees the inclusions (5.72) and (5.73). The identities (5.75) and (5.76) now follow from the computation

$$
\begin{array}{rcl}
\Delta_L(z) R_{L;\alpha}(z) & = & I - \frac{z - Le^{z\cdot}}{z-\alpha} - \frac{(z - Le^{z\cdot})(Le^{z\cdot} - \alpha)}{(z-\alpha)^2} \\
& = & \frac{(Le^{z\cdot} - \alpha)^2}{(z-\alpha)^2}.
\end{array}
\tag{5.79}
$$

□

We have now studied all of the terms in (5.28) in considerable detail. Combining these results, we arrive at the following characterization of the operator $\mathcal{T}_{L;\eta}$.

Proposition 5.13. *Consider a bounded linear operator* $L : C([r_{\min}, r_{\max}]; \mathcal{B}) \to \mathcal{B}$ *that satisfies* $(\mathrm{hF})_{L;\mathcal{B}}$ *and suppose that* $\Delta_L(z) \in \mathcal{L}(\mathcal{B};\mathcal{B})$ *is invertible for all* $z \in \mathbb{C}$ *with* $\mathrm{Re}\, z = \eta$*. Then for any sufficiently small* $\epsilon > 0$*, the map*

$$
L^2([r_{\min}, r_{\max}]; \mathcal{B}) \times \mathcal{B} \ni (\widehat{\phi}, v) \mapsto \mathcal{T}_{L;\eta}[\widehat{\phi}, v] \in BC^-_{\eta+\epsilon}(\mathcal{B}) \cap BC^+_{\eta-\epsilon}(\mathcal{B})
\tag{5.80}
$$

is well-defined and bounded.

Upon fixing $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$ and $v \in \mathcal{B}$, we have the jump discontinuity

$$\mathcal{T}_{L;\eta}[\widehat{\phi}, v](0^+) - \mathcal{T}_{L;\eta}[\widehat{\phi}, v](0^-) = v. \tag{5.81}$$

In addition, $\mathcal{T}_{L;\eta}[\widehat{\phi}, v]$ is continuously differentiable on $\mathbb{R} \setminus \mathcal{R}$. For $r_j < 0$, the derivative has the jump

$$\mathcal{T}'_{L;\eta}[\widehat{\phi}, v](-r_j^+) - \mathcal{T}'_{L;\eta}[\widehat{\phi}, v](-r_j^-) = A_j v - A_j \widehat{\phi}(0^-), \tag{5.82}$$

while for $r_j > 0$ we have

$$\mathcal{T}'_{L;\eta}[\widehat{\phi}, v](-r_j^+) - \mathcal{T}'_{L;\eta}[\widehat{\phi}, v](-r_j^-) = A_j v - A_j \widehat{\phi}(0^+). \tag{5.83}$$

On the other hand, we have

$$\begin{array}{rcl} \mathcal{T}'_{L;\eta}[\widehat{\phi}, v](0^+) - \mathcal{T}'_{L;\eta}[\widehat{\phi}, v](0^-) & = & \sum_{r_j < 0} A_j \widehat{\phi}(r_j) + \sum_{r_j > 0} A_j \widehat{\phi}(r_j) + \sum_{r_j = 0} A_j v \\ & & + \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\vartheta) \, d\vartheta. \end{array} \tag{5.84}$$

Finally, consider for any $\xi \in \mathbb{R} \setminus \mathcal{R}$ the expression

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) = \mathcal{T}_{L;\eta}[\widehat{\phi}, v]'(\xi) - \widehat{L} \, \widehat{\mathrm{ev}}_{\xi} \, \mathcal{T}_{L;\eta}[\widehat{\phi}, v]. \tag{5.85}$$

Then for any $\xi \in (-\infty, 0) \setminus \mathcal{R}$ we have

$$\begin{array}{rcl} \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) & = & - \sum_{r_j > 0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{-r_j < \xi < 0} \\ & & - \sum_{s_j^+ > 0} \int_{\max\{s_j^-, -\xi\}}^{\max\{s_j^+, -\xi\}} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \, d\vartheta, \end{array} \tag{5.86}$$

while for any $\xi \in (0, \infty) \setminus \mathcal{R}$ we have

$$\begin{array}{rcl} \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) & = & + \sum_{r_j < 0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{0 < \xi < -r_j} \\ & & + \sum_{s_j^- < 0} \int_{\min\{s_j^-, -\xi\}}^{\min\{s_j^+, -\xi\}} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \, d\vartheta. \end{array} \tag{5.87}$$

In particular, for $r_j < 0$ we have the jump

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](-r_j^+) - \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](-r_j^-) = -A_j \widehat{\phi}(0^-), \tag{5.88}$$

while for $r_j > 0$ we have the jump

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](-r_j^+) - \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](-r_j^-) = -A_j \widehat{\phi}(0^+). \tag{5.89}$$

In addition, at $\xi = 0$ we have the discontinuity

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](0^+) - \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](0^-) = \sum_{r_j < 0} A_j \widehat{\phi}(r_j) + \sum_{r_j > 0} A_j \widehat{\phi}(r_j) + \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\vartheta) \, d\vartheta. \tag{5.90}$$

Proof. For all $\xi \in \mathbb{R} \setminus \mathcal{R}$ we can compute

$$\begin{array}{rcl} \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) & = & - \sum_{j=0}^{N} A_j \mathrm{sign}(r_j) \widehat{\phi}(\xi + r_j) \mathbf{1}_{\min(-r_j, 0) < \xi < \max(-r_j, 0)} \\ & & - \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \mathrm{sign}(\vartheta) \widehat{\phi}(\xi + \vartheta) \mathbf{1}_{\min(-\vartheta, 0) < \xi < \max(-\vartheta, 0)} \, d\vartheta. \end{array} \tag{5.91}$$

For $\xi \in (0, \infty) \setminus \mathcal{R}$ this reduces to

$$\begin{array}{lcl}\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) & = & \sum_{r_j < 0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{0 < \xi < -r_j} \\ & & \qquad + \sum_{s_j^- < 0} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \mathbf{1}_{0 < \xi < -\vartheta} \, d\vartheta, \end{array} \tag{5.92}$$

while for $\xi \in (-\infty, 0) \setminus \mathcal{R}$ we obtain

$$\begin{array}{lcl}\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v](\xi) & = & - \sum_{r_j > 0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{-r_j < \xi < 0} \\ & & \qquad - \sum_{s_j^+ > 0} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \mathbf{1}_{-\vartheta < \xi < 0} \, d\vartheta. \end{array} \tag{5.93}$$

Further inspection of the integral terms leads to (5.86) and (5.87). □

Formally one can say that the functions $\mathcal{T}_{L;\eta}[\widehat{\phi}, v]'$ and $\mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, v]$ inherit the $\delta(\xi)v$ component at $\xi = 0$ from the functions $\mathcal{M}^1_\alpha[v]'$ and $\mathcal{H}_{\mathcal{M}^1_\alpha}[v]$.

Proof of Lemma 5.1. The statements follow immediately from Proposition 5.13. □

Proof of Lemma 5.2. Since $\mathcal{H}$ is finite-dimensional, the Ascoli-Arzela theorem can be used to obtain the desired compactness properties, exploiting the explicit expressions for $\mathcal{M}^2_\alpha[\widehat{\phi}]$ obtained in Lemma 5.11 together with the observation that

$$\widehat{\phi} \mapsto \mathrm{ev}_0 \mathcal{R}^2_\alpha[\widehat{\phi}]' \tag{5.94}$$

is a bounded map. □

Proof of Proposition 2.2. The Green's function $\widehat{G}_L(\eta)$ can be defined by writing

$$\widehat{G}_L(\eta)v = \mathcal{M}^1_\alpha[v] + \mathcal{R}^1_\alpha[v] = \mathcal{T}_{L;\eta}[0, v]. \tag{5.95}$$

The desired properties all follow directly from Proposition 5.13. □

5.2 Laplace transform

Our goal here is to use the Laplace transform to prove the representations in Proposition 5.3 for functions in $\widehat{P}_L(\eta)$ and $\widehat{Q}_L(\eta)$. For any $\widehat{y} \in \widehat{BC}^{\oplus}(\eta)$ and any $z \in \mathbb{C}$ with $\mathrm{Re}\, z > \eta$, we therefore introduce the Laplace transform

$$\widetilde{y}_+(z) = \int_0^\infty e^{-z\xi} \widehat{y}(\xi) \, d\xi. \tag{5.96}$$

In addition, for any $\widehat{x} \in \widehat{BC}^{\ominus}_\eta$ and any $z \in \mathbb{C}$ with $\mathrm{Re}\, z < \eta$ we write

$$\widetilde{x}_-(z) = \int_0^{-\infty} e^{-z\xi} \widehat{x}(\xi) \, d\xi. \tag{5.97}$$

Lemma 5.14. *Fix $\eta \in \mathbb{R}$ together with a Hilbert space $\mathcal{H}$. Then for any $\widehat{y} \in \widehat{BC}^{\oplus}_\eta(\mathcal{H})$ and $\gamma_+ > \eta$, we have*

$$\widehat{y}(\xi) = \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\gamma_+ - i\Omega}^{\gamma_+ + i\Omega} e^{\xi z} \widetilde{y}_+(z) \, dz \tag{5.98}$$

for almost all $\xi > 0$. *In addition, if* $\widehat{y}'$ *is continuous on* $(0,\infty) \setminus \mathcal{R}$, *then*

$$\int_0^\infty e^{-z\xi}\widehat{y}'(\xi) = z\widetilde{y}_+(z) - \widehat{y}(0^+) \tag{5.99}$$

for all $\mathrm{Re}\, z > \eta$.

On the other hand, for any $\widehat{x} \in \widehat{BC}_\eta^\ominus(\mathcal{H})$ *and* $\gamma_- < \eta$ *we have*

$$\widehat{x}(\xi) = -\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\gamma_- - i\Omega}^{\gamma_- + i\Omega} e^{\xi z}\widetilde{x}_-(z)\, dz \tag{5.100}$$

for almost all $\xi < 0$. *In addition, if* $\widehat{x}'$ *is continuous on* $(-\infty, 0) \setminus \mathcal{R}$, *then*

$$\int_0^{-\infty} e^{-z\xi}\widehat{x}'(\xi) = z\widetilde{x}_-(z) - \widehat{x}(0^-) \tag{5.101}$$

for all $\mathrm{Re}\, z < \eta$.

Proof. The statements for $\widehat{y}$ follow from the pointwise almost-everywhere convergence of inverse Fourier transforms for functions in $L^2(\mathbb{R};\mathcal{H})$; see e.g. [15]. To verify the statements for $\widehat{x}$, assume without loss that $r_{\min} = -r_{\max}$ and write

$$\widehat{w}(\xi) = \widehat{x}(-\xi). \tag{5.102}$$

Note that $\widehat{w} \in \widehat{BC}_{-\eta}^\oplus$, with

$$\begin{array}{rcl} \widetilde{w}_+(z) & = & \int_0^\infty e^{-\xi z}\widehat{w}(\xi)\, d\xi \\ & = & \int_0^\infty e^{-\xi z}\widehat{x}(-\xi)\, d\xi \\ & = & -\int_0^{-\infty} e^{\xi' z}\widehat{x}(\xi')\, d\xi' \\ & = & -\widetilde{x}_-(-z) \end{array} \tag{5.103}$$

after substituting $\xi' = -\xi$. Picking $\gamma_+ > -\eta$, we find that

$$\begin{array}{rcl} \widehat{x}(\xi) & = & w(-\xi) \\ & = & \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\gamma_+ - i\Omega}^{\gamma_+ + i\Omega} e^{-\xi z}\widetilde{w}_+(z)\, dz \\ & = & -\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\gamma_+ - i\Omega}^{\gamma_+ + i\Omega} e^{-\xi z}\widetilde{x}_-(-z)\, dz \\ & = & \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{-\gamma_+ + i\Omega}^{-\gamma_+ - i\Omega} e^{\xi z'}\widetilde{x}_-(z')\, dz' \\ & = & -\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\gamma_- - i\Omega}^{\gamma_- + i\Omega} e^{\xi z'}\widetilde{x}_-(z')\, dz' \end{array} \tag{5.104}$$

for almost all $\xi < 0$, in which we have used $z' = -z$ and $\gamma_- = -\gamma_+ < \eta$.

The expressions (5.99) and (5.101) follow in a standard fashion upon integrating by parts on the intervals where $\widehat{x}'$ and $\widehat{y}'$ are continuous. □

Let us now consider two functions $\widehat{q} \in \widehat{\mathfrak{Q}}_L(\eta)$ and $\widehat{p} \in \widehat{\mathfrak{P}}_L(\eta)$. Taking the appropriate Laplace transforms, we find

$$\begin{array}{rcl} z\widetilde{q}_+(z) - \widehat{q}(0^+) & = & \sum_{j=0}^N \Big[A_j \int_0^\infty e^{-z\xi}\widehat{q}(\xi + r_j)\, d\xi + \int_{s_j^-}^{s_j^+} B_j(\sigma)\int_0^\infty e^{-z\xi}\widehat{q}(\xi+\sigma)\, d\xi\, d\sigma\Big] \\ & = & \sum_{j=0}^N A_j e^{zr_j}\big(\widetilde{q}_+(z) + \int_{r_j}^0 e^{-z\tau}\widehat{q}(\tau)\, d\tau\big) \\ & & \quad + \sum_{j=0}^N \int_{s_j^-}^{s_j^+} B_j(\sigma)e^{z\sigma}\big(\widetilde{q}_+(z) + \int_\sigma^0 e^{-z\tau}\widehat{q}(\tau)\, d\tau\big)\, d\sigma, \end{array} \tag{5.105}$$

together with

$$
\begin{array}{rcl}
z\widetilde{p}_-(z) - \widehat{p}(0^-) & = & \sum_{j=0}^N \Big[A_j \int_0^{-\infty} e^{-z\xi} \widehat{p}(\xi + r_j)\, d\xi + \int_{s_j^-}^{s_j^+} B_j(\sigma) \int_0^{-\infty} e^{-z\xi} \widehat{p}(\xi + \sigma)\, d\xi\, d\sigma \Big] \\
& = & \sum_{j=0}^N A_j e^{z r_j} \big(\widetilde{p}_-(z) + \int_{r_j}^0 e^{-z\tau} \widehat{p}(\tau)\, d\tau \big) \\
& & \quad + \sum_{j=0}^N \int_{s_j^-}^{s_j^+} B_j(\sigma) e^{z\sigma} \big(\widetilde{p}_-(z) + \int_\sigma^0 e^{-z\tau} \widehat{p}(\tau)\, d\tau \big)\, d\sigma.
\end{array}
\tag{5.106}
$$

Rearranging, we obtain

$$
\begin{array}{rcl}
\Delta_L(z) \widetilde{q}_+(z) & = & \widehat{q}(0^+) + \sum_{j=0}^N \big[A_j e^{z r_j} \int_{r_j}^0 e^{-z\tau} \widehat{q}(\tau)\, d\tau + \int_{s_j^-}^{s_j^+} B_j(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \widehat{q}(\tau)\, d\tau\, d\sigma \big] \\
& = & \Big\langle \big(e^{-z^* \cdot}, 1 \big), \big(\widehat{\mathrm{ev}}_0\, \widehat{q}, \widehat{q}(0^+) \big) \Big\rangle_L ,
\end{array}
\tag{5.107}
$$

together with

$$
\begin{array}{rcl}
\Delta_L(z) \widetilde{p}_-(z) & = & \widehat{p}(0^-) + \sum_{j=0}^N \big[A_j e^{z r_j} \int_{r_j}^0 e^{-z\tau} \widehat{p}(\tau)\, d\tau + \int_{s_j^-}^{s_j^+} B_j(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} \widehat{p}(\tau)\, d\tau\, d\sigma \big] \\
& = & \Big\langle \big(e^{-z^* \cdot}, 1 \big), \big(\widehat{\mathrm{ev}}_0\, \widehat{p}, \widehat{p}(0^-) \big) \Big\rangle_L .
\end{array}
\tag{5.108}
$$

Proof of Proposition 5.3. First of all, we note that $\Delta_L(z) = z + O(1)$ as $\mathrm{Im}\, z \to \infty$, uniformly in vertical strips of the complex plane. In particular, there is $\epsilon > 0$ so that $\Delta_L(z)$ is invertible for all z in the vertical strip $|\mathrm{Re}\, z - \eta| < 2\epsilon$.

Possibly excluding a set $\xi \in \mathcal{E}$ of measure zero, the identities (5.9) and (5.10) with $\mathcal{T}_{L;\eta}$ replaced by $\mathcal{T}_{L;\eta+\epsilon}$ respectively $\mathcal{T}_{L;\eta-\epsilon}$ now follow from Lemma 5.14 and the identities (5.107) and (5.108) for $\widetilde{q}_+(z)$ and $\widetilde{p}_-(z)$.

To show that $\mathcal{E} = \emptyset$, we can invoke Lemma 5.1 to argue that the left and right hand sides of (5.9) and (5.10) are both continuous. In addition, the bound

$$
\big\| \Delta_L(z)^{-1} \big\|_{\mathcal{L}(\mathcal{H};\mathcal{H})} = O(|z|^{-1}) \text{ as } \mathrm{Im}\, z \to \infty,
\tag{5.109}
$$

which holds uniformly in vertical strips of the complex plane, implies that the integration paths in (5.98) and (5.100) can both be shifted to the line $\mathrm{Re}\, z = \eta$. □

5.3 Projection operators

The preparations in §5.1 and §5.2 allow us to establish the remaining technical results of this section. In particular, we can use the explicit form of the extension operators $E_{\widehat{Q}_L(\eta)}$ and $E_{\widehat{P}_L(\eta)}$ to show that they play a dual role as the projection operators associated to the desired exponential splitting of the state space $C([r_{\min}, r_{\max}]; \mathcal{H})$.

Proof of Proposition 5.5. Fix a $\widehat{\phi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B})$ and write

$$
\mathcal{H}_{\widehat{Q}_L(\eta)}(\xi) = [E_{\widehat{Q}_L(\eta)} \widehat{\phi}]'(\xi) - \widehat{L}\, \widehat{\mathrm{ev}}_\xi E_{\widehat{Q}_L(\eta)}
\tag{5.110}
$$

for $\xi \in (0, \infty) \setminus \mathcal{R}$, together with

$$
\mathcal{H}_{\widehat{P}_L(\eta)}(\xi) = [E_{\widehat{P}_L(\eta)} \widehat{\phi}]'(\xi) - \widehat{L}\, \widehat{\mathrm{ev}}_\xi E_{\widehat{P}_L(\eta)}
\tag{5.111}
$$

for $\xi \in (-\infty, 0) \setminus \mathcal{R}$. Inspection of the identities (5.12) and (5.87) shows that for all $\xi \in (0, \infty) \setminus \mathcal{R}$ we have

$$\begin{array}{rcl} \mathcal{H}_{\widehat{Q}_L(\eta)}(\xi) & = & \mathcal{H}_{\mathcal{T}_{L;\eta}}[\widehat{\phi}, \widehat{\phi}(0^+)](\xi) \\ & & \qquad - \sum_{r_j<0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{0<\xi<-r_j} - \sum_{s_j^-<0} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \mathbf{1}_{0<\xi<-\vartheta} \, d\vartheta \\ & = & 0, \end{array} \tag{5.112}$$

which together with the bounds in Proposition 5.13 yields the inclusion $E_{\widehat{Q}_L(\eta)}\widehat{\phi} \in \widehat{\mathfrak{Q}}_{L;\mathcal{B}}(\eta)$ and the bounds (5.17). In addition, inspection of (5.13) and (5.86) shows that

$$\begin{array}{rcl} \mathcal{H}_{\widehat{P}_L(\eta)}(\xi) & = & -\mathcal{H}_{\mathcal{T}_\eta}[\widehat{\phi}, \widehat{\phi}(0^-)](\xi) \\ & & \qquad - \sum_{r_j>0} A_j \widehat{\phi}(\xi + r_j) \mathbf{1}_{-r_j<\xi<0} - \sum_{s_j^+>0} \int_{s_j^-}^{s_j^+} B_j(\vartheta) \widehat{\phi}(\xi + \vartheta) \mathbf{1}_{-\vartheta<\xi<0} \, d\vartheta \\ & = & 0, \end{array} \tag{5.113}$$

which now guarantees the inclusion $E_{\widehat{P}_L(\eta)}\widehat{\phi} \in \widehat{\mathfrak{P}}_{L;\mathcal{B}}(\eta)$ together with the bounds (5.18). $\square$

Proof of Proposition 5.6. Combining Corollary 5.4 and Proposition 5.5, we have the characterizations

$$\begin{array}{rcl} P_L(\eta) & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) : \phi = \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi\}, \\ Q_L(\eta) & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathcal{H}) : \phi = \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\phi\}. \end{array} \tag{5.114}$$

This immediately implies that $P_L(\eta)$ and $Q_L(\eta)$ are closed.

In addition, we can inspect (5.12) and (5.13) to find that for any $\phi \in C([r_{\min}, r_{\max}]; \mathcal{H})$ we have

$$\phi(\xi) = \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\phi + \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi. \tag{5.115}$$

This shows that $P_L(\eta) + Q_L(\eta) = C([r_{\min}, r_{\max}]; \mathcal{H})$. In addition, if $\phi \in P_L(\eta) \cap Q_L(\eta)$, this identity gives $\phi = 2\phi$ which is only possible if $\phi = 0$. $\square$

Proof of Theorem 2.1. The statements are a subset of those in Proposition 5.6. $\square$

Proof of Theorem 2.3. The statements follow directly from the fact that for any $v \in \mathcal{H}$, we have $\widehat{G}_L(\eta) \in \widehat{\mathfrak{Q}}_L(\eta)$ and $\widehat{G}_L(\eta) \in \widehat{\mathfrak{P}}_L(\eta)$. In particular, for any $\widehat{q} \in \widehat{Q}_L(\eta)$ we have that

$$\widehat{q} - \widehat{\mathrm{ev}}_0 \, \widehat{G}_L[\widehat{q}(0^+) - \widehat{q}(0^-)] \in Q_L(\eta). \tag{5.116}$$

$\square$

6 Finite dimensional MFDEs

In this section we set out to prove the results in §3.1 concerning the restriction operators (3.5) in the finite dimensional case $\mathcal{H} = \mathbb{R}^n$. In §6.1 we establish Proposition 3.1, developing a technique that exploits our explicit representation of the symbol $\mathcal{T}_{L;\eta}$ introduced in §5. This provides an alternative for the more abstract arguments employed in [20]. In §6.2 we investigate the non-degeneracy of the Hale inner product and use it to establish the characterizations of $Q_L(\eta)$ and $P_L(\eta)$ stated in Proposition 3.4. We also prove Theorem 3.5, up to an index formula that mixes L with its formal adjoint L_*. This formula is derived in §6.3, where we exploit the ideas introduced in [20] for non-autonomous MFDEs.

6.1 Fredholm properties

It turns out that the desired Fredholm properties for the restriction operators (3.5) all follow easily from the corresponding properties for

$$\widehat{\pi}^-_{\widehat{Q}_L(\eta)} : \widehat{Q}_L(\eta) \to C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n, \qquad \widehat{\pi}^+_{\widehat{P}_L(\eta)} : \widehat{P}_L(\eta) \to \mathbb{R}^n \times C([0, r_{\max}]; \mathbb{R}^n). \tag{6.1}$$

We hence focus on these two operators here. For convenience, we recall the definitions

$$\begin{array}{lcl} \widehat{R}^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Range}\big(\widehat{\pi}^-_{\widehat{Q}_L}(\eta)\big) \subset C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n, \\ \widehat{K}^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Ker}\big(\widehat{\pi}^-_{\widehat{Q}_L}(\eta)\big) \subset \widehat{Q}_L(\eta) \end{array} \tag{6.2}$$

together with

$$\begin{array}{lcl} \widehat{R}^+_{\widehat{P}_L(\eta)} & = & \mathrm{Range}\big(\widehat{\pi}^+_{\widehat{P}_L}(\eta)\big) \subset \mathbb{R}^n \times C([0, r_{\max}]; \mathbb{R}^n), \\ \widehat{K}^+_{\widehat{P}_L(\eta)} & = & \mathrm{Ker}\big(\widehat{\pi}^+_{\widehat{P}_L}(\eta)\big) \subset \widehat{P}_L(\eta). \end{array} \tag{6.3}$$

Lemma 6.1. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies* $(\mathrm{HF})_L$. *Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the kernels $\widehat{K}^+_{\widehat{P}_L(\eta)}$ and $\widehat{K}^-_{\widehat{Q}_L(\eta)}$ are both finite dimensional and the ranges $\widehat{R}^+_{\widehat{P}_L(\eta)}$ and $\widehat{R}^-_{\widehat{Q}_L(\eta)}$ are both closed.*

Proof. We focus here on $\widehat{\pi}^-_{\widehat{Q}_L(\eta)}$, noting that the statements for $\widehat{\pi}^+_{\widehat{P}_L(\eta)}$ follow analogously. For convenience, we introduce the notation

$$\mathrm{ev}_0^- = \pi^- \widehat{\mathrm{ev}}_0, \qquad \mathrm{ev}_0^+ = \pi^+ \widehat{\mathrm{ev}}_0, \tag{6.4}$$

together with $\widehat{\phi} = (\phi^-, \phi^+)$ for $\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)$ and

$$\phi^- = \mathrm{ev}_0^- \widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n), \qquad \phi^+ = \mathrm{ev}_0^+ \widehat{\phi} \in C([0, r_{\max}]; \mathbb{R}^n). \tag{6.5}$$

In such cases we often replace the expression $\mathcal{T}_{L;\eta}[\widehat{\phi}, v]$ defined in (5.3) by $\mathcal{T}_{L;\eta}[\phi^-, \phi^+, v]$ for explicitness.

Inspection of (5.12) shows that $\widehat{q} = (q^-, q^+) \in \widehat{Q}_L(\eta)$ if and only if

$$q^- \quad = \quad q^- + \mathrm{ev}_0^- \mathcal{T}_{L;\eta}[q^-, q^+, q^+(0)] \tag{6.6}$$

holds, together with

$$q^+ \quad = \quad \mathrm{ev}_0^+ \mathcal{T}_{L;\eta}[q^-, q^+, q^+(0)]. \tag{6.7}$$

Observe that (6.7) is equivalent to

$$\big[I - \mathrm{ev}_0^+ \mathcal{T}_{L;\eta}[0, \cdot, 0]\big](q^+) = \mathrm{ev}_0^+ \mathcal{T}_{L;\eta}[q^-, 0, q^+(0)]. \tag{6.8}$$

Since $\mathrm{ev}_0^+ \mathcal{T}_{L;\eta}$ is a compact operator by Lemma 5.2, we see that the bounded linear map

$$\big[I - \mathrm{ev}_0^+ \mathcal{T}_{L;\eta}[0, \cdot, 0]\big] : C([0, r_{\max}]; \mathbb{R}^n) \to C([0, r_{\max}]; \mathbb{R}^n) \tag{6.9}$$

is Fredholm. In particular, for some integer $d \geq 0$ there exist compact bounded linear operators

$$L_1 : C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n \to C([0, r_{\max}]; \mathbb{R}^n), \qquad L_2 : \mathbb{R}^d \to C([0, r_{\max}]; \mathbb{R}^n) \tag{6.10}$$

so that a pair $\widehat{q} = (q^-, q^+) \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)$ satisfies (6.7) if and only if

$$q^+ = L_1\big(q^-, q^+(0)\big) + L_2\kappa \tag{6.11}$$

holds for some $\kappa \in \mathbb{R}^d$.

In summary, we see that $\widehat{q} = (q^-, q^+) \in \widehat{Q}_L(\eta)$ if and only if there exists $\kappa \in \mathbb{R}^d$ so that

$$\begin{array}{lcl} q^+ & = & L_1\big(q^-, q^+(0)\big) + L_2\kappa, \\ 0 & = & \mathrm{ev}_0^- \mathcal{T}_{L;\eta}[q^-, L_1\big(q^-, q^+(0)\big) + L_2\kappa, q^+(0)] \end{array} \tag{6.12}$$

both hold.

We now introduce the bounded linear operator $L_3 : \mathbb{R}^d \to C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n$ that acts as

$$L_3\kappa = \Big(\mathrm{ev}_0^- \mathcal{T}_{L;\eta}[0, L_2\kappa, 0], [L_2\kappa](0)\Big). \tag{6.13}$$

Write $K_{L_3} = \mathrm{Ker}\big(L_3\big)$ and pick a subspace $K_{L_3}^\perp \subset \mathbb{R}^d$ so that

$$\mathbb{R}^d = K_{L_3} \oplus K_{L_3}^\perp. \tag{6.14}$$

By inspection we readily obtain the characterization

$$\widehat{K}^-_{\widehat{Q}_L(\eta)} = \mathrm{Ker}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}) = \big(0, L_2(K_{L_3})\big). \tag{6.15}$$

In addition, for any $(\phi^-, v) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n$ we see that $(\phi^-, v) \in \widehat{R}^-_{\widehat{Q}_L(\eta)}$ if and only if

$$\begin{array}{lcl} \mathrm{ev}_0^- \mathcal{T}_{L;\eta}[\phi^-, L_1(\phi^-, v) + L_2\kappa^\perp, v] & = & 0, \\ \big[L_1(\phi^-, v) + L_2\kappa^\perp\big](0) & = & v \end{array} \tag{6.16}$$

both hold for some $\kappa^\perp \in K_{L_3}^\perp$. In particular, we then have

$$(\phi^-, v) = \widehat{\pi}^- \, \widehat{q} \tag{6.17}$$

with

$$\widehat{q} = \big(\phi^-, L_1(q^-, v) + L_2\kappa^\perp\big) \in \widehat{Q}_L(\eta). \tag{6.18}$$

Suppose now that (6.16) holds for $\kappa^\perp = \kappa_1^\perp \in K_{L_3}^\perp$ and also $\kappa^\perp = \kappa_2^\perp \in K_{L_3}^\perp$. Inspecting (6.13) we then see $L_3(\kappa_1^\perp) = L_3(\kappa_2^\perp)$, which implies that $\kappa_1^\perp - \kappa_2^\perp \in K_{L_3}$ and hence $\kappa_1^\perp = \kappa_2^\perp$. In particular, if it exists, $\kappa^\perp \in K_{L_3}^\perp$ in (6.16) depends uniquely and linearly on (ϕ^-, v).

We now claim that this dependence is also bounded. To see this, consider a sequence $\{\phi_j^-, v_j, \kappa_j^\perp\}$ of solutions to (6.16) with $\phi_j^- \in C([r_{\min}, 0]; \mathbb{R}^n)$, $v_j \in \mathbb{R}^n$ and $\kappa_j^\perp \in K_{L_3}^\perp$. Suppose also that $\big\|\phi_j^-\big\| + |v_j| = 1$. It now suffices to show that $\kappa_j^\perp$ is bounded. Supposing to the contrary that $\big|\kappa_j^\perp\big| \to \infty$, there exists a non-zero $\kappa_* \in K_{L_3}^\perp$ so that a subsequence of the bounded set $\{\big|\kappa_j^\perp\big|^{-1} \kappa_j^\perp\} \subset K_{L_3}^\perp$ converges to $\kappa_* \in K_{L_3}^\perp$, while $(0, 0, \kappa_*)$ is a solution to (6.16). This contradicts the uniqueness claim above, as $(0, 0, 0)$ also satisfies (6.16).

To see that $\widehat{R}^-_{\widehat{Q}_L(\eta)}$ is closed, consider a sequence $\{(\phi_j, v_j)\} \in \widehat{R}^-_{\widehat{Q}_L(\eta)}$ for which

$$(\phi_j, v_j) \to (\phi_*, v_*) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n. \tag{6.19}$$

The discussion above allows us to pick a bounded sequence $\{\kappa_j^\perp\}_{j=1}^\infty \subset K_{L_3}^\perp$ in such a way that $(\phi_j, v_j, \kappa_j^\perp)$ satisfies (6.16). After passing to a subsequence, we have the convergence $\kappa_j^\perp \to \kappa_*^\perp$, which by continuity implies that $(\phi_*, v_*, \kappa_*^\perp)$ also satisfies (6.16). This shows that $(\phi_*, v_*) \in \widehat{R}^-_{\widehat{Q}_L(\eta)}$. □

The result above implies that we can find spaces

$$\widehat{T}^+_{\widehat{P}_L(\eta)} \subset \widehat{P}_L(\eta), \qquad \widehat{T}^-_{\widehat{Q}_L(\eta)} \subset \widehat{Q}_L(\eta) \tag{6.20}$$

so that we have the decompositions

$$\widehat{P}_L(\eta) = \widehat{K}^+_{\widehat{P}_L(\eta)} \oplus \widehat{T}^+_{\widehat{P}_L(\eta)}, \qquad \widehat{Q}_L(\eta) = \widehat{K}^-_{\widehat{Q}_L(\eta)} \oplus \widehat{T}^-_{\widehat{Q}_L(\eta)}. \tag{6.21}$$

The maps

$$\widehat{\pi}^+_{\widehat{P}_L(\eta)} : \widehat{T}^+_{\widehat{P}_L(\eta)} \to \widehat{R}^+_{\widehat{P}_L(\eta)}, \qquad \widehat{\pi}^-_{\widehat{Q}_L(\eta)} : \widehat{T}^-_{\widehat{Q}_L(\eta)} \to \widehat{R}^-_{\widehat{Q}_L(\eta)} \tag{6.22}$$

are now invertible with bounded inverses.

Lemma 6.2. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the operators*

$$\begin{array}{rcl} \pi^-[\widehat{\pi}^+_{\widehat{P}_L(\eta)}]^{-1} & : & \widehat{R}^+_{\widehat{P}_L(\eta)} \to C([r_{\min}, 0]; \mathbb{R}^n), \\ \pi^+[\widehat{\pi}^-_{\widehat{Q}_L(\eta)}]^{-1} & : & \widehat{R}^-_{\widehat{Q}_L(\eta)} \to C([0, r_{\max}]; \mathbb{R}^n) \end{array} \tag{6.23}$$

are both compact.

Proof. The result follows by inspection of (6.17)-(6.18), recalling from the proof of Lemma 6.1 that $\kappa^\perp$ is a bounded linear function of (q^-, v) and that L_1 and L_2 are compact operators. □

Lemma 6.3. *Suppose that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are three Banach spaces and that $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{Y} \to \mathcal{Z}$ are Fredholm operators. Then $BA : \mathcal{X} \to \mathcal{Z}$ is also Fredholm with*

$$\mathrm{ind}(BA) = \mathrm{ind}(A) + \mathrm{ind}(B). \tag{6.24}$$

In particular, if $\mathcal{X} \subset \mathcal{Y}$ is a closed subspace of finite codimension, then the restriction $B_{|\mathcal{X}} : \mathcal{X} \to \mathcal{Z}$ is Fredholm with

$$\mathrm{ind}(B_{|\mathcal{X}}) = \mathrm{ind}(B) - \mathrm{codim}_{\mathcal{Y}}(\mathcal{X}). \tag{6.25}$$

Proof. See [20, Eq. (3.10)]. □

In order to obtain information concerning the codimensions of $\widehat{R}^+_{\widehat{P}_L(\eta)}$ and $\widehat{R}^-_{\widehat{Q}_L(\eta)}$, we adapt a technique developed by Mallet-Paret and Verduyn-Lunel in the proof of [20, Thm. 3.4].

Lemma 6.4. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the inclusions*

$$\widehat{R}^+_{\widehat{P}_L(\eta)} \subset \mathbb{R}^n \times C([0, r_{\max}]; \mathbb{R}^n), \qquad \widehat{R}^-_{\widehat{Q}_L(\eta)} \subset C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n \tag{6.26}$$

both have finite codimension, which means that the two operators in (6.1) are both Fredholm. In addition, we have the identity

$$\mathrm{ind}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}) + \mathrm{ind}(\widehat{\pi}^+_{\widehat{P}_L(\eta)}) = -n. \tag{6.27}$$

Proof. Consider the Banach space

$$\mathcal{X} = C([r_{\min}, r_{\max}]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \tag{6.28}$$

together with the bounded linear map

$$\mathcal{J}_* : \widehat{Q}_L(\eta) \oplus \widehat{P}_L(\eta) \to \mathcal{X} \tag{6.29}$$

that acts as

$$\begin{aligned} \mathcal{J}_*(\widehat{q}, 0) &= \Big(\widehat{q} - \widehat{G}_L(\eta)[\widehat{q}(0^+) - \widehat{q}(0^-)], [\widehat{q}(0^+) - \widehat{q}(0^-)], 0\Big), \\ \mathcal{J}_*(0, \widehat{p}) &= \Big(\widehat{p} - \widehat{G}_L(\eta)[\widehat{p}(0^+) - \widehat{p}(0^-)], 0, [\widehat{p}(0^+) - \widehat{p}(0^-)]\Big). \end{aligned} \tag{6.30}$$

Exploiting the fact that $P_L(\eta) \oplus Q_L(\eta) = C([r_{\min}, r_{\max}]; \mathbb{R}^n)$, the representations (2.32) imply that $\mathcal{J}_*$ is a bijection.

We now introduce the bounded linear map

$$T : \widehat{R}^-_{\widehat{Q}_L(\eta)} \times \widehat{R}^+_{\widehat{P}_L(\eta)} \to \mathcal{X} \tag{6.31}$$

that acts as

$$T\big((\phi^-, v), (w, \phi^+)\big) = \mathcal{J}_*\Big([\widehat{\pi}^-_{\widehat{Q}_L(\eta)}]^{-1}(\phi^-, v), [\widehat{\pi}^+_{\widehat{P}_L(\eta)}]^{-1}(w, \phi^+)\Big). \tag{6.32}$$

We see that T is injective because $\mathcal{J}_*$ and the inverses $[\widehat{\pi}^-_{\widehat{Q}_L(\eta)}]^{-1}$ and $[\widehat{\pi}^+_{\widehat{P}_L(\eta)}]^{-1}$ are all injective. In particular, the Fredholm index of T is

$$\mathrm{ind}(T) = -\mathrm{codim}\,\mathrm{Range}(T) = -\dim \widehat{K}^-_{\widehat{Q}_L(\eta)} - \dim \widehat{K}^+_{\widehat{P}_L(\eta)}. \tag{6.33}$$

Introducing the Banach space

$$\mathcal{Y} = C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \tag{6.34}$$

together with the operators

$$\mathcal{I}_1 : \mathcal{X} \to \mathcal{Y}, \qquad \mathcal{I}_2 : \mathcal{X} \to \mathcal{Y} \tag{6.35}$$

that act as

$$\begin{aligned} \mathcal{I}_1(\phi, v, w) &= \Big(\mathrm{ev}_0^- \phi, \mathrm{ev}_0^+ \phi, v, w\Big), \\ \mathcal{I}_2(\phi, v, w) &= \mathcal{I}_1(\phi, v, w) + \Big(\big[\mathrm{ev}_0^- \widehat{G}_L(\eta)\big](v + w), \big[\mathrm{ev}_0^+ \widehat{G}_L(\eta)\big](v + w), 0, 0\Big), \end{aligned} \tag{6.36}$$

we see that $\mathcal{I}_2 - \mathcal{I}_1$ is compact and that

$$\mathcal{I}_2(\phi, v, w) = \Big(\mathrm{ev}_0^- \phi + \big[\mathrm{ev}_0^- \widehat{G}_L(\eta)\big](v + w), \mathrm{ev}_0^+ \phi + \big[\mathrm{ev}_0^+ \widehat{G}_L(\eta)\big](v + w), v, w\Big). \tag{6.37}$$

In particular, since $\mathcal{I}_1$ is injective and $\mathcal{I}_1(\mathcal{X})$ has codimension n in $\mathcal{Y}$, we see that

$$\mathrm{ind}\big(\mathcal{I}_2\big) = \mathrm{ind}\big(\mathcal{I}_1\big) = -n. \tag{6.38}$$

Notice that

$$\begin{aligned} \mathcal{I}_2 \mathcal{J}_*(\widehat{q}, 0) &= \Big(\mathrm{ev}_0^- \,\widehat{q}, \mathrm{ev}_0^+ \,\widehat{q}, \big(\widehat{q}(0^+) - \widehat{q}(0^-)\big), 0\Big), \\ \mathcal{I}_2 \mathcal{J}_*(0, \widehat{p}) &= \Big(\mathrm{ev}_0^- \,\widehat{p}, \mathrm{ev}_0^+ \,\widehat{p}, 0, \big(\widehat{p}(0^+) - \widehat{p}(0^-)\big)\Big), \end{aligned} \tag{6.39}$$

which yields

$$\begin{array}{rcl}\mathcal{I}_2 T\Big((\phi^-, v), (w, \phi^+)\Big) & = & \Big(\phi^- + \pi^-[\widehat{\pi}^+_{\widehat{P}_L(\eta)}]^{-1}(w, \phi^+), \phi^+ + \pi^+[\widehat{\pi}^-_{\widehat{Q}_L(\eta)}]^{-1}(\phi^-, v), \\ & & \qquad v - \phi^-(0), \phi^+(0) - w\Big).\end{array} \tag{6.40}$$

Let us rewrite this as

$$\mathcal{I}_2 T\Big((\phi^-, v), (w, \phi^+)\Big) \quad = \quad L_4[(\phi^-, v), (w, \phi^+)] + L_5[(\phi^-, v), (w, \phi^+)] \tag{6.41}$$

for two operators

$$L_4 : \widehat{R}_{\widehat{Q}_L(\eta)} \times \widehat{R}_{\widehat{P}_L(\eta)} \to \mathcal{Y}, \qquad L_5 : \widehat{R}_{\widehat{Q}_L(\eta)} \times \widehat{R}_{\widehat{P}_L(\eta)} \to \mathcal{Y}, \tag{6.42}$$

in which L_4 acts as

$$L_4[(\phi^-, v), (w, \phi^+)] = \big(\phi^-, \phi^+, v, w\big). \tag{6.43}$$

Inspecting (6.40) and applying Lemma 6.2, we see that L_5 is a compact operator. In addition, it follows from (6.43) that L_4 is injective with Fredholm index

$$\mathrm{ind}(L_4) = -\mathrm{codim}\, \widehat{R}^-_{\widehat{Q}_L(\eta)} - \mathrm{codim}\, \widehat{R}^+_{\widehat{P}_L(\eta)}. \tag{6.44}$$

In particular, we may compute

$$\mathrm{ind}(T) - n = \mathrm{ind}\big(\mathcal{I}_2 T\big) = \mathrm{ind}(L_4) = -\mathrm{codim}\, \widehat{R}^-_{\widehat{Q}_L(\eta)} - \mathrm{codim}\, \widehat{R}^+_{\widehat{P}_L(\eta)}. \tag{6.45}$$

Comparing (6.33) with (6.45), we find

$$\dim \widehat{K}^-_{\widehat{Q}_L(\eta)} + \dim \widehat{K}^+_{\widehat{P}_L(\eta)} = -\mathrm{ind}(T) = -n + \mathrm{codim}\, \widehat{R}^-_{\widehat{Q}_L(\eta)} + \mathrm{codim}\, \widehat{R}^+_{\widehat{P}_L(\eta)} \tag{6.46}$$

and so

$$\mathrm{ind}\big(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}\big) + \mathrm{ind}\big(\widehat{\pi}^+_{\widehat{P}_L(\eta)}\big) = -n. \tag{6.47}$$

□

Lemma 6.5. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the operators*

$$\begin{array}{rcl}\pi^-_{\widehat{Q}_L(\eta)} : \widehat{Q}_L(\eta) & \to & C([r_{\min}, 0]; \mathbb{R}^n), \\ \pi^-_{Q_L(\eta)} : Q_L(\eta) & \to & C([r_{\min}, 0]; \mathbb{R}^n)\end{array} \tag{6.48}$$

and

$$\begin{array}{rcl}\pi^+_{\widehat{P}_L(\eta)} : \widehat{P}_L(\eta) & \to & C([0, r_{\max}]; \mathbb{R}^n), \\ \pi^+_{P_L(\eta)} : P_L(\eta) & \to & C([0, r_{\max}]; \mathbb{R}^n)\end{array} \tag{6.49}$$

are all Fredholm. In addition, we have the identities

$$\mathrm{ind}(\pi^-_{Q_L(\eta)}) = \mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) - n = \mathrm{ind}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}), \tag{6.50}$$

together with

$$\mathrm{ind}(\pi^+_{P_L(\eta)}) = \mathrm{ind}(\pi^+_{\widehat{P}_L(\eta)}) - n = \mathrm{ind}(\widehat{\pi}^+_{\widehat{P}_L(\eta)}). \tag{6.51}$$

Proof. Note first that the projection

$$\pi_1 : C([r_{\min},0];\mathbb{R}^n)\times\mathbb{R}^n \to C([r_{\min},0];\mathbb{R}^n), \qquad (\phi,v)\mapsto\phi \tag{6.52}$$

is Fredholm with index n. Since $\pi^-_{\widehat{Q}_L(\eta)} = \pi_1\widehat{\pi}^-_{\widehat{Q}_L(\eta)}$, Lemma 6.3 implies that $\pi^-_{\widehat{Q}_L(\eta)}$ is Fredholm with index

$$\mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) = n + \mathrm{ind}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}). \tag{6.53}$$

In addition, since the inclusion $Q_L(\eta)\subset\widehat{Q}_L(\eta)$ has codimension n, we can again use Lemma 6.3 to conclude that

$$\pi^-_{Q_L(\eta)} = [\pi^-_{\widehat{Q}_L(\eta)}]_{|Q_L(\eta)} \tag{6.54}$$

is Fredholm with index

$$\mathrm{ind}(\pi^-_{Q_L(\eta)}) = \mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) - n. \tag{6.55}$$

The statements concerning $P_L(\eta)$ and $\widehat{P}_L(\eta)$ follow in a similar fashion. □

Proof of Proposition 3.1. The statements follow directly from Lemma's 6.4 and 6.5. □

6.2 The Hale inner product

Our first focus here is the non-degeneracy of the Hale inner product. We cannot directly follow the approach in [20, §5] because $(\mathrm{HRnk})_L$ is weaker then the atomicity condition employed there. As a preparation, we need to rule out non-zero elements of $\mathfrak{P}_L(\eta)$ and $\mathfrak{Q}_L(\eta)$ that decay at a rate faster than any exponential.

Lemma 6.6. *Write $\mathcal{H}=\mathbb{R}^n$ for some integer $n\geq 1$ and consider a bounded linear operator $L: C([r_{\min},r_{\max}];\mathcal{H})\to\mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HRnk})_L$. Then we have the trivial intersections*

$$\bigcap_{\eta\in\mathbb{R}} P_L(\eta) = \{0\}, \qquad \bigcap_{\eta\in\mathbb{R}} Q_L(\eta) = \{0\}. \tag{6.56}$$

Proof. Consider any $q\in\bigcap_{\eta\in\mathbb{R}} Q_L(\eta)$. By uniqueness of extensions there is a

$$y_q \in \bigcap_{\eta\in\mathbb{R}} BC^{\oplus}_{\eta}(\mathbb{R}^n) \tag{6.57}$$

for which $y_q'(\xi) = L\mathrm{ev}_\xi\, y_q$ holds for all $\xi\geq 0$. In particular, we have $\lim_{\xi\to\infty} e^{\eta\xi}y_q(\xi)=0$ for all $\eta\in\mathbb{R}$. The proof of [20, Lem 5.6] can be repeated to show that one must have $y_q=0$. Indeed, this proposition uses an atomicity condition that is stricter than $(\mathrm{HRnk})_L$. However, this stricter condition [20, Eq. (2.3)] is only used to verify the conditions associated with a Phragmén-Lindelöf theorem [17, Thm. I.21]. This theorem asserts that entire functions that grow at most exponentially on $\mathbb{C}$ and polynomially on the real and imaginary axes, are in fact polynomials. Allowing $s_\pm>0$ in (3.29)-(3.30) does not destroy these required growth estimates. □

Our explicit characterization of $P_L(\eta)$ and $Q_L(\eta)$ in §5 is the key that allows us to exploit the result above to obtain the non-degeneracy of the Hale inner product.

Lemma 6.7. *Write $\mathcal{H}=\mathbb{R}^n$ for some integer $n\geq 1$ and consider a bounded linear operator $L: C([r_{\min},r_{\max}];\mathcal{H})\to\mathcal{H}$ that both satisfies $(\mathrm{HF})_L$ and $(\mathrm{HRnk})_L$. Then the Hale inner product is nondegenerate, in the sense that $\phi=0$ is the only $\phi\in C([r_{\min},r_{\max}];\mathbb{R}^n)$ for which*

$$\langle\psi,\phi\rangle_L = 0 \tag{6.58}$$

holds for all $\psi\in C([-r_{\max},-r_{\min}];\mathbb{R}^n)$.

Proof. For any $\phi \in C([r_{\min}, r_{\max}];\mathbb{R}^n)$ that satisfies the conditions in the statement, we see that

$$\langle e^{-z^* \cdot}, \phi\rangle = \Big\langle \big(e^{-z^* \cdot}, 1\big), \big(\phi, \phi(0^-)\big)\Big\rangle_L = \Big\langle \big(e^{-z^* \cdot}, 1\big), \big(\phi, \phi(0^+)\big)\Big\rangle_L = 0 \tag{6.59}$$

for every $z \in \mathbb{C}$. In particular, if the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$, then we have $\mathcal{T}_{L;\eta}[\phi, \phi(0)] = 0$. In view of Proposition 5.5, this means that

$$\mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\phi = (\mathrm{ev}_0^- \phi, 0) \in Q_L(\eta), \qquad \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi = (0, \mathrm{ev}_0^+ \phi) \in P_L(\eta) \tag{6.60}$$

for all such η. When $\eta_1 < \eta_2$ we naturally have the inclusions

$$Q_L(\eta_1) \subset Q_L(\eta_2), \qquad P_L(\eta_2) \subset P_L(\eta_1), \tag{6.61}$$

which means that (6.60) in fact holds for all $\eta \in \mathbb{R}$. In particular, we see that

$$(\mathrm{ev}_0^- \phi, 0) \in \bigcap_{\eta\in\mathbb{R}} Q_L(\eta), \qquad (0, \mathrm{ev}_0^+ \phi) \in \bigcap_{\eta\in\mathbb{R}} P_L(\eta), \tag{6.62}$$

which in view of Lemma 6.6 implies that $\phi = 0$. □

Proof of Proposition 3.3. The statements follow by applying Lemma 6.7 to the operators L, $L_{>0}$ and $L_{<0}$. □

The following technical result clarifies the relation between L and L_* induced by the Hale inner product. For convenience, we introduce the notation $\widehat{\mathrm{ev}}_\xi^*$ to refer to the evaluation operator that arises by making the substitutions $r_{\min} \mapsto -r_{\max}$ and $r_{\max} \mapsto -r_{\min}$ in the definition (2.21). This accounts for the fact that the natural statespace for L_* is $C([-r_{\max}, -r_{\min}];\mathbb{R}^n)$.

Lemma 6.8. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies $(\mathrm{HF})_L$. Consider two functions*

$$\widehat{y} \in C([r_{\min}, 0];\mathbb{R}^n) \times C\big([0,\infty);\mathbb{R}^n\big), \qquad \widehat{z} \in C([-r_{\max}, 0];\mathbb{R}^n) \times C\big([0,\infty);\mathbb{R}^n\big) \tag{6.63}$$

and write $H_{\widehat{y},\widehat{z}} : [0,\infty) \to \mathbb{R}$ for the function defined by

$$H_{\widehat{y},\widehat{z}}(\xi) = \Big\langle \big(\widehat{\mathrm{ev}}_\xi^* \,\widehat{z}, \widehat{z}(\xi^+)\big), \big(\widehat{\mathrm{ev}}_\xi \,\widehat{y}, \widehat{y}(\xi^+)\big)\Big\rangle_L. \tag{6.64}$$

Then $H_{\widehat{y},\widehat{z}}$ is continuous on $[0,\infty)$. In addition, if $\widehat{y}$ and $\widehat{z}$ are also in $C^1\big((0,\infty)\setminus\mathcal{R};\mathbb{R}^n\big)$, then in fact $H_{\widehat{y},\widehat{z}}$ is differentiable whenever $\xi \in (0,\infty)\setminus\mathcal{R}$. For any such ξ, we have

$$H'_{\widehat{y},\widehat{z}}(\xi) = \widehat{z}(\xi)^*[\widehat{y}'(\xi) - \widehat{L}\,\widehat{\mathrm{ev}}_\xi\,\widehat{y}] + [\widehat{z}'(\xi) - \widehat{L}_*\,\widehat{\mathrm{ev}}_\xi^*\,\widehat{z}]^*\widehat{y}(\xi). \tag{6.65}$$

Proof. We note that for $\xi \geq 0$ we have

$$\begin{array}{lcl} H_{\widehat{y},\widehat{z}}(\xi) & = & \widehat{z}(\xi^+)^*\widehat{y}(\xi^+) - \sum_{j=0}^N \int_\xi^{\xi+r_j} \widehat{z}(\tau - r_j)^* A_j \widehat{y}(\tau)\,d\tau \\ & & \qquad - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} \int_\xi^{\xi+\sigma} \widehat{z}(\tau-\sigma)^* B_j(\sigma)\widehat{y}(\tau)\,d\tau\,d\sigma, \end{array} \tag{6.66}$$

which implies that $H_{\widehat{y},\widehat{z}}$ is indeed continuous on $[0,\infty)$. In addition, for very $\xi \in (0,\infty)\setminus\mathcal{R}$ we may compute

$$\begin{array}{lcl} H'_{\widehat{y},\widehat{z}}(\xi) & = & \widehat{z}'(\xi)^*\widehat{y}(\xi) + \widehat{z}(\xi)^*\widehat{y}'(\xi) \\ & & \qquad - \sum_{j=0}^N \big[\widehat{z}(\xi)^* A_j \widehat{y}(\xi + r_j) - \widehat{z}(\xi - r_j)^* A_j \widehat{y}(\xi)\big] \\ & & \qquad - \sum_{j=0}^N \int_{s_j^-}^{s_j^+} \Big(\widehat{z}(\xi)^* B_j(\sigma)\widehat{y}(\xi+\sigma) - \widehat{z}(\xi-\sigma)^* B_j(\sigma)\widehat{y}(\xi)\Big)\,d\sigma, \end{array} \tag{6.67}$$

which reduces to the desired expression. □

Using the non-degeneracy of the Hale inner product, we can establish the representations (3.36)-(3.37) in Proposition 3.4. In order to appreciate the exponents $-\eta$ appearing in these expressions, we note that

$$\begin{array}{rcl} \Delta_{L_*}(z) & = & z + \sum_{j=0}^{N} A_j^* e^{-z r_j} + \int_{-r_{\max}}^{-r_{\min}} B(-\sigma)^* e^{z\sigma} d\sigma \\ & = & z + \sum_{j=0}^{N} A_j^* e^{-z r_j} + \int_{r_{\min}}^{r_{\max}} B(\sigma)^* e^{-z\sigma} d\sigma \\ & = & \big[z^* + \sum_{j=0}^{N} A_j e^{-z^* r_j} + \int_{r_{\min}}^{r_{\max}} B(\sigma) e^{-z^*\sigma} d\sigma \big]^* \\ & = & -\Delta_L(-z^*)^*. \end{array} \tag{6.68}$$

Proof of Proposition 3.4. Applying Lemma 6.8 with $\widehat{z} = E_{\widehat{Q}_{L_*}(-\eta)} \widehat{\psi}$ and $\widehat{y} = E_{\widehat{Q}_L(\eta)} \widehat{\phi}$ shows that (3.37) holds if one replaces the equality signs by the left inclusion $\subset$. In order to show the identities for $Q_L(\eta)$ in (3.37), it hence suffices to show that the inclusions

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset Q_L(\eta) \tag{6.69}$$

hold for the sets

$$\begin{array}{rcl} \mathcal{S}_1 & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\phi, \phi(0)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{Q}_{L_*}(-\eta)\}, \\ \mathcal{S}_2 & = & \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \langle \psi, \phi \rangle_L = 0 \text{ for all } \psi \in Q_{L_*}(-\eta)\}. \end{array} \tag{6.70}$$

The first inclusion is a consequence of $Q_{L_*}(-\eta) \subset \widehat{Q}_{L_*}(-\eta)$. For any $\phi \in \mathcal{S}_2$, we may write $\phi = p + q$ with $p \in P_L(\eta)$ and $q \in Q_L(\eta)$. By the remarks above for (3.37) we know that $Q_L(\eta) \subset \mathcal{S}_2$, which implies by linearity that also $p = \phi - q \in \mathcal{S}_2$. The same remarks but now applied to $P_L(\eta)$ and (3.36) show that $\langle p, \psi \rangle_L = 0$ for all $\psi \in P_{L_*}(-\eta)$. Since $Q_{L_*}(-\eta) \oplus P_{L_*}(-\eta) = C([-r_{\max}, -r_{\min}]; \mathbb{R}^n)$, the non-degeneracy of the Hale inner product implies that we must have $p = 0$, which gives $\phi \in Q_L(\eta)$.

Turning to the identities for $\widehat{Q}_L(\eta)$ in (3.37), it suffices to show that the inclusions

$$\widehat{\mathcal{S}}_1 \subset \widehat{\mathcal{S}}_2 \subset \widehat{Q}_L(\eta) \tag{6.71}$$

hold for the sets

$$\begin{array}{rcl} \widehat{\mathcal{S}}_1 & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \quad \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\widehat{\phi}, \widehat{\phi}(0^+)\big) \Big\rangle_L = 0 \text{ for all } \widehat{\psi} \in \widehat{Q}_{L_*}(-\eta)\}, \\ \widehat{\mathcal{S}}_2 & = & \{\widehat{\phi} \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) : \\ & & \quad \Big\langle \big(\psi, \psi(0)\big), \big(\widehat{\phi}, \widehat{\phi}(0^+)\big) \Big\rangle_L = 0 \text{ for all } \psi \in Q_{L_*}(-\eta)\}. \end{array} \tag{6.72}$$

The first inclusion again follows from $Q_{L_*}(-\eta) \subset \widehat{Q}_{L_*}(-\eta)$. Any $\widehat{\phi} \in \mathcal{S}_2$ can be decomposed as $\widehat{\phi} = p + \widehat{q}$ with $p \in P_L(\eta)$ and $\widehat{q} \in \widehat{Q}_L(\eta)$. Since $\widehat{Q}_L(\eta) \subset \widehat{\mathcal{S}}_2$, we see that also $p = \widehat{\phi} - \widehat{q} \in \widehat{\mathcal{S}}_2$. As above, we have $\langle p, \psi \rangle_L = 0$ for all $\psi \in P_{L_*}(-\eta)$, which again allows us to conclude $p = 0$ and hence $\widehat{\phi} \in \widehat{Q}_L(\eta)$. The identities (3.36) can be obtained in a similar fashion. □

We now turn our attention to Theorem 3.5, which we prove up to the index formula stated in Proposition 6.10 below. As a reminder, we recall the shorthands

$$\begin{array}{rcl} R^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Range}\big(\pi^-_{\widehat{Q}_L}(\eta)\big) \subset C([r_{\min}, 0]; \mathbb{R}^n), \\ K^-_{\widehat{Q}_L(\eta)} & = & \mathrm{Ker}\big(\pi^-_{\widehat{Q}_L}(\eta)\big) \subset \widehat{Q}_L(\eta), \end{array} \tag{6.73}$$

together with

$$\begin{array}{rcl} R^-_{Q_L(\eta)} & = & \mathrm{Range}\big(\pi^-_{Q_L}(\eta)\big) \subset C([r_{\min}, 0]; \mathbb{R}^n), \\ K^-_{Q_L(\eta)} & = & \mathrm{Ker}\big(\pi^-_{Q_L}(\eta)\big) \subset Q_L(\eta). \end{array} \tag{6.74}$$

Lemma 6.9. *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that both satisfies* $(\mathrm{HF})_L$ *and* $(\mathrm{HRnk})_L$*. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation* $\det \Delta_L(z) = 0$ *admits no roots with* $\mathrm{Re}\, z = \eta$*. We then have the inclusion*

$$R^-_{Q_L(\eta)} \subset \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \langle \pi^+ \widehat{\psi}, \phi \rangle_{L<0} = 0 \text{ for all } \widehat{\psi} \in K^-_{\widehat{Q}_{L_*}(-\eta)}\}, \tag{6.75}$$

together with

$$\begin{array}{rcl} R^-_{\widehat{Q}_L(\eta)} & \subset & \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \langle \pi^+ \psi, \phi \rangle_{L<0} = 0 \text{ for all } \psi \in K^-_{Q_{L_*}(-\eta)}\} \\ & = & \{\phi \in C([r_{\min}, 0]; \mathbb{R}^n) : \Big\langle \big(\pi^+ \widehat{\psi}, 0\big), \big(\phi, \phi(0)\big) \Big\rangle_{L<0} = 0 \text{ for all } \widehat{\psi} \in \widehat{K}^-_{\widehat{Q}_{L_*}(-\eta)}\} \end{array} \tag{6.76}$$

and finally

$$\begin{array}{rcl} \widehat{R}^-_{\widehat{Q}_L(\eta)} & \subset & \{(\phi, v) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n : \\ & & \quad \Big\langle \big(\pi^+ \widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\phi, v\big) \Big\rangle_{L<0} = 0 \text{ for all } \widehat{\psi} \in K^-_{\widehat{Q}_{L_*}(-\eta)}\}. \end{array} \tag{6.77}$$

Proof. To see (6.75), pick any $\phi \in R^-_{Q_L(\eta)}$ and choose an arbitrary $y \in \mathfrak{Q}_L(\eta)$ that has $\mathrm{ev}_0^-\, y = \phi$. For any $\widehat{\psi} = (\psi^-, \psi^+) \in K^-_{\widehat{Q}_{L_*}(-\eta)}$ we have

$$0 = \Big\langle \big(\widehat{\psi}, \psi^+(0)\big), \big(\mathrm{ev}_0\, y, y(0)\big) \Big\rangle_L = \langle \psi^+, \phi \rangle_{L<0}, \tag{6.78}$$

where the first identity follows from Proposition 3.4 and the second identity follows from the fact that $\psi^- = 0$.

To see (6.76), first observe that $\widehat{K}^-_{\widehat{Q}_{L_*}(-\eta)} = K^-_{Q_{L_*}(-\eta)}$, which allows us to focus on the first line. Let us therefore pick any $\phi \in R^-_{\widehat{Q}_L(\eta)}$ and choose an arbitrary $\widehat{y} \in \widehat{\mathfrak{Q}}_L(\eta)$ that has $\mathrm{ev}_0^-\, \widehat{y} = \phi$. For any $\psi = (\psi^-, \psi^+) \in K_{Q_{L_*}}(-\eta)$, Proposition 3.4 together with $\psi^- = 0$ and $\psi^+(0) = 0$ implies that

$$0 = \Big\langle \big(\psi, \psi^+(0)\big), \big(\mathrm{ev}_0\, \widehat{y}, \widehat{y}(0^+)\big) \Big\rangle_L = \langle \psi^+, \phi \rangle_{L<0}. \tag{6.79}$$

Finally, to establish (6.77), pick any $(\phi, v) \in \widehat{R}^-_{\widehat{Q}_L(\eta)}$ and an accompanying $\widehat{y} \in \widehat{\mathfrak{Q}}_L(\eta)$ with $\mathrm{ev}_0^-\, \widehat{y} = \phi$ and $\widehat{y}(0^+) = v$. For any $\widehat{\psi} = (\psi^-, \psi^+) \in K_{\widehat{Q}_{L_*}(-\eta)}$, we see that

$$0 = \Big\langle \big(\widehat{\psi}, \widehat{\psi}(0^+)\big), \big(\widehat{\mathrm{ev}}_0\, \widehat{y}, \widehat{y}(0^+)\big) \Big\rangle_L = \Big\langle \big(\psi^+, \psi^+(0)\big), \big(\phi, v\big) \Big\rangle_{L<0}, \tag{6.80}$$

in which the first identity follows from Proposition 3.4 and the second identity follows from the fact that $\psi^- = 0$. □

Proposition 6.10 (see §6.3). *Write $\mathcal{H} = \mathbb{R}^n$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HRnk})_L$*. Choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation* $\det \Delta_L(z) = 0$ *admits no roots with* $\mathrm{Re}\, z = \eta$*. Then we have the identities*

$$\begin{array}{rcl} 0 & = & \mathrm{ind}(\pi^-_{\widehat{Q}_L(\eta)}) + \mathrm{ind}(\pi^-_{Q_{L_*}(-\eta)}), \\ 0 & = & \mathrm{ind}(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}) + \mathrm{ind}(\pi^-_{\widehat{Q}_{L_*}(-\eta)}). \end{array} \tag{6.81}$$

Proof of Theorem 3.5. The inclusion (6.76) together with the non-degeneracy of $\langle , \rangle_{L<0}$, yields the inequality

$$\mathrm{codim}\, R_{\widehat{Q}_L(\eta)} \geq \dim K_{Q_{L_*}(-\eta)}. \tag{6.82}$$

On the other hand, applying (6.75) to L_* yields

$$\operatorname{codim} R_{Q_{L_*}(-\eta)} \geq \dim K_{\widehat{Q}_L(\eta)}. \tag{6.83}$$

In particular, we see that

$$\begin{array}{lcl} \operatorname{ind}\big(\pi^-_{\widehat{Q}_L(\eta)}\big) & = & \dim K_{\widehat{Q}_L(\eta)} - \operatorname{codim} R_{\widehat{Q}_L(\eta)} \\ & \leq & \operatorname{codim} R_{Q_{L_*}(-\eta)} - \dim K_{Q_{L_*}(-\eta)} \\ & = & -\operatorname{ind}\big(\pi^-_{Q_{L_*}(-\eta)}\big). \end{array} \tag{6.84}$$

On account of Proposition 6.10 we see that all inequalities above are in fact equalities. Applying the same argument to L_*, we may conclude that the inclusions in (6.75) and (6.76) are in fact identities.

In a similar fashion, we can use (6.76) and (6.77) to obtain

$$\begin{array}{lcl} \operatorname{ind}\big(\widehat{\pi}^-_{\widehat{Q}_L(\eta)}\big) & = & \dim \widehat{K}_{\widehat{Q}_L(\eta)} - \operatorname{codim} \widehat{R}_{\widehat{Q}_L(\eta)} \\ & \leq & \operatorname{codim} R_{\widehat{Q}_{L_*}(-\eta)} - \dim K_{\widehat{Q}_{L_*}(-\eta)} \\ & = & -\operatorname{ind}\big(\pi^-_{\widehat{Q}_{L_*}(-\eta)}\big). \end{array} \tag{6.85}$$

On account of Proposition 6.10 we again see that the inequality above is in fact an equality, which in turn shows that (6.76) and (6.77) are in fact identities. □

6.3 Index equations

Here we set out to establish Proposition 6.10. In order to further explore the relation between L and L_*, we introduce two operators

$$L^+_{\mathrm{mx}} : C([r_{\min}, r_{\max}]; \mathbb{R}^n) \to \mathbb{R}^n, \qquad L^-_{\mathrm{mx}} : C([r_{\min}, r_{\max}]; \mathbb{R}^n) \to \mathbb{R}^n, \tag{6.86}$$

that act as

$$\begin{array}{lcl} L^+_{\mathrm{mx}}\phi & = & \sum_{j=0}^N \big[A_j \phi(r_j) + \int_{s_j^-}^{s_j^+} B_j(\sigma)\phi(\sigma)\, d\sigma\big], \\ L^-_{\mathrm{mx}}\phi & = & \sum_{j=0}^N \big[A_j^* \phi(r_j) + \int_{s_j^-}^{s_j^+} B_j^*(\sigma)\phi(\sigma)\, d\sigma\big]. \end{array} \tag{6.87}$$

Writing

$$\begin{array}{lcl} L^\infty_\eta(\mathbb{R}; \mathbb{R}^n) & = & \{x \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) : e^{-\eta \cdot} x(\cdot) \in L^\infty(\mathbb{R}; \mathbb{R}^n)\}, \\ W^{1,\infty}_\eta(\mathbb{R}; \mathbb{R}^n) & = & \{x \in L^\infty_\eta(\mathbb{R}; \mathbb{R}^n) : x' \in L^\infty_\eta(\mathbb{R}; \mathbb{R}^n)\}, \end{array} \tag{6.88}$$

with norms

$$\|x\|_{L^\infty_\eta} = \big\|e^{-\eta \cdot} x(\cdot)\big\|_\infty, \qquad \|x\|_{W^{1,\infty}_\eta} = \|x\|_{L^\infty_\eta} + \|x'\|_{L^\infty_\eta}, \tag{6.89}$$

we now combine the two operators (6.87) into a single non-autonomous mixed operator

$$\mathcal{L}_{\mathrm{mx}} : W^{1,\infty}_\eta(\mathbb{R}; \mathbb{R}^n) \to L^\infty_\eta(\mathbb{R}; \mathbb{R}^n) \tag{6.90}$$

that acts as

$$[\mathcal{L}_{\mathrm{mx}} v](\xi) = v'(\xi) - L^-_{\mathrm{mx}} \operatorname{ev}_\xi v \tag{6.91}$$

whenever $\xi < 0$ and

$$[\mathcal{L}_{\mathrm{mx}} v](\xi) = v'(\xi) - L^+_{\mathrm{mx}} \operatorname{ev}_\xi v \tag{6.92}$$

for $\xi > 0$. In addition, we introduce the solution spaces

$$\begin{array}{lcl} \mathfrak{P}_{\mathrm{mx}}(\eta) & = & \big\{x \in BC_{\eta}^{\ominus}(\mathbb{R}^n) : x'(\xi) = L_{\mathrm{mx}}^{-}\,\mathrm{ev}_{\xi}\, x \text{ for all } \xi \leq 0\big\}, \\ \mathfrak{Q}_{\mathrm{mx}}(\eta) & = & \big\{y \in BC_{\eta}^{\oplus}(\mathbb{R}^n) : y'(\xi) = L_{\mathrm{mx}}^{+}\,\mathrm{ev}_{\xi}\, y \text{ for all } \xi \geq 0\big\}, \\ \mathfrak{B}_{\mathrm{mx}}(\eta) & = & \big\{b \in W_{\eta}^{1,\infty}(\mathbb{R};\mathbb{R}^n) : [\mathcal{L}_{\mathrm{mx}} b](\xi) = 0 \text{ for all } \xi \in \mathbb{R} \setminus \{0\}\big\}, \end{array} \tag{6.93}$$

together with the initial segment spaces

$$\begin{array}{lcl} P_{\mathrm{mx}}(\eta) & = & \big\{\phi \in C([r_{\min}, r_{\max}];\mathbb{R}^n) : \phi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{P}_{\mathrm{mx}}(\eta)\big\}, \\ Q_{\mathrm{mx}}(\eta) & = & \big\{\phi \in C([r_{\min}, r_{\max}];\mathbb{R}^n) : \phi = \mathrm{ev}_0\, y \text{ for some } y \in \mathfrak{Q}_{\mathrm{mx}}(\eta)\big\}, \\ B_{\mathrm{mx}}(\eta) & = & \big\{\phi \in C([r_{\min}, r_{\max}];\mathbb{R}^n) : \phi = \mathrm{ev}_0\, b \text{ for some } b \in \mathfrak{B}_{\mathrm{mx}}(\eta)\big\}. \end{array} \tag{6.94}$$

Finally, for any interval $\mathcal{I} \subset \mathbb{R}$ and any function $x : \mathcal{I} \to \mathbb{R}^n$, we write

$$\big[\mathrm{Rev}(x)\big](\xi) = x(-\xi) \tag{6.95}$$

for all $\xi \in -\mathcal{I}$.

Lemma 6.11. *Consider the setting of Proposition 6.10. Then $\mathcal{L}_{\mathrm{mx}}$ is Fredholm as a map from $W_{\eta}^{1,\infty}(\mathbb{R};\mathbb{R}^n)$ into $L_{\eta}^{\infty}(\mathbb{R};\mathbb{R}^n)$, with*

$$\mathrm{ind}(\mathcal{L}_{\mathrm{mx}}) = 0. \tag{6.96}$$

In addition, we have the identifications

$$Q_{L_*}(-\eta) = \mathrm{Rev}\big(P_{\mathrm{mx}}(\eta)\big), \qquad Q_L(\eta) = Q_{\mathrm{mx}}(\eta). \tag{6.97}$$

Proof. Seeking to employ a spectral flow argument, we introduce the expression

$$\begin{array}{lcl} \Delta_{\mu}(z) & = & z - \sum_{j=0}^{N} (\mu A_j + (1-\mu) A_j^*) e^{z r_j} \\ & & \quad - \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} \big(\mu B_j(\sigma) + (1-\mu) B_j^*(\sigma)\big) e^{z\sigma} d\sigma \end{array} \tag{6.98}$$

for $0 \leq \mu \leq 1$ and $z \in \mathbb{C}$. Note that

$$\Delta_0(z) = \Delta_{L_{\mathrm{mx}}^-}(z), \qquad \Delta_1(z) = \Delta_{L_{\mathrm{mx}}^+}(z), \tag{6.99}$$

while also

$$\Delta_{1-\mu}(z^*) = \Delta_{\mu}(z)^*. \tag{6.100}$$

In particular, the net number of roots of $\det \Delta_{\mu}(z) = 0$ that crosses the line $\mathrm{Re}\, z = \eta$ as μ is increased from zero to one is precisely zero. The spectral flow formula [19, Thm. C] hence yields (6.96).

The second identity in (6.97) is immediate as $L_{\mathrm{mx}}^{+} = L$. To see the first identity, pick any $v \in \widehat{\mathfrak{Q}}_{L_*}(-\eta)$ and write $w = \mathrm{Rev}(v)$. We then see that $w \in BC_{\eta}^{\ominus}(\mathbb{R}^n)$. In addition, for any $\xi < 0$ we may compute

$$\begin{array}{lcl} w'(\xi) & = & -v'(-\xi) \\ & = & \sum_{j=0}^{N} \big[A_j^* v(-\xi - r_j) + \int_{s_j^-}^{s_j^+} B_j(\sigma)^* v(-\xi - \sigma)\, d\sigma\big] \\ & = & \sum_{j=0}^{N} \big[A_j^* w(\xi + r_j) + \int_{s_j^-}^{s_j^+} B_j(\sigma)^* w(\xi + \sigma)\, d\sigma\big], \end{array} \tag{6.101}$$

which shows that $w \in \mathfrak{P}_{\mathrm{mx}}(\eta)$. □

In order to account for the possibility that the space $B_{\rm mx}(\eta) = Q_{\rm mx}(\eta) \cap P_{\rm mx}(\eta)$ is non-trivial, we need to introduce the normalized spaces

$$\begin{array}{lcl} \mathfrak{P}_{\mathrm{mx};\perp}(\eta) & = & \big\{ x \in \mathfrak{P}_{\rm mx}(\eta) : \int_{-\infty}^{0} e^{-2\eta\xi} b(\xi)^* x(\xi) d\xi = 0 \text{ for all } b \in \mathfrak{B}_{\rm mx}(\eta) \big\}, \\ \mathfrak{Q}_{\mathrm{mx};\perp}(\eta) & = & \big\{ y \in \mathfrak{Q}_{\rm mx}(\eta) : \int_{0}^{\infty} e^{-2\eta\xi} b(\xi)^* y(\xi) d\xi = 0 \text{ for all } b \in \mathfrak{B}_{\rm mx}(\eta) \big\}, \end{array} \tag{6.102}$$

together with the initial segment spaces

$$\begin{array}{lcl} P_{\mathrm{mx};\perp}(\eta) & = & \big\{ \phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \phi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{P}_{\mathrm{mx};\perp}(\eta) \big\}, \\ Q_{\mathrm{mx};\perp}(\eta) & = & \big\{ \phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n) : \phi = \mathrm{ev}_0\, y \text{ for some } y \in \mathfrak{Q}_{\mathrm{mx};\perp}(\eta) \big\}. \end{array} \tag{6.103}$$

Our goal here is to mimic the non-autonomous theory developed in [20, §4] and apply it to the operator $\mathcal{L}_{\rm mx}$. As before, special care needs to be taken here because our non-degeneracy condition $(\mathrm{HRnk})_L$ is weaker than its counterpart in [20].

Proposition 6.12. *Consider the setting of Proposition 6.10. Then the spaces $P_{\mathrm{mx};\perp}(\eta)$, $Q_{\mathrm{mx};\perp}(\eta)$ and $B_{\rm mx}(\eta)$ are all closed subsets of $C([r_{\min}, r_{\max}]; \mathbb{R}^n)$ and the direct sum*

$$S_{\rm mx}(\eta) = P_{\mathrm{mx};\perp}(\eta) \oplus Q_{\mathrm{mx};\perp}(\eta) \oplus B_{\rm mx}(\eta) \tag{6.104}$$

is well-defined. In addition, $S_{\rm mx}(\eta)$ is a closed subset of $C([r_{\min}, r_{\max}]; \mathbb{R}^n)$, with

$$\mathrm{codim}\, S_{\rm mx}(\eta) = \dim B_{\rm mx}(\eta). \tag{6.105}$$

Finally, we have the identities

$$P_{\rm mx}(\eta) = P_{\mathrm{mx};\perp}(\eta) \oplus B_{\rm mx}(\eta), \qquad Q_{\rm mx}(\eta) = Q_{\mathrm{mx};\perp}(\eta) \oplus B_{\rm mx}(\eta) \tag{6.106}$$

and the index formula

$$\mathrm{ind}(\pi^+_{P_{\rm mx}(\eta)}) + \mathrm{ind}(\pi^-_{Q_{\rm mx}(\eta)}) = -n. \tag{6.107}$$

Proof. The idea is to apply the results from [20, §4] that lead up to [20, Cor. 4.7]. In our setting we note that the Hale inner product associated to $\mathcal{L}_{\rm mx}$ at $\xi = 0$ is given by

$$\begin{array}{lcl} \langle \psi, \phi \rangle_{\rm mx} & = & \psi(0)^* \phi(0) - \sum_{r_j > 0} \int_0^{r_j} \psi(\tau - r_j)^* A_j \phi(\tau)\, d\tau - \sum_{r_j < 0} \int_0^{r_j} \psi(\tau - r_j)^* A_j^* \phi(\tau)\, d\tau \\ & & \quad - \sum_{s_j^+ > 0} \int_{\max\{0, s_j^-\}}^{s_j^+} \int_0^{\sigma} \psi(\tau - r_j)^* B_j(\sigma) \phi(\tau)\, d\tau\, d\sigma, \\ & & \quad - \sum_{s_j^- < 0} \int_{s_j^-}^{\min\{0, s_j^+\}} \int_0^{\sigma} \psi(\tau - r_j)^* B_j^*(\sigma) \phi(\tau)\, d\tau\, d\sigma, \end{array} \tag{6.108}$$

for $\psi \in C([-r_{\max}, -r_{\min}]; \mathbb{R}^n)$ and $\phi \in C([r_{\min}, r_{\max}]; \mathbb{R}^n)$. In particular, this can be written as $\langle \psi, \phi \rangle_{\widetilde{L}}$ for a suitably chosen $\widetilde{L}$ that satisfies $(\mathrm{HF})_{\widetilde{L}}$ and $(\mathrm{HRnk})_{\widetilde{L}}$. This implies that the Hale inner product $\langle \cdot, \cdot \rangle_{\rm mx}$ is non-degenerate in the sense of Proposition 3.3, which gives the analogue of [20, Prop 4.16].

We now claim that there are no non-zero functions $x \in \mathfrak{B}_{\rm mx}(\eta)$ that vanish on an interval of the form $[\xi_0 + r_{\min}, \xi_0 + r_{\max}]$ for some $\xi_0 \in \mathbb{R}$. To see this, note first that if $\xi_0 \geq 0$, the function $\overline{x}$ that has

$$\overline{x}(\xi) = \left\{ \begin{array}{ll} x(\xi), & \xi \leq \xi_0 + r_{\min}, \\ 0, & \xi_0 + r_{\min} \leq \xi \leq \xi_0 + r_{\max} + T \\ x(\xi - T), & \xi \geq \xi_0 + r_{\max} + T \end{array} \right. \tag{6.109}$$

is also an element of $\mathfrak{B}_{\mathrm{mx}}(\eta)$ for arbitrary $T \geq 0$. The finite dimensionality of $\mathfrak{B}_{\mathrm{mx}}(\eta)$ now implies that in fact $x(\xi) = 0$ for all $\xi \geq \xi_0$. In case $\xi_0 \leq 0$, the same reasoning can be used to show that $x(\xi) = 0$ for all $\xi \leq \xi_0$.

Furthermore, we note that the identifications (6.97) together with Lemma 6.6 imply that

$$\bigcap_{\eta \in \mathbb{R}} P_{\mathrm{mx}}(\eta) = \{0\}, \qquad \bigcap_{\eta \in \mathbb{R}} Q_{\mathrm{mx}}(\eta) = \{0\}. \tag{6.110}$$

In particular, we see that if $\xi_0 \geq 0$, the function x must vanish on $[r_{\min}, \infty)$, which allows us to take $\xi_0 = 0$. Similarly, if $\xi_0 \leq 0$, the function x must vanish on $(-\infty, r_{\max}]$, which again allows us to take $\xi_0 = 0$. In particular, we obtain the contradiction $x = 0$. This gives the analogue of [20, Prop 4.9].

With these obstacles removed, the relevant theory developed in [20, §4] can be generalized to our setting, which readily yields the desired properties. In the codimension formula (6.105) and the index formula (6.107), we exploit the fact that $\mathrm{ind}(\mathcal{L}_{\mathrm{mx}}) = 0$. □

Proof of Proposition 6.10. Using Lemma 6.11 we see that

$$\mathrm{ind}\big(\pi^-_{Q_{L_*}(-\eta)}\big) = \mathrm{ind}\big(\pi^+_{P_{\mathrm{mx}}(\eta)}\big), \qquad \mathrm{ind}\big(\pi^-_{Q_L(\eta)}\big) = \mathrm{ind}\big(\pi^-_{Q_{\mathrm{mx}}(\eta)}\big). \tag{6.111}$$

In particular, (6.107) gives

$$\mathrm{ind}(\pi^-_{Q_{L_*}(-\eta)}) + \mathrm{ind}(\pi^-_{Q_L(\eta)}) = -n. \tag{6.112}$$

The desired expressions now follow directly from Lemma 6.5. □

7 Algebraic Systems

In this section we study the differential-algebraic system (2.35), allowing both $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$. We start by studying the associated characteristic functions in §7.1, focussing on explicit techniques to divide and multiply such functions by factors of $(z - \alpha)$. As can be seen from §4, these results are useful by their own right.

In §7.2 however we exploit these root extraction techniques to establish the exponential splittings for (2.35) described in Theorem 2.5, slightly generalizing the approach in [4]. Finally, in §7.3 we study scalar algebraic equations and show how the Wiener-Hopf factorizations for differential systems can be coupled to the techniques from §7.1, allowing us to establish the results stated in §3.3.

7.1 Characteristic equations

Throughout this section we fix two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We are interested in bounded linear operators

$$L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2 \tag{7.1}$$

that satisfy one of the two conditions below.

$(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ We have $r_{\min} \leq 0 \leq r_{\max}$. There exists an integer $N \geq 0$ together with real numbers

$$r_{\min} = r_0 < r_1 < \ldots < r_N = r_{\max}, \qquad r_{\min} \leq s_j^- \leq s_j^+ \leq r_{\max} \tag{7.2}$$

and operators

$$A_{L;j} \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2), \qquad B_{L;j} \in C\big([s_j^-, s_j^+]; \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)\big), \tag{7.3}$$

defined for $0 \leq j \leq N$, so that

$$L\phi = \sum_{j=0}^{N} \big[A_{L;j}\phi(r_j) + \int_{s_j^-}^{s_j^+} B_{L;j}(\sigma)\phi(\sigma)\, d\sigma\big]. \tag{7.4}$$

$(\mathrm{hFin})_{\mathcal{H}_1,\mathcal{H}_2}$ We have $r_{\min} \leq 0 \leq r_{\max}$ and there exist integers $m > 0$ and $n > 0$ so that $\mathcal{H}_1 = \mathbb{R}^n$ and $\mathcal{H}_2 = \mathbb{R}^m$.

Let us recall the set $\mathrm{NBV}([r_{\min}, r_{\min}]; \mathbb{R}^{m\times n})$ that consists of all $\mathbb{R}^{m\times n}$-valued functions ζ that are right-continuous on $(r_{\min}, r_{\max})$, are normalized to have $\zeta(r_{\min}) = 0$ and have bounded variation on $[r_{\min}, r_{\max}]$; see [5, App. I]. If $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ is satisfied, there exists a unique

$$\zeta_L \in \mathrm{NBV}([r_{\min}, r_{\max}]; \mathbb{R}^{m\times n}) \tag{7.5}$$

so that

$$L\phi = \int_{r_{\min}}^{r_{\max}} d\zeta_L(\sigma)\phi(\sigma). \tag{7.6}$$

We start out with three preparatory results concerning the expression $L\phi$, which discuss how two frequently occurring transformations on ϕ can be transferred to the linear operator L. In particular, for any $\alpha \in \mathbb{R}$ we introduce the bounded linear operators

$$L_\alpha : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2, \qquad \mathcal{I}_{L;\alpha} : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2 \tag{7.7}$$

that act as

$$L_\alpha\phi = Le^{\alpha\cdot}\phi, \tag{7.8}$$

together with

$$\mathcal{I}_{L;\alpha}\phi = L\big[\sigma \mapsto e^{\alpha\sigma} \int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma'\big]. \tag{7.9}$$

We discuss the exponentially shifted operator L_α in Lemma 7.1 and the integrated operator $\mathcal{I}_{L;\alpha}$ in Lemma's 7.2 and 7.3.

Lemma 7.1. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ and suppose that $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ is satisfied. We then have*

$$\zeta_{L_\alpha}(\sigma) = e^{\alpha\sigma}\zeta_L(\sigma) - \alpha \int_{r_{\min}}^{\sigma} e^{\alpha\sigma'}\zeta_L(\sigma')\, d\sigma'. \tag{7.10}$$

In particular, we have

$$\zeta_{L_\alpha}(r_{\max}) = Le^{\alpha\cdot}. \tag{7.11}$$

Proof. A direct computation yields

$$\begin{array}{rcl} \int_{r_{\min}}^{r_{\max}} d\zeta_{L_\alpha}(\sigma)\phi(\sigma) & = & \int_{r_{\min}}^{r_{\max}} \big[\alpha e^{\alpha\sigma}\zeta_L(\sigma) - \alpha e^{\alpha\sigma}\zeta_L(\sigma)\big]\phi(\sigma)\, d\sigma \\ & & \quad + \int_{r_{\min}}^{r_{\max}} d\zeta_L(\sigma)e^{\alpha\sigma}\phi(\sigma) \\ & = & \int_{r_{\min}}^{r_{\max}} d\zeta_L(\sigma)e^{\alpha\sigma}\phi(\sigma) \\ & = & Le^{\alpha\cdot}\phi(\cdot). \end{array} \tag{7.12}$$

The final statement $\zeta_{L_\alpha}(r_{\max}) = Le^{\alpha\cdot}$ follows after substituting $\phi = 1$ and noting $\zeta_{L_\alpha}(0) = \zeta_L(0) = 0$. □

Lemma 7.2. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ and suppose that $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ is satisfied. Then we have the representation*

$$\zeta_{\mathcal{I}_{L;\alpha}}(\sigma) = \int_{r_{\min}}^{\sigma} [\zeta_{L_\alpha}(r_{\max}) - \zeta_{L_\alpha}(\sigma')]e^{-\alpha\sigma'} d\sigma'. \tag{7.13}$$

Proof. We compute

$$\begin{array}{rcl}\int_{r_{\min}}^{r_{\max}} d\zeta_{\mathcal{I}_{L;\alpha}}(\sigma)\phi(\sigma) & = & \int_{r_{\min}}^{r_{\max}} [\zeta_{L_\alpha}(r_{\max}) - \zeta_{L_\alpha}(\sigma)] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma \\ & = & [\zeta_{L_\alpha}(r_{\max}) - \zeta_{L_\alpha}(r_{\max})] \int_{r_{\min}}^{r_{\max}} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma' \\ & & \quad + \int_{r_{\min}}^{r_{\max}} d\zeta_{L_\alpha}(\sigma) \int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma' \\ & = & L_\alpha\big[\sigma \mapsto \int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma'\big] \\ & = & L\big[\sigma \mapsto e^{\alpha\sigma}\int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma'\big]\end{array} \tag{7.14}$$

as desired. □

Lemma 7.3. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ and suppose that $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ is satisfied. Then we have the representation*

$$\begin{array}{rcl}\mathcal{I}_{L;\alpha}\phi & = & \sum_{j=0}^{N} \int_{r_{\min}}^{r_j} A_{L;j} e^{\alpha(r_j - \sigma)}\phi(\sigma)\, d\sigma \\ & & + \sum_{j=0}^{N} \int_{r_{\min}}^{s_j^-} \big[\int_{s_j^-}^{s_j^+} e^{\alpha\sigma'} B_{L;j}(\sigma')\, d\sigma'\big] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma \\ & & + \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} \big[\int_{\sigma}^{s_j^+} e^{\alpha\sigma'} B_{L;j}(\sigma')\, d\sigma'\big] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma.\end{array} \tag{7.15}$$

In particular, we see that $(\mathrm{hF})_{\mathcal{I}_{L;\alpha};\mathcal{H}_1,\mathcal{H}_2}$ is satisfied.

Proof. A direct computation yields

$$\begin{array}{rcl}\mathcal{I}_{L;\alpha}\phi & = & \sum_{j=0}^{N} A_{L;j} e^{\alpha r_j} \int_{r_{\min}}^{r_j} e^{-\alpha\sigma}\phi(\sigma)\, d\sigma \\ & & + \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} B_{L;j}(\sigma) e^{\alpha\sigma} \int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\, d\sigma'\, d\sigma \\ & = & \sum_{j=0}^{N} \int_{r_{\min}}^{r_j} A_{L;j} e^{\alpha(r_j - \sigma)}\phi(\sigma)\, d\sigma \\ & & + \sum_{j=0}^{N} \int_{r_{\min}}^{s_j^+} \big[\int_{\max\{\sigma, s_j^-\}}^{s_j^+} e^{\alpha\sigma'} B_{L;j}(\sigma')\, d\sigma'\big] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma \\ & = & \sum_{j=0}^{N} \int_{r_{\min}}^{r_j} A_{L;j} e^{\alpha(r_j - \sigma)}\phi(\sigma)\, d\sigma \\ & & + \sum_{j=0}^{N} \int_{r_{\min}}^{s_j^-} \big[\int_{s_j^-}^{s_j^+} e^{\alpha\sigma'} B_{L;j}(\sigma')\, d\sigma'\big] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma \\ & & \quad + \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} \big[\int_{\sigma}^{s_j^+} e^{\alpha\sigma'} B_{L;j}(\sigma')\, d\sigma'\big] e^{-\alpha\sigma}\phi(\sigma)\, d\sigma,\end{array} \tag{7.16}$$

as desired. □

In Propositions 7.4-7.6 we show how and when factors of $(z - \alpha)$ can be divided out and factored into the characteristic functions associated to differential operators L respectively differential-algebraic operators M. We frequently use the fact that L and M are uniquely determined by their representations $z \mapsto Le^{z\cdot}$ and $z \mapsto Me^{z\cdot}$. This is a consequence of the fact that sums of exponential functions are dense in $C([r_{\min}, r_{\max}]; \mathbb{R})$.

Proposition 7.4. *Consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$, suppose that either $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ or $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ is satisfied and pick $J \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$.*

Then if $Le^{\alpha\cdot} = \alpha J$, there is a unique bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ such that

$$-(z - \alpha)Me^{z\cdot} = Jz - Le^{z\cdot} \tag{7.17}$$

holds for all $z \in \mathbb{C}$. This operator is given by

$$\begin{array}{rcl} M\phi & = & -J\phi(0) - J\alpha \int_{r_{\min}}^{0} e^{-\alpha\sigma}\phi(\sigma)\, d\sigma + \mathcal{I}_{L;\alpha}\phi \\ & = & -J\phi(0) + L\big[\sigma \mapsto e^{\alpha\sigma} \int_0^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')d\sigma'\big]. \end{array} \tag{7.18}$$

Proof. A short computation exploiting $Le^{\alpha\cdot} = J\alpha$ shows that the two identities for M are equal. In addition, for $z \neq \alpha$ we may compute

$$\begin{array}{rcl} Me^{z\cdot} & = & -J - \alpha J \int_{r_{\min}}^{0} e^{(z-\alpha)\sigma} d\sigma \\ & & \quad +L\big[\sigma \mapsto e^{\alpha\sigma} \int_{r_{\min}}^{\sigma} e^{(z-\alpha)\sigma'} d\sigma'\big] \\ & = & -J - \alpha J (z-\alpha)^{-1}[1 - e^{(z-\alpha)r_{\min}}] \\ & & \quad +L\big[\sigma \mapsto (z-\alpha)^{-1} e^{\alpha\sigma}[e^{(z-\alpha)\sigma} - e^{(z-\alpha)r_{\min}}]\big] \\ & = & (z-\alpha)^{-1}\big[-J(z-\alpha) - \alpha J[1 - e^{(z-\alpha)r_{\min}}]\big] \\ & & \quad +(z-\alpha)^{-1}\big[Le^{z\cdot} - e^{(z-\alpha)r_{\min}} Le^{\alpha\cdot}\big] \\ & = & (z-\alpha)^{-1}\big[Le^{z\cdot} - Jz - e^{(z-\alpha)r_{\min}}[Le^{\alpha\cdot} - \alpha J]\big], \end{array} \tag{7.19}$$

as desired. □

If $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ is satisfied in the result above, then Lemma 7.3 guarantees that also $(\mathrm{hF})_{M;\mathcal{H}_1,\mathcal{H}_2}$ is satisfied for the operator (7.18).

Proposition 7.5. *Consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ and suppose that $(\mathrm{hF})_{M;\mathcal{H}_1,\mathcal{H}_2}$ is satisfied. In addition, assume that M can be represented in such a way that the following three properties hold.*

- *(a) We have $r_0 = 0$ and $A_{M;0} = -J$ for some $J \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$.*
- *(b) For all $1 \leq j \leq N$ we have $A_{M;j} = 0$.*
- *(c) For all $0 \leq j \leq N$ we have*

$$B_{M;j} \in C^1\big([s_j^-, s_j^+]; \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)\big). \tag{7.20}$$

Then for any $\alpha \in \mathbb{R}$, there is a unique bounded linear operator

$$L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2 \tag{7.21}$$

for which

$$-(z-\alpha)Me^{z\cdot} = Jz - Le^{z\cdot} \tag{7.22}$$

holds for all $z \in \mathbb{C}$. This operator acts as

$$\begin{array}{rcl} L\phi & = & \alpha J\phi(0) - \sum_{j=0}^{N} \Big[B_{M;j}(s_j^-)\phi(s_j^-) - B_{M;j}(s_j^+)\phi(s_j^+)\Big] \\ & & \quad - \sum_{j=0}^{N} \int_{s_j^-}^{s_j^+} [B_{M;j}(\cdot)e^{\alpha\cdot}]'(\sigma)e^{-\alpha\sigma}\phi(\sigma)\, d\sigma, \end{array} \tag{7.23}$$

which means that $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ is satisfied.

Proof. Using the explicit expression for L, we simply verify

$$\begin{array}{lcl} Le^{z\cdot} & = & \alpha J - \sum_{j=0}^{N} \Big[- B_{M;j}(s_j^+)e^{zs_j^+} + B_{M;j}(s_j^-)e^{zs_j^-} + \int_{s_j^-}^{s_j^+} [B_{M;j}(\cdot)e^{\alpha\cdot}]'(\sigma)e^{(z-\alpha)\sigma}\, d\sigma \Big] \\ & = & \alpha J - \sum_{j=0}^{N} \Big[- B_{M;j}(s_j^+)e^{zs_j^+} + B_{M;j}(s_j^-)e^{zs_j^-} \\ & & \quad + B_{M;j}(s_j^+)e^{\alpha s_j^+}e^{(z-\alpha)s_j^+} - B_{M;j}(s_j^-)e^{\alpha s_j^-}e^{(z-\alpha)s_j^-} \\ & & \quad -(z-\alpha)\int_{s_j^-}^{s_j^+} B_{M;j}(\sigma)e^{\alpha\sigma}e^{(z-\alpha)\sigma}\, d\sigma \Big] \\ & = & \alpha J - \sum_{j=0}^{N} \Big[-(z-\alpha)\int_{s_j^-}^{s_j^+} B_{M;j}(\sigma)e^{z\sigma}\, d\sigma \Big] \\ & = & \alpha J + (z-\alpha)[Me^{z\cdot} + J] \\ & = & (z-\alpha)Me^{z\cdot} + zJ. \end{array} \tag{7.24}$$

□

Proposition 7.6. *Consider a bounded linear operator* $M : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2$ *and suppose that* $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ *is satisfied. In addition, assume that* M *can be represented in such a way that the following two properties hold.*

(a) There exists $J \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)$ *so that*

$$\zeta_M + JH(\cdot) \in W^{1,1}_{\mathrm{loc}}\big([r_{\min}, r_{\max}]; \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)\big), \tag{7.25}$$

in which H *is the Heaviside function defined in (5.29).*

(b) There exists $\nu \in \mathrm{NBV}\big([r_{\min}, r_{\max}]; \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2)\big)$ *so that*

$$[\zeta_M + JH(\cdot)]'(\sigma) = \nu(\sigma) \tag{7.26}$$

for almost all $\sigma \in [r_{\min}, r_{\max}]$.

Then for any $\alpha \in \mathbb{R}$ *there is a unique bounded linear operator*

$$L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2 \tag{7.27}$$

for which

$$-(z-\alpha)Me^{z\cdot} = Jz - Le^{z\cdot} \tag{7.28}$$

holds for all $z \in \mathbb{C}$. *Normalizing*[4] ν *so that* $\nu(r_{\max}) = 0$, *the operator* L *is then given by*

$$L\phi = \alpha J\phi(0) + \int_{r_{\min}}^{r_{\max}} d\zeta(\sigma)\phi(\sigma) \tag{7.29}$$

in which the NBV function ζ *is given by*

$$\zeta(\sigma) = -\big[\nu(\sigma) + \alpha \int_{r_{\min}}^{\sigma} \nu(\sigma')\, d\sigma'\big]. \tag{7.30}$$

[4]This is possible because the right hand point of an NBV function can be modified at will without destroying the NBV property.

Proof. Considering the operator L defined in (7.29), we write $\widetilde{L}\phi = L\phi - \alpha J\phi(0)$ and note that

$$\zeta_{\widetilde{L}} = \zeta. \tag{7.31}$$

In particular, comparing the definition (7.30) with the expression (7.10), we see that

$$\zeta_{\widetilde{L}_\alpha} = -e^{\alpha \cdot}\nu(\cdot). \tag{7.32}$$

This allows us to compute

$$\begin{array}{lcl} Le^{z\cdot} & = & \alpha J + \widetilde{L}e^{z\cdot} \\ & = & \alpha J + \widetilde{L}_\alpha e^{(z-\alpha)\cdot} \\ & = & \alpha J - \int_{r_{\min}}^{r_{\max}} d[e^{\alpha\cdot}\nu(\cdot)](\sigma)e^{(z-\alpha)\sigma} \\ & = & \alpha J - \int_{r_{\min}}^{r_{\max}} [\alpha e^{\alpha\sigma}\nu(\sigma)]e^{(z-\alpha)\sigma}\,d\sigma - \int_{r_{\min}}^{r_{\max}} e^{\alpha\sigma}d\nu(\sigma)e^{(z-\alpha)\sigma}. \end{array} \tag{7.33}$$

Integrating by parts, we obtain

$$\begin{array}{lcl} Le^{z\cdot} & = & \alpha J - \alpha\int_{r_{\min}}^{r_{\max}}[\zeta_M(\cdot) + JH(\cdot)]'(\sigma)e^{z\sigma}\,d\sigma \\ & & \quad -\nu(r_{\max})e^{zr_{\max}} + \nu(r_{\min})e^{zr_{\min}} + z\int_{r_{\min}}^{r_{\max}} \nu(\sigma)e^{z\sigma}\,d\sigma \\ & = & -\alpha\int_{r_{\min}}^{r_{\max}} d\zeta_M(\sigma)e^{z\sigma} \\ & & \quad -\nu(r_{\max})e^{zr_{\max}} + \nu(r_{\min})e^{zr_{\min}} + z\int_{r_{\min}}^{r_{\max}}[\zeta_M(\cdot) + JH(\cdot)]'(\sigma)e^{z\sigma}\,d\sigma \\ & = & -\alpha\int_{r_{\min}}^{r_{\max}} d\zeta_M(\sigma)e^{z\sigma} \\ & & \quad -\nu(r_{\max})e^{zr_{\max}} + \nu(r_{\min})e^{zr_{\min}} + zJ + z\int_{r_{\min}}^{r_{\max}} d\zeta_M(\sigma)e^{z\sigma} \\ & = & -\alpha Me^{z\cdot} + zJ + zMe^{z\cdot} \end{array} \tag{7.34}$$

as desired. In the last step we have exploited the normalizations $\nu(r_{\min}) = \nu(r_{\max}) = 0$. □

Our final results here explore some useful relations between L and M that we exploit in §7.2. The proofs are based heavily on the explicit factorizations obtained above.

Corollary 7.7. *Consider three bounded linear operators*

$$L : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2, \qquad M : C([r_{\min}, r_{\max}]; \mathcal{H}_1) \to \mathcal{H}_2, \qquad J : \mathcal{H}_1 \to \mathcal{H}_2 \tag{7.35}$$

for which

$$-(z-\alpha)Me^{z\cdot} = Jz - Le^{z\cdot} \tag{7.36}$$

holds for all $z \in \mathbb{C}$. Suppose furthermore that either $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ holds or that $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ and $(\mathrm{hF})_{M;\mathcal{H}_1,\mathcal{H}_2}$ both hold.

Then for any pair $\tau_1 \le \tau_2$ and any function $x \in C([\tau_1 + r_{\min}, \tau_2 + r_{\max}]; \mathcal{H}_1)$ for which $Jx \in C^1([\tau_1, \tau_2]; \mathcal{H}_2)$, the function

$$f(\xi) = M\mathrm{ev}_\xi x \tag{7.37}$$

satisfies the identity

$$(D-\alpha)f(\xi) = -Jx'(\xi) + L\,\mathrm{ev}_\xi\,x \tag{7.38}$$

for all $\tau_1 \le \xi \le \tau_2$.

Proof. Proposition 7.4 implies that

$$\begin{array}{rcl} f(\xi) & = & -Jx(\xi) + L\big[\sigma \mapsto e^{\alpha\sigma}\int_{r_{\min}}^{\sigma} e^{-\alpha\sigma'}x(\xi+\sigma')\,d\sigma'\big] - \alpha J\int_{r_{\min}}^{0} e^{-\alpha\sigma}x(\xi+\sigma)\,d\sigma \\ & = & -Jx(\xi) + e^{\alpha\xi}L\big[\sigma \mapsto e^{\alpha\sigma}\int_{\xi+r_{\min}}^{\xi+\sigma} e^{-\alpha\sigma'}x(\sigma')\,d\sigma'\big] - \alpha e^{\alpha\xi}J\int_{\xi+r_{\min}}^{\xi} e^{-\alpha\sigma}x(\sigma)\,d\sigma. \end{array} \tag{7.39}$$

We hence compute

$$\begin{array}{rcl} (D-\alpha)f(\xi) & = & -Jx'(\xi) + \alpha Jx(\xi) \\ & & +e^{\alpha\xi}L\Big[\sigma \mapsto e^{\alpha\sigma}\big[e^{-\alpha(\xi+\sigma)}x(\xi+\sigma) - e^{-\alpha(\xi+r_{\min})}x(\xi+r_{\min})\big]\Big] \\ & & -\alpha e^{\alpha\xi}J\big(e^{-\alpha\xi}x(\xi) - e^{-\alpha(\xi+r_{\min})}x(\xi+r_{\min})\big) \\ & = & -Jx'(\xi) + L[\sigma \mapsto x(\xi+\sigma)] - e^{\alpha r_{\min}}[Le^{\alpha\cdot}]x(\xi+r_{\min}) + \alpha Je^{-\alpha r_{\min}}x(\xi+r_{\min}) \\ & = & -Jx'(\xi) + L\,\mathrm{ev}_{\xi}\,x - e^{\alpha r_{\min}}[Le^{\alpha\cdot} - \alpha J]x(\xi+r_{\min}) \end{array} \tag{7.40}$$

and recall that $Le^{\alpha\cdot} = \alpha J$. □

Corollary 7.8. *Consider three bounded linear operators*

$$L : C([r_{\min}, r_{\max}];\mathcal{H}_1) \to \mathcal{H}_2, \qquad M : C([r_{\min}, r_{\max}];\mathcal{H}_1) \to \mathcal{H}_2, \qquad J : \mathcal{H}_1 \to \mathcal{H}_2 \tag{7.41}$$

for which

$$-(z-\alpha)Me^{z\cdot} = Jz - Le^{z\cdot} \tag{7.42}$$

holds for all $z \in \mathbb{C}$. *Suppose furthermore that either* $(\mathrm{HFin})_{\mathcal{H}_1,\mathcal{H}_2}$ *holds or that* $(\mathrm{hF})_{L;\mathcal{H}_1,\mathcal{H}_2}$ *and* $(\mathrm{hF})_{M;\mathcal{H}_1,\mathcal{H}_2}$ *both hold.*

Then for any $z \in \mathbb{C}$ *and* $\phi \in C([r_{\min}, r_{\max}];\mathcal{H}_1)$*, we have the identities*

$$\begin{array}{rcl} -(z-\alpha)M\big[\sigma \mapsto e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big] & = & L\big[\sigma \mapsto e^{\alpha\sigma}\int_0^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\,d\sigma'\big] \\ & & -L\big[\sigma \mapsto e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big] \\ & = & M\phi + J\phi(0) - L\big[\sigma \mapsto e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big]. \end{array} \tag{7.43}$$

Proof. As a preparation, note that for $z \neq \alpha$ we have

$$\begin{array}{rcl} e^{\alpha\sigma}\int_0^{\sigma} e^{-\alpha\sigma'}e^{z\sigma'}\int_0^{\sigma'} e^{-z\sigma''}\phi(\sigma'')\,d\sigma''\,d\sigma' & = & e^{\alpha\sigma}\int_0^{\sigma}\Big[\int_{\sigma''}^{\sigma} e^{(z-\alpha)\sigma'}\,d\sigma'\Big]e^{-z\sigma''}\phi(\sigma'')\,d\sigma'' \\ & = & (z-\alpha)^{-1}e^{\alpha\sigma}\int_0^{\sigma}[e^{(z-\alpha)\sigma} - e^{(z-\alpha)\sigma''}]e^{-z\sigma''}\phi(\sigma'')\,d\sigma'' \\ & = & (z-\alpha)^{-1}e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma' \\ & & -(z-\alpha)^{-1}e^{\alpha\sigma}\int_0^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\,d\sigma'. \end{array} \tag{7.44}$$

Proposition 7.4 now allows us to compute

$$\begin{array}{rcl} M\big[\sigma \mapsto e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big] & = & L\big[\sigma \mapsto e^{\alpha\sigma}\int_0^{\sigma} e^{-\alpha\sigma'}e^{z\sigma'}\int_0^{\sigma'} e^{-z\sigma''}\phi(\sigma'')\,d\sigma''\,d\sigma'\big] \\ & = & (z-\alpha)^{-1}L\big[\sigma \mapsto e^{z\sigma}\int_0^{\sigma} e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big] \\ & & -(z-\alpha)^{-1}L\big[\sigma \mapsto e^{\alpha\sigma}\int_0^{\sigma} e^{-\alpha\sigma'}\phi(\sigma')\,d\sigma'\big]. \end{array} \tag{7.45}$$

□

7.2 Exponential splittings for differential-algebraic systems

Our aim here is to prove Lemma 2.4 and Theorem 2.5, following the approach developed in [4, §3]. Our starting point is the identity

$$\mathcal{J}_\alpha(z)\delta_{\mathcal{I},M}(z) = \Delta_L(z) \tag{7.46}$$

formulated in $(\mathrm{HAlg})_{\mathcal{I},M}$. We introduce the expansion

$$\mathcal{J}_\alpha(z) = J_0 + J_1(z-\alpha) + \ldots + J_{\ell_*}(z-\alpha)^{\ell_*} \tag{7.47}$$

with $\ell_* = \max\{\ell_1, \ldots, \ell_n\}$. Here the matrices $J_i \in \mathbb{R}^{n\times n}$ satisfy $J_i^2 = J_i$ for $0 \le i \le \ell_*$ and $J_i J_j = 0$ for $i \neq j$. In addition, we have

$$I = J_0 + J_1 + \ldots + J_{\ell_*}. \tag{7.48}$$

Using the identity $\mathcal{J}_\alpha(\alpha) = \mathcal{I}$, we see that $\mathcal{I} = J_0$. In particular, (7.46) can be stated as

$$\big[J_0 + J_1(z-\alpha) + \ldots + J_{\ell_*}(z-\alpha)^{\ell_*}\big][J_0 z - Me^{z\cdot}] = z - Le^{z\cdot}. \tag{7.49}$$

Multiplying (7.49) by J_0 gives

$$J_0 z - J_0 M e^{z\cdot} = J_0 z - J_0 L e^{z\cdot}, \tag{7.50}$$

which implies that $J_0 M = J_0 L$. On the other hand, multiplying (7.49) by J_k gives

$$-(z-\alpha)^k J_k M e^{z\cdot} = J_k z - J_k L e^{z\cdot} \tag{7.51}$$

for $1 \le k \le \ell_*$. Repeatedly applying Proposition 7.4 and appropriately padding with zeroes, we find that for each $1 \le k \le \ell_*$ and $0 \le s \le k-1$, there is a bounded linear operator

$$M_{k,s} : C\big([r_{\min}, r_{\max}]; \mathcal{H}\big) \to \mathcal{H} \tag{7.52}$$

so that

$$-(z-\alpha)^s J_k M e^{z\cdot} = -M_{k,s} e^{z\cdot} \tag{7.53}$$

holds for all $z \in \mathbb{C}$. We now record a number of useful facts concerning these operators.

Lemma 7.9. *Consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ and suppose that $(\mathrm{HF})_M$, (HS) and $(\mathrm{HAlg})_{\mathcal{I},M}$ are all satisfied. Then for all $1 \le k \le \ell_*$, we have the identities*

$$J_k M \phi \quad = \quad M_{k,0}\phi \tag{7.54}$$

together with

$$\begin{aligned} J_k L[\sigma \mapsto e^{z\sigma}\textstyle\int_0^\sigma e^{-z\sigma'}\phi(\sigma')\,d\sigma'] \quad &= \quad M_{k,k-1}\phi + J_k\phi(0) \\ &\qquad +(z-\alpha)M_{k,k-1}\big[\sigma \mapsto e^{z\sigma}\textstyle\int_0^\sigma e^{-z\sigma'}\phi(\sigma')\,d\sigma'\big]. \end{aligned} \tag{7.55}$$

Proof. The first identity follows from inspection of (7.53). The second identity follows upon applying Corollary 7.8 to the identity

$$-(z-\alpha)M_{k,k-1}e^{z\cdot} = J_k z - J_k L e^{z\cdot}, \tag{7.56}$$

which is a direct consequence of (7.53). □

Lemma 7.10. *Consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ and suppose that* $(\mathrm{HF})_M$*,* (HS) *and* $(\mathrm{HAlg})_{\mathcal{I},M}$ *are all satisfied. Then for all $2 \le k \le \ell_*$ and $1 \le s \le k-1$, we have the identity*

$$\begin{array}{lcl} M_{k,s}[\sigma \mapsto e^{z\sigma} \int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'] & = & (z-\alpha) M_{k,s-1}\big[\sigma \mapsto e^{z\sigma} \int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ & & \quad + M_{k,s-1}\phi. \end{array} \tag{7.57}$$

Proof. This follows upon applying Corollary 7.8 to the identity

$$-(z-\alpha) M_{k,s} e^{z\cdot} = -M_{k,s+1} e^{z\cdot}. \tag{7.58}$$

□

Proof of Lemma 2.4. Applying Lemma 7.9 with $k = 1$ and $z = \alpha$, we find

$$J_1 M\phi = -J_1\phi(0) + J_1 L\big[\sigma \mapsto e^{\alpha\sigma} \int_0^\sigma e^{-\alpha\sigma'} \phi(\sigma')\, d\sigma'\big]. \tag{7.59}$$

For $2 \le k \le \ell_*$, we may apply Lemma 7.10 with $s = 1$ and $z = \alpha$ to find

$$J_k M\phi = M_{k,1}\big[\sigma \mapsto e^{\alpha\sigma} \int_0^\sigma e^{-\alpha\sigma'} \phi(\sigma')\, d\sigma'\big]. \tag{7.60}$$

In particular, we have the identity

$$\begin{array}{lcl} [I - J_0]M\phi & = & (J_1 + \ldots + J_{\ell_*})M\phi \\ & = & (J_1 L + M_{2,1} + \ldots + M_{\ell_*,1})\big[\sigma \mapsto e^{\alpha\sigma} \int_0^\sigma e^{-\alpha\sigma'} \phi(\sigma')\, d\sigma'\big] - J_1\phi(0). \end{array} \tag{7.61}$$

The continuity claim follows directly from this representation. Indeed, the only term that can cause trouble is $J_1\phi(0)$, but this is avoided by using $\widehat{M}_+$ on $[0, \infty)$ and $\widehat{M}_-$ on $(-\infty, 0]$. □

For any integer $1 \le \ell \le \ell_*$, we now define the function $\delta_\ell : \mathbb{C} \to \mathcal{L}(\mathcal{H}; \mathcal{H})$ in such a way that

$$\Delta_{L(z)} = (J_0 + J_1(z-\alpha) + \ldots + J_{\ell-1}(z-\alpha)^{\ell-1})\delta_{\mathcal{I},M}(z) + (z-\alpha)^\ell \delta_\ell(z) \tag{7.62}$$

holds for all $z \in \mathbb{Z}$. As usual, we assume that we have complexified $\mathcal{H}$ here. In addition, for all such integers we introduce the bounded linear operator

$$K_\ell : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H} \tag{7.63}$$

that acts as

$$K_\ell = M_{\ell,0} + M_{\ell+1,1} + \ldots + M_{\ell_*, \ell_*-\ell} = \sum_{k=\ell}^{\ell_*} M_{k,k-\ell}. \tag{7.64}$$

Lemma 7.11. *Consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ and suppose that* $(\mathrm{HF})_M$*,* (HS) *and* $(\mathrm{HAlg})_{\mathcal{I},M}$ *are all satisfied. Then for any $1 \le \ell \le \ell_*$ we have*

$$\delta_\ell(z) = -K_\ell e^{z\cdot} \tag{7.65}$$

for all $z \in \mathbb{C}$.

Proof. Rewriting (7.62) in the form

$$(z-\alpha)^\ell \delta_\ell(z) \quad = \quad -(z-\alpha)^\ell \big[J_\ell + J_{\ell+1}(z-\alpha) + \ldots + J_{\ell_*}(z-\alpha)^{\ell_*-\ell} \big] M e^{z\cdot}, \tag{7.66}$$

we compute

$$\begin{aligned} \delta_l(z) &= -\big[J_\ell + J_{\ell+1}(z-\alpha) + \ldots + J_{\ell_*}(z-\alpha)^{\ell_*-\ell} \big] M e^{z\cdot}, \\ &= -[M_{\ell,0} + M_{\ell+1,1} + \ldots + M_{\ell_*,\ell_*-\ell}] e^{z\cdot} \\ &= -K_\ell e^{z\cdot}. \end{aligned} \tag{7.67}$$

□

Lemma 7.12. *Consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ and suppose that* $(\mathrm{HF})_M$, (HS) *and* $(\mathrm{HAlg})_{\mathcal{I},M}$ *are all satisfied. Then we have the identity*

$$\begin{aligned} L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] &= (I - J_0)\phi(0) \\ &\quad + \mathcal{J}_\alpha(z) M[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &\quad + \textstyle\sum_{i=1}^{\ell_*} (z-\alpha)^{i-1} K_i \phi. \end{aligned} \tag{7.68}$$

Proof. For any integer $1 \le k \le \ell_*$, Lemma 7.9 implies

$$\begin{aligned} J_k L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] &= J_k \phi(0) + M_{k,k-1}\phi + (z-\alpha) M_{k,k-1}[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &= J_k \phi(0) + \textstyle\sum_{i=1}^{k} (z-\alpha)^{i-1} M_{k,k-i}\phi \\ &\quad + (z-\alpha)^k M_{k,0}[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big]. \end{aligned} \tag{7.69}$$

Recalling (7.48), the fact that $M_{k,0} = J_k M$ for $1 \le k \le \ell_*$ and the identity $J_0 M = J_0 L$, we hence see

$$\begin{aligned} L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] &= J_0 L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &\quad + \textstyle\sum_{k=1}^{\ell_*} J_k L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &= J_0 M[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &\quad + \textstyle\sum_{k=1}^{\ell_*} J_k \phi(0) \\ &\quad + \textstyle\sum_{k=1}^{\ell_*} \sum_{i=1}^{k} (z-\alpha)^{i-1} M_{k,k-i}\phi \\ &\quad + \textstyle\sum_{k=1}^{\ell_*} (z-\alpha)^k J_k M[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big]. \end{aligned} \tag{7.70}$$

Rearranging, we find

$$\begin{aligned} L[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] &= \mathcal{J}_\alpha(z) M[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &\quad + (I - J_0)\phi(0) \\ &\quad + \textstyle\sum_{i=1}^{\ell_*} (z-\alpha)^{i-1} \sum_{k=i}^{\ell_*} M_{k,k-i}\phi \\ &= \mathcal{J}_\alpha(z) M[\sigma \mapsto e^{z\sigma} \textstyle\int_0^\sigma e^{-z\sigma'} \phi(\sigma')\, d\sigma'\big] \\ &\quad + (I - J_0)\phi(0) \\ &\quad + \textstyle\sum_{i=1}^{\ell_*} (z-\alpha)^{i-1} K_i \phi \end{aligned} \tag{7.71}$$

as desired. □

Proof of Theorem 2.5. We can apply the same techniques as in the proof of Thm. 3.16 in [4]. Indeed, the crucial identity (7.68) above is the analogue of [4, Eq. 5.132]. This allows the computations in Lemma 5.8 - 5.11 from [4] to be copied almost verbatim, linking the Laplace transform of the differential-algebraic system (2.35) to that of the associated differential system (2.9). □

7.3 Algebraic Wiener-Hopf factorizations

In this subsection we establish the results stated in §3.3, using the explicit factorizations in §7.1 to transfer the techniques from (3.2) to the scalar differential-algebraic setting.

Proof of Proposition 3.10. Pick $\eta \in \mathbb{R}$ in such a way that $\delta_{0,M}(z) = 0$ has no roots with $\mathrm{Re}\, z = \eta$. Propositions 7.4 and 7.6 allow us to construct a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ for which $\delta_M(z) = (z - \eta)^{-\ell}\Delta_L(z)$ holds. In addition, Proposition 3.6 allows us to find

$$L_- \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{C}); \mathbb{C}\big), \qquad L_+ \in \mathcal{L}\big(C([0, r_{\max}]; \mathbb{C}); \mathbb{C}\big) \tag{7.72}$$

for which

$$(z - \eta)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z) \tag{7.73}$$

holds for all $z \in \mathbb{C}$. Writing $\ell_\pm \geq 0$ for the order of $z = \eta$ as a root of $\Delta_{L_\pm}(z) = 0$, we see that $\ell_- + \ell_+ = \ell + 1 \geq 2$. By root-swapping in the sense of [20, Lem. 5.7], we can hence ensure that $\ell_\pm \geq 1$. Applying Proposition 7.4 once more, we find

$$M_- \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{C}); \mathbb{C}\big), \qquad M_+ \in \mathcal{L}\big(C([0, r_{\max}]; \mathbb{C}); \mathbb{C}\big), \tag{7.74}$$

with

$$\delta_{0,M_\pm}(z) = (z - \eta)^{-\ell_\pm}\Delta_{L_\pm}(z), \tag{7.75}$$

which implies that $(\mathrm{HAlgSc})_{M_\pm}$ both hold. This gives the desired factorization

$$\delta_{0,M}(z) = \delta_{0,M_-}(z)\delta_{0,M_+}(z). \tag{7.76}$$

□

We note that if M satisfies $(\mathrm{HAlgSc})_M$, then Propositions 7.4 and 7.6 imply that for any $\alpha \in \mathbb{R}$ there is a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ so that

$$\delta_{0,M}(z) = (z - \alpha)^{-\ell}\Delta_{L[M;\alpha,\ell]}(z). \tag{7.77}$$

Throughout the remainder of this section we will use the notation

$$L = L[M; \alpha, \ell] \tag{7.78}$$

to refer to this operator.

Lemma 7.13. *Fix $r_{\min} \leq 0 \leq r_{\max}$, consider a bounded linear operator $M : C([r_{\min}, r_{\max}]; \mathbb{C}) \to \mathbb{C}$ that satisfies both $(\mathrm{HRnk})_M$ and $(\mathrm{HAlgSc})_M$ and suppose that $(M_-, \ell_-, M_+, \ell_+)$ is a Wiener-Hopf set for M. Pick $\eta \in \mathbb{R}$ in such a way that $\delta_{0,M}(z) = 0$ has no roots with $\mathrm{Re}\, z = \eta$.*

Then for all sufficiently small $\epsilon > 0$ we have

$$m^\sharp_M(\eta) = n^\sharp_{L[M;\eta,\ell]}(\eta - \epsilon) + \frac{1}{2}\ell = n^\sharp_{L[M;\eta,\ell]}(\eta + \epsilon) - \frac{1}{2}\ell. \tag{7.79}$$

In addition, for every $\gamma > \eta$, we have

$$m^\sharp_M(\eta) = n^\sharp_{L[M;\gamma,\ell]}(\eta) + \frac{1}{2}\ell, \tag{7.80}$$

while for every $\gamma < \eta$ we have

$$m^\sharp_M(\eta) = n^\sharp_{L[M;\gamma,\ell]}(\eta) - \frac{1}{2}\ell. \tag{7.81}$$

Proof. Writing

$$L = L[M;\eta,\ell], \qquad L_+ = L_+[M_+;\eta,\ell_+], \qquad L_- = L_-[M_-;\eta,\ell_-], \tag{7.82}$$

we see that

$$(z-\eta)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z). \tag{7.83}$$

Notice that for all sufficiently small $\epsilon > 0$ we have

$$\begin{array}{rcl} n^-_{L_+}(\eta-\epsilon) & = & m^-_{M_+}(\eta), \\ n^-_{L_+}(\eta+\epsilon) & = & \ell_+ + m^-_{M_+}(\eta), \\ n^+_{L_-}(\eta-\epsilon) & = & \ell_- + m^+_{M_-}(\eta), \\ n^+_{L_-}(\eta+\epsilon) & = & m^+_{M_-}(\eta), \end{array} \tag{7.84}$$

together with

$$\begin{array}{rcl} n^+_\eta(\eta-\epsilon) & = & 1 \\ n^+_\eta(\eta+\epsilon) & = & 0. \end{array} \tag{7.85}$$

We may hence explicitly compute

$$\begin{array}{rcl} n^\sharp_L(\eta-\epsilon) & = & m^-_{M_+}(\eta) - (\ell_- + m^+_{M_-}) + 1 \\ & = & m^\sharp_M(\eta) + \frac{1}{2} - \frac{1}{2}(\ell_+ + \ell_-) \\ & = & m^\sharp_M(\eta) - \frac{1}{2}\ell, \end{array} \tag{7.86}$$

together with

$$\begin{array}{rcl} n^\sharp_L(\eta+\epsilon) & = & \ell_+ + m^-_{M_+}(\eta) - m^+_{M_-}(\eta) + 0 \\ & = & m^\sharp_M(\eta) - \frac{1}{2} + \frac{1}{2}(\ell_+ + \ell_-). \\ & = & m^\sharp_M(\eta) + \frac{1}{2}\ell. \end{array} \tag{7.87}$$

The identities concerning $n^\sharp_{L[M;\gamma,\ell]}$ follow in a similar fashion. □

Proof of Proposition 3.11. Any Wiener-Hopf set for M leads to a factorization for $L[M;\eta,\ell]$ via (7.82)-(7.83). The invariance of $m^\sharp_M(\eta)$ hence follows directly from the invariance of $n^\sharp_L(\eta)$. □

Proof of Proposition 3.13. Write $L = L[M;\eta,\ell]$ and pick $\epsilon > 0$ sufficiently small. Combining Theorem 2.5 with Proposition 3.9, we obtain

$$\begin{array}{rclcrcl} \dim K^+_{p_{0,M}(\eta)} & = & \dim K^+_{P_L(\eta+\epsilon)} & & \operatorname{codim} R^+_{p_{0,M}(\eta)} & = & \operatorname{codim} R^+_{P_L(\eta+\epsilon)} \\ & = & \max\{-n^\sharp_L(\eta+\epsilon), 0\}, & & & = & \max\{n^\sharp_L(\eta+\epsilon), 0\}, \\ \dim K^+_{\widehat{p}_{0,M}(\eta)} & = & \dim K^+_{\widehat{P}_L(\eta+\epsilon)} & & \operatorname{codim} R^+_{\widehat{p}_{0,M}(\eta)} & = & \operatorname{codim} R^+_{\widehat{P}_L(\eta+\epsilon)} \\ & = & \max\{1 - n^\sharp_L(\eta+\epsilon), 0\}, & & & = & \max\{n^\sharp_L(\eta+\epsilon) - 1, 0\}, \end{array} \tag{7.88}$$

together with

$$\begin{array}{rclcrcl} \dim K^-_{q_{0,M}(\eta)} & = & \dim K^-_{Q_L(\eta-\epsilon)} & & \operatorname{codim} R^-_{q_{0,M}(\eta)} & = & \operatorname{codim} R^-_{Q_L(\eta-\epsilon)} \\ & = & \max\{n^\sharp_L(\eta-\epsilon)-1, 0\}, & & & = & \max\{1-n^\sharp_L(\eta-\epsilon), 0\}, \\ \dim K^-_{\widehat{q}_{0,M}(\eta)} & = & \dim K^-_{\widehat{Q}_L(\eta-\epsilon)} & & \operatorname{codim} R^-_{\widehat{q}_{0,M}(\eta)} & = & \operatorname{codim} R^-_{\widehat{Q}_L(\eta-\epsilon)} \\ & = & \max\{n^\sharp_L(\eta-\epsilon), 0\}, & & & = & \max\{-n^\sharp_L(\eta-\epsilon), 0\}. \end{array} \tag{7.89}$$

In addition, we have

$$\dim \widehat{K}^+_{\widehat{p}_{0,M}} = \dim K^+_{p_{0,M}}, \qquad \operatorname{codim} \widehat{R}^+_{\widehat{p}_{0,M}} = \operatorname{codim} R^+_{p_{0,M}}, \tag{7.90}$$

together with

$$\dim \widehat{K}^-_{\widehat{q}_{0,M}} = \dim K^-_{q_{0,M}}, \qquad \operatorname{codim} \widehat{R}^-_{\widehat{q}_{0,M}} = \operatorname{codim} R^-_{q_{0,M}}. \tag{7.91}$$

The desired expressions now follow immediately from Lemma 7.13. □

Proof of Proposition 3.12. Let us first suppose that $(\alpha_0 - \eta)(\alpha_1 - \eta) \neq 0$. Lemma 7.13 then implies that

$$m^\sharp_{M_1}(\eta) \quad = \quad n^\sharp_{\Gamma(1)}(\eta) + \operatorname{sign}(\alpha_1 - \eta)\tfrac{1}{2}\ell, \tag{7.92}$$

in which $\operatorname{sign}(x) = 1$ for $x > 0$ and -1 for $x < 0$. Similarly, we have

$$m^\sharp_{M_0}(\eta) \quad = \quad n^\sharp_{\Gamma(0)}(\eta) + \operatorname{sign}(\alpha_0 - \eta)\tfrac{1}{2}\ell. \tag{7.93}$$

We also know

$$n^\sharp_{\Gamma(1)}(\eta) - n^\sharp_{\Gamma(0)}(\eta) = -\operatorname{cross}(\Gamma;\eta), \tag{7.94}$$

which gives

$$m^\sharp_{M_1}(\eta) - m^\sharp_{M_0}(\eta) = -\operatorname{cross}(\Gamma;\eta) + \frac{1}{2}\ell[\operatorname{sign}(\alpha_1 - \eta) - \operatorname{sign}(\alpha_0 - \eta)]. \tag{7.95}$$

This is equivalent to the stated result.

If $\min(\alpha_1, \alpha_2) \geq \eta$, we choose $\epsilon > 0$ sufficiently small to ensure that $\delta_{0,M_i}(z) = 0$ has no roots with $\eta - \epsilon \leq \operatorname{Re} z \leq \eta$ for both $i = 0$ and $i = 1$. Applying the computation above with $\eta \mapsto \eta - \epsilon$, we find

$$\begin{array}{rcl} m^\sharp_{M_1}(\eta) - m^\sharp_{M_0}(\eta) & = & m^\sharp_{M_1}(\eta-\epsilon) - m^\sharp_{M_0}(\eta-\epsilon) \\ & = & -\operatorname{cross}(\Gamma;\eta-\epsilon). \end{array} \tag{7.96}$$

A similar computation covers the case $\max(\alpha_1, \alpha_2) \leq \eta$. □

8 Fourier decompositions

In this section we prove the main results stated in §3.4. We start in §8.1 by showing how solutions to the differential system (2.9) posed on $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ can be approximated by solutions taking values in $\ell^1(\mathbb{Z};\mathbb{R}^n)$. The latter class of solutions is easier to handle in frequency space as the Fourier transform is well-posed in a pointwise fashion.

We proceed in §8.2 by studying the frequency dependence of the restriction operators (3.5) associated to the Fourier components $L(\omega)$. In particular, we will obtain frequency-independent bounds on the inverses of these restriction operators that reference L^2-based norms in a sense similar to Proposition 5.5.

These bounds are subsequently used in §8.3 to show that the range $\widehat{R}^-_{\widehat{Q}_L(\eta)}$ can be written as the closure of a set of functions that all have smooth Fourier transforms with frequency components that belong to the appropriate $\widehat{R}^-_{\widehat{Q}_{L(\omega)}(\eta)}$. We characterize the kernel $\widehat{K}^-_{\widehat{Q}_L(\eta)}$ in a similar fashion, allowing us to obtain the direct sum decomposition of $\widehat{Q}_L(\eta)$ described in (3.88). We conclude in §8.4 by describing some minor adjustments that allow the remaining operators $\pi^-_{\widehat{Q}_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$ to be incorporated into the framework developed here.

8.1 Preparations

We start by considering the invertability of the characteristic functions $\Delta_L(z)$ and the relation with the Fourier components $L(\omega)$. We consider $\Delta_L(z)$ as operators in both $\mathcal{L}\big(\ell^1(\mathbb{Z};\mathbb{R}^n)\big)$ and $\mathcal{L}\big(\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$. In the latter case the Fourier transform readily provides the link with $\Delta_{L(\omega)}(z)$, while in the former case the following technical result from the field of Banach algebras plays a key role.

Proposition 8.1 (see [16, Thm. 3]). *Let $h \in \ell^1(\mathbb{Z};\mathbb{R}^{n\times n})$. Then the map $T_h : \ell^1(\mathbb{Z};\mathbb{R}^n) \to \ell^1(\mathbb{Z};\mathbb{R}^n)$ defined by*

$$[T_h v]_i = \sum_{j\in\mathbb{Z}} h_{i-j} v_j \tag{8.1}$$

is invertible if and only if $\det[\mathcal{F}h](\omega) \neq 0$ for all $\omega \in [-\pi,\pi]$.

Corollary 8.2. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Pick $\eta \in \mathbb{R}$. Then the following two statements are equivalent.*

(i) The characteristic operator $\Delta_L(z) \in \mathcal{L}\big(\ell^2(\mathbb{Z};\mathbb{R}^n);\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$ is invertible for all $\operatorname{Re} z = \eta$.

(ii) The characteristic function $\det \Delta_{L(\omega)}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$ for all $-\pi \leq \omega \leq \pi$.

In addition, if either (i) or (ii) holds, then there exists $\epsilon > 0$ and $K \geq 1$ so that for all $z \in \mathbb{C}$ with $|\operatorname{Re} z - \eta| < \eta$ the characteristic operator $\Delta_L(z)$ is invertible both in $\mathcal{L}\big(\ell^1(\mathbb{Z};\mathbb{R}^n)\big)$ and $\mathcal{L}\big(\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$, with

$$\big\|\Delta_L(z)^{-1}\big\|_{\mathcal{L}\left(\ell^1(\mathbb{Z};\mathbb{R}^n);\ell^1(\mathbb{Z};\mathbb{R}^n)\right)} + \big\|\Delta_L(z)^{-1}\big\|_{\mathcal{L}\left(\ell^2(\mathbb{Z};\mathbb{R}^n);\ell^2(\mathbb{Z};\mathbb{R}^n)\right)} \leq \frac{K}{1+|z|}. \tag{8.2}$$

Proof. The equivalence between (i) and (ii) follows from the fact that for all $v \in \ell^2(\mathbb{R}^n)$ we have

$$\mathcal{F}[\Delta_L(z)v](\omega) = \Delta_{L(\omega)}(z)[\mathcal{F}v](\omega) \tag{8.3}$$

and the fact that $\omega \mapsto \Delta_{L(\omega)}(z)^{-1} \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$ is continuous and hence bounded.

The final statement follows from Proposition 8.1, utilizing the continuity of the map

$$\eta + i\mathbb{R} \ni z \mapsto \Delta_L(z) \in \mathcal{L}\big(\ell^1(\mathbb{Z};\mathbb{R}^n);\ell^1(\mathbb{Z};\mathbb{R}^n)\big) \cap \mathcal{L}\big(\ell^2(\mathbb{Z};\mathbb{R}^n);\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \tag{8.4}$$

together with the fact that $\Delta_L(z) - zI$ can be uniformly bounded with respect to both norms on vertical strips in the complex plane. □

In view of the result above, we introduce the following assumption.

$(\mathrm{h}\omega)_{L;\eta}$ For each $\omega \in [-\pi, \pi]$, the equation $\det \Delta_{L(\omega)}(z) = 0$ has no roots with $\mathrm{Re}\, z = \eta$.

We now show how $\ell^2(\mathbb{Z}; \mathbb{R}^n)$-valued solutions can be approximated by $\ell^1(\mathbb{Z}; \mathbb{R}^n)$-valued ones. Proposition 5.5 provides the key to this result, as it shows how to extract solutions in $\widehat{\mathfrak{Q}}_{L;\ell^1(\mathbb{Z};\mathbb{R})}(\eta)$ from arbitrary functions

$$\widehat{\phi} \in C\big([r_{\min}, 0]; \ell^1(\mathbb{Z}; \mathbb{R})\big) \times C\big([0, r_{\max}]; \ell^1(\mathbb{Z}; \mathbb{R})\big). \tag{8.5}$$

In order to state this result, we introduce the notation

$$\widehat{Q}_{L;\mathcal{B}}(\eta) = \big\{\widehat{\psi} \in C([r_{\min}, 0]; \mathcal{B}) \times C([0, r_{\max}]; \mathcal{B}) : \widehat{\psi} = \widehat{\mathrm{ev}}_0\, \widehat{y} \text{ for some } \widehat{y} \in \widehat{\mathfrak{Q}}_{L;\mathcal{B}}(\eta)\big\}. \tag{8.6}$$

Lemma 8.3. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Pick $\eta \in \mathbb{R}$ in such a way that $(\mathrm{h}\omega)_{L;\eta}$ is satisfied.*

Then for any $\widehat{q} \in \widehat{Q}_{L;\ell^2(\mathbb{Z};\mathbb{R}^n)}(\eta)$ there exists a sequence $\{\widehat{q}_j\}_{j=1}^{\infty} \subset \widehat{Q}_{L;\ell^1(\mathbb{Z};\mathbb{R}^n)}(\eta)$ for which

$$\big\|q^- - q_j^-\big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \big\|q^+ - q_j^+\big\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} \to 0 \tag{8.7}$$

as $j \to \infty$.

Proof. Fix $(q^-, q^+) \in \widehat{Q}_{L;\ell^2(\mathbb{Z};\mathbb{R}^n)}(\eta)$. The density of $\ell^1(\mathbb{Z}; \mathbb{R}^n)$ in $\ell^2(\mathbb{Z}; \mathbb{R}^n)$ allows us to construct a sequence

$$(\phi_j^-, \phi_j^+) \in C\big([r_{\min}, 0]; \ell^1(\mathbb{Z}; \mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^1(\mathbb{Z}; \mathbb{R}^n)\big) \tag{8.8}$$

for which

$$\big\|\phi_j^- - q^-\big\|_{C([0,r_{\min};\ell^2(\mathbb{Z};\mathbb{R}^n))} + \big\|\phi_j^+ - q^+\big\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} \to 0 \text{ as } j \to \infty. \tag{8.9}$$

We now write

$$(q_j^-, q_j^+) = \widehat{\mathrm{ev}}_0 E_{\widehat{Q}_L(\eta)}[\phi_j^-, \phi_j^+], \tag{8.10}$$

which by Proposition 5.5 implies $(q_j^-, q_j^+) \in \widehat{Q}_{L;\ell^1(\mathbb{Z};\mathbb{R}^n)}(\eta)$.

We now compute

$$\begin{array}{lcl} \big\|q_j^- - q^-\big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} & = & \Big\|q^- - \mathrm{ev}_0^- E_{\widehat{Q}_L(\eta)}[\phi_j^-, \phi_j^+]\Big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} \\ & = & \Big\|\mathrm{ev}_0^- E_{\widehat{Q}_L(\eta)}[q^-, q^+] - \mathrm{ev}_0^- E_{\widehat{Q}_L(\eta)}[\phi_j^-, \phi_j^+]\Big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} \\ & = & \Big\|\mathrm{ev}_0^- E_{\widehat{Q}_L(\eta)}\big[q^- - \phi_j^-, q^+ - \phi_j^+\big]\Big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))}. \end{array} \tag{8.11}$$

Lemma 5.1 implies that

$$\mathrm{ev}_0^- E_{\widehat{Q}_L(\eta)} \in \mathcal{L}\Big(C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big); C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big)\Big), \tag{8.12}$$

which shows that $\big\|q_j^- - q^-\big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} \to 0$ as $j \to \infty$. The proof can be completed by obtaining the analogous estimates for q^+ and q_j^+. □

Our next task is to show how a class of elements of $\widehat{Q}_{L;\ell^2(\mathbb{Z};\mathbb{R}^n)}$ can be constructed from suitably prepared Fourier transforms. As a preparation, we fix $\eta \in \mathbb{R}$, introduce the shorthands

$$\widehat{Q}_\omega = \widehat{Q}_{L(\omega)}(\eta), \qquad P_\omega = P_{L(\omega)}(\eta) \tag{8.13}$$

and study how these spaces vary with the frequency ω.

Lemma 8.4. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Pick $\eta \in \mathbb{R}$ in such a way that also $(\mathrm{h}\omega)_{L;\eta}$ is satisfied.*

Then for each $\omega \in [-\pi, \pi]$ we have the decomposition

$$C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n) = \widehat{Q}_\omega \oplus P_\omega, \tag{8.14}$$

with associated projections

$$\begin{array}{rcl} \Pi_{\widehat{Q}_\omega} & : & C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n) \to \widehat{Q}_\omega, \\ \Pi_{P_\omega} & : & C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n) \to P_\omega. \end{array} \tag{8.15}$$

This decomposition varies continuously in ω, in the sense that for each fixed $\omega_0 \in [-\pi, \pi]$ there exists $\delta_{\omega_0} > 0$ together with continuous maps

$$\begin{array}{rcl} \omega & \mapsto & u^*_{\widehat{Q}_{\omega_0}}(\omega) \in \mathcal{L}\big(\widehat{Q}_{\omega_0}, C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n)\big), \\ \omega & \mapsto & u^*_{P_{\omega_0}}(\omega) \in \mathcal{L}\big(P_{\omega_0}, C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n)\big), \end{array} \tag{8.16}$$

defined for all $\omega \in [-\pi, \pi]$ that have $|\omega - \omega_0| < \delta_{\omega_0}$, so that

$$\widehat{Q}_\omega = u^*_{\widehat{Q}_{\omega_0}}(\omega)\big(\widehat{Q}_{\omega_0}\big), \qquad P_\omega = u^*_{P_{\omega_0}}(\omega)\big(P_{\omega_0}\big), \tag{8.17}$$

with

$$\Pi_{\widehat{Q}_{\omega_0}} u^*_{\widehat{Q}_{\omega_0}}(\omega) = I, \qquad \Pi_{P_{\omega_0}} u^*_{P_{\omega_0}}(\omega) = I. \tag{8.18}$$

Proof. The decomposition (8.14) follows from Theorems 2.1 and 2.3. Exploiting the continuity of the family $\omega \mapsto L(\omega)$, the remaining statements follow from the results in [14, §5]. □

Lemma 8.5. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Pick $\eta \in \mathbb{R}$ in such a way that $(\mathrm{h}\omega)_{L;\eta}$ is satisfied.*

Consider any function

$$\widehat{\phi} = (\phi^-, \phi^+) \in C([r_{\min}, 0] \times [-\pi, \pi];\mathbb{R}^n) \times C([0, r_{\max}] \times [-\pi, \pi];\mathbb{R}^n) \tag{8.19}$$

with the property that

$$\widehat{\phi}(\cdot, \omega) \in \widehat{Q}_\omega \subset C([r_{\min}, 0];\mathbb{R}^n) \times C([0, r_{\max}];\mathbb{R}^n) \tag{8.20}$$

for each ω. Then the inverse Fourier transforms

$$\begin{array}{rcll} q^-(\sigma) & = & \mathcal{F}_{\mathrm{inv}}\, \phi^-(\sigma, \cdot), & \qquad r_{\min} \leq \sigma \leq 0, \\ q^+(\sigma) & = & \mathcal{F}_{\mathrm{inv}}\, \phi^+(\sigma, \cdot), & \qquad 0 \leq \sigma \leq r_{\max} \end{array} \tag{8.21}$$

satisfy

$$\widehat{q} = (q^-, q^+) \in \widehat{Q}_L(\eta). \tag{8.22}$$

Proof. The compactness of the rectangle implies that

$$\begin{array}{rcl} \phi^- & \in & C\big([r_{\min}, 0]; C([-\pi, \pi];\mathbb{R}^n)\big) \cap C\big([-\pi, \pi]; C([r_{\min}, 0];\mathbb{R}^n)\big), \\ \phi^+ & \in & C\big([0, r_{\max}]; C([-\pi, \pi];\mathbb{R}^n)\big) \cap C\big([-\pi, \pi]; C([0, r_{\max}];\mathbb{R}^n)\big), \end{array} \tag{8.23}$$

with norms that are independent of the above-mentioned spaces. For every $\omega \in [-\pi, \pi]$ we now define a function $\widehat{y}(\cdot, \omega) \in \widehat{BC}_\eta(\mathbb{R}^n)$ that has

$$\widehat{\mathrm{ev}}_0\, \widehat{y} = \widehat{\phi}(\cdot, \omega) \tag{8.24}$$

together with

$$\widehat{y}(\xi, \omega) = \big[E_{\widehat{Q}_\omega} \widehat{\phi}(\cdot, \omega)\big](\xi) \tag{8.25}$$

for $\xi > r_{\max}$. The second inclusions above for $\phi^{\pm}$, the continuity of the map

$$(\xi, \omega) \to \big[E_{\widehat{Q}_\omega} \cdot\big](\xi) \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n); \mathbb{R}^n\big) \tag{8.26}$$

for $\xi > 0$ and the fact that $\widehat{\phi}(\cdot, \omega) \in \widehat{Q}_\omega$ imply that $\widehat{y}(\cdot, \omega) \in \widehat{\mathfrak{Q}}_{L(\omega)}(\eta)$ for all $\omega \in [-\pi, \pi]$, but also that the map

$$(0, \infty) \ni \xi \mapsto \big(\omega \mapsto \widehat{y}(\xi, \omega)\big) \in C([-\pi, \pi]; \mathbb{R}^n) \tag{8.27}$$

is continuous. In addition, the estimate (5.17) yields the bound

$$\|\widehat{y}(\xi, \cdot)\|_{C([-\pi,\pi];\mathbb{R}^n)} \le C e^{\eta \xi} \, \|\widehat{y}\|_{C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n)} \,. \tag{8.28}$$

Extending $\widehat{q}$ by writing

$$\widehat{q}(\xi) = \mathcal{F}_{\mathrm{inv}}\, \widehat{y}(\xi, \cdot) \tag{8.29}$$

for $\xi > r_{\max}$, we hence see that this expression is well-defined with

$$\widehat{q} \in \widehat{BC}_\eta^{\oplus}(\mathcal{H}). \tag{8.30}$$

Similarly arguments allow us to show that $\widehat{q}'$ and $\widehat{L}\, \widehat{\mathrm{ev}}_\xi\, \widehat{q}$ are bounded continuous functions on $(0, \infty) \setminus \mathcal{R}$ and that $\widehat{q}$ satisfies

$$\widehat{q}'(\xi) = \widehat{L}\, \widehat{\mathrm{ev}}_\xi\, \widehat{q} \tag{8.31}$$

for all $\xi \in (0, \infty) \setminus \mathcal{R}$. This allows us to conclude that in fact $\widehat{q} \in \widehat{\mathfrak{Q}}_L(\eta)$. □

8.2 Frequency dependent restriction operators

In this subsection we concentrate on the ω-dependence of the two restriction operators

$$\widehat{\pi}^-_{\widehat{Q}_\omega} : \widehat{Q}_\omega \to C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n, \qquad \pi^-_{\widehat{Q}_\omega} : \widehat{Q}_\omega \to C([r_{\min}, 0]; \mathbb{R}^n). \tag{8.32}$$

In particular, we set out to show that we can invert these operators in a fashion that depends continuously on ω. In addition, we obtain frequency independent L^2-based bounds on these inverses.

In order to project out the kernels $\widehat{K}^-_{\widehat{Q}_\omega}$ and $K^-_{\widehat{Q}_\omega}$ of the restriction operators (8.32), we introduce for any $\omega \in [-\pi, \pi]$ the subspaces

$$\begin{aligned} \widehat{T}^-_{\widehat{Q}_\omega} &= \{\widehat{\phi} \in \widehat{Q}_\omega : \textstyle\int_0^{r_{\max}} \widehat{k}^*(\sigma)\widehat{\phi}(\sigma)\, d\sigma = 0 \text{ for all } \widehat{k} \in \widehat{K}^-_{\widehat{Q}_\omega}\}, \\ T^-_{\widehat{Q}_\omega} &= \{\widehat{\phi} \in \widehat{Q}_\omega : \textstyle\int_0^{r_{\max}} \widehat{k}^*(\sigma)\widehat{\phi}(\sigma)\, d\sigma = 0 \text{ for all } \widehat{k} \in K^-_{\widehat{Q}_\omega}\} \end{aligned} \tag{8.33}$$

In addition, most of our results will require the following non-degeneracy condition on the Fourier components $L(\omega)$.

(hr) For each $\omega \in [-\pi, \pi]$ the condition $(\mathrm{HRnk})_{L(\omega)}$ is satisfied.

The following two propositions are the main results of this subsection, constructing two branches of inverse functions $\widehat{\pi}_{\omega}^{-1}$ and π_{ω}^{-1}. We note that these operators are defined on the whole function space $C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n$ respectively $C([r_{\min}, 0]; \mathbb{R}^n)$, allowing us to avoid the usual complications that occur when discussing the continuity of maps with varying domains of definition.

Proposition 8.6. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ together with $\eta \in \mathbb{R}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{h}\omega)_{L;\eta}$ and (hr) are satisfied and that $\dim \widehat{K}^-_{\widehat{Q}_\omega}$ does not depend on ω.*

Then there exists a constant $K \geq 1$ together with a continuous map

$$[-\pi, \pi] \ni \omega \quad \mapsto \quad \widehat{\pi}_{\omega}^{-1} \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n; C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)\big) \tag{8.34}$$

so that for all $\omega \in [-\pi, \pi]$ and $(\phi, v) \in \widehat{R}_{\widehat{Q}_\omega}$, we have the inclusion $\widehat{\pi}_{\omega}^{-1}(\phi, v) \in \widehat{T}^-_{\widehat{Q}_\omega}$, the identity $\widehat{\pi}^- \widehat{\pi}_{\omega}^{-1}(\phi, v) = (\phi, v)$ and the estimate

$$\big\| \pi^+ [\widehat{\pi}_\omega]^{-1}(\phi, v) \big\|_{C([0, r_{\max}]; \mathbb{R}^n)} \leq K \big[\, \|\phi\|_{L^2([0, r_{\max}]; \mathbb{R}^n)} + |v| \, \big]. \tag{8.35}$$

Proposition 8.7. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ together with $\eta \in \mathbb{R}$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{h}\omega)_{L;\eta}$ and (hr) are satisfied and that $\dim K^-_{\widehat{Q}_\omega}$ does not depend on ω.*

Then there exists a constant $K \geq 1$ together with a continuous map

$$[-\pi, \pi] \ni \omega \quad \mapsto \quad \pi_{\omega}^{-1} \in \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{R}^n); C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)\big) \tag{8.36}$$

so that for all $\omega \in [-\pi, \pi]$ and $\phi \in R_{\widehat{Q}_\omega}$ we have the inclusion $\pi_{\omega}^{-1}\phi \in T^-_{\widehat{Q}_\omega}$, the identity $\pi^- \pi_{\omega}^{-1}\phi = \phi$ and the estimate

$$\big\| \pi^+ [\pi_\omega]^{-1} \phi \big\|_{C([0, r_{\max}]; \mathbb{R}^n)} \leq K \, \|\phi\|_{L^2([0, r_{\max}]; \mathbb{R}^n)} \,. \tag{8.37}$$

We note that if $(\mathrm{h}\omega)_{L;\eta}$ is satisfied, Proposition 3.1 guarantees the decomposition

$$\widehat{Q}_\omega = \widehat{K}^-_{\widehat{Q}_\omega} \oplus \widehat{T}^-_{\widehat{Q}_\omega}. \tag{8.38}$$

Our first goal is to study how this decomposition varies with the parameter ω.

Lemma 8.8. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Pick $\eta \in \mathbb{R}$ in such a way that also $(\mathrm{h}\omega)_{L;\eta}$ is satisfied.*

Then the Fredholm index of the restriction operator $\widehat{\pi}^-_{\widehat{Q}_\omega}$ does not depend on ω. In addition, for each fixed ω_0 there exists $\delta_{\omega_0} > 0$ together with continuous maps

$$\begin{array}{lllll} \omega & \mapsto & u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega) & \in & \mathcal{L}\big(\widehat{K}^-_{\widehat{Q}_{\omega_0}}; C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)\big), \\ \omega & \mapsto & u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega) & \in & \mathcal{L}\big(\widehat{T}^-_{\widehat{Q}_{\omega_0}}; C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n)\big), \end{array} \tag{8.39}$$

defined for all $\omega \in [-\pi, \pi]$ that have $|\omega - \omega_0| < \delta_{\omega_0}$, so that

$$u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega_0) = I, \qquad u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega_0) = I, \tag{8.40}$$

together with

$$\widehat{K}^-_{\widehat{Q}_\omega} \subset u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{K}^-_{\widehat{Q}_{\omega_0}}\big), \qquad u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{T}^-_{\widehat{Q}_{\omega_0}}\big) \subset \widehat{T}^-_{\widehat{Q}_\omega} \tag{8.41}$$

and

$$\widehat{Q}_\omega = u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{K}^-_{\widehat{Q}_{\omega_0}}\big) \oplus u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{T}^-_{\widehat{Q}_{\omega_0}}\big). \tag{8.42}$$

Proof. The representation (8.17) shows that for all ω sufficiently close to ω_0 we have

$$\mathrm{ind}\big(\widehat{\pi}^-_{\widehat{Q}_\omega}\big) = \mathrm{ind}_{\mathcal{L}(\widehat{Q}_{\omega_0};C([r_{\min},0];\mathbb{R}^n)\times\mathbb{R}^n)}\Big(\widehat{\pi}^- u^*_{\widehat{Q}_{\omega_0}}(\omega)\Big). \tag{8.43}$$

The latter index varies continuously in ω and hence must be constant.

We now recall from Proposition 3.1 the decomposition

$$C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n) = \widehat{K}^-_{\widehat{Q}_{\omega_0}} \oplus \widehat{T}^-_{\widehat{Q}_{\omega_0}} \oplus P_{\omega_0}, \tag{8.44}$$

together with

$$C([r_{\min},0];\mathbb{R}^n) \times \mathbb{R}^n = \widehat{R}^-_{\widehat{Q}_{\omega_0}} \oplus \widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}. \tag{8.45}$$

We write $\Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}}$ and $\Pi_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}}$ for the projection operators corresponding to the latter decomposition. In view of (8.17) we have the characterization

$$\widehat{K}^-_{\widehat{Q}_\omega} \quad = \quad \{u^*_{\widehat{Q}_{\omega_0}}[\widehat{k}+\widehat{t}] : (\widehat{k},\widehat{t}) \in \widehat{K}^-_{\widehat{Q}_{\omega_0}} \times \widehat{T}^-_{\widehat{Q}_{\omega_0}} \text{ with } \widehat{\pi}^- u^*_{\widehat{Q}_{\omega_0}}(\omega)[\widehat{k}+\widehat{t}] = 0\}. \tag{8.46}$$

This last condition is equivalent to requiring that both

$$\Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[\widehat{k}+\widehat{t}] + \Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[u^*_{\widehat{Q}_{\omega_0}}(\omega) - I][\widehat{k}+\widehat{t}] = 0 \tag{8.47}$$

and

$$\Pi_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^- u^*_{\widehat{Q}_{\omega_0}}(\omega)[\widehat{k}+\widehat{t}] = 0 \tag{8.48}$$

are satisfied. Since $\widehat{\pi}^-\widehat{k} = 0$, we note that (8.47) can be rewritten as

$$\widehat{\pi}^-\widehat{t} + \Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[u^*_{\widehat{Q}_{\omega_0}}(\omega) - I]\widehat{t} = -\Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[u^*_{\widehat{Q}_{\omega_0}}(\omega) - I]\widehat{k}, \tag{8.49}$$

which is equivalent to

$$\widehat{t} + [\widehat{\pi}^-_{\widehat{Q}_{\omega_0}}]^{-1}\Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[u^*_{\widehat{Q}_{\omega_0}}(\omega) - I]\widehat{t} = -[\widehat{\pi}^-_{\widehat{Q}_{\omega_0}}]^{-1}\Pi_{\widehat{R}^-_{\widehat{Q}_{\omega_0}}} \widehat{\pi}^-[u^*_{\widehat{Q}_{\omega_0}}(\omega) - I]\widehat{k}. \tag{8.50}$$

For ω sufficiently close to ω_0, the left hand side can be considered as an invertible linear operator on $\widehat{T}^-_{\widehat{Q}_{\omega_0}}$. In particular, for all such ω there is a bounded linear map

$$\widehat{t}_\omega : \widehat{K}^-_{\widehat{Q}_{\omega_0}} \to \widehat{T}^-_{\widehat{Q}_{\omega_0}}, \tag{8.51}$$

which depends continuously on ω, so that (8.47) is satisfied if and only if $\widehat{t} = \widehat{t}_\omega[\widehat{k}]$. Notice that $\widehat{t}_{\omega_0} = 0$.

For any $\widehat{k} \in \widehat{K}^-_{\widehat{Q}_{\omega_0}}$, we now define

$$u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{k} = u^*_{\widehat{Q}_{\omega_0}}(\omega)\big[\widehat{k} + \widehat{t}_\omega[\widehat{k}]\big]. \tag{8.52}$$

This allows to write

$$\widehat{K}^-_{\widehat{Q}_\omega} = \{\phi \in u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)(\widehat{K}^-_{\widehat{Q}_{\omega_0}}) : \Pi_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}}\widehat{\pi}^-\phi = 0\}, \tag{8.53}$$

establishing the first inclusion in (8.41).

In addition, for any $\widehat{t} \in \widehat{T}^-_{\widehat{Q}_{\omega_0}}$ we define $\widehat{k}_\omega[\widehat{t}] \in \widehat{K}^-_{\widehat{Q}_{\omega_0}}$ in such a way that

$$\int_{r_{\min}}^{r_{\max}} \big[u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{k}\big](\sigma)\big[u^*_{\widehat{Q}_{\omega_0}}(\omega)\widehat{t}\big](\sigma)\,d\sigma = \int_{r_{\min}}^{r_{\max}} \big[u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{k}\big](\sigma)\big[u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{k}_\omega[\widehat{t}]\big](\sigma)\,d\sigma \tag{8.54}$$

holds for all $\widehat{k} \in \widehat{K}_{\widehat{Q}_{\omega_0}}$. This is possible because one can choose a basis for the finite dimensional space $\widehat{K}_{\widehat{Q}_{\omega_0}}$ that is orthonormal under the integration above and $u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega) - I = O(\omega - \omega_0)$. Note that

$$\big\|\widehat{k}_\omega\big\|_{\mathcal{L}(\widehat{T}^-_{\widehat{Q}_{\omega_0}};\widehat{K}^-_{\widehat{Q}_{\omega_0}})} = O(\omega - \omega_0), \tag{8.55}$$

since the left-hand side of (8.54) vanishes at $\omega = \omega_0$ by the definition of $\widehat{T}^-_{\widehat{Q}_{\omega_0}}$.

For any $\widehat{t} \in \widehat{T}^-_{\widehat{Q}_{\omega_0}}$, this allows us to define

$$u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{t} = u^*_{\widehat{Q}_{\omega_0}}(\omega)\widehat{t} - u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\widehat{k}_\omega[\widehat{t}], \tag{8.56}$$

which yields the second inclusion in (8.41).

Finally, to establish (8.42), we note that for any pair $(\widehat{k},\widehat{t}) \in \widehat{K}^-_{\widehat{Q}_{\omega_0}} \times \widehat{T}^-_{\widehat{Q}_{\omega_0}}$ it is possible to find a pair $(\tilde{k},\tilde{t}) \in \widehat{K}^-_{\widehat{Q}_{\omega_0}} \times \widehat{T}^-_{\widehat{Q}_{\omega_0}}$ so that

$$(\widehat{k},\widehat{t}) = (\tilde{k},\tilde{t}) + \big(-\widehat{k}_\omega[\tilde{t}], \widehat{t}_\omega[\tilde{k}] - \widehat{t}_\omega\big[\widehat{k}_\omega[\tilde{t}]\big]\big). \tag{8.57}$$

In particular, we have

$$\begin{array}{rcl} u^*_{\widehat{Q}_{\omega_0}}(\omega)[\widehat{k} + \widehat{t}] & = & u^*_{\widehat{Q}_{\omega_0}}(\omega)\big[\tilde{k} + \widehat{t}_\omega[\tilde{k}]\big] + u^*_{\widehat{Q}_{\omega_0}}(\omega)\tilde{t} - u^*_{\widehat{Q}_{\omega_0}}(\omega)\Big[\widehat{k}_\omega[\tilde{t}] + \widehat{t}_\omega\big[\widehat{k}_\omega[\tilde{t}]\big]\Big] \\ & = & u^*_{\widehat{K}^-_{\widehat{Q}_{\omega_0}}}(\omega)\tilde{k} + u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega)\tilde{t}, \end{array} \tag{8.58}$$

from which the decomposition (8.42) easily follows. □

Corollary 8.9. *Consider the setting of Lemma 8.8. If* $\dim \widehat{K}^-_{\widehat{Q}_\omega}$ *does not depend on* ω*, then the inclusions (8.41) are identities and the associated projections*

$$\begin{array}{rcl} \Pi_{\widehat{K}^-_{\widehat{Q}_\omega}} & : & C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n) \to \widehat{K}^-_{\widehat{Q}_\omega}, \\ \Pi_{\widehat{T}^-_{\widehat{Q}_\omega}} & : & C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n) \to \widehat{T}^-_{\widehat{Q}_\omega} \end{array} \tag{8.59}$$

depend continuously on ω *as elements of* $\mathcal{L}\big(C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n)\big)$.

Proof. Our assumption directly implies that the first inclusion (8.41) is an identity. In view of (8.42), the second inclusion must hence also be an identity. The continuity of the projections follows directly from (8.58). ◻

A second consequence of Proposition 3.1 and $(\mathrm{h}\omega)_{L;\eta}$ is that one can pick finite-dimensional spaces $\widehat{R}^{-;\perp}_{\widehat{Q}_\omega}$ for which the decomposition

$$C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n = \widehat{R}^{-}_{\widehat{Q}_\omega} \oplus \widehat{R}^{-;\perp}_{\widehat{Q}_\omega} \tag{8.60}$$

holds for each $\omega \in [-\pi, \pi]$. Our next result shows that the spaces $\widehat{R}^{-;\perp}_{\widehat{Q}_\omega}$ can be picked in a continuous fashion.

Lemma 8.10. *Fix $\eta \in \mathbb{R}$ and $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{h}\omega)_{L;\eta}$ and (hr) are satisfied and that $\dim \widehat{K}^{-}_{\widehat{Q}_\omega}$ does not depend on ω.*

Then one can choose finite-dimensional subspaces

$$\widehat{R}^{-;\perp}_{\widehat{Q}_\omega} \subset C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n \tag{8.61}$$

in such a way that the following properties are satisfied.

(i) The dimension $\dim \widehat{R}^{-;\perp}_{\widehat{Q}_\omega}$ does not depend on ω.

(ii) The decomposition (8.60) holds for all $\omega \in [-\pi, \pi]$.

(iii) For each fixed $\omega_0 \in [-\pi, \pi]$, there exists $\delta_{\omega_0} > 0$ together with maps

$$\begin{array}{rcl} \omega & \mapsto & u^*_{\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}}(\omega) \in \mathcal{L}\big(\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}; C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n\big), \\ \omega & \mapsto & v^*_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}}(\omega) \in \mathcal{L}\big(\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}; C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n\big), \end{array} \tag{8.62}$$

defined for $\omega \in [-\pi, \pi]$ that have $|\omega - \omega_0| < \delta_{\omega_0}$, so that

$$u^*_{\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}}(\omega_0) = I, \qquad v^*_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}}(\omega_0) = I \tag{8.63}$$

and

$$\widehat{R}^{-}_{\widehat{Q}_\omega} = u^*_{\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}\big), \qquad v^*_{\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}}(\omega)\big(\widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}\big) = \widehat{R}^{-;\perp}_{\widehat{Q}_\omega}. \tag{8.64}$$

In fact, in (iii) we can choose

$$u^*_{\widehat{R}^{-}_{\widehat{Q}_{\omega_0}}}(\omega)[\phi, v] = \widehat{\pi}^{-} u^*_{\widehat{T}^{-}_{\widehat{Q}_{\omega_0}}}(\omega)[\widehat{\pi}^{-}_{\widehat{Q}_{\omega_0}}]^{-1}[\phi, v] \tag{8.65}$$

for all $(\phi, v) \in \widehat{R}^{-}_{\widehat{Q}_{\omega_0}}$.

Proof. First of all, we note that Theorem 3.5 and the fact that the index of $\widehat{\pi}^{-}_{\widehat{Q}_\omega}$ does not depend on ω imply that there exists an integer $d \geq 0$ for which

$$d = \dim K^{-}_{\widehat{Q}_{L_*(\omega)}(-\eta)} \tag{8.66}$$

holds for all $\omega \in [-\pi, \pi]$. Exploiting continuity properties for $K^-_{\widehat{Q}_{L_*(\omega)}(-\eta)}$ that are similar to those stated in Lemma 8.8 for $\widehat{K}^-_{\widehat{Q}_\omega}$, it is possible to construct continuous mappings

$$[-\pi, \pi] \ni \omega \mapsto \widehat{\psi}_j(\omega) \in C([r_{\min}, 0]; \mathbb{R}^n) \times C([0, r_{\max}]; \mathbb{R}^n) \tag{8.67}$$

for $1 \le j \le d$ so that the set $\{\widehat{\psi}_j(\omega)\}_{j=1}^d$ forms a basis for $K_{\widehat{Q}_{L_*(\omega)}(-\eta)}$ for each $\omega \in [-\pi, \pi]$. This in turn can be used to construct continuous mappings

$$[-\pi, \pi] \ni \omega \mapsto \big(r_j^\perp(\omega), v_j^\perp(\omega)\big) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n \tag{8.68}$$

for $1 \le j \le d$ in such a way that

$$\Big\langle \big(\pi^+ \widehat{\psi}_j(\omega), \widehat{\psi}_j(\omega)(0^+)\big), \big(r_i^\perp(\omega), v_i^\perp(\omega)\big) \Big\rangle_{L_{<0}} = \delta_{ij} \tag{8.69}$$

holds for all $\omega \in [-\pi, \pi]$ and all pairs $(i, j) \in \{1, \ldots, d\}^2$. Upon writing

$$\widehat{R}^{-;\perp}_{\widehat{Q}_\omega} = \operatorname{span}\{\big(r_j^\perp(\omega), v_j^\perp(\omega)\big)\}_{j=1}^d, \tag{8.70}$$

the desired statements follow from Theorem 3.5 and the explicit expression (8.65). □

Proof of Proposition 8.6. We first set out to construct the branch $\widehat{\pi}_\omega^{-1}$. To this end, fix $\omega_0 \in [-\pi, \pi]$. Lemma 8.10 allows us to construct continuous maps

$$\begin{array}{lllll} \omega & \mapsto & \widehat{t}_\omega & \in & \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n; \widehat{T}^-_{\widehat{Q}_{\omega_0}}\big), \\ \omega & \mapsto & \widehat{r}^\perp_\omega & \in & \mathcal{L}\big(C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n; \widehat{R}^{-;\perp}_{\widehat{Q}_{\omega_0}}\big), \end{array} \tag{8.71}$$

defined for ω sufficiently close to ω_0, such that for any $(\phi, v) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n$ we have the decomposition

$$(\phi, v) = \widehat{\pi}^- u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega) \widehat{t}_\omega[\phi, v] + v^*_{\widehat{R}^\perp_{\widehat{Q}_{\omega_0}}}(\omega) \widehat{r}^\perp_\omega[\phi, v]. \tag{8.72}$$

In particular, we see that $(\phi, v) \in \widehat{R}^-_{\widehat{Q}_\omega}$ if and only if $\widehat{r}^\perp_\omega[\phi, v] = 0$. For any $(\phi, v) \in C([r_{\min}, 0]; \mathbb{R}^n) \times \mathbb{R}^n$, this allows us to define

$$\widehat{\pi}_\omega^{-1}(\phi, v) = u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega) \widehat{t}_\omega[\phi, v], \tag{8.73}$$

which can readily be verified to satisfy the required properties for all ω sufficiently close to ω_0. To see that this definition does not depend on the specific basis-point ω_0, we note that

$$\Pi_{\widehat{R}^-_\omega}(\phi, v) = \widehat{\pi}^- u^*_{\widehat{T}^-_{\widehat{Q}_{\omega_0}}}(\omega) \widehat{t}_\omega[\phi, v], \tag{8.74}$$

which yields the alternative representation

$$\widehat{\pi}_\omega^{-1} = [\widehat{\pi}^-_{\widehat{Q}_{L(\omega)}}]^{-1} \Pi_{\widehat{R}^-_\omega}. \tag{8.75}$$

We now turn to the estimate (8.35). If one cannot find a constant $K \ge 1$ for which this estimate holds, there exists a sequence

$$\{(\phi_j^-, \phi_j^+, \omega_j)\}_{j=1}^\infty \subset \widehat{T}^-_{\widehat{Q}_{\omega_j}} \times [-\pi, \pi] \tag{8.76}$$

such that $\left\|\phi_j^+\right\|_{C([0,r_{\max}];\mathbb{R}^n)} = 1$ and

$$\left\|\phi_j^-\right\|_{L^2([r_{\min},0];\mathbb{R}^n)} + \left|\phi_j^+(0)\right| \to 0 \text{ as } j \to \infty. \tag{8.77}$$

Inspecting (5.12), exploiting the continuity of $\omega \mapsto \mathrm{ev}_0^{\pm}\mathcal{T}_{L(\omega);\eta}$ and utilizing (5.80), we see

$$\begin{array}{rcl} -\mathrm{ev}_0^-\mathcal{T}_{L(\omega_j);\eta}[0,\phi_j^+,0] & = & \mathrm{ev}_0^-\mathcal{T}_{L(\omega_j);\eta}[\phi_j^-,0,\phi_j^+(0)] \to 0, \\ \big(I - \mathrm{ev}_0^+\mathcal{T}_{L(\omega_j);\eta}[0,\cdot,0]\big)\phi_j^+ & = & \mathrm{ev}_0^+\mathcal{T}_{L(\omega_j);\eta}[\phi_j^-,0,\phi_j^+(0)] \to 0 \end{array} \tag{8.78}$$

as $j \to \infty$. Passing to a subsequence, we may assume $\omega_j \to \omega_*$. In particular, we see that

$$\big(I - \mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,\cdot,0]\big)\phi_j^+ \to 0 \tag{8.79}$$

as $j \to \infty$. The compactness of $\mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,\cdot,0]$ allows us to pass to a further subsequence for which we have the convergence

$$\phi_j^+ \to \phi_*^+ \in C([0,r_{\max}];\mathbb{R}^n) \text{ as } j \to \infty. \tag{8.80}$$

Again exploiting continuity, we have $\phi_*^+(0) = 0$ and

$$\begin{array}{rcl} -\mathrm{ev}_0^-\mathcal{T}_{L(\omega_*);\eta}[0,\phi_*^+,0] & = & 0, \\ \big(I - \mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,\cdot,0]\big)\phi_*^+ & = & 0, \end{array} \tag{8.81}$$

which shows that $(0,\phi_*^+) \in \widehat{K}^-_{\widehat{Q}_{\omega_*}}$. Inspection of (8.33) shows that these normalization conditions survive the limit $j \to \infty$, which implies $(0,\phi_*^+) \in \widehat{T}^-_{\widehat{Q}_{\omega_*}}$. In particular, we must have $\phi_*^+ = 0$, contradicting our initial assumption that $\left\|\phi_j^+\right\| = 1$ for all $j \geq 1$. □

Proof of Proposition 8.7. The maps π_ω^{-1} can be constructed in the same fashion as the maps $\widehat{\pi}_\omega^{-1}$ in the proof of Proposition 8.6 above. If one cannot find a constant $K \geq 1$ for which the estimate (8.37) holds, there exists a sequence

$$\{(\phi_j^-,\phi_j^+,\omega_j)\}_{j=1}^{\infty} \subset T^-_{\widehat{Q}_{\omega_j}} \times [-\pi,\pi] \tag{8.82}$$

such that $\left\|\phi_j^+\right\|_{C([0,r_{\max}];\mathbb{R}^n)} = 1$ and

$$\left\|\phi_j^-\right\|_{L^2([r_{\min},0];\mathbb{R}^n)} \to 0 \text{ as } j \to \infty. \tag{8.83}$$

After passing to a subsequence, we may assume that $\phi_j^+(0) \to v_* \in \mathbb{R}^n$ and $\omega_j \to \omega_*$ as $j \to \infty$. Inspecting (5.12), exploiting the continuity of $\omega \mapsto \mathrm{ev}_0^{\pm}\mathcal{T}_{L(\omega);\eta}$ and utilizing (5.80), we see

$$\begin{array}{rcl} -\mathrm{ev}_0^-\mathcal{T}_{L(\omega_j);\eta}[0,\phi_j^+,0] & = & \mathrm{ev}_0^-\mathcal{T}_{L(\omega_j);\eta}[\phi_j^-,0,\phi_j^+(0)] \to \mathrm{ev}_0^-\mathcal{T}_{L(\omega_*);\eta}[0,0,v_*], \\ \big(I - \mathrm{ev}_0^+\mathcal{T}_{L(\omega_j);\eta}[0,\cdot,0]\big)\phi_j^+ & = & \mathrm{ev}_0^+\mathcal{T}_{L(\omega_j);\eta}[\phi_j^-,0,\phi_j^+(0)] \to \mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,0,v_*] \end{array} \tag{8.84}$$

as $j \to \infty$. In particular, we see that

$$\big(I - \mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,\cdot,0]\big)\phi_j^+ \to \mathrm{ev}_0^+\mathcal{T}_{L(\omega_*);\eta}[0,0,v_*] \tag{8.85}$$

as $j \to \infty$. One can now continue with the arguments in the proof of Proposition 8.6. □

8.3 Properties of $\widehat{\pi}^-_{\widehat{Q}_L(\eta)}$

In this subsection we set out to establish the statements in Theorem 3.14 concerning the restriction operator $\widehat{\pi}^-_{\widehat{Q}_L(\eta)}$. The main issues are to prove that the spaces $\widehat{K}^-_{\widehat{Q}_L(\eta)}$ and $\widehat{R}^-_{\widehat{Q}_L(\eta)}$ are closed and to construct a closed complement $\widehat{T}^-_{\widehat{Q}_L(\eta)}$ for $\widehat{K}^-_{\widehat{Q}_L(\eta)}$. Our approach will be to construct dense subsets of these spaces that consist of the inverse Fourier transforms of appropriate continuous functions in the frequency domain.

In particular, we define the spaces

$$\begin{aligned}\widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} &= \mathcal{F}_{\mathrm{inv}}\{(\phi^-,\phi^+) \in C([r_{\min},0]\times[-\pi,\pi];\mathbb{R}^n)\times C([0,r_{\max}]\times[-\pi,\pi];\mathbb{R}^n) \\ &\qquad : \big(\phi^-(\cdot,\omega),\phi^+(\cdot,\omega)\big) \in \widehat{K}^-_{\widehat{Q}_\omega} \text{ for every } -\pi\le\omega\le\pi\}, \\ \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} &= \mathcal{F}_{\mathrm{inv}}\{(\phi^-,\phi^+) \in C([r_{\min},0]\times[-\pi,\pi];\mathbb{R}^n)\times C([0,r_{\max}]\times[-\pi,\pi];\mathbb{R}^n) \\ &\qquad : \big(\phi^-(\cdot,\omega),\phi^+(\cdot,\omega)\big) \in \widehat{T}^-_{\widehat{Q}_\omega} \text{ for every } -\pi\le\omega\le\pi\}.\end{aligned} \tag{8.86}$$

As a consequence of Lemma 8.5, we have the inclusions

$$\widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \subset \widehat{Q}_L(\eta), \qquad \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \subset \widehat{Q}_L(\eta). \tag{8.87}$$

In addition, we define

$$\begin{aligned}\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} &= \mathcal{F}_{\mathrm{inv}}\{(\phi,v) \in C([r_{\min},0]\times[-\pi,\pi];\mathbb{R}^n)\times C([-\pi,\pi];\mathbb{R}^n) \\ &\qquad : \big(\phi(\cdot,\omega),v(\omega)\big) \in \widehat{R}^-_{\widehat{Q}_\omega} \text{ for every } -\pi\le\omega\le\pi\},\end{aligned} \tag{8.88}$$

which implies

$$\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \subset C\big([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big)\times\ell^2(\mathbb{Z};\mathbb{R}^n). \tag{8.89}$$

As a final preparation, we encode the frequency independence of the kernel dimensions in the following assumption.

(hk) The dimension of the kernel $\widehat{K}^-_{\widehat{Q}_\omega}$ does not depend on $\omega \in [-\pi,\pi]$.

The first step is to explore the relation between $\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$ and $\widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$. In particular, we lift the identities

$$\widehat{R}^-_{\widehat{Q}_\omega} = \widehat{\pi}^-\big(\widehat{T}^-_{\widehat{Q}_\omega}\big) \tag{8.90}$$

from the frequency level to the full system. The key ingredient is our use of L^2-based estimates, which allows us to effectively interchange the norms concerning ω and σ.

Lemma 8.11. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \ge 1$ and consider a bounded linear operator $L : C([r_{\min},r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{h}\omega)_{L;\eta}$, (hk) and (hr) are all satisfied.*

Then we have the identification

$$\widehat{\pi}^-\big(\widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}\big) = \widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}. \tag{8.91}$$

In addition, for every $(r,v) \in \widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$ there is a unique pair

$$(t^-,t^+) = \big(t^-[r,v],t^+[r,v]\big) \in \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \tag{8.92}$$

so that $(r,v) = \widehat{\pi}^-(t^-,t^+) = \big(t^-, t^+(0)\big)$.

Finally, there exist constants $K_1 \geq 1$ *and* $K_2 \geq 1$ *so that the estimates*

$$\begin{array}{lcl} \|t^+[r,v]\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}))} & \leq & K_1\big[\,\|r\|_{L^2([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}))} + \|v\|_{\ell^2(\mathbb{Z};\mathbb{R})}\,\big] \\ & \leq & K_2\big[\,\|r\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}))} + \|v\|_{\ell^2(\mathbb{Z};\mathbb{R})}\,\big] \end{array} \tag{8.93}$$

hold for all $(r,v) \in \widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$.

Proof. The $\subset$ inclusion in (8.91) is immediate. Let us therefore consider a pair $(r,v) \in \widehat{R}^{-;\mathrm{sm}}_{Q_L(\eta)}$. By construction, we have

$$\big([\mathcal{F}r](\cdot,\omega), [\mathcal{F}v](\omega)\big) \in \widehat{R}^-_\omega \tag{8.94}$$

for all $\omega \in [-\pi,\pi]$, which allows us to define

$$\widehat{t}(\cdot,\omega) = \widehat{\pi}^{-1}_\omega\big([\mathcal{F}r](\cdot,\omega), [\mathcal{F}v](\omega)\big). \tag{8.95}$$

On account of Proposition 8.6, we may conclude that

$$\widehat{t} \in C([r_{\min},0]\times[-\pi,\pi];\mathbb{R}^n) \times C([0,r_{\max}]\times[-\pi,\pi];\mathbb{R}^n), \tag{8.96}$$

which yields

$$\mathcal{F}_{\mathrm{inv}}\,\widehat{t} \in \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}. \tag{8.97}$$

By construction, we have $\widehat{\pi}^-\mathcal{F}_{\mathrm{inv}}\,\widehat{t} = (r,v)$.

We now turn to the estimate (8.93). The estimates in Proposition 8.6 imply that

$$\big|\widehat{t}(\sigma,\omega)\big| \leq K\big[\,\|[\mathcal{F}r](\cdot,\omega)\|_{L^2([r_{\min},0];\mathbb{R}^n)} + |[\mathcal{F}v](\omega)|\,\big] \tag{8.98}$$

for all $0 < \sigma \leq r_{\max}$ and $\omega \in [-\pi,\pi]$. In particular, for all such σ we obtain

$$\begin{array}{lcl} \big\|\mathcal{F}_{\mathrm{inv}}\,\widehat{t}(\sigma,\cdot)\big\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^n)} & = & \int_{-\pi}^{\pi}\big|\widehat{t}(\sigma,\omega)\big|^2\,d\omega \\ & \leq & 2K^2\int_{-\pi}^{\pi}\big[\,\|[\mathcal{F}r](\cdot,\omega)\|^2_{L^2([r_{\min},0];\mathbb{R}^n)} + |[\mathcal{F}v](\omega)|^2\,\big]\,d\omega \\ & \leq & 2K^2\int_{-\pi}^{\pi}\big[\,\int_{r_{\min}}^{0}[\mathcal{F}r](s,\omega)^2\,ds + |[\mathcal{F}v](\omega)|^2\,\big]\,d\omega \\ & = & 2K^2\big[\,\int_{r_{\min}}^{0}\int_{-\pi}^{\pi}[\mathcal{F}r](s,\omega)^2\,d\omega\,ds + \int_{-\pi}^{\pi}|[\mathcal{F}v](\omega)|^2\,d\omega\big] \\ & = & 2K^2\big[\,\|r\|^2_{L^2([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|v\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^n)}\,\big] \\ & \leq & K'\big[\,\|r\|^2_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|v\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^n)}\,\big] \end{array} \tag{8.99}$$

for some $K' \geq 1$, as desired. □

The estimate (8.93) can be exploited to construct Cauchy sequences in the closed space $Q_L(\eta)$ from Cauchy sequences in $\widehat{R}^-_{Q_L(\eta)}$. Together with the approximation technique described in Lemma 8.3, this allows us to show that $\widehat{R}^-_{Q_L(\eta)}$ is the closure of $\widehat{R}^{-;\mathrm{sm}}_{Q_L(\eta)}$.

Lemma 8.12. *Fix* $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ *for some integer* $n \geq 1$ *and consider a bounded linear operator* $L : C([r_{\min},r_{\max}];\mathcal{H}) \to \mathcal{H}$ *that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$. *Suppose furthermore that* $(\mathrm{h}\omega)_{L;\eta}$, (hk) *and* (hr) *are all satisfied. Then we have*

$$\mathrm{clos}_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))\times\ell^2(\mathbb{Z};\mathbb{R}^n)}\big(\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}\big) \subset \widehat{R}^-_{\widehat{Q}_L(\eta)}. \tag{8.100}$$

Proof. Consider a sequence $\{(r_j, v_j)\}_{j=1}^{\infty} \subset \widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$ and suppose that

$$(r_j, v_j) \to (r_*, v_*) \in C\big([r_{\min}, 0]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big) \times \ell^2(\mathbb{Z}; \mathbb{R}^n) \tag{8.101}$$

as $j \to \infty$. Lemma 8.11 implies that there is a sequence $\{(t_j^-, t_j^+)\}_{j=1}^{\infty} \in \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$ so that $t_j^- = r_j$ and $t_j^+(0) = v_j$. In addition, exploiting the linearity of $\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$, the estimate (8.93) yields

$$\big|t_{j_1}^+(\sigma) - t_{j_2}^+(\sigma)\big|^2_{\ell^2(\mathbb{Z};\mathbb{R}^n)} \le K_2\big[\, \|r_{j_1} - r_{j_2}\|^2_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|v_{j_1} - v_{j_2}\|^2_{\ell^2(\mathbb{Z};\mathbb{R}^n)}\,\big] \tag{8.102}$$

for $0 \le \sigma \le r_{\max}$. Since $\{(r_j, v_j)\}$ is a Cauchy sequence, we can use this estimate to conclude that $t_j^+ \to t_*^+ \in C\big([0, r_{\max}]; \ell^2(\mathbb{Z}; \mathbb{R}^n)\big)$ with $t_*^+(0) = v_*$. Since $\widehat{Q}_L(\eta)$ is closed, we find that $(r_*, t_*^+) \in \widehat{Q}_L(\eta)$ and hence

$$(r_*, v_*) = \widehat{\pi}^-(r_*, t_*^+) \in \widehat{R}^-_{\widehat{Q}_L(\eta)}. \tag{8.103}$$

□

Lemma 8.13. *Fix $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{R}^n)$ for some integer $n \ge 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$*. Suppose furthermore that* $(\mathrm{h}\omega)_{L;\eta}$, (hk) *and* (hr) *are all satisfied. Then we have*

$$\widehat{R}_{\widehat{Q}_L(\eta)} \subset \mathrm{clos}_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))\times \ell^2(\mathbb{Z};\mathbb{R}^n)}\big(\widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}\big). \tag{8.104}$$

Proof. Pick $(q^-, q^+) \in \widehat{Q}_L(\eta)$ and consider an approximating sequence $\{(q_j^-, q_j^+)\}_{j=1}^{\infty} \in \widehat{Q}_{L;\ell^1(\mathbb{Z};\mathbb{R}^n)}(\eta)$ as described in Lemma 8.3. Write

$$\widehat{y}_{q_j} = E_{\widehat{Q}_L(\eta)}(q_j^-, q_j^+) \in \widehat{\mathfrak{Q}}_{L;\ell^1(\mathbb{Z};\mathbb{R}^n)}(\eta) \tag{8.105}$$

and note that $\mathrm{ev}_0^{\pm}\widehat{y}_{q_j} = q_j^{\pm}$ by construction.

For every $\omega \in [-\pi, \pi]$ we now see that

$$[\mathcal{F}\widehat{y}_{q_j}](\cdot, \omega) \in \mathfrak{Q}_{L(\omega)}(\eta), \tag{8.106}$$

which directly implies that

$$\big(q_j^-, q_j^+(0)\big) \in \widehat{R}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}, \tag{8.107}$$

completing the proof. □

We now set out to obtain the desired direct sum decomposition of $\widehat{Q}_L(\eta)$. In particular, we define

$$\widehat{T}^-_{\widehat{Q}_L}(\eta) = \mathrm{clos}_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))\times C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))}\widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \tag{8.108}$$

and proceed to show that

$$\widehat{Q}_L(\eta) = \widehat{K}^-_{\widehat{Q}_L(\eta)} \oplus \widehat{T}^-_{\widehat{Q}_L}(\eta). \tag{8.109}$$

We start by identifying $\widehat{K}^-_{\widehat{Q}_L(\eta)}$ with the closure of $\widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}$. The approximation technique from Lemma 8.3 again plays an important role here.

Lemma 8.14. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both $(\mathrm{HF})_L$ and $(\mathrm{HFrr})_L$. Suppose furthermore that $(\mathrm{h}\omega)_{L;\eta}$, (hk) and (hr) are all satisfied. Then we have*

$$\widehat{K}^{-}_{\widehat{Q}_L(\eta)} = \mathrm{clos}_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))\times C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} \widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}. \tag{8.110}$$

Proof. It suffices to show the inclusion

$$\widehat{K}^{-}_{\widehat{Q}_L(\eta)} \subset \mathrm{clos}_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))\times C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} \widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}. \tag{8.111}$$

To this end, pick $\widehat{k} \in \widehat{K}_{\widehat{Q}_L(\eta)}$ and consider an approximating sequence

$$\{\widehat{q}_j\}_{j=1}^{\infty} = \{(q_j^-, q_j^+)\}_{j=1}^{\infty} \subset \widehat{Q}_{L;\ell^1(\mathbb{Z};\mathbb{R}^n)}(\eta) \tag{8.112}$$

as described in Lemma 8.3. This allows us exploit the decomposition (8.38) and define

$$\widehat{k}_j(\cdot,\omega) = \Pi_{\widehat{K}^-_{\widehat{Q}_\omega}}[\mathcal{F}\widehat{q}_j](\cdot,\omega), \qquad \widehat{t}_j(\cdot,\omega) = \Pi_{\widehat{T}^-_{\widehat{Q}_\omega}}[\mathcal{F}\widehat{q}_j](\cdot,\omega) \tag{8.113}$$

for each $\omega \in [-\pi,\pi]$. Writing

$$\begin{array}{lcl} \widehat{k}_j(\cdot,\omega) & = & \big(k_j^-(\cdot,\omega), k_j^+(\cdot,\omega)\big) \in C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n), \\ \widehat{t}_j(\cdot,\omega) & = & \big(t_j^-(\cdot,\omega), t_j^+(\cdot,\omega)\big) \in C([r_{\min},0];\mathbb{R}^n) \times C([0,r_{\max}];\mathbb{R}^n), \end{array} \tag{8.114}$$

Corollary 8.9 implies that the maps

$$\begin{array}{l} [r_{\min},0] \times [-\pi,\pi] \ni (\sigma,\omega) \mapsto \big(k_j^-(\sigma,\omega), t_j^-(\sigma,\omega)\big), \\ [0,r_{\max}] \times [-\pi,\pi] \ni (\sigma,\omega) \mapsto \big(k_j^+(\sigma,\omega), t_j^+(\sigma,\omega)\big) \end{array} \tag{8.115}$$

are both continuous. By construction, we hence have

$$\mathcal{F}_{\mathrm{inv}}\,\widehat{k}_j \in \widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}, \qquad \mathcal{F}_{\mathrm{inv}}\,\widehat{t}_j \in \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \tag{8.116}$$

with

$$\widehat{q}_j = \mathcal{F}_{\mathrm{inv}}\,\widehat{k}_j + \mathcal{F}_{\mathrm{inv}}\,\widehat{t}_j. \tag{8.117}$$

In particular, we have the convergence

$$\mathcal{F}_{\mathrm{inv}}(k_j^-, k_j^+) + \mathcal{F}_{\mathrm{inv}}(t_j^-, t_j^+) \to \widehat{k} \in C\big([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \tag{8.118}$$

as $j \to \infty$.

Since $k_j^- = 0$ and $k_j^+(0) = 0$, we see that

$$\big\|\mathcal{F}_{\mathrm{inv}}\,t_j^-\big\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \big|[\mathcal{F}_{\mathrm{inv}}\,t_j^+](0)\big|_{\ell^2(\mathbb{Z};\mathbb{R}^n)} \to 0 \tag{8.119}$$

as $j \to \infty$. The estimate (8.93) implies that also $t_j^+ \to 0$ in $C\big([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n)\big)$ as $j \to \infty$, which shows that in fact

$$\mathcal{F}_{\mathrm{inv}}(k_j^-, k_j^+) \to \widehat{k} \in C\big([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n)\big), \tag{8.120}$$

as desired. □

Lemma 8.15. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$. *Suppose furthermore that* $(\mathrm{h}\omega)_\eta$, (hk) *and* (hr) *are all satisfied. Then we have*

$$\widehat{K}^-_{\widehat{Q}_L(\eta)} \cap \widehat{T}^-_{\widehat{Q}_L(\eta)} = \{0\}. \tag{8.121}$$

Proof. If the statement is false, then there exists a non-zero

$$\widehat{\phi} = (\phi^-, \phi^+) \in C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \tag{8.122}$$

together with sequences

$$\{(k_j^-, k_j^+)\}_{j=1}^\infty \subset \widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)}, \qquad \{(t_j^-, t_j^+)\}_{j=1}^\infty \subset \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \tag{8.123}$$

so that both

$$\begin{array}{rcl} \|k_j^- - \phi^-\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|t_j^- - \phi^-\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} & \to & 0, \\ \|k_j^+ - \phi^+\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|t_j^+ - \phi^+\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} & \to & 0 \end{array} \tag{8.124}$$

as $j \to \infty$.

The estimate (8.93) yields

$$\|t_j^+\|_{C([0,r_{\max}];\ell^2(\mathbb{Z};\mathbb{R}^n))} \leq K_2[\|t_j^-\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|t_j^+(0)\|_{\ell^2(\mathbb{Z};\mathbb{R}^n)}]. \tag{8.125}$$

On the other hand, since $k_j^- = 0$ and $k_j^+(0) = 0$, we find $\phi^- = 0$ and $\phi^+(0) = 0$, which gives

$$\|t_j^-\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|t_j^+(0)\| \to 0 \tag{8.126}$$

as $j \to \infty$. In particular, we see $t_j^+ \to 0$ in $C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big)$ and hence $\widehat{\phi} = 0$. □

Lemma 8.16. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}];\mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$. *Suppose furthermore that* $(\mathrm{h}\omega)_{L;\eta}$, (hk) *and* (hr) *are all satisfied. Then we have*

$$\widehat{Q}_L(\eta) \subset \widehat{K}^-_{\widehat{Q}_L(\eta)} + \widehat{T}^-_{\widehat{Q}_L(\eta)}. \tag{8.127}$$

Proof. Pick $(q^-, q^+) \in \widehat{Q}_L(\eta)$. Arguing as in the proof of Lemma 8.14, one can find sequences

$$\{\mathcal{F}_{\mathrm{inv}}(k_j^-, k_j^+)\}_{j=1}^\infty \subset \widehat{K}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \qquad \{\mathcal{F}_{\mathrm{inv}}(t_j^-, t_j^+)\}_{j=1}^\infty \subset \widehat{T}^{-;\mathrm{sm}}_{\widehat{Q}_L(\eta)} \tag{8.128}$$

for which we have the convergence

$$\mathcal{F}_{\mathrm{inv}}(k_j^-, k_j^+) + \mathcal{F}_{\mathrm{inv}}(t_j^-, t_j^+) \to (q^-, q^+) \in C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \tag{8.129}$$

as $j \to \infty$.

Since $k_j^- = 0$ and $k_j^+(0) = 0$, we have

$$\|\mathcal{F}_{\mathrm{inv}}\, t_j^- - q^-\|_{C([r_{\min},0];\ell^2(\mathbb{Z};\mathbb{R}^n))} + \|[\mathcal{F}_{\mathrm{inv}}\, t_j^+](0) - q^+(0)\|_{\ell^2(\mathbb{Z};\mathbb{R}^n)} \to 0 \tag{8.130}$$

as $j \to \infty$. In particular, the sequence

$$\{\big(\mathcal{F}_{\mathrm{inv}}\, t_j^-, [\mathcal{F}_{\mathrm{inv}}\, t_j^+](0)\big)\}_{j=1}^\infty \subset C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times \ell^2(\mathbb{Z};\mathbb{R}^n) \tag{8.131}$$

is a Cauchy sequence. The estimate (8.93) implies that the same holds for the sequence

$$\{\mathcal{F}_{\rm inv}\, t_j^+\}_{j=1}^{\infty} \subset C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big). \tag{8.132}$$

In particular, there exist (t_*^-, t_*^+) and (k_*^-, k_*^+) so that we have the separate convergences

$$\begin{array}{lcl} \mathcal{F}_{\rm inv}(t_j^-, t_j^+) \to (t_*^-, t_*^+) & \in & C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big), \\ \mathcal{F}_{\rm inv}(k_j^-, k_j^+) \to (k_*^-, k_*^+) & \in & C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times C\big([0, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \end{array} \tag{8.133}$$

as $j \to \infty$. By construction, we have $(t_*^-, t_*^+) \in \widehat{T}^-_{\widehat{Q}_L(\eta)}$, while $(k_*^-, k_*^+) \in \widehat{K}^-_{\widehat{Q}_L(\eta)}$ by Lemma 8.14. □

8.4 Proof of main results

We are now ready to prove Theorems 3.14 and 3.16. We note that the restriction operator $\pi^-_{\widehat{Q}_L(\eta)}$ can be treated exactly as in §8.3, provided one uses Proposition 8.7 rather than Proposition 8.6. For the remaining operator $\pi^-_{Q_L(\eta)}$ we develop the following more direct approach.

Lemma 8.17. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$*. Suppose furthermore that* $(\mathrm{h}\omega)_{L;\eta}$, (hk) *and* (hr) *are all satisfied. Then the range $R^-_{Q_L(\eta)}$ is closed in $C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big)$.*

Proof. Consider a sequence $r_j \in R^-_{Q_L(\eta)}$ for which $r_j \to r_* \in C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big)$. Naturally, we also have

$$\widehat{R}^-_{\widehat{Q}_L(\eta)} \ni \big(r_j, r_j(0)\big) \to \big(r_*, r_*(0)\big) \in C\big([r_{\min}, 0]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big) \times \ell^2(\mathbb{Z};\mathbb{R}^n). \tag{8.134}$$

The closedness of $\widehat{R}^-_{\widehat{Q}_L(\eta)}$ now implies that

$$\big(r_*, r_*(0)\big) = \widehat{\pi}^- \widehat{q} \tag{8.135}$$

for some $\widehat{q} \in \widehat{Q}_L(\eta)$, which hence must have $\widehat{q}(0^+) = \widehat{q}(0^-)$. In particular, we may write $q = \widehat{q} \in Q_L(\eta)$ and $r_* = \pi^- q$, as desired. □

Lemma 8.18. *Fix $\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{R}^n)$ for some integer $n \geq 1$ and consider a bounded linear operator $L : C([r_{\min}, r_{\max}]; \mathcal{H}) \to \mathcal{H}$ that satisfies both* $(\mathrm{HF})_L$ *and* $(\mathrm{HFrr})_L$*. Suppose furthermore that* $(\mathrm{h}\omega)_{L;\eta}$, (hk) *and* (hr) *are all satisfied. Then the subspace*

$$T^-_{Q_L(\eta)} = \{\widehat{t} \in \widehat{T}^-_{\widehat{Q}_L(\eta)} : \widehat{t}(0^-) = \widehat{t}(0^+)\} \tag{8.136}$$

is closed in $C\big([r_{\min}, r_{\max}]; \ell^2(\mathbb{Z};\mathbb{R}^n)\big)$ and we have the direct sum decomposition

$$Q_L(\eta) = K^-_{Q_L(\eta)} \oplus T^-_{Q_L(\eta)}. \tag{8.137}$$

Proof. The closedness of $T_{Q_L(\eta)}$ follows from the closedness of $\widehat{T}^-_{\widehat{Q}_L(\eta)}$ and the fact that the defining property is preserved through limits. In addition, remembering that $\widehat{K}^-_{\widehat{Q}_L(\eta)} = K^-_{Q_L(\eta)}$, we see that

$$K^-_{Q_L(\eta)} \cap T^-_{Q_L(\eta)} \subset \widehat{K}^-_{\widehat{Q}_L(\eta)} \cap \widehat{T}^-_{\widehat{Q}_L(\eta)} = \{0\}. \tag{8.138}$$

It hence remains to show that $Q_L(\eta) \subset K^-_{Q_L(\eta)} + T^-_{Q_L(\eta)}$. To this end, pick any $q \in Q_L(\eta)$ and write $q = \widehat{k} + \widehat{t}$ with $\widehat{k} \in \widehat{K}^-_{\widehat{Q}_L(\eta)} = K^-_{Q_L(\eta)}$ and $\widehat{t} \in \widehat{T}^-_{\widehat{Q}_L}$. Since $\widehat{k}(0^+) = \widehat{k}(0^-) = 0$, the identity $\widehat{t} = q - \widehat{k}$ implies that $\widehat{t}(0^-) = \widehat{t}(0^+)$ and hence $\widehat{t} \in T_{Q_L(\eta)}$, as desired. □

Proof of Theorem 3.14. The results for $\widehat{\pi}^-_{\widehat{Q}_L(\eta)}$ follow from Lemma's 8.12-8.16. On the other hand, the results for $\pi^-_{\widehat{Q}_L(\eta)}$ can be obtained by almost identical arguments, substituting Proposition 8.7 for Proposition 8.6 where appropriate. Finally, $\pi^-_{Q_L(\eta)}$ can be analyzed using Lemma's 8.17 and 8.18, while the spaces $P_L(\eta)$ and $\widehat{P}_L(\eta)$ can be treated in an analogous fashion. □

Proof of Theorem 3.16. The statements follow from Theorems 2.5 and 3.14, using the observation that the identity

$$\mathcal{J}_\alpha(z)\delta_{\mathcal{I},M}(z) = \Delta_{L(z)} \tag{8.139}$$

implies that

$$\mathcal{J}_\alpha(z)\delta_{\mathcal{I},M(\omega)}(z) = \Delta_{L(\omega)}(z) \tag{8.140}$$

holds for each $\omega \in [-\pi, \pi]$. □

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