Leiden 13th April 2007

Invariant Manifolds and Applications for
Functional Differential Equations of Mixed Type

Hermen Jan Hupkes
Universiteit Leiden

(Joint work with S.M. Verduyn Lunel and E. Augeraud-Véron)
Prototype MFDE is the differential equation given by

\[ \dot{x}(t) = f(x(t)) + x(t - 1) + x(t + 1). \]
Mixed Type Functional Differential Equations (MFDEs)

Prototype MFDE is the differential equation given by

\[
\dot{x}(t) = f(x(t)) + x(t - 1) + x(t + 1).
\]

- Delay equations have been used for more than half a century.
- They arise naturally in many modelling applications.
- Appropriate functional-analytic setup developed in past two decades, finally enabling extensions of classic ODE results (stable / unstable manifolds, Hopf bifurcations, etc.)
- Recently theory for linear MFDEs has started to develop (Mallet-Paret and Verduyn-Lunel (2001), Härterich et al. (2002))
Main (historical) motivation for study of MFDEs comes from lattice differential equations, e.g., the discrete reaction-diffusion equation

\[ \dot{u}_{i,j} = \alpha L_D u_{i,j} - f(u_{i,j}), \quad (i, j) \in \mathbb{Z}^2. \]

\( L_D \) is a discrete Laplacian, which could be given by

\[ L_D u_{i,j} = (\Delta^+ u)_{i,j} \equiv u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}. \]

Bistable nonlinearity, typically

\[ f_{\text{cub}}(u) = u(u-a)(u-1). \]
Lattice equations: Anisotropy

We consider travelling wave solutions which propagate at an angle $\theta$.

$$u_{i,j}(t) = \phi(i \cos \theta + j \sin \theta - ct).$$

We require $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. Substitution yields the MFDE

$$-c\phi' (\xi) = \alpha L_\theta(\phi)(\xi) - f_{\text{cub}}(\phi(\xi), a)$$

where

$$L_\theta(\phi) = \phi(\xi + \cos \theta) + \phi(\xi - \cos \theta) + \phi(\xi + \sin \theta) + \phi(\xi - \sin \theta) - 4\phi(\xi).$$

- Notice the $\theta$ - dependence in the discrete case, which is absent in PDE case.
- Propogation direction dependent!
Lattice equations: Anisotropy

The lattice anisotropy can be illustrated by studying the $c(\theta)$ relation. Example LDE: $\dot{u}_{i,j} = \frac{1}{4}(\Delta^+ u)_{i,j} + (\Delta \times u)_{i,j}) - 10 f_{\text{cub}}(u_{i,j}, a)$.

 Propagation failure.
• Anisotropy.
Consider the MFDE

\[ \dot{x}(\xi) = x(\xi - 1) + x(\xi + 1). \]
MFDEs vs Delay Equations

Consider the MFDE

\[ \dot{x}(\xi) = x(\xi - 1) + x(\xi + 1). \]

• Continuity lost \( \implies \) problem not well-defined \( \implies \) no semi-group techniques.

\[ \dot{x}(t) = 0, \ x(t-1) = 1 \implies x(t+1) = -1 \]
MFDE: What is going on?

Substitution of $e^{z\xi}$ into
\[
\dot{x}(\xi) = x(\xi - 1) + x(\xi + 1),
\]
yields the characteristic equation
\[
\Delta(z) := z - e^{-z} - e^{z} = 0.
\]

• The problem is infinite dimensional (as for delay equations).
• There is no exponential bound possible for solutions, at both $\pm \infty$ (unlike delay equations)!
MFDE: Nonlinear

Now include nonlinear term

\[ \dot{x}(\xi) = A_-x(\xi - 1) + A_+x(\xi + 1) + f(x(\xi)), \]

with \( f(0) = Df(0) = 0 \), so \( x \equiv 0 \) is an equilibrium solution.

For applications we are interested in bounded solutions near the equilibrium.

**Main Result** (H. + Verduyn-Lunel, 2006)

In the neighbourhood of the equilibrium, the dynamics of the MFDE can be completely captured on a finite dimensional object, called the center manifold. The dynamics on this manifold is described by an ODE, which can be explicitely calculated.

This reduction

\[ \text{MFDE} \quad \longrightarrow \quad \text{ODE} \]

\[ \infty \text{dim} \quad \longrightarrow \quad n \text{dim} \]

opens up the full toolbox available for ODEs!
Overlapping Generations

- Population consists of a continuum of individuals, that have fixed lifespan $T_{\text{life}}$. 

![Diagram showing overlapping generations with individuals and population axes labeled.](image-url)
• Each individual wishes to maximize his lifetime consumption, but may not die in debt.
Overlapping Generations: The Collective

- Assumption: Perfect foresight. Consumer can accurately predict interest rate.
Overlapping Generations: The Dynamics

- Equilibrium at $t_0$ requires summation over all living indv (blue line).
- Condition for $r(t_0)$ involves all values $r(t_0 + \theta)$ for $\theta = -T_{\text{life}} \ldots T_{\text{life}}$. 
Overlapping Generations: The Mathematics

Equilibrium condition for the interest rate given by

\[
1 = \int_{t-T_{\text{life}}}^{t} \frac{\int_{s}^{s+1} \exp\left[-\int_{t}^{v} r(u) du\right] dv}{\int_{s}^{s+T_{\text{life}}} \exp\left[-(1-\sigma)\int_{t}^{v} r(u) du\right] dv} ds.
\]

Economists are very interested in periodic solutions!

- Twofold differentiation leads to MFDE.
- Linearization leads to characteristic equation (with \( T = T_{\text{life}} \))

\[
\Delta(z) \sim [-T\omega e^{z} + (1-\sigma)e^{zT} + (\omega e^{z} - T + 1 - \sigma)e^{-zT} + (T - 2 + 2\sigma) + \sigma T^{2} z^{2}].
\]

- Look for pairs \((\sigma_{0}, T_{0})\) such that \(\Delta(i\omega) = 0\) for some \(\omega \in \mathbb{R}\).
- At these pairs Hopf bifurcations occur, producing periodic orbits for \((\sigma, T)\) near \((\sigma_{0}, T_{0})\), with period \(T_{\text{period}} \approx \frac{2\pi}{\omega}\).
Overlapping Generations: Hopf bifurcations

Period of periodic orbit $T_{\text{period}} = T_0(T_{\text{life}})/k$. 

\[ k = 1, 06, 11 \]
\[ k = 2, 07, 12 \]
\[ k = 3, 08, 13 \]
\[ k = 4, 09, 14 \]
\[ k = 5, 10, 15 \]
Overlapping Generations

One might worry about the differentiation needed to turn equilibrium condition into MFDE.

\[ 1 = \int_{t-T_{\text{life}}}^{t} \frac{\int_{s}^{s+T_{\text{life}}} \exp[-\int_{t}^{v} r(u)du]dv}{\int_{s}^{s+T_{\text{life}}} \exp[-(1 - \sigma)\int_{t}^{v} r(u)du]dv} ds. \]

- Question: how do original equation and associated MFDE relate?
  - Can we lift solutions of MFDE back to original equation?
  - Can we still perform reduction to an ODE?

- H+VL+AV (2007): Can construct center manifold directly for original equation, again (locally) reducing dynamics to an ODE.

- This should be compared to theory of DAEs (Differential-Algebraic equations), where ODEs are coupled to algebraic equations.
Overview

- MFDEs arise naturally in different disciplines.
  - Indirectly, through travelling wave solutions to lattice equations.
  - Indirectly, through optimal control problems with delays.
  - Directly, through economic overlapping-generation models.

- MFDEs are infinite dimensional problems.

- Near-equilibrium dynamics can be reduced to an ODE on a center manifold.

- Can invoke ODE theory to study periodic orbits etc.

- Works for algebraic mixed type equations as well.
Consider a simple, closed economy.

- Production capacity $k(t)$ split between consumption $c(t)$ and investments $u(t)$. 
Optimal Control with delays

Maximize

\[ \text{Consumption } c = C(u, k) \]
\[ \text{Investments } u \]
\[ \text{Production capacity } k \]
\[ \text{Time} \]

- Want to maximize the total welfare

\[ \int_{0}^{\infty} e^{-\rho t} \ln C(\dot{k}(t) + gk(t - \tau), k(t - \tau)) dt. \]

- Parameter \( \rho \) relates appreciation of future welfare to present welfare.
- Parameter \( g \) describes natural decay of machines, factories etc.
Optimal Control: MFDE

Periodic solutions are interesting from economic point of view.

(1968) Hughes: solving optimal control problem with delays leads to a MFDE.

(1979) Benhabib & Nishimura: periodic solutions for model with $\tau = 0$ (no delay). However, need at least two different types of ”machines” (capital goods).

(1989) Rustichini: ”formal” analysis of models with $\tau \neq 0$. However, no rigorous results possible.

(2006) H+VL: Center manifold construction, allowing reduction MFDE $\rightarrow$ ODE.

- Allows use of Hopf bifurcation theorem to establish periodic solutions.
- Provides explicit conditions for Hopf bifurcations to occur in terms of parameters $\rho$ and $g$.
- Need to look for parameters $(\rho, g)$ such that the linear part of the MFDE admits periodic solutions.
Optimal control: Hopf bifurcations

Curves in parameter space where Hopf bifurcations occur.
Examples of periodic orbits for $k(t)$ at different parameter values.
Optimal control: Periodic Orbits

\[\omega \approx 2.8\]
\[\omega \approx 13.7\]
\[\omega \approx 15.3\]
\[\omega \approx 16.9\]
References Continued