Travelling Pulse Solutions for the Discrete FitzHugh-Nagumo System

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Abstract

The existence of fast travelling pulses of the discrete FitzHugh–Nagumo equation is obtained in the weak-recovery regime. This result extends to the spatially discrete setting the well-known theorem that states that the FitzHugh–Nagumo PDE exhibits a branch of fast waves that bifurcates from a singular pulse solution. The key technical result that allows for the extension to the discrete case is the Exchange Lemma that we establish here for functional differential equations of mixed type.

Key words: lattice differential equations, travelling waves, singular perturbation theory, exchange lemma, Lin's method, discrete FitzHugh-Nagumo.

1 Introduction

In this paper we consider the discrete FitzHugh–Nagumo equation

$$\begin{array}{rcl} \dot{u}_i(t) & = & \alpha[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] + g\big(u_i(t)\big) - w_i(t), \\ \dot{w}_i(t) & = & \epsilon\big(u_i(t) - \gamma w_i(t)\big), \end{array} \tag{1.1}$$

where $u_i, w_i \in \mathbb{R}$ for each $i \in \mathbb{Z}$. The nonlinearity g is taken from a class of bistable nonlinearities that includes the cubic polynomial $g(u;a) = u(1-u)(u-a)$ for some $0 < a < \frac{1}{2}$. We consider arbitrary positive coupling coefficients $\alpha > 0$, take $0 < \epsilon \ll 1$ to be small, and assume that $\gamma > 0$ is not too large so that $\{(u_i, w_i)\}_{i \in \mathbb{Z}} = (0,0)$ is the only i-independent rest state of (1.1); this requires that $g(\gamma w) \neq w$ for all $w \neq 0$.

Our primary reason for looking at the spatially discrete FitzHugh–Nagumo equation is its relevance in modelling. For example, when studying the propagation of electrical signals through nerve fibers, it turns out to be more natural to study the discrete system (1.1) instead of its continuous counterpart that is traditionally used for this purpose. This is related to the fact that a nerve axon is almost entirely surrounded by an insulating myeline coating that admits small gaps at regular

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intervals. These nodes were first observed in 1878 by Ranvier [42] and now carry his name. The insulation induced by the myeline causes excitations of the nerve at these nodes of Ranvier to effectively jump from one node to the next, through a process called saltatory conduction. This mechanism was first suggested in 1925 by Lillie [37] and demonstrated convincingly in 1949 by Huxley and Stämpfli [29]. Other discrete lattice models have appeared in a wide range of scientific disciplines, including chemical reaction theory [19, 36], material science [2, 5] and image processing and pattern recognition [13].

Our goal is to show that (1.1) admits travelling pulse solutions

$$(u_i, w_i)(t) = (u_*, w_*)(i + ct)$$

for some positive wave speed $c > 0$, where the profiles (u_*, w_*) are localized so that $(u_*, w_*)(\xi) \to 0$ as $\xi \to \pm\infty$. Substituting our ansatz into (1.1), we see that these profiles must satisfy the system

$$\begin{array}{rcl} cu_*'(\xi) & = & \alpha[u_*(\xi+1) + u_*(\xi-1) - 2u_*(\xi)] + g\big(u_*(\xi)\big) - w_*(\xi), \\ cw_*'(\xi) & = & \epsilon\big(u_*(\xi) - \gamma w_*(\xi)\big). \end{array} \tag{1.2}$$

Such equations are referred to as functional differential equations of mixed type (MFDEs), since they contain both advanced and retarded terms. This class of equations is notoriously difficult to analyse.

Previous work on the discrete FitzHugh–Nagumo equation and variants thereof can be split into two main directions. On the one hand, rigorous results have been obtained for specially tailored nonlinearities g. Tonnelier [45] and Elmer and Van Vleck [18], for example, considered the McKean sawtooth caricature of the cubic, while Chen and Hastings [10] studied a discrete Morris–Lecar type system with a nonlinearity that vanishes identically on certain critical regions of u and w. On the other hand, using asymptotic techniques, formal results have been obtained for (1.1) by Carpio and coworkers [7–9]. We are, however, not aware of any rigorous results for (1.1) that hold for the cubic polynomial or more general bistable nonlinearities, and it is this issue that we shall address in this paper. Before outlining our result in more detail, we briefly discuss the spatially continuous case, for which a large body of literature exists.

1.1 Travelling waves for the FitzHugh–Nagumo PDE

Let us therefore consider the spatially continuous FitzHugh–Nagumo system

$$\begin{array}{rcl} u_t & = & u_{xx} + g(u) - w, \\ w_t & = & \epsilon(u - \gamma w), \end{array} \tag{1.3}$$

where $x \in \mathbb{R}$. This partial differential equation (PDE) plays an important role as a tractable simplification of the Hodgkin–Huxley equations that are widely used to model the propagation of signals through myelinated nerve fibers [25]. As a consequence, (1.3) has been analysed extensively in the literature. A large portion of the results that have been obtained concern travelling-wave solutions to (1.3), that is, solutions of the form $(u, w)(x, t) = (u, w)(x + ct)$ that depend on the single argument $\xi = x + ct$. Such solutions must satisfy the ordinary differential equation (ODE)

$$\begin{array}{rcl} u' & = & v, \\ v' & = & cv - g(u) + w, \\ w' & = & \frac{\epsilon}{c}(u - \gamma w). \end{array} \tag{1.4}$$

For simplicity, we take g to be the cubic nonlinearity $g(u; a) = u(1 - u)(u - a)$. Note that the origin $p_0 = (0, 0, 0)$ is an equilibrium for (1.4) regardless of the precise values of a, c and ϵ. Finding travelling pulses of (1.3) then amounts to constructing homoclinic orbits for (1.4) that are bi-asymptotic to p_0.

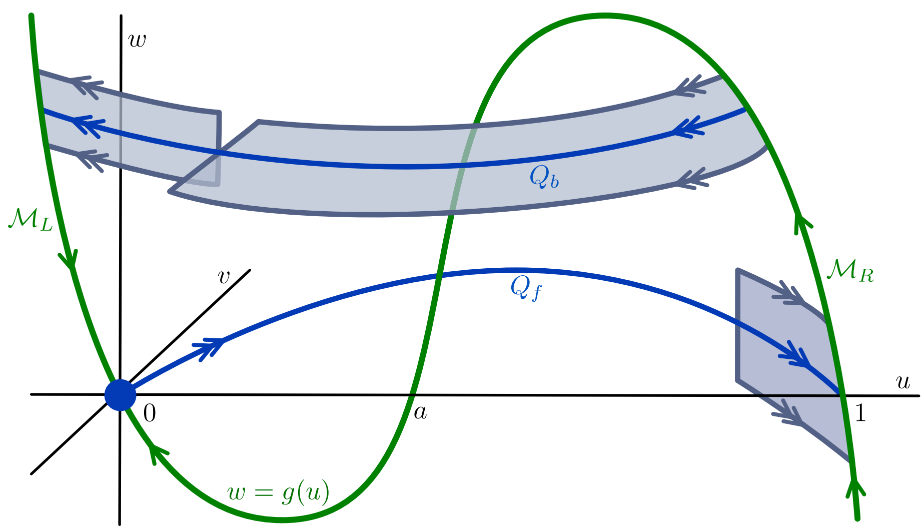

Fig. 1: *Phase diagram for the travelling-wave equation (1.4) of the FitzHugh–Nagumo PDE.*

In the regime $0 < \epsilon \ll 1$, this question can be answered using geometric singular perturbation theory. We will now outline this approach and refer to Figure 1 for an illustration. First, we set $\epsilon = 0$ in (1.4) to get the system

$$\begin{array}{rcl} u' &=& v, \\ v' &=& cv - g(u) + w, \\ w' &=& 0, \end{array} \tag{1.5}$$

which admits a manifold $\mathcal{M}$ of equilibria that consists of all points $(u, 0, w)$ that have $w = g(u)$. Obviously, this manifold contains the points p_0 and $p_1 = (1, 0, 0)$. One can now choose a neighborhood $\mathcal{M}_L \subset \mathcal{M}$ around p_0 together with a neighborhood $\mathcal{M}_R \subset \mathcal{M}$ around p_1. If these neighborhoods do not contain the knees of the cubic polynomial, they are normally hyperbolic invariant manifolds that hence persist as locally invariant sets for small $\epsilon > 0$ as a consequence of Fenichel's first theorem [20]. It is well-known that, for each fixed $0 < a < \frac{1}{2}$, there exists a heteroclinic solution $Q_f = (q_f, q_f', 0)$ to (1.5) that connects p_0 to p_1 and has wave speed $c = c_*$ for some $c_* > 0$: indeed, these solutions correspond to travelling fronts of the Nagumo equation

$$u_t = u_{xx} + g(u). \tag{1.6}$$

In addition, for any such a there exists a $w_* > 0$ such that (1.4) with $\epsilon = 0$ and $c = c_*$ admits a heteroclinic solution $Q_b = (q_b, q_b', w_*)$ that connects $\mathcal{M}_R$ to $\mathcal{M}_L$. We can now write Γ_0^{fs} for the singular orbit that arises by combining these orbits with the segments of $\mathcal{M}_R$ and $\mathcal{M}_L$ that connect $w = 0$ to $w = w_*$. The superscript fs is used in view of the fact that we are considering fast waves with speed $c_* > 0$. The following well-known result is the ODE analogue of the result we set out to obtain for the functional differential equation (1.2).

Proposition 1.1 *Consider (1.4) with the cubic nonlinearity $g = g(\cdot\,; a)$ for any fixed $0 < a < \frac{1}{2}$, then there exists a unique curve in the (ϵ, c)-plane emanating from the point $(0, c_*)$ that consists of homoclinic solutions to (1.4) that are bi-asymptotic to 0, while being $\mathrm{O}(\epsilon)$-close to Γ_0^{fs} and winding around Γ_0^{fs} exactly once.*

The first proofs establishing the existence of the branch of homoclinics described in the result above are due to Carpenter [6] and Hastings [24], who obtained their results independently using

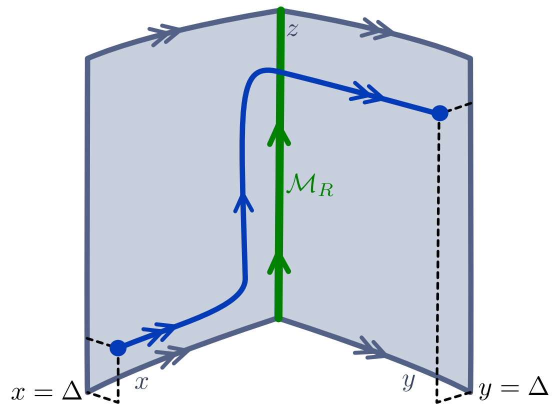

Fig. 2: *Illustration of the geometric setting of the Exchange Lemma.*

classical singular perturbation theory [6] and the Conley index [24]. A more streamlined proof of Proposition 1.1 that also gives transversality and local uniqueness is based on a geometrical construction developed by Jones and coworkers [33]. The idea is to construct the desired homoclinic orbits as an intersection of the unstable manifold $W^u(0)$ and the stable manifold $W^s(\mathcal{M}_L)$. The main difficulty is to track the unstable manifold $W^u(0)$ as it passes close to $\mathcal{M}_R$, since it spends time of order $\mathrm{O}(\epsilon^{-1})$ here. The tool developed in [33] to deal with this tracking is referred to as the Exchange Lemma. We refer to Figure 2 for an illustration of the geometric setting of this result, which we describe here briefly.

The statement of the Exchange Lemma can be explained most easily in terms of the Fenichel normal form [20, 32]: in a neighborhood of $\mathcal{M}_R$, the ODE (1.4) can be put into the form

$$\begin{aligned} x' &= -A^s(x,y,z)x, \\ y' &= A^u(x,y,z)y, \\ z' &= \epsilon[1+B(x,y,z)xy], \end{aligned} \tag{1.7}$$

where the new coordinates x, y and z are real-valued, the functions A^s, A^u and B are smooth, and A^s and A^u are bounded from below by some constant $\eta > 0$. The Exchange Lemma then states that (1.7) has, for each $z_0 \in \mathbb{R}$, each sufficiently large T and each sufficiently small $\epsilon > 0$ and $\Delta > 0$, a solution that satisfies the boundary conditions $x(0) = \Delta$, $z(0) = z_0$ and $y(T) = \Delta$. Furthermore, the norms $|y(0)|$, $|x(T)|$ and $|z(T) - z_0 - \epsilon T|$ and their derivatives with respect to T, ϵ, z_0 and any other parameters that may appear in the problem are of order $\mathrm{e}^{-\eta T}$ as $T \to \infty$.

Instead of attempting to analyse the intersection of $\mathcal{W}^u(0)$ with $\mathcal{W}^s(\mathcal{M}_R)$ directly, one can now decouple the problem for large T and study separately how $\mathcal{W}^u(0)$ and $\mathcal{W}^s(\mathcal{M}_L)$ behave near $x = \Delta$ and $y = \Delta$, respectively, which are far easier to analyse and lead to a two-dimensional nonlinear system that involves the three variables ϵ, c and T [35]. This system can be solved to yield the branch of homoclinic orbits described in Proposition 1.1.

A great deal more is known about (1.4). For example, the PDE stability of the resulting fast travelling pulses was proved independently by Jones [30] and Yanagida [46]. It is also known that there is a second slow travelling wave that exists, for fixed $0 < a < \frac{1}{2}$, in the limit $c \to 0$ and $\epsilon/c \to 0$. The resulting singular homoclinic orbit Γ_0^{sl} for $\epsilon = \epsilon/c = 0$ is actually a regular homoclinic orbit to the origin that lives in the plane $w = 0$. Since Γ_0^{sl} does not contain any segments of $\mathcal{M}_L$ and $\mathcal{M}_R$, perturbations are easier to analyse and one may show that a branch of slow homoclinic solutions can be constructed near Γ_0^{sl} for small $c > 0$ and $\epsilon > 0$ [44].

A conjecture due to Yanagida [46] states that these branches of fast and slow waves connect to each other. At the moment, this has only been confirmed for a near the critical point $a = \frac{1}{2}$, where the two singular orbits Γ_0^{fs} and Γ_0^{sl} coalesce [35]. We remark that [35] also contains a proof that, somewhere along this connecting curve, the homoclinic orbits undergo an inclination-flip bifurcation. The presence of such a bifurcation makes it very likely to find homoclinic solutions that wind around the singular orbit an arbitrary number of times. To establish this rigorously, a specific non-degeneracy condition needs to be verified. At the moment, this is only feasible when considering orbits that have winding number two, in which case a result due to Nii [41] can be invoked.

1.2 The discrete Nagumo equation

The construction of fast pulses of the continuous FitzHugh–Nagumo equation relied on gluing travelling fronts and backs of appropriate Nagumo equations (1.6) together. Thus, it is natural to start our discussion of the discrete FitzHugh–Nagumo system by summarizing a few key features of the discrete Nagumo equation

$$\dot{u}_i(t) = \alpha[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] + g(u_i(t); a) \tag{1.8}$$

with $g(u; a) = u(1-u)(u-a)$. Travelling waves $u_i(t) = u_*(i+ct)$ of (1.8) satisfy the scalar functional differential equation

$$cu_*'(\xi) = \alpha[u_*(\xi+1) + u_*(\xi-1) - 2u_*(\xi)] + g(u_*(\xi); a) \tag{1.9}$$

of mixed type. The first numerical study of travelling fronts of (1.1) was conducted by Chi, Bell and Hassard [11]. Since that early paper, the discrete Nagumo equation and the associated travelling-wave MFDE (1.9) have served as prototype systems for investigating the properties of lattice differential equations.

In contrast to the continuous case where travelling fronts with positive wave speeds exist for each $0 < a < \frac{1}{2}$, the discrete Nagumo equation may not support travelling fronts for each such a. The reason is that the limit $c \to 0$ in (1.9) is highly singular. Indeed, the limiting system is a map which may admit transverse heteroclinic orbits that preclude the existence of travelling fronts. More precisely, the combined results of Keener [34] and Mallet-Paret [39, Theorem 2.6] give the following: for each sufficiently small $\alpha > 0$, there exists an $0 < a_0 < \frac{1}{2}$ such that, for each $a \in [a_0, \frac{1}{2}]$, heteroclinic solutions to (1.9) that connect the two equilibria $u = 0$ and $u = 1$ exist if and only if $c = 0$. This feature is called propagation failure and distinguishes (1.8) from its continuous counterpart (1.6). By now there is an abundance of numerical evidence showing that this phenomenon may occur in an extremely robust fashion throughout a wide range of discrete systems [1, 15–17]. One implication of this feature for the present work is that we need to assume that a does not lie inside the region of propagation failure for the discrete Nagumo equation.

We remark that propagation failure in the underlying discrete Nagumo equation is the reason why slow waves do not exist for the discrete FitzHugh–Nagumo equation in the same way as they do for the continuous case. Hence, we focus on fast waves in this paper.

1.3 Travelling waves for the discrete FitzHugh–Nagumo system

We now turn to the travelling-wave equation (1.2),

$$\begin{array}{rcl} cu'(\xi) & = & \alpha[u(\xi+1) + u(\xi-1) - 2u(\xi)] + g\big(u(\xi)\big) - w_*(\xi), \\ cw'(\xi) & = & \epsilon\big(u(\xi) - \gamma w(\xi)\big), \end{array} \tag{1.10}$$

associated with the discrete FitzHugh–Nagumo equation (1.1). Our goal is to find an appropriate value of $c > 0$ and construct solutions $(u, w)(\xi)$ of this MFDE for $0 < \epsilon \ll 1$ that converge to zero as $|\xi| \to \infty$.

Similar to the case of delay equations, the state space associated with (1.10) will necessarily be infinite-dimensional, and we will consequently work with $(u, w) \in Y = C([-1, 1], \mathbb{R}) \times \mathbb{R}$ in this paper. In contrast to the case of delay equations, however, the initial-value problem associated with

(1.10) on the space Y is ill-posed[1] due to the presence of both advanced and retarded terms. This issue prevents us from using the semigroup techniques developed for retarded differential equations [14]. An alternative strategy is to utilize Fredholm properties and exponential dichotomies, which were developed for MFDEs by Mallet-Paret [38], Verduyn Lunel [40] and Härterich, Sandstede and Scheel [23]. This approach was recently used successfully by Hupkes and Verduyn Lunel to extend Lin's method to MFDEs [27].

The key complication that needs to be overcome before homoclinic solutions to (1.10) can be constructed in the singular limit $\epsilon \to 0$, is that geometric singular perturbation theory is not readily available for MFDEs. Indeed, this theory relies heavily on the existence of semiflows which, as we outlined above, do not exist in our MFDE setting. For instance, almost all proofs of Fenichel's first theorem [20] about the persistence of normally hyperbolic slow manifolds are based on Hadamard's graph transform technique [21].

The approach that we use in this paper to resolve these issues is based on a combination of the ideas contained in [26, 27, 35, 43]. First, the work of Sakamoto [43] uses analytic techniques to establish Fenichel's first theorem for ODEs through a systematic use of the concept of slowly varying coefficients. Combining this approach with our recent results [26] concerning linear MFDEs that have slowly varying coefficients we construct one-dimensional slow manifolds $\mathcal{M}_L$ and $\mathcal{M}_R$ for (1.10) that persist for small $\epsilon > 0$. To prove an appropriate version of the Exchange Lemma, we exploit the ideas in [35] in which an analytic proof was given that is based on Lin's method. The key feature of this approach is that, unlike earlier proofs using differential forms [31] or boundary-value techniques [4], the construction of the stable and unstable fibers of $\mathcal{M}_R$ can be done globally, thereby allowing us to immediately reduce the existence problem to a finite set of nonlinear equations, similar to those that we need to solve. Borrowing the techniques used in [27] to generalize Lin's method to MFDEs and again applying the slowly-varying coefficient framework developed in [26], we can imitate this construction in the current setting.

While we concentrate in this paper on the concrete discrete FitzHugh–Nagumo equation, we believe that our techniques can be used in a much wider context than in just the construction of pulses for the specific system (1.1). For example, we expect that after some minor adaptations it should be possible to study travelling multi-pulse solutions or long-period wave train solutions to general MFDEs in which a slow time-scale can be identified. In addition, we currently use our singular perturbation framework to assess the stability of the fast waves constructed here with respect to the dynamics of the underlying lattice equation.

The rest of this paper is organized as follows. We state our main result in §2 and give a detailed outline of the main steps that are need to prove this result in §3, while hiding most of the technical details behind a sequence of propositions. The invariant slow manifolds $\mathcal{M}_L(c, \epsilon)$ and $\mathcal{M}_R(c, \epsilon)$ are constructed throughout §4. We then study the stable and unstable foliations of these slow manifolds in §5 and develop a suitable version of the Exchange Lemma in §6. Section 7 contains a brief discussion.

2 Main result

Recall that travelling waves of the discrete FitzHugh–Nagumo equation (1.1) can be found as solutions of the system

$$\begin{array}{rcl} cu'(\xi) & = & \alpha[u(\xi+1) + u(\xi-1) - 2u(\xi)] + g\big(u(\xi)\big) - w(\xi), \\ cw'(\xi) & = & \epsilon\big(u(\xi) - \gamma w(\xi)\big). \end{array} \tag{2.1}$$

Throughout this paper we will assume that $\alpha > 0$ and $\gamma > 0$. The prototype nonlinearity that we have in mind is given by the cubic polynomial $g = g(u; a) = u(1-u)(u-a)$ for some $0 < a < \frac{1}{2}$. However, we will focus on a broader class of bistable nonlinearities in order to illustrate the generality

[1] In general, given an initial condition in Y, we cannot solve (1.10) forward or backward in ξ.

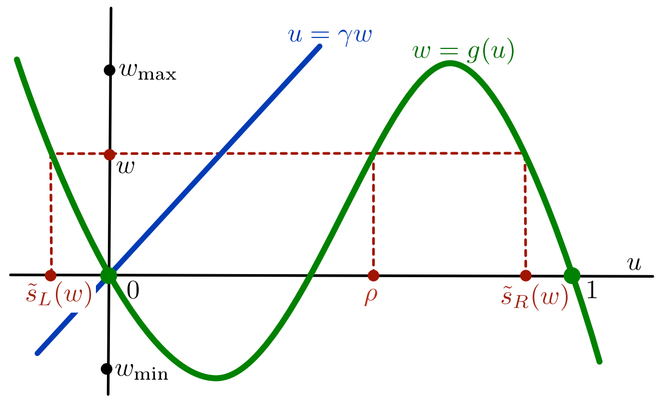

Fig. 3: *Illustration of the assumptions on the nonlinearity $g(u)$ and the constant γ.*

of our approach. We therefore impose the following generic assumptions on the nonlinearity g, which are also illustrated in Figure 3.

Hypothesis (H1) *The nonlinearity g is C^{r+3}-smooth for some integer $r \geq 2$.*

Hypothesis (H2) *We have $g(0) = g(1) = 0$, $g'(0) < 0$ and $g'(1) < 0$.*

On account of condition (H2), we may choose closed intervals I_L and I_R, with $0 \in I_L$ and $1 \in I_R$, that have non-empty interior and in addition have $g'(u) < 0$ for all $u \in I_L \cup I_R$. We pick constants $w_{\min} < 0$ and $w_{\max} > 0$ in such a way that both $w_{\min}, w_{\max} \in g(I_L) \cap g(I_R)$. The implicit function theorem can now be used to define two C^{r+3}-smooth functions $\widetilde{s}_L : [w_{\min}, w_{\max}] \to I_L$ and $\widetilde{s}_R : [w_{\min}, w_{\max}] \to I_R$ in such a way that

$$g\big(\widetilde{s}_L(w)\big) = g\big(\widetilde{s}_R(w)\big) = w$$

for all $w \in [w_{\min}, w_{\max}]$. Notice that $\widetilde{s}_L(0) = 0$ and $\widetilde{s}_R(0) = 1$. Our next assumption roughly states that $-g$ is N-shaped, admitting precisely one extra solution to $g(u) = w$.

Hypothesis (H3) *For any $w \in [w_{\min}, w_{\max}]$, there exists a $\rho \in \big(\widetilde{s}_L(w), \widetilde{s}_R(w)\big)$ such that $g(\rho) = w$. In addition, we have*

$$\begin{array}{ll} g(u) > w, & u \in \big(-\infty, \widetilde{s}_L(w)\big) \cup \big(\rho, \widetilde{s}_R(w)\big), \\ g(u) < w, & u \in \big(\widetilde{s}_L(w), \rho\big) \cup \big(\widetilde{s}_R(w), \infty\big). \end{array}$$

In order to ensure that (2.1) admits a suitable singular orbit, we will need to assume that this equation with $\epsilon = 0$ admits a front and a back solution that propagate at the same wave speed.

Hypothesis (H4) *There exist two constants $w_* \in (0, w_{\max})$ and $c_* > 0$ such that (2.1) with $\epsilon = 0$ and $c = c_*$ admits two solutions $(q_f, 0)$ and (q_b, w_*) that satisfy the limits*

$$\begin{array}{lclclcl} \lim_{\xi \to -\infty} q_f(\xi) & = & 0, & \quad & \lim_{\xi \to \infty} q_f(\xi) & = & 1, \\ \lim_{\xi \to -\infty} q_b(\xi) & = & \widetilde{s}_R(w_*), & \quad & \lim_{\xi \to \infty} q_b(\xi) & = & \widetilde{s}_L(w_*). \end{array} \tag{2.2}$$

We remark here that [38, Proposition 5.3] in combination with the fact that $g'(0) \neq 0$, $g'(1) \neq 0$, $g'\big(\widetilde{s}_L(w_*)\big) \neq 0$ and $g'\big(\widetilde{s}_R(w_*)\big) \neq 0$ allows us to conclude that q_f and q_b approach their limits at $\pm\infty$ at an exponential rate. Such an argument is made explicit in the proof of [39, Theorem 2.2].

In contrast to the setting of Proposition 1.1, the possibility of propagation failure prevents us from obtaining results that hold for the cubic polynomial $g(\cdot\,; a)$ with arbitrary $0 < a < \frac{1}{2}$.

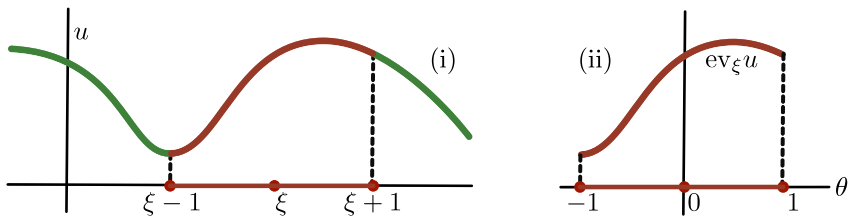

Fig. 4: *Panel (i) shows the graph of a given function u, while panel (ii) illustrates the associated function* $\mathrm{ev}_\xi u : [-1, 1] \to \mathbb{R}$ *for a fixed* ξ.

Lemma 2.1 *Fix any positive coupling coefficient $\alpha > 0$, then the conditions (H1)-(H4) are satisfied for the cubic nonlinearity $g(u) = u(1-u)(u-a)$ provided $a > 0$ is sufficiently small.*

Proof. The conditions (H1) through (H3) are obviously satisfied. Using [39, Theorem 2.6], one can conclude the existence of a wave speed $c_* > 0$ and a wave profile q_f such that the pair $(q_f, 0)$ satisfies (2.1) with $\epsilon = 0$ and $c = c_*$, while satisfying the limits given in the first line of (2.2). The requirement that a is sufficiently small is needed here to ensure that the wave speed c_* does not vanish. To obtain the existence of the pair (q_b, w_*) that solves (2.1) at the same speed c_*, one may exploit the mirror symmetry of cubic polynomials. ∎

Our final assumption concerns the parameter γ and ensures that, for any $\epsilon > 0$ and $c \neq 0$, the only equilibrium solution to (2.1) is given by $(0, 0)$.

Hypothesis (H5) *The parameter $\gamma > 0$ is so small that $g(\gamma w) \neq w$ for all $w \neq 0$.*

Let us now write Γ_0^{fs} for the singular homoclinic orbit that arises by following the heteroclinic connection q_f from $(0, 0)$ to $(1, 0)$, moving along the manifold $\mathcal{M}_R := \{(\widetilde{s}_R(w), w)\}$ from $(1, 0)$ to $(\widetilde{s}_R(w_*), w_*)$, following q_b from $(\widetilde{s}_R(w_*), w_*)$ to $(\widetilde{s}_L(w_*), w_*)$ and finally moving back to $(0, 0)$ along the manifold $\mathcal{M}_L := \{(\widetilde{s}_L(w), w)\}$. Our main result is concerned with homoclinic solutions to (2.1) that bifurcate off Γ_0^{fs} as ϵ moves away from zero and wind around this singular homoclinic exactly once.

In order to define this winding number properly, we need to have a notion of transversality that will allow us to construct Poincaré sections. The winding number can then be related to the number of times a homoclinic orbit passes through these sections. Let us therefore write $X = C([-1, 1], \mathbb{R})$ for the state space associated with the first component of (2.1). The state of a function $u \in C(\mathbb{R}, \mathbb{R})$ at $\xi \in \mathbb{R}$ will be denoted by $\mathrm{ev}_\xi u \in X = C([-1, 1], \mathbb{R})$ and is defined by

$$[\mathrm{ev}_\xi u](\theta) := u(\xi + \theta), \qquad \theta \in [-1, 1];$$

see Figure 4. We can now pick two subspaces $\widehat{X}_f$ and $\widehat{X}_b$ of X such that

$$X = \mathrm{span}\{\mathrm{ev}_0 q_f'\} \oplus \widehat{X}_f, \qquad X = \mathrm{span}\{\mathrm{ev}_0 q_b'\} \oplus \widehat{X}_b. \tag{2.3}$$

We are now ready to describe the type of solutions to (2.1) that we are interested in and refer to Figure 5 for an illustration.

Definition 2.2 (Homoclinic solution) *For each $0 < \delta \ll 1$ and $\xi_* \gg 1$, we say that a pair $(u, w) \in C(\mathbb{R}, \mathbb{R}^2)$ is a (δ, ξ_*)-homoclinic solution if (u, w) satisfies (2.1) for all $\xi \in \mathbb{R}$ and meets the following conditions:*

(i) There exists exactly one $\xi_f \in \mathbb{R}$ for which $\left\| \mathrm{ev}_{\xi_f} u - \mathrm{ev}_0\, q_f \right\| < \delta$, $|w(\xi_f)| < \delta$, and $\mathrm{ev}_{\xi_f} u \in \mathrm{ev}_0\, q_f + \widehat{X}_f$.

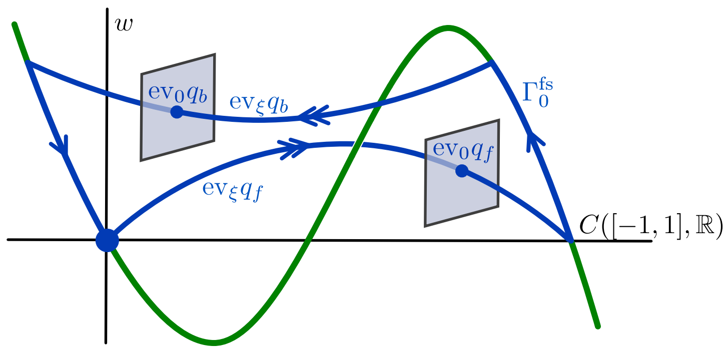

Fig. 5: *Shown are the singular homoclinic orbit* Γ_0^{fs} *and the location of the two Poincaré sections along the front* $(\mathrm{ev}_\xi q_f, 0)$ *and the back* $(\mathrm{ev}_\xi q_b, w_*)$ *in the underlying phase space* $C([-1,1],\mathbb{R}) \times \mathbb{R}$.

(ii) There exists exactly one $\xi_b \in \mathbb{R}$ *for which* $\|\mathrm{ev}_{\xi_b} u - \mathrm{ev}_0\, q_b\| < \delta$, $|w(\xi_b) - w_*| < \delta$, *and* $\mathrm{ev}_{\xi_b} u \in \mathrm{ev}_0\, q_b + \widehat{X}_f$.

(iii) We have $\lim_{\xi\to\pm\infty} \big(u(\xi), w(\xi)\big) = 0$.

(iv) The solution (u,w) *is close to* Γ_0^{fs} *in the sense that*

$$\begin{array}{llll} |u(\xi) - q_f(\xi - \xi_f)| < \delta & \text{and} \quad |w(\xi)| < \delta & \text{for} & \xi \le \xi_f + \xi_*, \\ \big|u(\xi) - \widetilde{s}_R\big(w(\xi)\big)\big| < \delta & & \text{for} & \xi_f + \xi_* \le \xi \le \xi_b - \xi_*, \\ |u(\xi) - q_b(\xi - \xi_b)| < \delta & \text{and} \quad |w(\xi) - w_*| < \delta & \text{for} & \xi_b - \xi_* \le \xi \le \xi_b + \xi_*, \\ \big|u(\xi) - \widetilde{s}_L\big(w(\xi)\big)\big| < \delta & & \text{for} & \xi_b + \xi_* \le \xi. \end{array}$$

Our main result shows that by varying the wave speed c, one may obtain a one-parameter branch of such solutions that bifurcates away from Γ_0^{fs}.

Theorem 1 *Consider the nonlinear system (2.1) and suppose that Hypotheses (H1)-(H5) hold, then there are constants* $0 < \delta \ll 1$ *and* $\xi_* \ll 1$ *with the following property: for each* $c < c_*$ *that is sufficiently close to* c_*, *there exists a unique* $\epsilon = \epsilon(c) > 0$ *for which (2.1) admits a* (δ, ξ_*)*-homoclinic solution* (u,w). *This pair* (u,w) *is* $\mathrm{O}(c - c_*)$*-close to* Γ_0^{fs} *and unique up to translations.*

3 Proof of Theorem 1

Our proof of Theorem 1 is split into four main parts. In this section we will describe each of these steps, hiding the technical details behind four propositions that will be proved throughout the remainder of this paper. At the end of this section, our main claim will have been reduced to a statement concerning the roots of a two-dimensional nonlinear system involving three variables. The desired one-parameter branch of (δ, ξ_*)-homoclinic solutions to (2.1) can subsequently be read off from these equations.

The four main parts of our argument can be outlined roughly as follows. First, we consider the equilibrium manifolds $\mathcal{M}_L = \{(\widetilde{s}_L(w), w)\}$ and $\mathcal{M}_R = \{(\widetilde{s}_R(w), w)\}$. We show that these curves can be perturbed to yield slow manifolds $\mathcal{M}_L(c,\epsilon)$ and $\mathcal{M}_R(c,\epsilon)$ that remain invariant when considering (2.1) with small $\epsilon > 0$ and $c \approx c_*$.

In the next step, we show that, for each $\epsilon \approx 0$ and $c \approx c_*$, there are two unique solutions near the front $(q_f, 0)$ so that the first orbit lies in the infinite-dimensional unstable manifold of $(0,0)$, the second solution lies in the infinite-dimensional stable manifold of $\mathcal{M}_R(c,\epsilon)$, and their difference at $\xi = 0$ is contained in a certain one-dimensional subspace. Thus, up to this one-dimensional jump, these manifolds have a unique intersection near the front. We refer to such connections as quasi-front solutions and refer to Figure 7 below for an illustration. Similarly, for each such ϵ and c, and for each choice of w_0, there are unique quasi-back solutions in the unstable manifold of $\mathcal{M}_R(c,\epsilon)$ and the strong stable fiber of $\mathcal{M}_L(c,\epsilon)$ belonging to $w = w_0$, respectively, so that their difference at $\xi = 0$ again lies in an appropriate fixed one-dimensional subspace. Using the Hale inner product, which is tailored specifically for functional differential equations, the derivatives of the aforementioned jumps with respect to the three free parameters can be related to Melnikov-type integrals whose signs we can evaluate.

In the third step, we prove an Exchange Lemma for MFDEs that allows us to match quasi-fronts and quasi-backs as they pass near the manifold $\mathcal{M}_R(c,\epsilon)$. This can be done up to two extra jumps that lie in the same one-dimensional spaces that we discussed above. These extra jumps turn out to be C^1-exponentially small with respect to the time spent near the slow manifold $\mathcal{M}_R(c,\epsilon)$. This allows us to set up and analyse the resulting two-dimensional nonlinear system that describes the size of the remaining gaps in the final step.

3.1 Step 1 - The slow manifolds

We now describe the slow manifolds $\mathcal{M}_L(c,\epsilon)$ and $\mathcal{M}_R(c,\epsilon)$ in more detail. In order to avoid complications that arise when w leaves the region $[w_{\min}, w_{\max}]$, we will need to modify (2.1). To this end, we choose a C^∞-smooth cut-off function $\chi_{\mathrm{sl}} : \mathbb{R} \to \mathbb{R}$ as shown in Figure 6 and consider the system

$$\begin{array}{rcl} cu'(\xi) & = & \alpha[u(\xi+1) + u(\xi-1) - 2u(\xi)] + g\big(u(\xi)\big) - w(\xi), \\ cw'(\xi) & = & \epsilon\big(u(\xi) - \gamma w(\xi)\big)\chi_{\mathrm{sl}}\big(w(\xi)\big), \end{array} \tag{3.1}$$

instead of working directly with (2.1). The following result will be established in §4.

Proposition 3.1 *Consider the nonlinear system (3.1) and suppose that (H1)-(H3) are satisfied, then there exist constants $\delta_\epsilon > 0$ and $\delta_c > 0$, together with two C^{r+2}-smooth functions*

$$s_R, s_L : [w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to \mathbb{R},$$

such that the following is true:

(i) For each $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, we have the identities $s_R(\vartheta, c, 0) = \widetilde{s}_R(\vartheta)$ and $s_L(\vartheta, c, 0) = \widetilde{s}_L(\vartheta)$.

(ii) For each $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, consider the unique solution of the ODE

$$c\theta'(\xi) = \epsilon\big(s_R(\theta(\xi), c, \epsilon) - \gamma\theta(\xi)\big)\chi_{\mathrm{sl}}\big(\theta(\xi)\big), \qquad \theta(0) = \vartheta, \tag{3.2}$$

then the pair (u, w) defined by $u(\xi) = s_R(\theta(\xi), c, \epsilon)$ and $w(\xi) = \theta(\xi)$ satisfies (3.1). The same statement holds upon replacing the subscript R by L.

(iii) There exists a constant $\delta > 0$ such that any solution (u, w) to (3.1) with $|c - c_*| < \delta_c$ and $0 \le \epsilon \le \delta_\epsilon$ that has both $w_{\min} \le w(\xi) \le w_{\max}$ and $\big|u(\xi) - \widetilde{s}_R\big(w(\xi)\big)\big| < \delta$ for all $\xi \in \mathbb{R}$ must in fact satisfy $u(\xi) = s_R(w(\xi), c, \epsilon)$ for all $\xi \in \mathbb{R}$. The same statement holds for the subscript L.

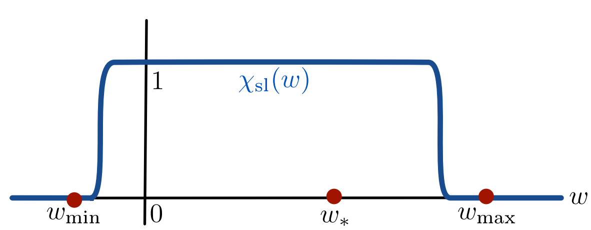

Fig. 6: *The definition of the cut-off function $\chi_{\mathrm{sl}}(w)$.*

The functions s_L and s_R can be used to define the invariant manifolds $\mathcal{M}_L(c,\epsilon)$ and $\mathcal{M}_R(c,\epsilon)$ mentioned at the start of this section. In particular, we take $\mathcal{M}_L(c,\epsilon) = \{(s_L(w,c,\epsilon), w)\}$ and $\mathcal{M}_R(c,\epsilon) = \{(s_R(w,c,\epsilon), w)\}$, letting w run through the interval $[w_{\min}, w_{\max}]$.

In the sequel we will often need to refer to the flow on these manifolds $\mathcal{M}_L$ and $\mathcal{M}_R$, so we will introduce some notation here for convenience. Recall the constants $\delta_c > 0$ and $\delta_\epsilon > 0$ that appear in Proposition 3.1 and introduce the functions

$$\mathcal{T}_R, \mathcal{T}_L : [w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to \mathbb{R}$$

that are given by

$$\begin{array}{rcl} \mathcal{T}_R(\vartheta, c, \epsilon) & = & [s_R(\vartheta, c, \epsilon) - \gamma\vartheta]\chi_{\mathrm{sl}}(\vartheta), \\ \mathcal{T}_L(\vartheta, c, \epsilon) & = & [s_L(\vartheta, c, \epsilon) - \gamma\vartheta]\chi_{\mathrm{sl}}(\vartheta). \end{array}$$

For each $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, we write $\Theta_R^{\mathrm{fs}}(\vartheta, c, \epsilon) \in C(\mathbb{R}, \mathbb{R})$ to denote the unique solution of the ODE

$$c\theta'(\xi) = \epsilon\mathcal{T}_R(\theta(\xi), c, \epsilon), \qquad \theta(0) = \vartheta. \tag{3.3}$$

Similarly, we introduce the notation $\Theta_L^{\mathrm{fs}}(\vartheta, c, \epsilon) \in C(\mathbb{R}, \mathbb{R})$ to denote the unique solution of the ODE

$$c\theta'(\xi) = \epsilon\mathcal{T}_L(\theta(\xi), c, \epsilon), \qquad \theta(0) = \vartheta. \tag{3.4}$$

The superscript fs refers to the fact that (3.3) and (3.4) are written in terms of the fast time scale. In contrast, we will write $\Theta_R^{\mathrm{sl}}(\vartheta, c, \epsilon)$ for the unique solution of the ODE

$$c\theta'(\zeta) = \mathcal{T}_R(\theta(\zeta), c, \epsilon), \qquad \theta(0) = \vartheta,$$

where the superscript now indicates that we solve with respect to the slow time scale. Note that $\Theta_R^{\mathrm{fs}}(\vartheta, c, \epsilon)(\xi) = \Theta_R^{\mathrm{sl}}(\vartheta, c, \epsilon)(\epsilon\xi)$.

3.2 Step 2 - Quasi-fronts and quasi-backs: stable and unstable foliations

We now construct the quasi-front connections between $(0,0)$ and $\mathcal{M}_R(c,\epsilon)$ that are illustrated in Figure 7. As shown there, the construction depends on the choice of the one-dimensional subspace Γ_f that contains the difference of two solutions of the underlying MFDE. Thus, we first focus on outlining our choice of Γ_f and of the space Γ_b that we shall use to construct quasi-back solutions which connect $\mathcal{M}_R(c,\epsilon)$ back to $(0,0)$. We will use the decomposition (2.3),

$$X = \mathrm{span}\{\mathrm{ev}_0 q_f'\} \oplus \widehat{X}_f = \mathrm{span}\{\mathrm{ev}_0 q_b'\} \oplus \widehat{X}_b,$$

of the phase space $X = C([-1,1], \mathbb{R})$ and the associated Poincaré sections that we introduced in Definition 2.2(i)-(ii) for this purpose. Our goal is to find suitable subspaces Γ_f and Γ_b of $\widehat{X}_f$ and $\widehat{X}_b$ that contain the jumps.

As a preparation, we substitute the ansatz $u(\xi) = q_f(\xi) + v(\xi)$ into the first equation of (2.1) and set $w = 0$ and $\epsilon = 0$. We obtain the variational MFDE

$$cv'(\xi) = \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] + g\big(q_f(\xi) + v(\xi)\big) - g\big(q_f(\xi)\big)$$

whose linearization about $v = 0$ gives the operator $\Lambda_f : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ with

$$[\Lambda_f v](\xi) = cv'(\xi) - \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] - g'\big(q_f(\xi)\big)v(\xi).$$

We also define the formal adjoint $\Lambda_f^* : W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \to L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ of Λ via

$$[\Lambda_f^* v](\xi) = cv'(\xi) + \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] + g'\big(q_f(\xi)\big)v(\xi).$$

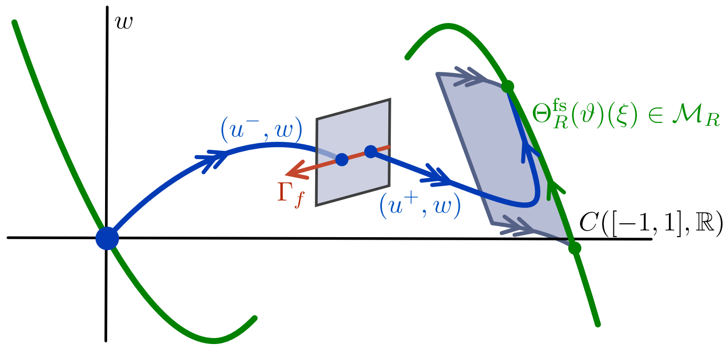

Fig. 7: *Shown is a quasi-front solution which consists of two solutions that lie respectively in the unstable manifold of the equilibrium $(u, w) = 0$ and the stable foliation of the slow manifold $\mathcal{M}_R$. These solutions will, in general, not coincide but we will show that they can be chosen so that their difference at $\xi = 0$ lies in the one-dimensional subspace Γ_f of the phase space $C([-1, 1], \mathbb{R})$. The new equilibria inside $\mathcal{M}_R$ are created by the cut-off function in (3.1).*

The dual product between Λ_f and Λ_f^* is provided through the Hale inner product [22], which is given by

$$\langle \psi, \phi \rangle = c\psi(0)\phi(0) - \alpha \left[\int_0^1 \psi(\sigma - 1)\phi(\sigma)\,\mathrm{d}\sigma + \int_0^{-1} \psi(\sigma + 1)\phi(\sigma)\,\mathrm{d}\sigma \right]$$

for any pair $\phi, \psi \in X$. It was established in [40] that the Hale inner product is non-degenerate in the sense that if $\langle \psi, \phi \rangle = 0$ for all $\psi \in X$ then necessarily $\phi = 0$. A key feature of the Hale inner product is the identity

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \langle \mathrm{ev}_\xi \psi, \mathrm{ev}_\xi \phi \rangle = \psi(\xi)[\Lambda_f \phi](\xi) + [\Lambda_f^* \psi](\xi)\phi(\xi), \tag{3.5}$$

which holds for any pair $\psi, \phi \in C^1(\mathbb{R}, \mathbb{R})$ and $\xi \in \mathbb{R}$. Indeed, if we pick ψ in such a way that $\Lambda_f^* \psi = 0$, one readily sees that integration of (3.5) will yield Melnikov-type identities. In view of these considerations, it is important to understand the kernels

$$\begin{array}{rcl} \mathbb{K}_f & = & \{\phi \in C^1(\mathbb{R}, \mathbb{R}) \mid \Lambda_f \phi = 0 \text{ and } \|\phi\|_\infty < \infty\}, \\ \mathbb{K}_f^* & = & \{\psi \in C^1(\mathbb{R}, \mathbb{R}) \mid \Lambda_f^* \psi = 0 \text{ and } \|\psi\|_\infty < \infty\}. \end{array} \tag{3.6}$$

In addition, we will also need to consider the kernels $\mathbb{K}_b$ and $\mathbb{K}_b^*$ that arise in the exact same fashion after substituting the ansatz $u(\xi) = q_b(\xi) + v(\xi)$ into (2.1), while keeping $w = w_*$ and $\epsilon = 0$ fixed. The following result follows directly from [39, Theorem 4.1].

Lemma 3.2 *Consider the nonlinear system (2.1) and suppose that (H1)-(H4) are satisfied, then we have $q_f'(\xi) > 0$ and $q_b'(\xi) < 0$ for all $\xi \in \mathbb{R}$, together with*

$$\mathbb{K}_f = \mathrm{span}\{q_f'\}, \qquad \mathbb{K}_b = \mathrm{span}\{q_b'\}.$$

In addition, there exist two bounded functions d_f and d_b that decay exponentially at both $\pm\infty$ and have $d_f(\xi) > 0$ and $d_b(\xi) > 0$ for all $\xi \in \mathbb{R}$, such that

$$\mathbb{K}_f^* = \mathrm{span}\{d_f\}, \qquad \mathbb{K}_b^* = \mathrm{span}\{d_b\}.$$

Let us consider any non-zero $d_f \in \mathbb{K}_f^*$, write

$$\widehat{X}_f^\perp = \{\phi \in \widehat{X}_f \mid \langle \mathrm{ev}_0 d_f, \phi \rangle = 0\}$$

and define $\widehat{X}_b^{\perp}$ in the analogous fashion. Note that $\widehat{X}_f^{\perp} \subset \widehat{X}_f$ is closed and of codimension one and that the same holds for the inclusion $\widehat{X}_b^{\perp} \subset \widehat{X}_b$. This allows us to choose appropriate one-dimensional spaces $\Gamma_f \subset X$ and $\Gamma_b \subset X$ and write

$$X = \mathrm{span}\{\mathrm{ev}_0 q_f'\} \oplus \widehat{X}_f^{\perp} \oplus \Gamma_f = \mathrm{span}\{\mathrm{ev}_0 q_b'\} \oplus \widehat{X}_b^{\perp} \oplus \Gamma_b. \tag{3.7}$$

By construction, any $\phi \in \Gamma_f$ satisfies $\phi = 0$ if and only if $\langle \mathrm{ev}_0 d_f, \phi \rangle = 0$, which in combination with (3.5) ensures that Γ_f and Γ_b are ideally suited to capture the jumps that the quasi-fronts and quasi-backs make when they pass through the hyperplanes $\mathrm{ev}_0\, q_f \oplus \widehat{X}_f$ and $\mathrm{ev}_0\, q_b \oplus \widehat{X}_b$.

As a final preparation, let us consider the homogeneous MFDEs

$$cv'(\xi) = \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] - g'\big(\widetilde{s}_j(\vartheta)\big)v(\xi),$$

for $j = L, R$, where the quantity ϑ is taken from $[w_{\min}, w_{\max}]$. Looking for solutions of the form $v(\xi) = \mathrm{e}^{z\xi}$ we obtain the characteristic equations $\Delta_{j,\vartheta}(z) = 0$, with

$$\Delta_{j,\vartheta}(z) = cz - \alpha[\mathrm{e}^{z} + \mathrm{e}^{-z} - 2] - g'\big(\widetilde{s}_j(\vartheta)\big). \tag{3.8}$$

Notice that $\mathrm{Im}\,\Delta_{j,\vartheta}(\mathrm{i}\kappa) = c\kappa$ for any $\kappa \in \mathbb{R}$, while $\mathrm{Re}\,\Delta_{j,\vartheta}(0) = -g'\big(\widetilde{s}_j(\vartheta)\big)$. Our choice of the constants $w_{\min}$ and $w_{\max}$ hence ensures that we can pick $\eta_* > 0$ in such a way that the characteristic equations $\Delta_{j,\vartheta}(z) = 0$ have no roots with $|\mathrm{Re}\, z| \le \eta_*$ for any $c \neq 0$, any $\vartheta \in [w_{\min}, w_{\max}]$ and $j = L, R$. This constant η_* will be used ubiquitously throughout this paper.

We are now ready to define the concept of a quasi-front solution; see again Figure 7. Recall the quantities $\delta_c > 0$ and $\delta_\epsilon > 0$ that appear in Proposition 3.1 and fix $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$.

Definition 3.3 (Quasi-front solution) *For each $0 < \delta \ll 1$ and $\xi_* \gg 1$, we say that the quadruplet*

$$(u^-, u^+, w, \vartheta) \in C((-\infty, 1], \mathbb{R}) \times C([-1, \infty), \mathbb{R}) \times C(\mathbb{R}, \mathbb{R}) \times [-\delta, \delta]$$

is a (δ, ξ_*)-quasi-front solution if the following is true:

(i) The pair $(u^\pm, w)$ satisfies (3.1) on the interval $\mathbb{R}^\pm$.

(ii) We have $\lim_{\xi \to -\infty}(u^-(\xi), w(\xi)) = 0$ and

$$\begin{array}{llll} |u(\xi) - q_f(\xi)| < \delta & \text{and} \quad |w(\xi)| < \delta & \text{for} & \xi \le \xi_* \\ \big|u^+(\xi) - \widetilde{s}_R\big(w(\xi)\big)\big| < \delta & & \text{for} & \xi_* \le \xi, \end{array}$$

where $u(\xi)$ should be read as $u^-(\xi)$ for $\xi \le -1$, as $u^+(\xi)$ for $\xi \ge 1$, and as both $u^\pm(\xi)$ in the region $-1 \le \xi \le 1$.

(iii) We have $\lim_{\xi \to \infty} \mathrm{e}^{\eta_* \xi}[w(\xi) - \Theta_R^{\mathrm{fs}}(\vartheta, c, \epsilon)(\xi)] = 0$.

(iv) We have $\mathrm{ev}_0 u^- \in \mathrm{ev}_0\, q_f \oplus \widehat{X}_f$, $\mathrm{ev}_0 u^+ \in \mathrm{ev}_0\, q_f \oplus \widehat{X}_f$, and $\mathrm{ev}_0[u^- - u^+] \in \Gamma_f$.

Roughly speaking, these properties imply that $u^\pm$ and w can be combined to build a solution to (3.1) that remains δ-close to the portion of the singular orbit Γ_0^{fs} that consists of q_f and $\mathcal{M}_R$ and that is continuous everywhere except on the interval $[-1, 1]$. On this interval the solution is double-valued, with a difference that is contained in Γ_f.

We need one more definition before we can state our result concerning the existence of quasi-front-solutions. To this end, consider any interval $\mathcal{I} \subset \mathbb{R}$. We introduce the following family of Banach spaces, parametrized by $\eta \in \mathbb{R}$,

$$BC_\eta(\mathcal{I}, \mathbb{R}) \quad = \quad \Big\{ x \in C(\mathcal{I}, \mathbb{R}) \mid \|x\|_\eta := \sup_{\xi \in \mathcal{I}} \mathrm{e}^{-\eta|\xi|}\, |x(\xi)| < \infty \Big\}.$$

In the sequel, we will also use the spaces $BC_\eta^1(\mathcal{I}, \mathbb{R}) = \{ y \in BC_\eta(\mathcal{I}, \mathbb{R}) \mid y' \in BC_\eta(\mathcal{I}, \mathbb{R}) \}$.

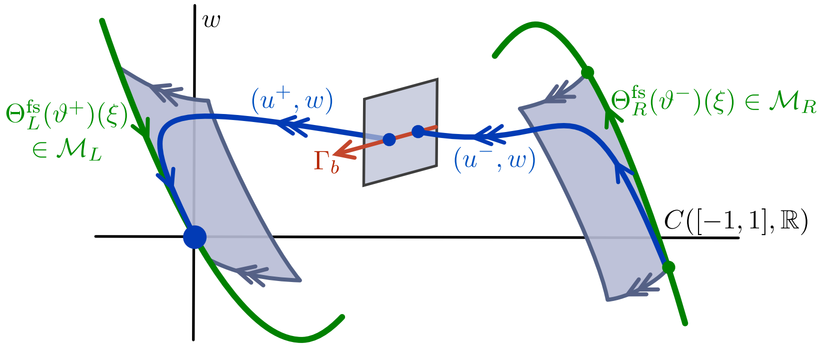

Fig. 8: *Shown is a quasi-back solution that connects $\mathcal{M}_R$ to $(0,0)$ and has a discontinuity at $\xi = 0$ with a jump that lies in the one-dimensional Γ_b.*

Proposition 3.4 *Consider the nonlinear equation (3.1) and assume that (H1)-(H4) are satisfied, then there are constants $\xi_* \gg 1$ and $0 < \delta, \delta_c, \delta_\epsilon \ll 1$ and a set of maps*

$$\begin{array}{lcl} u_f^- & : & [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C((-\infty, 1], \mathbb{R}), \\ u_f^+ & : & [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C([-1, \infty), \mathbb{R}), \\ w_f & : & [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C(\mathbb{R}, [w_{\min}, w_{\max}]), \\ \vartheta_f & : & [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to [-\delta, \delta] \end{array}$$

that satisfies the following properties.

(i) For any $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, the quadruplet $\big(u_f^-(c,\epsilon), u_f^+(c,\epsilon), w_f(c,\epsilon), \vartheta_f(c,\epsilon)\big)$ is the unique (δ, ξ_*)-quasi-front solution to (3.1).

(ii) Write $\xi_f^\infty(c,\epsilon) := \mathrm{ev}_0[u_f^-(c,\epsilon) - u_f^+(c,\epsilon)] \in \Gamma_f$ and pick a non-zero $d_f \in \mathbb{K}_f^*$ with $d_f(0) > 0$, then the following Melnikov inequalities hold,

$$\begin{array}{lcl} D_c[\langle \mathrm{ev}_0 d_f, \xi_f^\infty(c,\epsilon)\rangle]_{c=c_*,\epsilon=0} & < & 0, \\ D_\epsilon[\langle \mathrm{ev}_0 d_f, \xi_f^\infty(c,\epsilon)\rangle]_{c=c_*,\epsilon=0} & < & 0. \end{array} \tag{3.9}$$

(iii) The maps $(c,\epsilon) \mapsto \vartheta_f(c,\epsilon) \in \mathbb{R}$ and

$$(c,\epsilon) \longmapsto \left\{ \begin{array}{ll} u_f^-(c,\epsilon) - q_f & \in BC_{-\eta_*}((-\infty,1],\mathbb{R}) \\ w_f(c,\epsilon)_{|\mathbb{R}_-} & \in BC_{-\eta_*}((-\infty,0],\mathbb{R}) \\ w_f(c,\epsilon)_{|\mathbb{R}_+} - \Theta_R^{\text{fs}}(\vartheta_f(c,\epsilon),c,\epsilon) & \in BC_{-\eta_*}([0,\infty),\mathbb{R}) \\ u_f^+(c,\epsilon) - s_R(w_f(c,\epsilon)(\cdot),c,\epsilon) & \in BC_{-\eta_*}([-1,\infty),\mathbb{R}) \end{array} \right.$$

are C^r-smooth with values in the spaces indicated above, where r appeared in (H1).

Moving on to study the connections between $\mathcal{M}_R(c,\epsilon)$ and $\mathcal{M}_L(c,\epsilon)$, we now define quasi-back solutions, which are illustrated in Figure 8.

Definition 3.5 (Quasi-back solution) *For each $0 < \delta \ll 1$ and $\xi_* \gg 1$, we say that the quintuplet*

$$(u^-, u^+, w, \vartheta^-, \vartheta^+) \in C((-\infty,1],\mathbb{R}) \times C([-1,\infty),\mathbb{R}) \times C(\mathbb{R},\mathbb{R}) \times [w_* - \delta, w_* + \delta]^2$$

is a (δ, ξ_*)-quasi-back solution if the following is true:

(i) The pair $(u^\pm, w)$ satisfies (3.1) on the interval $\mathbb{R}^\pm$.

(ii) *We have* $\lim_{\xi\to\infty}(u^+(\xi), w(\xi)) = 0$ *and*

$$\begin{array}{rlll} \big|u^-(\xi) - \widetilde{s}_R\big(w(\xi)\big)\big| < \delta & & \text{for} & -\infty \le \xi \le -\xi_*, \\ |u(\xi) - q_b(\xi)| < \delta & \text{and} \quad |w(\xi) - w_*| < \delta & \text{for} & -\xi_* \le \xi \le \xi_*, \\ \big|u^+(\xi) - \widetilde{s}_L\big(w(\xi)\big)\big| < \delta & & \text{for} & \xi_* \le \xi, \end{array}$$

where $u(\xi)$ *should be read as* $u^-(\xi)$ *for* $\xi \le -1$*, as* $u^+(\xi)$ *for* $\xi \ge 1$*, and as both* $u^\pm(\xi)$ *in the region* $-1 \le \xi \le 1$*.*

(iii) *We have*

$$\begin{array}{rcl} \lim_{\xi\to-\infty} \mathrm{e}^{\eta_*|\xi|}[w(\xi) - \Theta_R^{\mathrm{fs}}(\vartheta^-, c, \epsilon)(\xi)] & = & 0, \\ \lim_{\xi\to\infty} \mathrm{e}^{\eta_*\xi}[w(\xi) - \Theta_L^{\mathrm{fs}}(\vartheta^+, c, \epsilon)(\xi)] & = & 0. \end{array}$$

(iv) *We have* $\mathrm{ev}_0 u^- \in \mathrm{ev}_0\, q_b \oplus \widehat{X}_b$*,* $\mathrm{ev}_0 u^+ \in \mathrm{ev}_0\, q_b \oplus \widehat{X}_b$*, and* $\mathrm{ev}_0[u^- - u^+] \in \Gamma_b$*.*

Compared to the existence result for the quasi-fronts, an additional degree of freedom arises when constructing quasi-back solutions to (3.1). This freedom is used in the following result to fix $w(0)$.

Proposition 3.6 *Consider the nonlinear equation (3.1) and suppose that (H1)-(H5) are satisfied. Then there exist constants* $\xi_* \gg 1$ *and* $0 < \delta, \delta_\vartheta, \delta_c, \delta_\epsilon \ll 1$*, together with a set of maps*

$$\begin{array}{rcl} u_b^- & : & [w_* - \delta_\vartheta, w_* + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C((-\infty, 1], \mathbb{R}), \\ u_b^+ & : & [w_* - \delta_\vartheta, w_* + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C([-1, \infty), \mathbb{R}), \\ w_b & : & [w_* - \delta_\vartheta, w_* + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C(\mathbb{R}, [w_{\min}, w_{\max}]), \\ \vartheta_b^- & : & [w_* - \delta_\vartheta, w_* + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to [w_{\min}, w_{\max}], \\ \vartheta_b^+ & : & [w_* - \delta_\vartheta, w_* + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to [w_{\min}, w_{\max}] \end{array}$$

that satisfies the following properties.

(i) *For any* $\vartheta^0 \in [w_* - \delta_\vartheta, w_* + \delta_\vartheta]$*,* $c \in [c_* - \delta_c, c_* + \delta_c]$ *and* $\epsilon \in [0, \delta_\epsilon]$*, the quintuplet*

$$\big(u_b^-(\vartheta^0, c, \epsilon), u_b^+(\vartheta^0, c, \epsilon), w_b(\vartheta^0, c, \epsilon), \vartheta_b^-(\vartheta^0, c, \epsilon), \vartheta_b^+(\vartheta^0, c, \epsilon)\big)$$

is the unique (δ, ξ_*)*-quasi-back solution to (3.1) that has* $w(0) = \vartheta^0$*.*

(ii) *Write* $\xi_b^\infty(\vartheta^0, c, \epsilon) := \mathrm{ev}_0[u_b^-(\vartheta^0, c, \epsilon) - u_b^+(\vartheta^0, c, \epsilon)] \in \Gamma_b$ *and pick a nonzero* $d_b \in \mathbb{K}_b^*$ *that has* $d_b(0) > 0$*, then the following Melnikov inequalities hold,*

$$\begin{array}{rcl} D_c[\langle \mathrm{ev}_0 d_b, \xi_b^\infty(\vartheta^0, c, \epsilon)\rangle_0]_{\vartheta^0 = w_*, c = c_*, \epsilon = 0} & > & 0, \\ D_{\vartheta^0}[\langle \mathrm{ev}_0 d_b, \xi_b^\infty(\vartheta^0, c, \epsilon)\rangle_0]_{\vartheta^0 = w_*, c = c_*, \epsilon = 0} & < & 0. \end{array}$$

In addition, we have $D_{\vartheta^0}\vartheta_b^\pm(w_*, c_*, 0) \neq 0$*.*

(iii) *The maps* $(\vartheta^0, c, \epsilon) \mapsto \vartheta_b^\pm(\vartheta^0, c, \epsilon) \in \mathbb{R}$ *and*

$$(\vartheta^0, c, \epsilon) \longmapsto \left\{ \begin{array}{ll} w_b(\vartheta^0, c, \epsilon)_{|\mathbb{R}_-} - \Theta_R^{\mathrm{fs}}(\vartheta_b^-(\vartheta^0, c, \epsilon), c, \epsilon) & \in BC_{-\eta_*}((-\infty, 0], \mathbb{R}) \\ u_b^-(\vartheta^0, c, \epsilon) - s_R(w_b(\vartheta^0, c, \epsilon)(\cdot), c, \epsilon) & \in BC_{-\eta_*}((-\infty, 1], \mathbb{R}) \\ w_b(\vartheta^0, c, \epsilon)_{|\mathbb{R}_+} - \Theta_L^{\mathrm{fs}}(\vartheta_b^+(\vartheta^0, c, \epsilon), c, \epsilon) & \in BC_{-\eta_*}([0, \infty), \mathbb{R}) \\ u_b^+(\vartheta^0, c, \epsilon) - s_L(w_b(\vartheta^0, c, \epsilon)(\cdot), c, \epsilon) & \in BC_{-\eta_*}([-1, \infty), \mathbb{R}) \end{array} \right.$$

are C^r*-smooth with values in the spaces indicated above, where* r *appeared in (H1).*

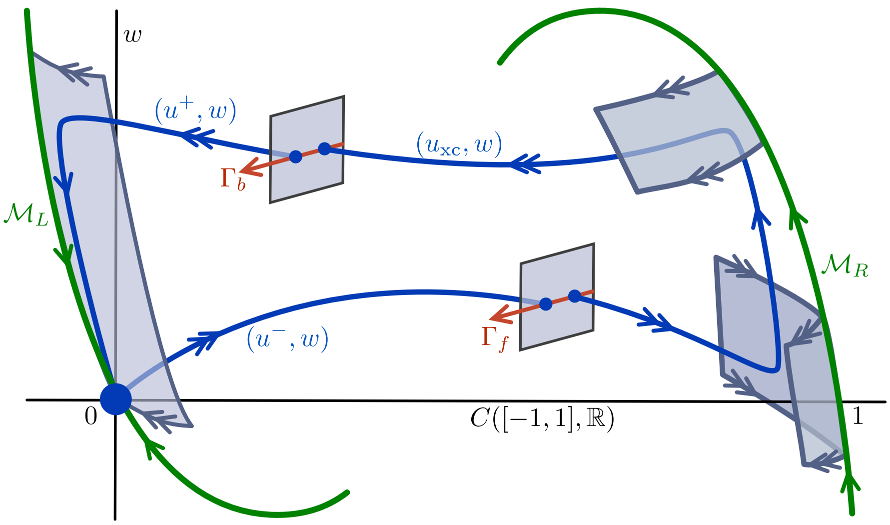

Fig. 9: *An illustration of quasi-solutions and their passage near the slow manifold $\mathcal{M}_R$.*

3.3 Step 3 - The passage near $\mathcal{M}_R$: the Exchange Lemma

We now proceed to connect the quasi-front solutions to the quasi-back solutions somewhere near the manifold $\mathcal{M}_R(c, \epsilon)$: Figure 9 illustrates the solutions we shall construct in this section.

We will use the time T that solutions spend near $\mathcal{M}_R$ as our primary parameter. We note that (H5) allows us to define the slow time $T_*^{\rm sl} > 0$ as the unique time for which

$$\Theta_R^{\rm sl}(0, c_*, 0)(T_*^{\rm sl}) = w_*. \tag{3.10}$$

Since we need solutions to follow the back q_b with $w \approx w_*$, we require that $\epsilon T \approx T_*^{\rm sl}$. In particular, this means that ϵ and the fast time T cannot be treated as independent parameters. To accommodate this requirement, we introduce the slow time variable $T^{\rm sl} = \epsilon T$ and treat c, $T^{\rm sl}$ and T as the independent parameters. We therefore introduce the parameter space

$$\Omega = \Omega(\delta_c, \delta_{\rm sl}, T_*) = [c_* - \delta_c, c_* + \delta_c] \times [T_*^{\rm sl} - \delta_{\rm sl}, T_*^{\rm sl} + \delta_{\rm sl}] \times [T_*, \infty). \tag{3.11}$$

Recall the functions ϑ_f and ϑ_b^+ that appear in Propositions 3.4 and 3.6: these functions select the specific fiber of $\mathcal{M}_R(c, \epsilon)$ that quasi-fronts and quasi-backs approach as $\xi \to \infty$ or $\xi \to -\infty$. We will use the additional parameter ϑ^0 that is available when selecting a quasi-back to ensure that these fibers match up properly after the time T spent near $\mathcal{M}_R(c, \epsilon)$. Specifically, we introduce the function $\vartheta_\infty^0 : \Omega \to [w_{\min}, w_{\max}]$ that is uniquely defined by the requirement that

$$\vartheta_b^-(\vartheta_\infty^0(\omega), c, T^{\rm sl}/T) = \Theta_R^{\rm sl}(\vartheta_f^+(c, T^{\rm sl}/T), c, T^{\rm sl}/T)(T^{\rm sl})$$

for all $\omega = (c, T^{\rm sl}, T) \in \Omega$. On account of Proposition 3.6(ii), the function ϑ_∞^0 is well-defined provided that T_* is chosen to be sufficiently large and $\delta_c > 0$ and $\delta_{\rm sl} > 0$ are chosen to be sufficiently small. In addition, we have the expansion

$$\vartheta_\infty^0(\omega) - w_* = \kappa_1(\omega)[T^{\rm sl} - T_*^{\rm sl}] + \kappa_2(\omega)/T + \kappa_3(\omega)(c - c_*), \tag{3.12}$$

in which κ_1, κ_2 and κ_3 are of class C_b^r on Ω, with $\kappa_1(c_*, T_*^{\rm sl}, \infty) \neq 0$.

Definition 3.7 (Quasi-solution) *Pick $T_* \gg 1$ and $0 < \delta_{sl}, \delta_c \ll 1$, choose $\omega = (c, T^{\rm sl}, T) \in \Omega(\delta_c, \delta_{\rm sl}, T_*)$, and consider (3.1) with $\epsilon := T^{\rm sl}/T$. For each $0 < \delta \ll 1$ and $\xi_* \gg 1$, we say that the quadruplet*

$$(u_f, u_b, u_{\rm xc}, w) \in C((-\infty, 1], \mathbb{R}) \times C([T-1, \infty), \mathbb{R}) \times C([-1, T+1], \mathbb{R}) \times C(\mathbb{R}, \mathbb{R})$$

is a (δ, ξ_*)*-quasi-solution if the following holds:*

(i) The pairs (u_f, w), (u_{xc}, w), *and* (u_b, w) *satisfy (3.1) on the intervals* $(-\infty, 0]$, $[0, T]$, *and* $[T, \infty)$, *respectively.*

(ii) We have $\lim_{\xi \to -\infty}(u_f(\xi), w(\xi)) = 0$ *and* $\lim_{\xi \to \infty}(u_b(\xi), w(\xi)) = 0$.

(iii) We have

$$\begin{array}{rlll} |u(\xi) - q_f(\xi)| < \delta & \text{and} & |w(\xi)| < \delta & \text{for} \quad \xi \leq \xi_*, \\ \big|u(\xi) - \widetilde{s}_R\big(w(\xi)\big)\big| < \delta & & & \text{for} \quad \xi_* \leq \xi \leq T - \xi_*, \\ |u(\xi) - q_b(\xi - T)| < \delta & \text{and} & |w(\xi) - w_*| < \delta & \text{for} \quad T - \xi_* \leq \xi \leq T + \xi_*, \\ \big|u(\xi) - \widetilde{s}_L\big(w(\xi)\big)\big| < \delta & & & \text{for} \quad T + \xi_* \leq \xi, \end{array}$$

where $u(\xi)$ *should be read as* $u_f(\xi)$ *for* $\xi \leq -1$, $u_{\mathrm{xc}}(\xi)$ *for* $1 \leq \xi \leq T-1$, $u_b(\xi)$ *for* $\xi \geq T+1$, *both* $u_f(\xi)$ *and* $u_{\mathrm{xc}}(\xi)$ *in the region* $-1 \leq \xi \leq 1$, *and both* $u_b(\xi)$ *and* $u_{\mathrm{xc}}(\xi)$ *in the region* $T - 1 \leq \xi \leq T + 1$.

(iv) We have

$$\mathrm{ev}_0 u_f,\ \mathrm{ev}_0 u_{\mathrm{xc}} \in \mathrm{ev}_0\, q_f \oplus \widehat{X}_f, \quad \mathrm{ev}_T u_b,\ \mathrm{ev}_T u_{\mathrm{xc}} \in \mathrm{ev}_0\, q_b \oplus \widehat{X}_b,$$

(v) We have $\mathrm{ev}_0[u_f - u_{\mathrm{xc}}] \in \Gamma_f$ *and* $\mathrm{ev}_T[u_b - u_{\mathrm{xc}}] \in \Gamma_b$.

Our next result, which can be interpreted as an extension of the Exchange Lemma to MFDEs, is concerned with the existence of quasi-solutions.

Proposition 3.8 *Consider the nonlinear equation (3.1) and suppose that (H1)-(H5) are satisfied, then there are constants* $\xi_* \gg 1$, $T_* \gg 1$ *and* $0 < \delta, \delta_c, \delta_{\mathrm{sl}} \ll 1$ *with the following property. For each* $\omega = (c, T^{\mathrm{sl}}, T) \in \Omega = \Omega(\delta_c, \delta_{\mathrm{sl}}, T_*)$, *there exists a quadruplet* $\big(u_f(\omega), u_b(\omega), u_{\mathrm{xc}}(\omega), w(\omega)\big)$ *with*

$$\begin{array}{rcl} u_f(\omega) & \in & C((-\infty, 1], \mathbb{R}), \\ u_b(\omega) & \in & C([T-1, \infty), \mathbb{R}), \\ u_{\mathrm{xc}}(\omega) & \in & C([-1, T+1], \mathbb{R}), \\ w(\omega) & \in & C(\mathbb{R}, [w_{\min}, w_{\max}]), \end{array}$$

that satisfies the following properties.

(i) For any $\omega \in \Omega$, *the quadruplet* $\big(u_f(\omega), u_b(\omega), u_{\mathrm{xc}}(\omega), w(\omega)\big)$ *is the unique* (δ, ξ_*)*-quasi-solution to (3.1) with* $\epsilon = T^{\mathrm{sl}}/T$.

(ii) The maps $\omega \mapsto \xi_f(\omega)$ *and* $\omega \mapsto \xi_b(\omega)$ *defined by*

$$\begin{array}{rcl} \xi_f(\omega) & := & \mathrm{ev}_0[u_f(\omega) - u_{\mathrm{xc}}(\omega)] \in \Gamma_f, \\ \xi_b(\omega) & := & \mathrm{ev}_T[u_b(\omega) - u_{\mathrm{xc}}(\omega)] \in \Gamma_b \end{array}$$

are C^r*-smooth, where the integer* r *appeared in (H1).*

(iii) Consider the maps

$$\begin{array}{rcl} \widetilde{\xi}_f & : & \omega \mapsto \xi_f(\omega) - \xi_f^{\infty}(c, T^{\mathrm{sl}}/T), \\ \widetilde{\xi}_b & : & \omega \mapsto \xi_f(\omega) - \xi_f^{\infty}(\vartheta^0_{\infty}(\omega), c, T^{\mathrm{sl}}/T), \end{array}$$

then there exists a constant $C > 0$ *such that, for any integer* $0 \leq \ell \leq r$ *and any* $\omega \in \Omega$, *we have the estimate*

$$\Big|D^{\ell}_{\omega}\widetilde{\xi}_f(\omega)\Big| + \Big|D^{\ell}_{\omega}\widetilde{\xi}_b(\omega)\Big| \leq C\mathrm{e}^{-\eta_* T}. \tag{3.13}$$

With this result in hand we have gathered all the ingredients we need to establish our main theorem.

3.4 Step 4 - Proof of Theorem 1

The remaining arguments are almost identical to those used in the proof of [35, Theorem 1]. Let Ω be as in Proposition 3.8. Finding (δ, ξ_*)-homoclinic solutions to (3.1) has now been reduced to finding $\omega \in \Omega$ that have $\xi_f(\omega) = \xi_b(\omega) = 0$. This leads to the system

$$\begin{array}{rcl} b_1(\omega)\mathrm{e}^{-\eta_* T} & = & -c_1(\omega)(c - c_*) - c_2(\omega)/T, \\ b_2(\omega)\mathrm{e}^{-\eta_* T} & = & c_3(\omega)(c - c_*) - c_4(\omega)[\vartheta^0_\infty(\omega) - w_*] + c_5(\omega)/T, \end{array} \tag{3.14}$$

in which the functions b_1, b_2 and c_1 through c_5 and their derivatives are bounded on Ω. In addition, setting $\omega_0 = (c_*, T_*^{\mathrm{sl}}, \infty)$, we have $c_1(\omega_0) \neq 0$, $c_2(\omega_0) \neq 0$, $c_4(\omega_0) \neq 0$ and $\mathrm{sign}\big(c_1(\omega_0)\big) = \mathrm{sign}\big(c_2(\omega_0)\big)$. Using (3.12) and solving the second equation in (3.14), we find

$$\frac{T^{\mathrm{sl}}}{T} = \frac{T_*^{\mathrm{sl}}}{T} + \mathrm{O}\left(\frac{1}{T^2} + (c - c_*)\frac{1}{T}\right).$$

Substituting this expression into the first line of (3.14) and solving, we obtain

$$\frac{1}{T} = -\frac{c_1(\omega_0)}{c_2(\omega_0)T_*^{\mathrm{sl}}}(c - c_*) + \mathrm{O}\big((c - c_*)^2\big),$$

which, using $\epsilon = T^{\mathrm{sl}}/T$, yields the desired expansion

$$\epsilon = -\frac{c_1(\omega_0)}{c_2(\omega_0)}(c - c_*) + \mathrm{O}\big((c - c_*)^2\big).$$

This completes the proof of our main result subject to proving the propositions that we stated in the preceding sections. Their proofs will occupy the remainder of this paper.

4 Persistence of slow manifolds

In this section we set out to prove Proposition 3.1. The approach in this section is based heavily on the construction developed in [43§2] to establish the persistence of slow manifolds in the context of singularly perturbed ODEs. At the appropriate points in the analysis, the machinery that was developed in [26§6] for MFDEs with slowly modulating coefficients will be put to work. We will focus on the construction of the function s_R, noting that s_L can be constructed in a similar fashion. Our approach will be to fix w_0, c and $\epsilon > 0$ and look for a bounded solution (u, w) to (3.1) that remains close to $\mathcal{M}_R$ and has $w(0) = w_0$. We will then write $s_R(w_0, c, \epsilon) = u(0)$ and show that this function has the desired properties. In essence, we are constructing a center manifold around $\mathcal{M}_R$.

Let us start by introducing the new variable v via

$$u(\xi) = \widetilde{s}_R\big(w(\xi)\big) + v(\xi). \tag{4.1}$$

Substituting this back into (3.1) and recalling the identity $g\big(\widetilde{s}_R(w)\big) = w$, we find that the pair (v, w) must satisfy

$$\begin{array}{rcl} cv'(\xi) & = & L\big(\widetilde{s}_R(w(\xi))\big)\mathrm{ev}_\xi v - \epsilon D\widetilde{s}_R\big(w(\xi)\big)\big[\widetilde{s}_R\big(w(\xi)\big) + v(\xi) - \gamma w(\xi)\big]\chi_1\big(w(\xi)\big) \\ & & +G\big(v(\xi), w(\xi)\big) + H(\mathrm{ev}_\xi w), \\ cw'(\xi) & = & \epsilon[\widetilde{s}_R\big(w(\xi)\big) + v(\xi) - \gamma w(\xi)]\chi_{\mathrm{sl}}\big(w(\xi)\big), \end{array} \tag{4.2}$$

in which the operator $L : \mathbb{R} \to \mathcal{L}(X, \mathbb{R})$ is given by

$$L(u)\mathrm{ev}_\xi v = \alpha[v(\xi + 1) + v(\xi - 1) - 2v(\xi)] + g'(u)v(\xi), \tag{4.3}$$

while the nonlinear operators $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $H : C([-1,1],[w_{\min}, w_{\max}]) \to \mathbb{R}$ are given by

$$\begin{array}{rcl} G(v,w) & = & g\big(\widetilde{s}_R(w)+v\big) - g\big(\widetilde{s}_R(w)\big) - g'\big(\widetilde{s}_R(w)\big)v, \\ H(\mathrm{ev}_\xi w) & = & \alpha[\widetilde{s}_R\big(w(\xi+1)\big) + \widetilde{s}_R\big(w(\xi-1)\big) - 2\widetilde{s}_R\big(w(\xi)\big)]. \end{array} \tag{4.4}$$

In order to stay as close as possible to the setting in [43] and in particular to reproduce the estimate [43, Equation (2.3)], we will split the operator H into a part H_{lin} that is linear in w and a nonlinear part H_{nl}. After using the differential equation for w to transform H_{lin}, we write

$$\begin{array}{rcl} H_{\mathrm{lin}}(\mathrm{ev}_\xi v, \mathrm{ev}_\xi w) & = & \frac{\epsilon\alpha}{c} D\widetilde{s}_R\big(w(\xi)\big)\Big[\int_\xi^{\xi+1} \big[\widetilde{s}_R\big(w(\xi')\big) + v(\xi') - \gamma w(\xi')\big]\chi_{\mathrm{sl}}\big(w(\xi')\big)\,\mathrm{d}\xi' \\ & & +\int_\xi^{\xi-1} \big[\widetilde{s}_R\big(w(\xi')\big) + v(\xi') - \gamma w(\xi')\big]\chi_{\mathrm{sl}}\big(w(\xi')\big)\,\mathrm{d}\xi'\Big], \\ H_{\mathrm{nl}}(\mathrm{ev}_\xi w) & = & \alpha[\widetilde{s}_R\big(w(\xi+1)\big) + \widetilde{s}_R\big(w(\xi-1)\big) - 2\widetilde{s}_R\big(w(\xi)\big)] \\ & & -\alpha D\widetilde{s}_R\big(w(\xi)\big)[w(\xi+1)+w(\xi-1)-2w(\xi)]. \end{array} \tag{4.5}$$

Since we are only interested in solutions for which v is small, we will add a cut-off to v. In addition, to bound the Lipschitz constant associated with H_{nl}, we will need to apply a special cut-off to w. To this end, let us introduce for any $w \in C(\mathbb{R},\mathbb{R})$, the notation

$$\mathrm{cev}_\xi w = \big(w(\xi+1) - w(\xi), w(\xi-1) - w(\xi)\big) \in \mathbb{R}^2. \tag{4.6}$$

We pick an arbitrary C^∞-smooth function $\chi : [0,\infty) \to \mathbb{R}$ that has $\chi(\zeta) = 1$ for $0 \le \zeta \le 1$ and $\chi(\zeta) = 0$ for $\zeta \ge 2$. For any $\delta > 0$, we write χ_δ for the function $\chi_\delta(\zeta) = \chi(\zeta/\delta)$. We are now ready to define, for small quantities $\delta_v > 0$ and $\delta_w > 0$, the cut-off nonlinearities

$$\begin{array}{rcl} G^c(v,w) & = & \chi_{\delta_v}(|v|)G(v,w), \\ H^c_{\mathrm{lin}}(\mathrm{ev}_\xi v, \mathrm{ev}_\xi w) & = & \frac{\epsilon\alpha}{c} D\widetilde{s}_R\big(w(\xi)\big)\Big[\int_\xi^{\xi+1} \big[\widetilde{s}_R\big(w(\xi')\big) + v(\xi') - \gamma w(\xi')\big]\chi_1(|v(\xi')|)\chi_{\mathrm{sl}}\big(w(\xi')\big)\,\mathrm{d}\xi' \\ & & +\int_\xi^{\xi-1} \big[\widetilde{s}_R\big(w(\xi')\big) + v(\xi') - \gamma \theta(\xi')\big]\chi_1(|v(\xi')|)\chi_{\mathrm{sl}}\big(w(\xi')\big)\,\mathrm{d}\xi'\Big], \\ H^c_{\mathrm{nl}}(\mathrm{ev}_\xi w) & = & \chi_{\delta_w}(|\mathrm{cev}_\xi w|)H_{\mathrm{nl}}(\mathrm{ev}_\xi w). \end{array} \tag{4.7}$$

Putting this together, we pick $\zeta \ge 0$, introduce the nonlinearity

$$\mathcal{R}^c_{\mathrm{cm}} : BC_\zeta(\mathbb{R},\mathbb{R}) \times C(\mathbb{R},[w_{\min}, w_{\max}]) \times \mathbb{R} \times \mathbb{R} \to BC_\zeta(\mathbb{R},\mathbb{R}) \tag{4.8}$$

that is given by

$$\begin{array}{rcl} \mathcal{R}^c_{\mathrm{cm}}(v,w,\epsilon,c)(\xi) & = & -\epsilon D\widetilde{s}_R\big(w(\xi)\big)\big[\widetilde{s}_R\big(w(\xi)\big) + v(\xi) - \gamma w(\xi)\big]\chi_1(|v(\xi)|)\chi_{\mathrm{sl}}\big(w(\xi)\big) \\ & & +G^c\big(v(\xi),\theta(\xi)\big) + H^c_{\mathrm{lin}}(\mathrm{ev}_\xi v, \mathrm{ev}_\xi w) + H^c_{\mathrm{nl}}(\mathrm{ev}_\xi w) \end{array} \tag{4.9}$$

and study the equation

$$cv'(\xi) = L\Big(\widetilde{s}_R\big(w(\xi)\big)\Big)\mathrm{ev}_\xi v + \mathcal{R}^c_{\mathrm{cm}}(v,w,\epsilon,c)(\xi), \tag{4.10a}$$

$$cw'(\xi) = \epsilon[\widetilde{s}_R\big(w(\xi)\big) + v(\xi) - \gamma w(\xi)]\chi_{\mathrm{sl}}\big(w(\xi)\big)\chi_1(|v(\xi)|). \tag{4.10b}$$

Let us pick small constants $\delta_c > 0$ and $\delta_\epsilon > 0$. For quantities a and b that depend on the various cut-offs δ_v, δ_w, δ_c and δ_ϵ that we have introduced, we will use the notation

$$a \le_* b \tag{4.11}$$

to express the fact that there exists a $C > 0$ that does not depend on these cut-offs, such that $a \le Cb$ holds for all $\delta_v \le 1$, $\delta_w \le 1$, $\delta_c \le 1$ and $\delta_\epsilon \le 1$.

Notice that for any $w, w_1, w_2 \in C(\mathbb{R}, [w_{\min}, w_{\max}])$, $v, v_1, v_2 \in BC_\zeta(\mathbb{R}, \mathbb{R})$, $\epsilon \in [0, \delta_\epsilon]$ and $c \in [c_* - \delta_c, c_* + \delta_c]$, we have the inequalities

$$\begin{array}{lcl} |\mathcal{R}^c_{\rm cm}(v, w, c, \epsilon)(\xi)| & \leq_* & \delta_\epsilon + \delta_v^2 + \delta_w^2, \\ \|\mathcal{R}^c_{\rm cm}(v_1, w_1, c, \epsilon) - \mathcal{R}^c_{\rm cm}(v_2, w_2, c, \epsilon)\|_\zeta & \leq_* & (\delta_\epsilon + \delta_v + \delta_w)[\|v_1 - v_2\|_\zeta + \|w_1 - w_2\|_\zeta]. \end{array} \tag{4.12}$$

We proceed with our analysis by considering the linear part of (4.10a). We will write this as

$$\Lambda(w, c)v = h, \tag{4.13}$$

in which the linear operator $\Lambda(w, c) : C^1(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ acts as

$$[\Lambda(w, c)v](\xi) = cv'(\xi) - L\Big(\widetilde{s}_R\big(w(\xi)\big)\Big)\mathrm{ev}_\xi v, \tag{4.14}$$

for any $w \in C(\mathbb{R}, [w_{\min}, w_{\max}])$ and $c \neq 0$. The next result shows that for any $\eta \in [-\eta_*, \eta_*]$, an inverse can be defined for the operator $\Lambda(w, c)$ on the space $BC_\eta(\mathbb{R}, \mathbb{R})$ provided that $\|w'\|_\infty$ is sufficiently small.

Lemma 4.1 *Consider the linear system (4.13) and suppose that (H1)-(H3) are satisfied. Then there exists a constant $\delta_c > 0$, together with a family of maps*

$$\mathcal{K}_\eta : C(\mathbb{R}, [w_{\min}, w_{\max}]) \times [c_* - \delta_c, c_* + \delta_c] \to \mathcal{L}\big(BC_\eta(\mathbb{R}, \mathbb{R}), BC_\eta(\mathbb{R}, \mathbb{R})\big) \tag{4.15}$$

defined for all $\eta \in [-\eta_*, \eta_*]$, such that the following properties are satisfied.

(i) There exists $\kappa > 0$, such that if $w \in C^1(\mathbb{R}, [w_{\min}, w_{\max}])$ and $|w'(\xi)| < \kappa$ for all $\xi \in \mathbb{R}$, then $v = \mathcal{K}_\eta(w, c)h$ satisfies $\Lambda(w, c)v = h$ for any $\eta \in [-\eta_*, \eta_*]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $h \in BC_\eta(\mathbb{R}, \mathbb{R})$.

(ii) The norm $\|\mathcal{K}_\eta(w, c)\|$ can be bounded independently of $\eta \in [-\eta_*, \eta_*]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $w \in C(\mathbb{R}, [w_{\min}, w_{\max}])$.

(iii) There exists a constant $C > 0$ such that for any $\eta_1 > 0$, any $\eta_2, \eta_3 \in [-\eta_*, \eta_*]$ that have $\eta_1 + \eta_2 \leq \eta_3$, any two functions $w_1, w_2 \in C(\mathbb{R}, [w_{\min}, w_{\max}])$, any two $c_1, c_2 \in [c_* - \delta_c, c_* + \delta_c]$ and any $h \in BC_{\eta_2}(\mathbb{R}, \mathbb{R})$, we have the estimate

$$\|\mathcal{K}_{\eta_2}(w_1, c_1)h - \mathcal{K}_{\eta_2}(w_2, c_2)h\|_{\eta_3} \leq C[\|w_1 - w_2\|_{\eta_1} + |c_1 - c_2|]\,\|h\|_{\eta_2}. \tag{4.16}$$

(iv) Consider a pair $\eta_1, \eta_2 \in [-\eta_*, \eta_*]$ together with a function

$$h \in BC_{\eta_1}(\mathbb{R}, \mathbb{R}) \cap BC_{\eta_2}(\mathbb{R}, \mathbb{R}). \tag{4.17}$$

Then for any $w \in C(\mathbb{R}, [w_{\min}, w_{\max}])$ and $c \in [c_* - \delta_c, c_* + \delta_c]$, we have

$$\mathcal{K}_{\eta_1}(w, c)h = \mathcal{K}_{\eta_2}(w, c)h. \tag{4.18}$$

(v) Recall the integer r that appears in (H1). Consider any integer $0 \leq \ell \leq r + 2$ and pick $\eta_1 > 0$ and $\eta_2, \eta_3 \in [-\eta_*, \eta_*]$ in such a way that $\eta_3 > \ell\eta_1 + \eta_2$. Then the map $(\theta, c) \mapsto \mathcal{K}(\theta, c)$ is C^ℓ-smooth when considered as a map from $BC_{\eta_1}(\mathbb{R}, [w_{\min}, w_{\max}]) \times [c_* - \delta_c, c_* + \delta_c]$ into $\mathcal{L}\big(BC_{\eta_2}(\mathbb{R}, \mathbb{R}), BC_{\eta_3}(\mathbb{R}, \mathbb{R})\big)$. In addition, for any pair of integers $p_1, p_2 \geq 0$ with $p_1 + p_2 = \ell$, the derivative $D_1^{p_1} D_2^{p_2}\mathcal{K}$ is well-defined even when interpreted as a map

$$\begin{array}{rcl} D_1^{p_1} D_2^{p_2}\mathcal{K} & : & BC_{\eta_1}(\mathbb{R}, [w_{\min}, w_{\max}]) \times [c_* - \delta_c, c_* + \delta_c] \\ & & \quad \to \mathcal{L}^{(\ell)}\Big(BC_{\eta_1}(\mathbb{R}, \mathbb{R})^{p_1} \times \mathbb{R}^{p_2}, \mathcal{L}\big(BC_{\eta_2}(\mathbb{R}, \mathbb{R}), BC_\eta(\mathbb{R}, \mathbb{R})\big)\Big) \end{array} \tag{4.19}$$

with $\eta = \ell\eta_1 + \eta_2$.

(vi) For any $\xi_0 \in \mathbb{R}$, $\eta \in [-\eta_*, \eta_*]$, $w \in C(\mathbb{R}, [w_{\min}, w_{\max}])$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $h \in BC_\eta(\mathbb{R}, \mathbb{R})$, we have

$$\mathcal{K}_\eta(T_{\xi_0} w, c) T_{\xi_0} h = T_{\xi_0} \mathcal{K}_\eta(w, c) h, \tag{4.20}$$

in which T_{ξ_0} denotes the shift $[T_{\xi_0} h](\xi') = h(\xi_0 + \xi')$.

Proof. We first consider the linear system $\Lambda(w, c)v = h$ in the special case that w is a constant, i.e., $w(\xi) = w_0$ for all $\xi \in \mathbb{R}$. In this case (4.13) reduces to a linear constant-coefficient inhomogeneous MFDE that has been studied extensively [28, 38]. The characteristic function Δ associated with this MFDE can be obtained by seeking a solution of the form $v(\xi) = \exp(z\xi)$ to the homogeneous system $\Lambda(w_0 \mathbf{1}, c)v = 0$. Recalling (3.8), we find that $\Delta(z) = \Delta_{R,w_0}(z)$, from which it follows that the characteristic equation $\Delta(z) = 0$ admits no roots with $|\mathrm{Re}\, z| \leq \eta_*$.

After picking δ_c to be sufficiently small, the constructions in [28§5] can be used to define, for any $w_0 \in [w_{\min}, w_{\max}]$, any $c \in [c_* - \delta_c, c_* + \delta_c]$ and any $\eta \in [-\eta_*, \eta_*]$, the operators

$$\mathcal{K}^{\mathrm{cs}}_\eta(w_0, c) : BC_\eta(\mathbb{R}, \mathbb{R}) \to BC^1_\eta(\mathbb{R}, \mathbb{R}) \tag{4.21}$$

that solve the constant coefficient system $\Lambda(w_0 \mathbf{1}, c)v = h$. More precisely, for any $v \in BC^1_\eta(\mathbb{R}, \mathbb{R})$ we have $\mathcal{K}^{\mathrm{cs}}_\eta(w_0, c)\Lambda(w_0 \mathbf{1}, c)v = v$ and for any $h \in BC_\eta(\mathbb{R}, \mathbb{R})$ we have $\Lambda(w_0 \mathbf{1}, c)\mathcal{K}^{\mathrm{cs}}_\eta(w_0, c)h = h$. One can now employ simplified versions of the arguments in [26§6] and use these operators $\mathcal{K}^{\mathrm{cs}}_\eta(w_0, c)$ to construct a family $\mathcal{K}_\eta$ that satisfies the properties (i) through (vi). ∎

Lemma 4.2 *Consider the linear system (4.13) and suppose that (H1)-(H3) are satisfied. Then there exist constants $\kappa > 0$ and $\delta_c > 0$, such that for any $c \in [c_* - \delta_c, c_* + \delta_c]$ and any $w \in C(\mathbb{R}, [w_{\min}, w_{\max}])$ that has $|w'(\xi)| < \kappa$ for all $\xi \in \mathbb{R}$, the homogeneous equation $\Lambda(w, c)v = 0$ admits no non-zero solutions $v \in BC^1_{\eta_*}(\mathbb{R}, \mathbb{R})$.*

Proof. In the special case that w is a constant function, the claim follows from [28, Proposition 5.2], in view of the observation contained in the proof of Lemma 4.1 that the characteristic function $\Delta(z) = 0$ admits no roots with $|\mathrm{Re}\, z| \leq \eta_*$. A simplified version of the proof of [26, Lemma 6.4] can now be used to generalize the claim to functions w that have a sufficiently small derivative. ∎

We now turn our attention to the equation for w given by (4.10b). For any fixed $v \in C(\mathbb{R}, \mathbb{R})$ and $c \neq 0$, this equation is an ODE with a smooth right-hand side. This allows us to introduce, for $\delta_c > 0$ sufficiently small and any $\delta_\epsilon > 0$, the operator

$$W : [w_{\min}, w_{\max}] \times C(\mathbb{R}, \mathbb{R}) \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to C(\mathbb{R}, [w_{\min}, w_{\max}]) \tag{4.22}$$

that is uniquely defined by the property that $w = W(w_0, v, c, \epsilon)$ solves (4.10b) with $w(0) = w_0$. Our next result is the equivalent of [43, Lemma 2.4] and can be proved using Gronwall's inequality and variational equations.

Lemma 4.3 *There exist constants $L_0 > 0$ and $L_1 > 0$ such that the following hold true.*

(i) For any $w_0 \in [w_{\min}, w_{\max}]$, $\epsilon \in [0, \delta_\epsilon]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $v \in C(\mathbb{R}, \mathbb{R})$, we have

$$|W(w_0, v, c, \epsilon)(\xi)| \leq |\xi| + \epsilon L_0 |\xi| . \tag{4.23}$$

(ii) Consider any $\epsilon \in [0, \delta_\epsilon]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and any $\eta > \epsilon L_1$. Then for any two pairs $(w^1_0, v^1), (w^2_0, v^2) \in [w_{\min}, w_{\max}] \times BC_\eta(\mathbb{R}, \mathbb{R})$, we have the estimate

$$\left|W(w^1_0, v^1, c, \epsilon)(\xi) - W(w^2_0, v^2, c, \epsilon)(\xi)\right| \leq \left|w^1_0 - w^2_0\right| \mathrm{e}^{\epsilon L_1 \xi} + \frac{\epsilon L_1}{\eta - \epsilon L_1} \left\|v^1 - v^2\right\|_\eta \mathrm{e}^{\eta \xi}. \tag{4.24}$$

(iii) *Recall the integer r that appears in (H1). Consider any $0 \le \ell \le r+3$ and pick $\eta_1 > \delta_\epsilon L_1$ and η_2 in such a way that $\eta_2 > \ell\eta_1$. Then the map $(w_0, v, c, \epsilon) \mapsto W(w_0, v, c, \epsilon)$ is C^ℓ-smooth when considered as a map from $[w_{\min}, w_{\max}] \times BC_{\eta_1}(\mathbb{R}, \mathbb{R}) \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon]$ into $BC_{\eta_2}(\mathbb{R}, \mathbb{R})$. In addition, if $1 \le \ell \le r+3$, then for any set of integers $p_1, p_2, p_3, p_4 \ge 0$ that have $p_1+p_2+p_3+p_4 = \ell$, the derivative $D_1^{p_1} D_2^{p_2} D_3^{p_3} D_4^{p_4} W$ is well-defined even when interpreted as a map*

$$\begin{array}{rcl} D_1^{p_1} D_2^{p_2} D_3^{p_3} D_4^{p_4} W & : & [w_{\min}, w_{\max}] \times BC_{\eta_1}(\mathbb{R}, \mathbb{R}) \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \\ & & \to \mathcal{L}^{(\ell)}\big(\mathbb{R}^{p_1} \times BC_{\eta_1}(\mathbb{R}, \mathbb{R})^{p_2} \times \mathbb{R}^{p_3} \times \mathbb{R}^{p_4}, BC_\eta(\mathbb{R}, \mathbb{R})\big) \end{array} \tag{4.25}$$

with $\eta = \ell\eta_1$.

(iv) *For any $\xi_0 \in \mathbb{R}$, $w_0 \in [w_{\min}, w_{\max}]$, $v \in C(\mathbb{R}, \mathbb{R})$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, we have*

$$W(W(w_0, v, c, \epsilon)(\xi_0), T_{\xi_0} v, c, \epsilon) = T_{\xi_0} W(w_0, v, c, \epsilon). \tag{4.26}$$

Proof of Proposition 3.1 We will only consider the statements concerning s_R. Possibly after decreasing $\delta_\epsilon > 0$, pick η in such a way that $\delta_\epsilon L_1 < \eta < \eta_*$. Fix any $w_0 \in [w_{\min}, w_{\max}]$, $\epsilon \in [0, \delta_\epsilon]$ and $c \in [c_* - \delta_c, c_* + \delta_c]$ and consider the fixed point equation

$$v = \mathcal{G}(w_0, v, c, \epsilon) := \mathcal{K}_\eta\big(W(w_0, v, c, \epsilon)\big)\mathcal{R}^c_{\mathrm{cm}}\big(v, W(w_0, v, c, \epsilon), c, \epsilon\big) \tag{4.27}$$

that is posed on the space $BC_\eta(\mathbb{R}, \mathbb{R})$. To see that the right hand side of this equation is a contraction, let us pick a pair $v_1, v_2 \in BC_\eta(\mathbb{R}, \mathbb{R})$ and write $\Delta_{\mathcal{G}} = \|\mathcal{G}(w_0, v_1, c, \epsilon) - \mathcal{G}(w_0, v_2, c, \epsilon)\|_\eta$. We may use Lemma 4.1(iii) with $\eta_1 = \eta_3 = \eta$ and $\eta_2 = 0$, together with Lemma 4.3(ii) to compute

$$\begin{array}{rcl} \Delta_{\mathcal{G}} & \le & \Big\|[\mathcal{K}_\eta\big(W(w_0, v_1, c, \epsilon)\big) - \mathcal{K}_\eta\big(W(w_0, v_2, c, \epsilon)\big)]\mathcal{R}^c_{\mathrm{cm}}\big(v, W(w_0, v_1, c, \epsilon), c, \epsilon\big)\Big\|_\eta \\ & & + \Big\|\mathcal{K}_\eta\big(W(w_0, v_2, c, \epsilon)\big) \\ & & \qquad [\mathcal{R}^c_{\mathrm{cm}}\big(v_1, W(w_0, v_1, c, \epsilon), c, \epsilon\big) - \mathcal{R}^c_{\mathrm{cm}}\big(v_2, W(w_0, v_2, c, \epsilon), c, \epsilon\big)]\Big\|_\eta \\ & \le_* & \|W(w_0, v_1, c, \epsilon) - W(w_0, v_2, c, \epsilon)\|_\eta \left\|\mathcal{R}^c_{\mathrm{cm}}\big(v, W(w_0, v_1, c, \epsilon), c, \epsilon\big)\right\|_0 \\ & & + \left\|\mathcal{R}^c_{\mathrm{cm}}\big(v_1, W(w_0, v_1, c, \epsilon), c, \epsilon\big) - \mathcal{R}^c_{\mathrm{cm}}\big(v_2, W(w_0, v_2, c, \epsilon), c, \epsilon\big)\right\|_\eta \\ & \le_* & [\delta_\epsilon + \delta_v + \delta_w] \|v_1 - v_2\|_\eta . \end{array} \tag{4.28}$$

Using the estimates (4.12) once more, we also find

$$\|\mathcal{G}(w_0, v, c, \epsilon)\|_\eta \le_* \delta_\epsilon + \delta_v^2 + \delta_w^2. \tag{4.29}$$

By choosing $\delta_v > 0$, $\delta_w > 0$ and $\delta_\epsilon > 0$ to be sufficiently small, we can hence use the contraction mapping principle to ensure that the fixed point equation (4.27) has a unique solution that we will write as $v = v^*(w_0, c, \epsilon)$. In addition, we may immediately read off the estimate

$$|v^*(w_0, c, \epsilon)(\xi)| \le_* \delta_\epsilon + \delta_v^2 + \delta_w^2 \tag{4.30}$$

by considering (4.27) in the space $BC_0(\mathbb{R}, \mathbb{R})$. After possibly decreasing δ_v, δ_w and δ_ϵ even further, we hence find that v^* remains unaffected by the cut-offs introduced in (4.7).

We are now ready to define s_R by means of

$$s_R(w_0, c, \epsilon) = \widetilde{s}_R(w_0) + v^*(w_0, c, \epsilon)(0). \tag{4.31}$$

Item (i) now follows from the observation that $v^*(w_0, c, 0) = 0$. To establish (ii), we introduce the notation $w^*(w_0, c, \epsilon) = W(w_0, v^*(w_0, c, \epsilon), c, \epsilon)$. Using Lemma 4.1(iv) and Lemma 4.3(iv), it is not hard to verify that

$$\mathcal{G}(w^*(w_0, c, \epsilon)(\xi_0), T_{\xi_0} v^*(w_0, c, \epsilon), c, \epsilon) = T_{\xi_0} v^*(w_0, c, \epsilon). \tag{4.32}$$

Due to the uniqueness of solutions to the fixed point equation (4.27), this implies that

$$v^*(w^*(w_0,c,\epsilon)(\xi_0),c,\epsilon) = T_{\xi_0} v^*(w_0,c,\epsilon). \tag{4.33}$$

In particular, we have

$$\widetilde{s}_R\big(w^*(w_0,c,\epsilon)(\xi)\big) + v^*(w_0,c,\epsilon)(\xi) = s_R(w^*(w_0,c,\epsilon)(\xi),c,\epsilon), \tag{4.34}$$

which shows that $w^*(w_0,c,\epsilon)$ in fact satisfies (3.2) and hence establishes (ii). Item (iii) together with the smoothness of s_R can be established exactly as in [43, Theorem 3.1]. ∎

5 Melnikov Computations

The goal of this section is to establish Propositions 3.4 and 3.6. In order to do this, we will need to understand the variational equations that arise when studying orbits that converge to the manifolds $\mathcal{M}_R(c,\epsilon)$ and $\mathcal{M}_L(c,\epsilon)$. This issue is studied in the first part of this section, which is inspired by [43§3]. In the second part we consider the variational equations that occur in a neighborhood of q_f and q_b, recalling material from [27, 40]. After these preparations, we will be able to construct the quasi-front and quasi-back solutions featured in Propositions 3.4 and 3.6 in the final two parts of this section.

5.1 Linearization around center manifolds

Let us proceed by studying the stable and unstable fibers associated with the center-like manifolds $\mathcal{M}_R(c,\epsilon)$ and $\mathcal{M}_L(c,\epsilon)$. We will focus for the moment on perturbations from $\mathcal{M}_R(c,\epsilon)$ and look for solutions to (3.1) on $\mathbb{R}_+$ that can be written in the form

$$\begin{array}{lcl} u(\xi) & = & s_R\big(\Theta_R^{\mathrm{fs}}(\vartheta,c,\epsilon)(\xi)+\theta(\xi),c,\epsilon\big)+v(\xi), \\ w(\xi) & = & \Theta_R^{\mathrm{fs}}(\vartheta,c,\epsilon)(\xi)+\theta(\xi), \end{array} \tag{5.1}$$

in which both v and θ should decay exponentially as $\xi \to \infty$. The variable ϑ encodes the fiber of $\mathcal{M}_R(c,\epsilon)$ around which we are linearizing. For convenience, we adopt the shorthand $\Theta = \Theta_R^{\mathrm{fs}}(\vartheta,c,\epsilon)$. In terms of these coordinates, the equation for θ can be written as

$$\begin{array}{lcl} c\theta'(\xi) & = & -c\Theta'(\xi)+\epsilon[s_R(\Theta(\xi)+\theta(\xi),c,\epsilon)+v(\xi)-\gamma\big(\Theta(\xi)+\theta(\xi)\big)]\chi_1\big(\Theta(\xi)+\theta(\xi)\big) \\ & = & \epsilon\mathcal{S}_R(\Theta(\xi),\theta(\xi),v(\xi),c,\epsilon), \end{array} \tag{5.2}$$

in which we have introduced the function

$$\mathcal{S}_R : [w_{\min},w_{\max}]^2 \times \mathbb{R} \times [c_* - \delta_c, c_* + \delta_c] \times [0,\delta_\epsilon] \to \mathbb{R} \tag{5.3}$$

that is given by

$$\begin{array}{lcl} \mathcal{S}_R(\Theta,\theta,v,c,\epsilon) & = & \epsilon[s_R(\Theta+\theta,c,\epsilon)-s_R(\Theta,c,\epsilon)+v-\gamma\theta]\chi_{\mathrm{sl}}(\Theta+\theta) \\ & & \quad +\epsilon[s_R(\Theta,c,\epsilon)-\gamma\Theta][\chi_{\mathrm{sl}}(\Theta+\theta)-\chi_{\mathrm{sl}}(\Theta)]. \end{array} \tag{5.4}$$

We now turn our attention to the equation for v. Direct substitution of (5.1) into (3.1) yields

$$\begin{array}{lcl} cv'(\xi) & = & -cD_1 s_R(\Theta(\xi)+\theta(\xi),c,\epsilon)[\Theta'(\xi)+\theta'(\xi)] \\ & & +\alpha\Big[s_R(\Theta(\xi+1)+\theta(\xi+1),c,\epsilon)+s_R(\Theta(\xi-1)+\theta(\xi-1),c,\epsilon) \\ & & \quad -2s_R(\Theta(\xi)+\theta(\xi),c,\epsilon)\Big] \\ & & +\alpha[v(\xi+1)+v(\xi-1)-2v(\xi)] \\ & & +g\big(s_R(\Theta(\xi)+\theta(\xi),c,\epsilon)+v\big)-\Theta(\xi)-\theta(\xi). \end{array} \tag{5.5}$$

Using Proposition 3.1(ii), one may verify the identity

$$\begin{array}{rcl} cD_1 s_R(\Theta(\xi),c,\epsilon)\Theta'(\xi) & = & \alpha[s_R(\Theta(\xi+1),c,\epsilon) + s_R(\Theta(\xi-1),c,\epsilon) - 2s_R(\Theta(\xi),c,\epsilon)] \\ & & \quad +g\big(s_R(\Theta(\xi),c,\epsilon)\big) - \Theta(\xi). \end{array} \tag{5.6}$$

Substituting this back into (5.5), we find

$$\begin{array}{rcl} cv'(\xi) & = & L\big(s_R(\Theta(\xi),c,\epsilon)\big)\mathrm{ev}_\xi v \\ & & +F(\Theta(\xi),\theta(\xi),v(\xi),c,\epsilon) + G(\Theta(\xi),\theta(\xi),v(\xi),c,\epsilon) \\ & & +H^0_{\mathrm{lin}}(\mathrm{ev}_\xi\Theta,\mathrm{ev}_\xi\theta,c,\epsilon) + H_{\mathrm{nl}}(\mathrm{ev}_\xi\Theta,\mathrm{ev}_\xi\theta,c,\epsilon), \end{array} \tag{5.7}$$

with

$$\begin{array}{rcl} H^0_{\mathrm{lin}}(\mathrm{ev}_\xi\Theta,\mathrm{ev}_\xi\theta,c,\epsilon) & = & \alpha[D_1 s_R(\Theta(\xi+1),c,\epsilon) - D_1 s_R(\Theta(\xi),c,\epsilon)]\theta(\xi+1) \\ & & +\alpha[D_1 s_R(\Theta(\xi-1),c,\epsilon) - D_1 s_R(\Theta(\xi),c,\epsilon)]\theta(\xi-1) \\ & & +\alpha D_1 s_R(\Theta(\xi),c,\epsilon)[\theta(\xi+1)+\theta(\xi-1)-2\theta(\xi)], \\ H_{\mathrm{nl}}(\mathrm{ev}_\xi\Theta,\mathrm{ev}_\xi\theta,c,\epsilon) & = & \alpha[s_R(\Theta(\xi+1)+\theta(\xi+1),c,\epsilon) - s_R(\Theta(\xi+1),c,\epsilon) \\ & & \qquad -D_1 s_R(\Theta(\xi+1),c,\epsilon)\theta(\xi+1)] \\ & & +\alpha[s_R(\Theta(\xi-1)+\theta(\xi-1),c,\epsilon) - s_R(\Theta(\xi-1),c,\epsilon) \\ & & \qquad -D_1 s_R(\Theta(\xi-1),c,\epsilon)\theta(\xi-1)] \\ & & -2\alpha[s_R(\Theta(\xi)+\theta(\xi),c,\epsilon) - s_R(\Theta(\xi),c,\epsilon) \\ & & \qquad -D_1 s_R(\Theta(\xi),c,\epsilon)\theta(\xi)], \end{array} \tag{5.8}$$

together with

$$\begin{array}{rcl} F(\Theta,\theta,v,c,\epsilon) & = & -\epsilon\big[D_1 s_R(\Theta+\theta,c,\epsilon) - D_1 s_R(\Theta,c,\epsilon)\big]\mathcal{T}_R(\Theta,c,\epsilon) \\ & & -\epsilon D_1 s_R(\Theta+\theta,c,\epsilon)\mathcal{S}_R(\Theta,\theta,v,c,\epsilon), \\ G(\Theta,\theta,v,c,\epsilon) & = & g\big(s_R(\Theta+\theta,c,\epsilon)+v\big) - g\big(s_R(\Theta,c,\epsilon)\big) \\ & & \qquad -g'\big(s_R(\Theta,c,\epsilon)\big)[v + D_1 s_R(\Theta,c,\epsilon)\theta] \\ & & +\Big[g'\big(s_R(\Theta,c,\epsilon)\big)D_1 s_R(\Theta,c,\epsilon) \\ & & \qquad -g'\big(s_R(\Theta,c,0)\big)D_1 s_R(\Theta,c,0)\Big]\theta. \end{array} \tag{5.9}$$

In the above computation we used the fact that for any $\vartheta \in [w_{\min}, w_{\max}]$, the identity

$$g'\big(s_R(\vartheta,c,0)\big)D_1 s_R(\vartheta,c,0) = 1 \tag{5.10}$$

holds. As in §4, we will modify the function H^0_{lin} to make the dependence on ϵ more explicit. We therefore introduce the new function

$$\begin{array}{rcl} H_{\mathrm{lin}}(\mathrm{ev}_\xi\Theta,\mathrm{ev}_\xi\theta,\mathrm{ev}_\xi v,c,\epsilon) & = & \frac{\alpha\epsilon}{c}\big[\int_\xi^{\xi+1} D_1^2 s_R(\Theta(\sigma),c,\epsilon)\mathcal{T}_R(\Theta(\sigma),c,\epsilon)\,\mathrm{d}\sigma\big]\theta(\xi+1) \\ & & +\frac{\alpha\epsilon}{c}\big[\int_\xi^{\xi-1} D_1^2 s_R(\Theta(\sigma),c,\epsilon)\mathcal{T}_R(\Theta(\sigma),c,\epsilon)\,\mathrm{d}\sigma\big]\theta(\xi-1) \\ & & +\frac{\alpha\epsilon}{c}D_1 s_R(\Theta(\xi),c,\epsilon)\big[\int_\xi^{\xi+1}\mathcal{S}_R(\Theta(\sigma),\theta(\sigma),v(\sigma),c,\epsilon)\,\mathrm{d}\sigma\big] \\ & & +\frac{\alpha\epsilon}{c}D_1 s_R(\Theta(\xi),c,\epsilon)\big[\int_\xi^{\xi-1}\mathcal{S}_R(\Theta(\sigma),\theta(\sigma),v(\sigma),c,\epsilon)d\sigma\big]. \end{array} \tag{5.11}$$

To write these definitions more concisely, let us consider any two functions $\theta \in C([-1,\infty),[w_{\min},w_{\max}])$ and $v \in C([-1,\infty),\mathbb{R})$. We now introduce two new functions by way of

$$\begin{array}{rcl} \mathcal{R}^{\mathrm{fb}}_R(\theta,v,\vartheta,c,\epsilon)(\xi) & = & F(\Theta^{\mathrm{fs}}_R(\vartheta,c,\epsilon)(\xi),\theta(\xi),v(\xi),c,\epsilon) \\ & & +G(\Theta^{\mathrm{fs}}_R(\vartheta,c,\epsilon)(\xi),\theta(\xi),v(\xi),c,\epsilon) \\ & & +H_{\mathrm{lin}}(\mathrm{ev}_\xi\Theta^{\mathrm{fs}}_R(\vartheta,c,\epsilon),\mathrm{ev}_\xi\theta,\mathrm{ev}_\xi v,c,\epsilon) \\ & & +H_{\mathrm{nl}}(\mathrm{ev}_\xi\Theta^{\mathrm{fs}}_R(\vartheta,c,\epsilon),\mathrm{ev}_\xi\theta,c,\epsilon), \\ \mathcal{S}^{\mathrm{fb}}_R(\theta,v,\vartheta,c,\epsilon)(\xi) & = & \mathcal{S}_R(\Theta^{\mathrm{fs}}_R(\vartheta,c,\epsilon)(\xi),\theta(\xi),v(\xi),c,\epsilon). \end{array} \tag{5.12}$$

In terms of these new functions, the variational equation for the pair (v, θ) can be written as

$$\begin{aligned} cv'(\xi) &= L\big(s_R(\Theta_R^{\mathrm{fs}}(\vartheta, c, \epsilon)(\xi), c, \epsilon)\big)\mathrm{ev}_\xi v + \mathcal{R}_R^{\mathrm{fb}}(\theta, v, \vartheta, c, \epsilon)(\xi), & (5.13a)\\ c\theta'(\xi) &= \epsilon \mathcal{S}_R^{\mathrm{fb}}(\theta, v, c, \epsilon)(\xi). & (5.13b)\end{aligned}$$

Let us now consider any $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$. It is easy to see that

$$\mathcal{R}_R^{\mathrm{fb}}(0, 0, \vartheta, c, \epsilon) = \mathcal{S}_R^{\mathrm{fb}}(0, 0, \vartheta, c, \epsilon) = 0. \tag{5.14}$$

In addition, let us fix $\zeta < 0$ and consider a pair $v_1, v_2 \in BC_\zeta([-1, \infty), \mathbb{R})$ together with a pair $\theta_1, \theta_2 \in BC_\zeta([-1, \infty), [w_{\min}, w_{\max}])$. We will assume that $\|v_i\|_\zeta \le \delta_v$ and $\|\theta_i\|_\zeta \le \delta_w$ for $i = 1, 2$. Then upon writing

$$\Delta_{\mathrm{fb}} = \big\|\mathcal{R}_R^{\mathrm{fb}}(\theta_1, v_1, \vartheta, c, \epsilon) - \mathcal{R}_R^{\mathrm{fb}}(\theta_2, v_2, \vartheta, c, \epsilon)\big\|_\zeta + \big\|\mathcal{S}_R^{\mathrm{fb}}(\theta_1, v_1, \vartheta, c, \epsilon) - \mathcal{S}_R^{\mathrm{fb}}(\theta_2, v_2, \vartheta, c, \epsilon)\big\|_\zeta, \tag{5.15}$$

we have the Lipschitz estimate

$$\Delta_{\mathrm{fb}} \quad \le_* \quad (\delta_\epsilon + \delta_v + \delta_w)[\|v_1 - v_2\|_\zeta + \|\theta_1 - \theta_2\|_\zeta]. \tag{5.16}$$

Using standard arguments as in [43, Lemmas 3.3-3.4], one may obtain the following two smoothness results. We note that $\mathcal{R}_R^{\mathrm{fb}}$ loses two orders of smoothness as a consequence of (5.11).

Lemma 5.1 *Recall the integer r appearing in (H1). There exists a constant $N_1 > 0$ such that for any integer $0 \le \ell \le r + 2$ and any $\eta > \ell \delta_\epsilon N_1$, the maps Θ_R^{fs} and Θ_L^{fs} are C^ℓ-smooth when considered as maps from $[w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon]$ into $BC_\eta(\mathbb{R}, \mathbb{R})$.*

Lemma 5.2 *Recall the integer r that appears in (H1). There exist constants $N_1 > 0$, $\delta_c > 0$ and $\delta_\epsilon > 0$, such that for any $-\eta_* \le \eta < -r\delta_\epsilon N_1$ and any integer $0 \le \ell \le r$, the nonlinearity $\mathcal{R}_R^{\mathrm{fb}}$ is C^ℓ-smooth when considered as a map*

$$\begin{aligned} \mathcal{R}_R^{\mathrm{fb}} \quad : \quad & BC_\eta([-1, \infty), [w_{\min}, w_{\max}]) \times BC_\eta([-1, \infty), \mathbb{R}) \times [w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \\ & \to BC_{\eta_2}([0, \infty), \mathbb{R}) \end{aligned} \tag{5.17}$$

with $\eta_2 > \eta + \ell \delta_\epsilon N_1$. The same result holds for $\mathcal{S}_R^{\mathrm{fb}}$.

Let us now consider the linear part of (5.13a), which we will write as

$$\Lambda_R^{\mathrm{fb}}(\vartheta, c, \epsilon)v = h, \tag{5.18}$$

in which $\Lambda_R^{\mathrm{fb}}(\vartheta, c, \epsilon)$ acts as

$$[\Lambda_R^{\mathrm{fb}}(\vartheta, c, \epsilon)v](\xi) = cv'(\xi) - L\big(s_R(\Theta_R^{\mathrm{fs}}(\vartheta, c, \epsilon)(\xi), c, \epsilon)\big)\mathrm{ev}_\xi v. \tag{5.19}$$

The following result shows that (5.18) can be solved when considered on appropriate function spaces. It can be obtained by combining Lemmas 4.1 and 5.1.

Lemma 5.3 *Consider the linear system (5.18) and suppose that (H1)-(H3) hold. Then there exist constants $\delta_c > 0$, $\delta_\epsilon > 0$ and $N_1 > 0$, together with maps*

$$\begin{aligned} \mathcal{K}_{R,\eta}^{\mathrm{fb}} \quad : \quad & [w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \\ & \to \mathcal{L}\big(BC_\eta([0, \infty), \mathbb{R}), BC_\eta([-1, \infty), \mathbb{R})\big) \end{aligned} \tag{5.20}$$

defined for $\eta \in [-\eta_*, \eta_*]$, such that the following properties are satisfied.

(i) Consider any $h \in BC_\eta([0,\infty),\mathbb{R})$ for some $\eta \in [-\eta_*, \eta_*]$. Then the function $v = \mathcal{K}^{\mathrm{fb}}_{R,\eta}(\vartheta, c, \epsilon)h$ satisfies $\Lambda^{\mathrm{fb}}_R(\vartheta, c, \epsilon)v = h$ for any $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$.

(ii) The norm $\left\|\mathcal{K}^{\mathrm{fb}}_{R,\eta}(\vartheta, c, \epsilon)\right\|$ can be bounded independently of $\eta \in [-\eta_*, \eta_*]$, $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$.

(iii) Consider a pair $\eta_1, \eta_2 \in [-\eta_*, \eta_*]$ together with a function

$$h \in BC_{\eta_1}([0,\infty),\mathbb{R}) \cap BC_{\eta_2}([0,\infty),\mathbb{R}). \tag{5.21}$$

Then for any $\vartheta \in [w_{\min}, w_{\max}]$, $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$, we have

$$\mathcal{K}^{\mathrm{fb}}_{R,\eta_1}(\vartheta, c, \epsilon)h = \mathcal{K}^{\mathrm{fb}}_{R,\eta_2}(\vartheta, c, \epsilon)h. \tag{5.22}$$

(iv) Recall the integer r that appears in (H1). Consider any integer $0 \le \ell \le r + 2$ and pick $\eta_1, \eta_2 \in [-\eta_*, \eta_*]$ in such a way that $\eta_2 > \ell\delta_\epsilon N_1 + \eta_1$. Then the map $(\vartheta, c, \epsilon) \mapsto \mathcal{K}^{\mathrm{fb}}(\vartheta, c, \epsilon)$ is C^ℓ-smooth when considered as a map from $[w_{\min}, w_{\max}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon]$ into $\mathcal{L}\big(BC_{\eta_1}([0,\infty),\mathbb{R}), BC_{\eta_2}([-1,\infty),\mathbb{R})\big)$.

Before we can study solutions to the homogeneous equation $\Lambda^{\mathrm{fb}}_R(\vartheta, c, \epsilon)v = 0$, we need to introduce some terminology. For any $w_0 \in [w_{\min}, w_{\max}]$, consider the autonomous homogeneous system

$$c_* v'(\xi) = L\big(\widetilde{s}_R(w_0)\big)\mathrm{ev}_\xi v. \tag{5.23}$$

The following result is based on [40, Theorems 3.1-3.2] and characterizes the set of solutions to (5.23) posed on half-lines.

Lemma 5.4 *Consider the linear system (5.23) and suppose that (H1)-(H3) are satisfied. Then for every $w_0 \in [w_{\min}, w_{\max}]$, there exist closed subspaces $P^{\mathrm{fb}}_R(w_0) \subset X$ and $Q^{\mathrm{fb}}_R(w_0) \subset X$ such that the following properties hold.*

(i) We have the splitting $X = P^{\mathrm{fb}}_R(w_0) \oplus Q^{\mathrm{fb}}_R(w_0)$ for all $w_0 \in [w_{\min}, w_{\max}]$.

(ii) Suppose that $\phi \in P^{\mathrm{fb}}_R(w_0)$ for some $w_0 \in [w_{\min}, w_{\max}]$. Then there exists $v \in C((-\infty, 1],\mathbb{R})$ that solves (5.23) and has $\mathrm{ev}_0 v = \phi$. In addition, there exists a constant $C > 0$, that does not depend on ϕ, such that the estimate

$$\|\mathrm{ev}_\xi v\| \le C\mathrm{e}^{-\eta_*|\xi|}\,\|\phi\| \tag{5.24}$$

holds for every $\xi \le 0$.

(iii) Suppose that $\phi \in Q^{\mathrm{fb}}_R(w_0)$ for some $w_0 \in [w_{\min}, w_{\max}]$. Then there exists $v \in C([-1,\infty),\mathbb{R})$ that solves (5.23) and has $\mathrm{ev}_0 v = \phi$. In addition, there exists a constant $C > 0$, that does not depend on ϕ, such that the estimate

$$\|\mathrm{ev}_\xi v\| \le C\mathrm{e}^{-\eta_*|\xi|}\,\|\phi\| \tag{5.25}$$

holds for every $\xi \ge 0$.

(iv) Any $v \in BC_0((-\infty, 1],\mathbb{R})$ that satisfies (5.23) for all $\xi \le 0$ must have $\mathrm{ev}_0 v \in P^{\mathrm{fb}}_R(w_0)$.

(v) Any $v \in BC_0([-1,\infty),\mathbb{R})$ that satisfies (5.23) for all $\xi \ge 0$ must have $\mathrm{ev}_0 v \in Q^{\mathrm{fb}}_R(w_0)$.

We will write $\Pi_{P^{\mathrm{fb}}_R(w_0)} : X \to P^{\mathrm{fb}}_R(w_0)$ and $\Pi_{Q^{\mathrm{fb}}_R(w_0)} : X \to Q^{\mathrm{fb}}_R(w_0)$ for the projections that can be associated with the splitting obtained in (i) above. Based on this result and the techniques developed in [26], we can now study the solutions to the slowly-modulating homogeneous system $\Lambda^{\mathrm{fb}}_R(\vartheta, c, \epsilon)v = 0$. We obtain the following characterization.

Lemma 5.5 *Consider the linear system (5.18) and assume that (H1)-(H3) are satisfied. Fix* $w_0 \in (w_{\min}, w_{\max})$*, then there exist constants* $N_1 > 0$*,* $\delta_\vartheta > 0$*,* $\delta_c > 0$ *and* $\delta_\epsilon > 0$*, together with a map*

$$E^{\mathrm{fb}}_{R,w_0} : [w_0 - \delta_\vartheta, w_0 + \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to \mathcal{L}\big(Q^{\mathrm{fb}}_R(w_0), BC_{-\eta_*}([-1,\infty),\mathbb{R})\big), \quad (5.26)$$

such that the following properties are satisfied.

(i) For any $\phi \in Q^{\mathrm{fb}}_R(w_0)$*,* $\vartheta \in [w_0 - \delta_\vartheta, w_0 + \delta_\vartheta]$*,* $c \in [c_* - \delta_c, c_* + \delta_c]$ *and* $\epsilon \in [0, \delta_\epsilon]$*, the function* $v = E^{\mathrm{fb}}_{R,w_0}(\vartheta, c, \epsilon)\phi$ *satisfies* $\Lambda^{\mathrm{fb}}_R(\vartheta, c, \epsilon)v = 0$*.*

(ii) We have $\Pi_{Q^{\mathrm{fb}}_R(w_0)} \mathrm{ev}_0 E^{\mathrm{fb}}_{R,w_0}(\vartheta, c, \epsilon)\phi = \phi$ *for all* $\phi \in Q^{\mathrm{fb}}_R(w_0)$*,* $\vartheta \in [w_0 - \delta_\vartheta, w_0 + \delta_\vartheta]$*,* $c \in [c_* - \delta_c, c_* + \delta_c]$ *and* $\epsilon \in [0, \delta_\epsilon]$*.*

(iii) Suppose that $v \in BC_\eta([-1,\infty),\mathbb{R})$ *satisfies* $\Lambda(\vartheta, c, \epsilon)v = 0$ *for some* $\eta \in [-\eta_*, \eta_*]$*,* $\vartheta \in [w_0 - \delta_\vartheta, w_0 + \delta_\vartheta]$*,* $c \in [c_* - \delta_c, c_* + \delta_c]$ *and* $\epsilon \in [0, \delta_\epsilon]$*. Then* v *satisfies the identity*

$$v = E^{\mathrm{fb}}_{R,w_0}(\vartheta, c, \epsilon)\Pi_{Q^{\mathrm{fb}}_R(w_0)} \mathrm{ev}_0 v. \quad (5.27)$$

(iv) Recall the integer r *that appears in (H1). Consider any integer* $0 \le \ell \le r + 2$ *and pick* $\eta > -\eta_* + \ell\delta_\epsilon N_1$*. Then the map* $(\vartheta, c, \epsilon) \mapsto E^{\mathrm{fb}}_{R,w_0}(\vartheta, c, \epsilon)$ *is* C^ℓ*-smooth when considered as a map from* $[w_0 - \delta_\vartheta, w_0 + \vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon]$ *into* $\mathcal{L}\big(Q^{\mathrm{fb}}_R(w_0), BC_\eta([-1,\infty),\mathbb{R})\big)$*.*

Proof. For each fixed $w_0 \in [w_{\min}, w_{\max}]$, one can mimic the construction in [26§6] to obtain an operator $E^{\mathrm{fb}}_{R,w_0}(w_0, \cdot, \cdot)$ that satisfies (i) through (iii) with $\delta_\vartheta = 0$. This setup can be generalized to include situations where $\delta_\vartheta > 0$ by recalling from [27] that for each ϑ sufficiently close to w_0, there exists a linear map $u^*_{w_0}(\vartheta) : Q(w_0) \to X$ that depends smoothly on ϑ, with the property that $\mathrm{Range}\big(u^*_{w_0}(\vartheta)\big) = Q^{\mathrm{fb}}_R(\vartheta)$, while $\Pi_{Q^{\mathrm{fb}}_R(w_0)} u^*_{w_0}(\vartheta) = I$. In view of equation [26, Equation (6.59)], the smoothness property (iv) can be read off from Lemma 5.3. ∎

We conclude this section by noting that the objects $\mathcal{R}^{\mathrm{fb}}_L$, $\mathcal{S}^{\mathrm{fb}}_L$, $\mathcal{K}^{\mathrm{fb}}_{L,\eta}$ and E^{fb}_{L,w_0} can be constructed in an analogous fashion. In addition, operators analogous to $\mathcal{K}^{\mathrm{fb}}_{R,\eta}$ and E^{fb}_{R,w_0} that are related to (5.18) posed on $(-\infty, 0]$ can also be constructed.

5.2 Linearization around front and back

We start out by considering the variational equations near the front q_f. Substituting the ansatz $u(\xi) = q_f(\xi) + v(\xi)$, $w(\xi) = \theta(\xi)$ into (3.1), we arrive at the system

$$\begin{aligned} c_* v'(\xi) &= L\big(q_f(\xi)\big)\mathrm{ev}_\xi v - w(\xi) + \mathcal{R}_f(\theta, v, c, \epsilon)(\xi), && (5.28\mathrm{a}) \\ c_* \theta'(\xi) &= \epsilon \mathcal{S}_f(\theta, v, c, \epsilon)(\xi), && (5.28\mathrm{b}) \end{aligned}$$

in which

$$\begin{aligned} \mathcal{R}_f(\theta, v, c, \epsilon)(\xi) &= g^f_{\mathrm{nl}}\big(\xi, v(\xi)\big) + \tfrac{c_* - c}{c}\big[L\big(q_f(\xi)\big)\mathrm{ev}_\xi v - \theta(\xi) + g^f_{\mathrm{nl}}\big(\xi, v(\xi)\big)\big] \\ &\quad + \tfrac{c_* - c}{c}\big[g\big(q_f(\xi)\big) + \alpha[q_f(\xi - 1) + q_f(\xi + 1) - 2q_f(\xi)]\big], \\ \mathcal{S}_f(\theta, v, c, \epsilon)(\xi) &= \tfrac{c_*}{c}[q_f(\xi) + v(\xi) - \gamma\theta(\xi)], \end{aligned} \quad (5.29)$$

with

$$g^f_{\mathrm{nl}}(\xi, v) = g(q_f(\xi) + v) - g'\big(q_f(\xi)\big)v - g\big(q_f(\xi)\big). \quad (5.30)$$

Moving on to the variational equations near q_b, we use the ansatz $u(\xi) = q_b(\xi) + v(\xi)$, $w(\xi) = w_* + \theta(\xi)$ and find

$$\begin{aligned} c_* v'(\xi) &= L\big(q_b(\xi)\big)\mathrm{ev}_\xi v + \mathcal{R}_b(\theta, v, c, \epsilon)(\xi), && (5.31\mathrm{a}) \\ c_* \theta'(\xi) &= \epsilon \mathcal{S}_b(\theta, v, c, \epsilon)(\xi), && (5.31\mathrm{b}) \end{aligned}$$

in which

$$
\begin{array}{rcl}
\mathcal{R}_b(\theta, v, c, \epsilon)(\xi) & = & g_{\rm nl}^b\big(\xi, v(\xi)\big) + \frac{c_* - c}{c}[L\big(q_b(\xi)\big)\mathrm{ev}_\xi v - \theta(\xi) + g_{\rm nl}^b\big(\xi, v(\xi)\big)] \\
& & + \frac{c_* - c}{c}\big[g\big(q_b(\xi)\big) + \alpha[q_b(\xi - 1) + q_b(\xi + 1) - 2q_b(\xi)] - w_*\big], \\
\mathcal{S}_b(\theta, v, c, \epsilon)(\xi) & = & \frac{c_*}{c}[q_b(\xi) + v(\xi) - \gamma w_* - \gamma\theta(\xi)],
\end{array}
\tag{5.32}
$$

with

$$
g_{\rm nl}^b(\xi, v) = g(q_b(\xi) + v) - g'\big(q_b(\xi)\big)v - g\big(q_b(\xi)\big). \tag{5.33}
$$

We recall from §3 the operators Λ_f and Λ_b that are associated to the linear parts of (5.28a) and (5.31a). These operators are elements of $\mathcal{L}\big(W^{1,\infty}(\mathbb{R}, \mathbb{R}), L^\infty(\mathbb{R}, \mathbb{R})\big)$ and act as

$$
\begin{array}{rcl}
[\Lambda_f v](\xi) & = & c_* v'(\xi) - L\big(q_f(\xi)\big)\mathrm{ev}_\xi v, \\
[\Lambda_b v](\xi) & = & c_* v'(\xi) - L\big(q_b(\xi)\big)\mathrm{ev}_\xi v.
\end{array}
\tag{5.34}
$$

We will focus on Λ_f for the moment. We recall the kernels $\mathbb{K}_f$ and $\mathbb{K}_f^*$ defined in (3.6). Let us write

$$
X_f^\perp(\xi) = \{\phi \in X \mid \langle \mathrm{ev}_\xi d, \phi\rangle = 0 \text{ for all } d \in \mathbb{K}_f^*\}, \tag{5.35}
$$

which is a closed subspace of codimension one in X for each $\xi \in \mathbb{R}$. In addition, we write

$$
B_f(\xi) = \{\phi \in X \mid \phi = \mathrm{ev}_\xi b \text{ for some } b \in \mathbb{K}_f\}. \tag{5.36}
$$

The first result concerns a precise splitting for $X_f^\perp(\xi)$ and was established in [40].

Lemma 5.6 *For any $\xi \in \mathbb{R}$, there exist closed subspaces $\widehat{P}_f(\xi) \subset X_f^\perp(\xi)$ and $\widehat{Q}_f(\xi) \subset X_f^\perp(\xi)$ together with constants $K > 0$ and $\alpha > 0$, such that the following properties hold.*

(i) For any $\xi \in \mathbb{R}$ we have the decomposition

$$
X_f^\perp(\xi) = \widehat{P}_f(\xi) \oplus \widehat{Q}_f(\xi) \oplus B_f(\xi). \tag{5.37}
$$

(ii) Consider any $\phi \in \widehat{P}_f(\xi) \oplus B_f(\xi)$. There exists a function $v = \widetilde{E}\phi \in BC_0((-\infty, \xi + 1], \mathbb{R})$ that has $[\Lambda_f v](\xi') = 0$ for all $\xi' \leq \xi$. In addition, in the special case that $\phi \in \widehat{P}_f(\xi)$, we have the bound

$$
|v(\xi')| \leq K\mathrm{e}^{-\alpha|\xi' - \xi|}\, \|\phi\| \qquad \text{for all } \xi' \leq \xi. \tag{5.38}
$$

(iii) Consider any $\phi \in \widehat{Q}_f(\xi) \oplus B_f(\xi)$. There exists a function $v = \widetilde{E}\phi \in BC_0([\xi - 1, \infty), \mathbb{R})$ that has $[\Lambda_f v](\xi') = 0$ for all $\xi' \geq \xi$. In addition, in the special case that $\phi \in \widehat{Q}_f(\xi)$, we have the bound

$$
|v(\xi')| \leq K\mathrm{e}^{-\alpha|\xi' - \xi|}\, \|\phi\| \qquad \text{for all } \xi' \geq \xi. \tag{5.39}
$$

(iv) Any $v \in BC_0((-\infty, \xi + 1], \mathbb{R})$ that has $[\Lambda_f v](\xi') = 0$ for all $\xi' \leq \xi$ must satisfy $\mathrm{ev}_\xi v \in \widehat{P}_f(\xi) \oplus B_f(\xi)$.

(v) Any $v \in BC_0([\xi - 1, \infty), \mathbb{R})$ that has $[\Lambda_f v](\xi') = 0$ for all $\xi' \geq \xi$ must satisfy $\mathrm{ev}_\xi v \in \widehat{Q}_f(\xi) \oplus B_f(\xi)$.

Recalling the one-dimensional subspaces $\Gamma_f \subset X$ and $\Gamma_b \subset X$ defined in §3 and noticing that $\text{span}\{\text{ev}_0 q_f'\} = B_f(0)$ and $\text{span}\{\text{ev}_0 q_b'\} = B_b(0)$, we may refine the splitting (3.7) and obtain

$$X = B_f(0) \oplus \widehat{P}_f(0) \oplus \widehat{Q}_f(0) \oplus \Gamma_f = B_b(0) \oplus \widehat{P}_b(0) \oplus \widehat{Q}_b(0) \oplus \Gamma_b. \tag{5.40}$$

We will write $\Pi_{B_f(0)}$, $\Pi_{\widehat{P}_f(0)}$ and $\Pi_{\widehat{Q}_f(0)}$ together with $\Pi_{B_b(0)}$, $\Pi_{\widehat{P}_b(0)}$ and $\Pi_{\widehat{Q}_b(0)}$ for the projections that are associated with this splittings.

As a final preparation, we need to consider perturbations to Λ_f and Λ_b. Let us therefore consider a parameter dependent operator $\Lambda_f(\xi_0, \mu) : W^{1,\infty}(\mathbb{R}, \mathbb{R}) \to L^\infty(\mathbb{R}, \mathbb{R})$ that is given by

$$[\Lambda_f(\xi_0, \mu)v](\xi) = c_* v'(\xi) - \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] - A(\xi_0 + \xi, \mu)v(\xi), \tag{5.41}$$

in which the parameters ξ_0 and μ are taken from an open neighborhood of $0 \in \mathbb{R}$ and $A(\xi, 0) = g'\big(q_f(\xi)\big)$. We will assume that the map $(\xi_0, \mu) \mapsto \Lambda_f(\xi_0, \mu)$ is C^k-smooth. With this requirement out of the way, we will subsequently drop the dependence on ξ_0 and simply write $\Lambda_f(\mu) = \Lambda_f(0, \mu)$. The theory developed in [27§3] shows that the homogeneous system $\Lambda_f(\mu)v = h$ can be solved on half-lines.

Lemma 5.7 *For any sufficiently small μ and any $\eta \in [-\eta_*, \eta_*]$, there exist operators*

$$\begin{array}{rcl} \Lambda^{\text{inv}}_{f,-}(\mu) & : & BC_\eta((-\infty, 0], \mathbb{R}) \to BC_\eta((-\infty, 1], \mathbb{R}), \\ \Lambda^{\text{inv}}_{f,+}(\mu) & : & BC_\eta([0, \infty), \mathbb{R}) \to BC_\eta([-1, \infty), \mathbb{R}), \end{array} \tag{5.42}$$

that depend C^k-smoothly on μ and satisfy the following two properties.

(i) For any $h \in BC_\eta((-\infty, 0], \mathbb{R})$, the function $v = \Lambda^{\text{inv}}_{f,-}(\mu)h$ satisfies

$$[\Lambda_f(\mu)v](\xi) = h(\xi) \tag{5.43}$$

for all $\xi \leq 0$.

(ii) For any $h \in BC_\eta([0, \infty), \mathbb{R})$, the function $v = \Lambda^{\text{inv}}_{f,+}(\mu)h$ satisfies

$$[\Lambda_f(\mu)v](\xi) = h(\xi) \tag{5.44}$$

for all $\xi \geq 0$.

The following result was obtained in [27§5] and establishes the existence of exponential dichotomies on the half-lines $\mathbb{R}_\pm$ for the homogeneous system $\Lambda_f(\mu)v = 0$.

Lemma 5.8 *For every sufficiently small μ, there exist a family of closed subspaces $Q_f(\xi, \mu) \subset X$ and $S_f(\xi, \mu) \subset X$ parametrized by $\xi \geq 0$, together with two constants $\alpha > 0$ and $K > 0$, such that the following properties are satisfied.*

(i) For every $\xi \geq 0$, we have the splitting

$$X = Q_f(\xi, \mu) \oplus S_f(\xi, \mu) \tag{5.45}$$

with accompanying projections $\Pi_{Q_f(\xi,\mu)} : X \to Q_f(\xi, \mu)$ and $\Pi_{S_f(\xi,\mu)} : X \to S_f(\xi, \mu)$.

(ii) Consider any $\phi \in Q_f(\xi, \mu)$. There exists a function $v = \widetilde{E}\phi \in C([\xi - 1, \infty), \mathbb{R})$ with $\text{ev}_\xi v = \phi$ that has $[\Lambda(\mu)v](\xi') = 0$ for all $\xi' \geq \xi$.

(iii) Consider any $\phi \in S_f(\xi, \mu)$. There exists a function $v = \widetilde{E}\phi \in C([-1, \xi + 1], \mathbb{R})$ with $\text{ev}_\xi v = \phi$ that has $[\Lambda(\mu)v](\xi') = 0$ for all $0 \leq \xi' \leq \xi$.

(iv) For any integer $0 \le \ell \le k$ we have the estimates

$$\begin{array}{lclr} \left\| D^{\ell} \mathrm{ev}_{\xi'} \widetilde{E} \Pi_{Q_f(\xi,\mu)} \right\| & \le & K \mathrm{e}^{-\alpha|\xi'-\xi|} & \text{for every } \xi' \ge \xi, \\ \left\| D^{\ell} \mathrm{ev}_{\xi'} \widetilde{E} \Pi_{S_f(\xi,\mu)} \right\| & \le & K \mathrm{e}^{-\alpha|\xi'-\xi|} & \text{for every } 0 \le \xi' \le \xi, \end{array} \tag{5.46}$$

in which the differentiation operator D acts with respect to the parameter μ.

Our final result shows how these parameter-dependent subspaces at $\xi = 0$ or $\xi \approx \infty$ can be written as graphs over $\widehat{Q}_f(0) \oplus B_f(0)$ and $P_R^{\mathrm{fb}}(0)$. For a proof, we again refer to [27§5].

Lemma 5.9 *For any sufficiently small μ, there exist a linear map $E_{Q_f}(\mu) : \widehat{Q}_f(0) \oplus B_f(0) \to C([-1,\infty),\mathbb{R})$ that depends C^k-smoothly on μ and satisfies the following properties.*

- *(i) We have $Q_f(0,\mu) = \mathrm{Range}\big(\mathrm{ev}_0 E_{Q_f}(\mu)\big)$ and $E_{Q_f}(\mu) = \widetilde{E}\mathrm{ev}_0 E_{Q_f}(\mu)$.*
- *(ii) We have $\Pi_{\widehat{Q}_f(0)} \oplus \Pi_{B_f(0)} \mathrm{ev}_0 E_{Q_f}(\mu) = I$ and $[\mathrm{ev}_0 E_{Q_f}(\mu) - I] = \mathrm{O}(|\mu|)$ as $\mu \to 0$.*
- *(iii) The map $\mu \mapsto \mathrm{ev}_0 E_{Q_f}(\mu)$ is C^k-smooth.*

In addition, for any sufficiently small μ and any sufficiently large ξ there exists a linear map $E_{S_f}(\xi,\mu) : P_R^{\mathrm{fb}}(0) \to C([-1,\xi+1],\mathbb{R})$, such that the following properties hold.

- *(iv) We have $S_f(\xi,\mu) = \mathrm{Range}\big(\mathrm{ev}_\xi E_{S_f}(\xi,\mu)\big)$ and $E_{S_f}(\xi,\mu) = \widetilde{E}\mathrm{ev}_\xi E_{S_f}(\xi,\mu)$.*
- *(v) We have $\Pi_{P_R^{\mathrm{fb}}(0)} \mathrm{ev}_\xi E_{S_f}(\xi,\mu) = I$ and $[\mathrm{ev}_\xi E_{S_f}(\mu) - I] = \mathrm{O}(|\mu| + \mathrm{e}^{-\eta_* \xi})$ as $(\mu,\xi) \to (0,\infty)$.*
- *(vi) The map $(\xi,\mu) \mapsto \mathrm{ev}_\xi E_{S_f}(\xi,\mu)$ is C^k-smooth.*

Very similar results can be obtained for the family of splittings $X = Q_b(\xi,\mu) \oplus S_b(\xi,\mu)$ with $\xi \ge 0$. In addition, for every $\xi \le 0$, we have the splitting

$$X = P_b(\xi,\mu) \oplus R_b(\xi,\mu) \tag{5.47}$$

that is accompanied by linear maps $E_{P_b}(\mu) : \widehat{P}_b(0) \oplus B_b(0) \to C((-\infty,1],\mathbb{R})$ and $E_{R_b}(\xi,\mu) : Q_R^{\mathrm{fb}}(w_*) \to C([\xi-1,1],\mathbb{R})$ in such a way that analogous versions of Lemmas 5.8 and 5.9 hold.

5.3 Construction of quasi-fronts

We set out to prove Proposition 3.4. Our approach is to choose a large constant $\xi_0 > 0$ and split the real line into the three intervals $(-\infty,0]$, $[0,\xi_0]$ and $[\xi_0,\infty)$ that we each consider separately. To aid us in this scheme, we introduce the families of function spaces

$$\begin{array}{rcl} BC_\alpha^{\ominus} & = & \{(v,\theta) \in C((-\infty,1],\mathbb{R}) \times C((-\infty,0],\mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC_\alpha^{\ominus}} := \sup_{\xi \le 1} \mathrm{e}^{-\alpha|\xi|} |v(\xi)| + \sup_{\xi \le 0} \mathrm{e}^{-\alpha|\xi|} |\theta(\xi)| < \infty\}, \\ BC^{\odot} & = & \{(v,\theta) \in C([-1,\xi_0+1],\mathbb{R}) \times C([0,\xi_0],\mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC^{\odot}} := \sup_{-1 \le \xi \le \xi_0+1} |v(\xi)| + \sup_{0 \le \xi \le \xi_0} |\theta(\xi)| < \infty\}, \\ BC_\alpha^{\oplus} & = & \{(v,\theta) \in C([\xi_0-1,\infty),\mathbb{R}) \times C([\xi_0-1,\infty),\mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC_\alpha^{\oplus}} := \sup_{\xi \ge \xi_0-1} \mathrm{e}^{-\alpha|\xi-\xi_0|}\big(|v(\xi)| + |\theta(\xi)|\big) < \infty\}, \end{array} \tag{5.48}$$

parametrized by $\alpha > 0$, together with the families

$$\begin{array}{rcl} BC_\alpha^{-} & = & \{g = (g_1,g_2) \in C((-\infty,0],\mathbb{R}) \times C((-\infty,0],\mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC_\alpha^{-}} := \sup_{\xi \le 0} \mathrm{e}^{-\alpha|\xi|}\big(|g_1(\xi)| + |g_2(\xi)|\big) < \infty\}, \\ BC^{\diamond} & = & \{(g_1,g_2) \in C([0,\xi_0],\mathbb{R}) \times C([0,\xi_0],\mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC^{\diamond}} := \sup_{0 \le \xi \le \xi_0} |g_1(\xi)| + \xi_0 |g_2(\xi)| < \infty\}, \\ BC_\alpha^{+} & = & \{(g_1,g_2) \in C([\xi_0,\infty),\mathbb{R}) \times C([\xi_0,\infty),\mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC_\alpha^{+}} := \sup_{\xi \ge \xi_0} \mathrm{e}^{-\alpha|\xi-\xi_0|}\big(|g_1(\xi)| + |g_2(\xi)|\big) < \infty\}. \end{array} \tag{5.49}$$

Notice the additional factor ξ_0 that appears in the second component of the norm defined on $BC^{\diamond}$.

We recall the constant $\eta_* > 0$ appearing in §3. Our goal is to find $\vartheta^+ \in [w_{\min}, w_{\max}]$ together with pairs $(v^-, \theta^-) \in BC^{\ominus}_{-\eta_*}$, $(v^{\diamond}, \theta^{\diamond}) \in BC^{\odot}$ and $(v^+, \theta^+) \in BC^{\oplus}_{-\eta_*}$, such that the choice

$$w_f(\xi) = \begin{cases} \theta^-(\xi) & \text{for } \xi \leq 0, \\ \theta^{\diamond}(\xi) & \text{for } 0 \leq \xi \leq \xi_0, \\ \Theta^{\mathrm{fs}}_R(\vartheta^+, c, \epsilon)(\xi) + \theta^+(\xi) & \text{for } \xi \geq \xi_0, \end{cases} \tag{5.50}$$

in combination with $u_f^-(\xi) = q_f(\xi) + v^-(\xi)$ for $\xi \leq 1$ and

$$u_f^+(\xi) \quad = \quad \begin{cases} q_f(\xi) + v^{\diamond}(\xi) & \text{for } -1 \leq \xi \leq \xi_0, \\ s_R(w_f(\xi), c, \epsilon) + v^+(\xi) & \text{for } \xi \geq \xi_0, \end{cases} \tag{5.51}$$

satisfies the conditions of Proposition 3.6.

Recalling the computations in the previous part of this section, we find that the pair (v^-, θ^-) must satisfy the equation

$$\Lambda^-(v^-, \theta^-) = (-\theta^-, 0) + \mathcal{N}^-(\theta^-, v^-, c, \epsilon), \tag{5.52}$$

in which $\Lambda^- : BC^{\ominus}_{-\eta_*} \to BC^-_{-\eta_*}$ is given by $\Lambda^-(v^-, \theta^-) = (\Lambda_1^- v^-, \Lambda_2^- \theta^-)$, with

$$\begin{array}{rcl} [\Lambda_1^- v](\xi) & = & c_* v'(\xi) - L\big(q_f(\xi)\big)\mathrm{ev}_\xi v, \\ [\Lambda_2^- \theta](\xi) & = & c_* \theta'(\xi), \end{array} \tag{5.53}$$

for $\xi \leq 0$, while $\mathcal{N}^- = (\mathcal{N}_1^-, \mathcal{N}_2^-)$ is given by

$$\begin{array}{rcl} \mathcal{N}_1^-(\theta, v, c, \epsilon)(\xi) & = & \mathcal{R}_f(\theta, v, c, \epsilon)(\xi), \\ \mathcal{N}_2^-(\theta, v, c, \epsilon)(\xi) & = & \epsilon \mathcal{S}_f(\theta, v, c, \epsilon)(\xi), \end{array} \tag{5.54}$$

again for $\xi \leq 0$. Similarly, we write

$$\Lambda^{\diamond}(v^{\diamond}, \theta^{\diamond}) = (-\theta^{\diamond}, 0) + \mathcal{N}^{\diamond}(v^{\diamond}, \theta^{\diamond}, c, \epsilon), \tag{5.55}$$

for the equation that the pair $(v^{\diamond}, \theta^{\diamond})$ must satisfy, noting that the operators $\Lambda^{\diamond} : BC^{\odot} \to BC^{\diamond}$ and $\mathcal{N}^{\diamond}$ differ only from Λ^- and $\mathcal{N}^-$ by the interval on which the relevant functions are defined.

Finally, the pair (v^+, θ^+) must satisfy

$$\Lambda^+(\vartheta^+, c, \epsilon)(v^+, \theta^+) = \mathcal{N}^+(\theta^+, v^+, \vartheta^+, c, \epsilon), \tag{5.56}$$

in which $\Lambda^+(\vartheta^+, c, \epsilon) : BC^{\oplus}_{-\alpha} \to BC^+_{-\alpha}$ is given by $\Lambda^+(\vartheta^+, c, \epsilon)(v^+, \theta^+) = (\Lambda_1^+(\vartheta^+, c, \epsilon)v^+, \Lambda_2^+(c)\theta^+)$ with

$$\begin{array}{rcl} [\Lambda_1^+(\vartheta^+, c, \epsilon)v](\xi) & = & [\Lambda^{\mathrm{fb}}_R(\vartheta^+, c, \epsilon)v](\xi), \\ [\Lambda_2^+(c)\theta](\xi) & = & c\theta'(\xi), \end{array} \tag{5.57}$$

while $\mathcal{N}^+ = (\mathcal{N}_1^+, \mathcal{N}_2^+)$ is given by

$$\begin{array}{rcl} \mathcal{N}_1^+(\theta, v, \vartheta^+, c, \epsilon)(\xi) & = & \mathcal{R}^{\mathrm{fb}}_R(\theta, v, \vartheta^+, c, \epsilon)(\xi), \\ \mathcal{N}_2^+(\theta, v, \vartheta^+, c, \epsilon)(\xi) & = & \epsilon \mathcal{S}^{\mathrm{fb}}_R(\theta, v, \vartheta^+, c, \epsilon)(\xi), \end{array} \tag{5.58}$$

after slightly modifying $\mathcal{R}^{\mathrm{fb}}_R$ and $\mathcal{S}^{\mathrm{fb}}_R$ to account for the fact that v^+ and θ^+ are defined on $[\xi_0 - 1, \infty)$ instead of $[-1, \infty)$.

For ease of notation, we introduce the family of function spaces

$$\mathcal{H}^{\circ}_{\zeta} \quad = \quad BC^{\ominus}_{-\eta_*} \times BC^{\odot} \times BC^{\oplus}_{-\eta_* + \zeta}, \tag{5.59}$$

parametrized by $\zeta \geq 0$. When $\zeta = 0$, we will also use the shorthand $\mathcal{H}^\circ = \mathcal{H}_0^\circ$. In addition, we write

$$\mathcal{H}_\zeta \quad = \quad BC^-_{-\eta_*} \times BC^\diamond \times BC^+_{-\eta_*+\zeta}, \tag{5.60}$$

again with $\mathcal{H} = \mathcal{H}_0$. For any $h = (v^-, \theta^-, v^\diamond, \theta^\diamond, v^+, \theta^+) \in \mathcal{H}^\circ$, we write $\pi_{v^-} h = v^-$ and define the projections $\pi_{v^\diamond}$, π_{v^+}, $\pi_{\theta^\diamond}$ and $\pi_{\theta^\pm}$ in a similar fashion.

Let us combine the parameters appearing in the equations above into a single quantity $p = (\vartheta^+, c, \epsilon)$. Choosing a set of small constants $\delta_\vartheta > 0$, $\delta_c > 0$ and $\delta_\epsilon > 0$, we write

$$\mathcal{D}_p = \mathcal{D}_p(\delta_\vartheta, \delta_c, \delta_\epsilon) = [-\delta_\vartheta, \delta_\vartheta] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \tag{5.61}$$

for the parameter space we are interested in. We also choose a large constant $\xi_* > 0$ and again write $a \leq_* b$ to express the fact that there exists a $C > 0$ that does not depend on $\xi_0 \geq \xi_*$, any of the cut-offs appearing in (5.61) or the cut-off δ_v that will be introduced in the sequel, such that $a \leq Cb$ for all quantities a and b that depend on these constants. In addition, for any $h \in \mathcal{H}^\circ$ and $p \in \mathcal{D}_p$, we will use the shorthand $\mathcal{N}^-(h,p) = \mathcal{N}^-(\pi_{v^-}h, \pi_{\theta^-}h, \vartheta^+, c, \epsilon)$, together with analogous definitions for $\mathcal{N}^\diamond(h,p)$ and $\mathcal{N}^+(h,p)$. Finally, we use the notation $\mathcal{N}(h,p)$ to denote the set $\big(\mathcal{N}^-(h,p), \mathcal{N}^\diamond(h,p), \mathcal{N}^+(h,p)\big) \in \mathcal{H}$.

The first step in the proof of Proposition 3.4 is to consider the linearized equations that the quasi-fronts must solve. Note that the quantity ϑ^+ is a free parameter in this step, but we do impose the shape conditions from Definition 3.3(iii)-(iv).

Lemma 5.10 *Fix a sufficiently large constant ξ_* and sufficiently small $\delta_c > 0$, $\delta_\vartheta > 0$ and $\delta_\epsilon > 0$. Choose any $\xi_0 \geq \xi_*$. Then for any $g = (g^-, g^\diamond, g^+) \in \mathcal{H}$, any boundary condition $\phi \in X$ and any $p = (\vartheta^+, c, \epsilon) \in \mathcal{D}_p(\delta_\vartheta, \delta_c, \delta_\epsilon)$, there exists a unique*

$$h = (v^-, \theta^-, v^\diamond, \theta^\diamond, v^+, \theta^+) \in \mathcal{H}^\circ \tag{5.62}$$

that satisfies the following properties.

(i) The linear systems

$$\begin{array}{lcl} \Lambda^-(v^-, \theta^-) & = & (-\theta^-, 0) + g^-, \\ \Lambda^\diamond(v^\diamond, \theta^\diamond) & = & (-\theta^\diamond, 0) + g^\diamond, \\ \Lambda^+(p)(v^+, \theta^+) & = & g^+ \end{array} \tag{5.63}$$

are all satisfied.

(ii) The identity $\theta^\diamond(0) = \theta^-(0)$ holds.

(iii) We have the inclusions

$$\begin{array}{lcl} \mathrm{ev}_0 v^- & \in & \widehat{P}_f(0) \oplus \widehat{Q}_f(0) \oplus \Gamma_f, \\ \mathrm{ev}_0 v^\diamond & \in & \widehat{P}_f(0) \oplus \widehat{Q}_f(0) \oplus \Gamma_f. \end{array} \tag{5.64}$$

(iv) The gap between v^- and $v^\diamond$ at zero satisfies $\mathrm{ev}_0[v^- - v^\diamond] \in \Gamma_f$.

(v) Upon writing $\widetilde{v}_\theta^+(\xi) = D_1 s_R\big(\Theta_R^{\mathrm{fs}}(p)(\xi)\big)\theta^+(\xi)$, the following boundary condition is satisfied,

$$\mathrm{ev}_{\xi_0}[v^\diamond - \widetilde{v}_\theta^+ - v^+] \quad = \quad \phi. \tag{5.65}$$

The function $h \in \mathcal{H}^\circ$ described above will be denoted by

$$h = L_1(p)(g, \phi). \tag{5.66}$$

Recalling the integer r appearing in (H1), there exists a constant $N_1 > 0$ such that for any integer $0 \le \ell \le r+2$ and any $\zeta > \ell\delta_\epsilon N_1$, the map $p \mapsto L_1(p)$ is C^ℓ-smooth when considered as a map

$$L_1 : \mathcal{D}_p \to \mathcal{L}\big(\mathcal{H} \times X, \mathcal{H}^\circ_\zeta\big), \tag{5.67}$$

with derivatives that can be bounded independently of $\xi_0 \ge \xi_*$. Finally, consider any $d \in \mathbb{K}_f^*$. Then the following identity holds for the gap at zero,

$$\begin{array}{rcl} \langle \mathrm{ev}_0 d, \mathrm{ev}_0[v^- - v^\diamond]\rangle & = & \int_{-\infty}^0 d(\xi')^* g_1^-(\xi')\,\mathrm{d}\xi' + \int_0^{\xi_0} d(\xi') g_1^\diamond(\xi')\,\mathrm{d}\xi' \\ & & -\frac{1}{c_*}\int_{-\infty}^0 d(\xi')\int_{-\infty}^{\xi'} g_2^-(\xi'')\,\mathrm{d}\xi''\,\mathrm{d}\xi' \\ & & -\frac{1}{c_*}\int_0^{\xi_0} d(\xi')[\int_{-\infty}^0 g_2^-(\xi'')\,\mathrm{d}\xi'' + \int_0^{\xi'} g_2^\diamond(\xi'')\,\mathrm{d}\xi'']\,\mathrm{d}\xi' \\ & & -\langle \mathrm{ev}_{\xi_0} d, \mathrm{ev}_{\xi_0} v^\diamond\rangle. \end{array} \tag{5.68}$$

Proof. First of all, we can use Lemma 5.3 to define an operator

$$[\Lambda_1^+(p)]^{-1} : BC_{-\eta_*}([\xi_0,\infty),\mathbb{R}) \to BC_{-\eta_*}([\xi_0 - 1,\infty),\mathbb{R}) \tag{5.69}$$

and Lemma 5.7 to define linear operators

$$\begin{array}{rcl} [\Lambda_1^-]^{-1} & : & BC_{-\eta_*}((-\infty,0],\mathbb{R}) \to BC_{-\eta_*}((-\infty,1],\mathbb{R}), \\ [\Lambda_1^\diamond]^{-1} & : & BC_0([0,\xi_0],\mathbb{R}) \to BC_0([-1,\xi_0+1],\mathbb{R}), \end{array} \tag{5.70}$$

such that the choice $h_0 = (v_0^-, \theta^-, v_0^\diamond, \theta^\diamond, v_0^+, \theta^+)$ with

$$\begin{array}{rclrcl} \theta^-(\xi) & = & \frac{1}{c_*}\int_{-\infty}^{\xi} g_2^-(\xi')\,\mathrm{d}\xi', & v_0^- & = & [\Lambda_1^-]^{-1}[g_1^- - \theta^-], \\ \theta^\diamond(\xi) & = & \theta^-(0) + \frac{1}{c_*}\int_0^\xi g_2^\diamond(\xi')\,\mathrm{d}\xi', & v_0^\diamond & = & [\Lambda_1^\diamond]^{-1}[g_1^\diamond - \theta^\diamond], \\ \theta^+(\xi) & = & \frac{1}{c}\int_\infty^\xi g_2^+(\xi')\,\mathrm{d}\xi', & v_0^+ & = & [\Lambda_1^+(p)]^{-1} g_1^+, \end{array} \tag{5.71}$$

satisfies items (i) and (ii).

We note here that the exponents -1 above are used suggestively, since the relevant homogeneous equations have non-zero solutions. We shall use this freedom to ensure that the remaining properties (iii) - (v) are also satisfied. In particular, we will modify v_0^-, $v_0^\diamond$ and v_0^+ by choosing $\psi^{B\diamond} \in B_f(0)$, $\psi^{B-} \in B_f(0)$, $\psi^{\widehat{P}-} \in \widehat{P}_f(0)$, $\psi^{\widehat{Q}\diamond} \in \widehat{Q}_f(0)$, $\psi^{S\diamond} \in P^{\mathrm{fb}}_{R,0}$ and $\psi^{Q+} \in Q^{\mathrm{fb}}_{R,0}$ and writing

$$\begin{array}{rcl} v^- & = & v_0^- + E^{P-}(\psi^{B-} + \psi^{\widehat{P}-}), \\ v^\diamond & = & v_0^\diamond + E^{Q\diamond}(\psi^{B\diamond} + \psi^{\widehat{Q}\diamond}) + E^{S\diamond}\psi^{S\diamond}, \\ v^+ & = & v_0^+ + E^{Q+}(p)\psi^{Q+}, \end{array} \tag{5.72}$$

in which the extension operators E^{P-}, $E^{Q\diamond}$ and $E^{S\diamond}$ are relabelled versions of those defined in Lemma 5.9, while E^{Q+} is constructed from the operator $E^{\mathrm{fb}}_{R,0}$ appearing in Lemma 5.5.

In terms of these new variables, the boundary condition in (v) can be written as

$$\begin{array}{rcl} \phi_P & = & \psi^{S\diamond} + \Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}E^{Q\diamond}(\psi^{B\diamond} + \psi^{\widehat{Q}\diamond}) - \Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}E^{Q+}(p)\psi^{Q+}, \\ \phi_Q & = & \psi^{Q+} - \Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}E^{Q\diamond}(\psi^{B\diamond} + \psi^{\widehat{Q}\diamond}), \end{array} \tag{5.73}$$

in which

$$\begin{array}{rcl} \phi_P & = & \Pi_{P^{\mathrm{fb}}_{R,0}}\phi - \Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}v_0^\diamond + \Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[\widetilde{v}_\theta^+ + v_0^+], \\ \phi_Q & = & -\Pi_{Q^{\mathrm{fb}}_{R,0}}\phi + \Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}v_0^\diamond - \Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[\widetilde{v}_\theta^+ + v_0^+]. \end{array} \tag{5.74}$$

Conditions (iii) and (iv) are equivalent to the system

$$\begin{array}{lcl} -\Pi_{B_f(0)}\mathrm{ev}_0 v_0^- & = & \psi^{B-}, \\ -\Pi_{B_f(0)}\mathrm{ev}_0 v_0^\diamond & = & \psi^{B\diamond} + \Pi_{B_f(0)}\mathrm{ev}_0 E^{S\diamond}\psi^{S\diamond}, \\ -\Pi_{\widehat{P}_f(0)}\mathrm{ev}_0\big(v_0^- - v_0^\diamond\big) & = & \psi^{\widehat{P}-} - \Pi_{\widehat{P}_f(0)}\mathrm{ev}_0 E^{S\diamond}\psi^{S\diamond}, \\ \Pi_{\widehat{Q}_f(0)}\mathrm{ev}_0\big(v_0^- - v_0^\diamond\big) & = & \psi^{\widehat{Q}\diamond} + \Pi_{\widehat{Q}_f(0)}\mathrm{ev}_0 E^{S\diamond}\psi^{S\diamond}. \end{array} \tag{5.75}$$

Inspection of the system (5.73)-(5.75) readily shows that for sufficiently large ξ_* and sufficiently small δ_c, δ_ϑ and δ_ϵ, the right hand side is close to the identity matrix. This allows the linear system to be solved, yielding the desired set $h = L_1(p)(g,\phi) \in \mathcal{H}^\circ$. To complete the proof, observe that the integral expression (5.68) follows from (3.5), while the smoothness of the map $p \mapsto L_1(p)$ follows from Lemmas 5.3 and 5.5. ■

As a second step towards establishing Proposition 3.4, we will need to study the nonlinear fixed point problem

$$h = \mathcal{F}(h,p) := L_1(p)\big(\mathcal{N}(h,p), \Phi(h,p)\big), \tag{5.76}$$

in which

$$\begin{array}{rcl} \Phi(h,p) & = & \mathrm{ev}_{\xi_0}\big[s_R(\Theta_R^{\mathrm{fs}}(p) + \theta^+, p) - D_1 s_R(\Theta_R^{\mathrm{fs}}(p), p)\theta^+ - s_R(\Theta_R^{\mathrm{fs}}(p), p)\big] \\ & & +\mathrm{ev}_{\xi_0}[s_R(\Theta_R^{\mathrm{fs}}(p), p) - q_f]. \end{array} \tag{5.77}$$

Let us introduce the set $\mathcal{B}^\circ_{\delta_v} = \{h \in \mathcal{H}^\circ \mid \|h\|_{\mathcal{H}^\circ} \le \delta_v\}$. It is not hard to see that for all $h \in \mathcal{H}^\circ_{\delta_v}$ and $p \in \mathcal{D}_p$, we have

$$\|\Phi(h,p)\| \le_* \delta_v^2 + \delta_\vartheta + \delta_c + \delta_\epsilon + \mathrm{e}^{-\eta_*\xi_0}. \tag{5.78}$$

In addition, for a pair $h_1, h_2 \in \mathcal{B}^\circ_{\delta_v}$ and $p \in \mathcal{D}_p$ we have the Lipschitz estimate

$$\|\Phi(h_1,p) - \Phi(h_2,p)\| \le_* \delta_v \|h_1 - h_2\|_{\mathcal{H}^\circ}. \tag{5.79}$$

Combining this estimate with (5.16) and inspecting (5.29), we find the estimates

$$\begin{array}{lcl} \|\mathcal{F}(h,p)\|_{\mathcal{H}^\circ} & \le_* & \delta_v^2 + \delta_\vartheta + \delta_c + \delta_\epsilon\xi_0 + \mathrm{e}^{-\eta_*\xi_0}, \\ \|\mathcal{F}(h_1,p) - \mathcal{F}(h_2,p)\|_{\mathcal{H}^\circ} & \le_* & (\delta_v + \delta_c + \delta_\epsilon\xi_0)\|h_1 - h_2\|_{\mathcal{H}^\circ}, \end{array} \tag{5.80}$$

which hold for all $p \in \mathcal{D}_p$ and $h, h_1, h_2 \in \mathcal{B}^\circ_{\delta_v}$. After writing

$$\delta_\vartheta = \delta_v^{3/2}, \qquad \delta_c = \delta_v^2, \qquad \delta_\epsilon = \delta_v^2/\xi_0 \tag{5.81}$$

and choosing δ_v to be sufficiently small, we hence see that $\mathcal{F}(\cdot,p)$ is a contraction mapping on the set $\mathcal{B}^\circ_{\delta_v}$ for all $p \in \mathcal{D}_p$. This shows that the fixed point problem (5.76) has a solution $h = h^*(p)$ that is unique in the set $\mathcal{H}_{\delta_v}$.

Before we can proceed further, we need to obtain estimates on the derivative $D_p h^*$. This can be done by writing $D_p h^*(p) = h^{(1)}$ and noting that $h^{(1)}$ must satisfy the linear fixed point problem

$$\begin{array}{rcl} h^{(1)} & = & D_p L_1(p)\big(\mathcal{N}(h^*(p),p), \Phi(h^*(p),p)\big) \\ & & +L_1(p)\big(D_h\mathcal{N}(h^*(p),p), D_h\Phi(h^*(p),p)\big)h^{(1)} \\ & & +L_1(p)\big(D_p\mathcal{N}(h^*(p),p), D_p\Phi(h^*(p),p)\big), \end{array} \tag{5.82}$$

posed on the space $\mathcal{H}_\zeta$ for some $\delta_\epsilon N_1 < \zeta < \eta_*$. To see that this well-defined, we observe that $L_1(p)$ can also be treated as a map from $\mathcal{H}_\zeta \times X$ into $\mathcal{H}^\circ_\zeta$. We have the estimate

$$\|D_h\mathcal{N}(h,p)\|_{\mathcal{L}(\mathcal{H}^\circ_\zeta, \mathcal{H}_\zeta)} \quad \le_* \quad \delta_v + \delta_c + \delta_\epsilon\xi_0 \tag{5.83}$$

for any $h \in \mathcal{H}_{\delta_v}$ and $p \in \mathcal{D}_p$. In addition, for such h and p we have

$$\|D_p \mathcal{N}^-(h,p)\|_{BC^-_{-\eta_*}} + \|D_p \mathcal{N}^+(h,p)\|_{BC^+_{-\eta_*+\zeta}} \quad \leq_* \quad 1, \tag{5.84}$$

together with

$$\begin{array}{lcl} \|D_\epsilon \mathcal{N}^\diamond(h,p)\|_{BC^\diamond} & \leq_* & \xi_0, \\ \|D_c \mathcal{N}^\diamond(h,p)\|_{BC^\diamond} + \|D_{\vartheta^+} \mathcal{N}^\diamond(h,p)\|_{BC^\diamond} & \leq_* & 1. \end{array} \tag{5.85}$$

After these preparations, it is clear that the fixed point problem (5.82) has a unique solution $h^{(1)} = h^{*(1)}(p)$ for all $p \in \mathcal{D}_p$. In addition, we find

$$\begin{array}{lcl} \|D_\epsilon h^*(p)\|_{\mathcal{H}^\diamond_\zeta} & \leq_* & \xi_0, \\ \|D_c h^*(p)\|_{\mathcal{H}^\diamond_\zeta} + \|D_{\vartheta^+} h^*(p)\|_{\mathcal{H}^\diamond_\zeta} & \leq_* & 1. \end{array} \tag{5.86}$$

Inspection of (5.71) shows that both $\pi_{\theta^\diamond} L_1(p)$ and $\pi_{\theta^+} L_1(p)$ do not depend on ϑ^+. We may therefore compute

$$\begin{array}{lcl} D_{\vartheta^+} \pi_{\theta^\diamond} h^*(p) & = & \pi_{\theta^\diamond} L_1(p) \Big(\big(0, D_h \mathcal{N}_2^-(h^*(p),p)\big), \big(0, D_h \mathcal{N}_2^\diamond(h^*(p),p)\big), \\ & & \qquad \big(0, D_h \mathcal{N}_2^+(h^*(p),p)\big), 0 \Big) D_{\vartheta^+} h^*(p) \\ & & + \pi_{\theta^\diamond} L_1(p) \Big(0, 0, \big(0, D_{\vartheta^+} \mathcal{N}_2^+(h^*(p),p)\big), 0 \Big). \end{array} \tag{5.87}$$

A similar computation can be performed for $D_{\vartheta^+} \pi_{\theta^+} h^*(p)$. Upon using (5.86), we may hence conclude

$$\|D_{\vartheta^+} \pi_{\theta^\diamond} h^*(p)\|_{-\eta_*+\zeta} + \|D_{\vartheta^\diamond} \pi_{\theta^\diamond} h^*(p)\|_0 \quad \leq_* \quad \delta_\epsilon \xi_0. \tag{5.88}$$

We have now gathered all the ingredients we need in order to find ϑ^+ as a function of c and ϵ. Indeed, the requirement that the function w_f constructed in (5.50) is continuous at ξ_0 leads to the fixed point problem

$$\vartheta^+ = \mathcal{F}_2(\vartheta^+, c, \epsilon) := [\pi_{\theta^\diamond} h^*(\vartheta^+, c, \epsilon)](\xi_0) - [\pi_{\theta^+} h^*(\vartheta^+, c, \epsilon)](\xi_0). \tag{5.89}$$

In view of the scalings (5.81), it is not hard to see that $|\mathcal{F}_2(p)| \leq_* \delta_v^2$, for all $p \in \mathcal{D}_p$, which implies that, possibly after decreasing $\delta_v > 0$, the operator $\mathcal{F}_2(\cdot, c, \epsilon)$ maps the interval $[-\delta_\vartheta, \delta_\vartheta]$ into itself. In addition, (5.88) implies that $\mathcal{F}_2(\cdot, c, \epsilon)$ is a contraction mapping, which implies that we can find a solution $\vartheta^+ = \vartheta^+(c, \epsilon)$ to the fixed point problem (5.89) for all $c \in [c_* - \delta_c, c_* + \delta_c]$ and $\epsilon \in [0, \delta_\epsilon]$. In addition, for such c and ϵ we find the bounds

$$\begin{array}{lcl} |D_c \vartheta^+(c,\epsilon)| & \leq_* & 1, \\ |D_\epsilon \vartheta^+(c,\epsilon)| & \leq_* & \xi_0. \end{array} \tag{5.90}$$

Finally, we set out to establish the Melnikov inequalities (3.9). Let us therefore fix a $d \in \mathbb{K}^*$ that is normalized to have $d(0) > 0$ and $\|d\|_\infty = 1$. We study the map $M : \mathcal{D}_p \to \mathbb{R}$ that is given by

$$M : p \mapsto \langle \mathrm{ev}_0 d, \mathrm{ev}_0 \pi_{v^-} h^*(p) - \mathrm{ev}_0 \pi_{v^\diamond} h^*(p) \rangle. \tag{5.91}$$

Using the identity (5.68), we may write

$$\begin{array}{lcl} M(p) & = & \int_{-\infty}^0 d(\xi') \mathcal{N}_1^-(h^*(p),p)(\xi')\, \mathrm{d}\xi' + \int_0^{\xi_0} d(\xi') \mathcal{N}_1^\diamond(h^*(p),p)(\xi')\, \mathrm{d}\xi' \\ & & - \frac{1}{c_*} \int_{-\infty}^0 d(\xi') \int_{-\infty}^{\xi'} \mathcal{N}_2^-(h^*(p),p)(\xi'')\, \mathrm{d}\xi''\, \mathrm{d}\xi' \\ & & - \frac{1}{c_*} \int_0^{\xi_0} d(\xi') [\int_{-\infty}^0 \mathcal{N}_2^-(h^*(p),p)(\xi'')\, \mathrm{d}\xi'' + \int_0^{\xi'} \mathcal{N}_2^\diamond(h^*(p),p)(\xi'')\, \mathrm{d}\xi''] \, \mathrm{d}\xi' \\ & & - \langle \mathrm{ev}_{\xi_0} d, \mathrm{ev}_{\xi_0} \pi_{v^\diamond} h^*(p) \rangle. \end{array} \tag{5.92}$$

Let us now write $\widetilde{M} : [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon] \to \mathbb{R}$ for the operator $\widetilde{M}(c, \epsilon) = M(\vartheta^+(c, \epsilon), c, \epsilon)$. Let us write $p_0 = (0, c_*, 0)$. Observe first that $D_h \mathcal{N}^-(h^*(p_0), p_0) = 0$ and $D_h \mathcal{N}^\diamond(h^*(p_0), p_0) = 0$. Using the estimates (5.86) and (5.90), we may hence compute

$$\begin{array}{rcl} D_c \widetilde{M}(c_*, 0) & = & \int_{-\infty}^0 d(\xi') D_c \mathcal{N}_1^-(0, p_0)(\xi') \, \mathrm{d}\xi' \\ & & + \int_0^{\xi_0} d(\xi') D_c \mathcal{N}_1^\diamond(0, p_0)(\xi') \, \mathrm{d}\xi' + O(\mathrm{e}^{-\eta_* \xi_0}) \\ & = & - \int_{-\infty}^{\xi_0} d(\xi') q_f'(\xi') \, \mathrm{d}\xi' + O(\mathrm{e}^{-\eta_* \xi_0}) \\ D_\epsilon \widetilde{M}(c_*, 0) & = & -\frac{1}{c_*} \int_{-\infty}^0 d(\xi') \int_{-\infty}^{\xi'} D_\epsilon \mathcal{N}_2^-(0, p_0)(\xi'') \, \mathrm{d}\xi'' \, \mathrm{d}\xi' \\ & & -\frac{1}{c_*} \int_0^{\xi_0} d(\xi') [\int_{-\infty}^0 D_\epsilon \mathcal{N}_2^-(0, p_0)(\xi'') + \int_0^{\xi'} D_\epsilon \mathcal{N}_2^\diamond(0, p_0)(\xi'')] \, \mathrm{d}\xi'' \, \mathrm{d}\xi' \\ & & + O(\xi_0 \mathrm{e}^{-\eta_* \xi_0}) \\ & = & -\frac{1}{c_*} \int_{-\infty}^{\xi_0} d(\xi') \int_{-\infty}^{\xi'} q_f(\xi'') \, \mathrm{d}\xi'' \, \mathrm{d}\xi' + O(\xi_0 \mathrm{e}^{-\eta_* \xi_0}) \end{array} \tag{5.93}$$

The inequalities (3.9) now follow from Lemma 3.2.

Proof of Proposition 3.4 We fix $\delta_\vartheta > 0$, $\delta_c > 0$ and $\delta_\epsilon > 0$ as in (5.81), for some sufficiently small $\delta_v > 0$. Let us choose any pair $(c, \epsilon) \in [c_* - \delta_c, c_* + \delta_c]$, write

$$(v^-, \theta^-, v^\diamond, \theta^\diamond, v^+, \theta^+) = h^*(\vartheta^+(c, \epsilon), c, \epsilon) \tag{5.94}$$

and define $u_f^\pm(c, \epsilon)$ and $w_f(c, \epsilon)$ as in (5.50)-(5.51). In addition, write

$$\vartheta_f(c, \epsilon) = \Theta_R^{\mathrm{sl}}(\vartheta^+(c, \epsilon), c, \epsilon)(-\xi_0). \tag{5.95}$$

The properties required in Definition 3.3 and in (ii) are satisfied as immediate consequences of the construction above. The smoothness properties in (iii) can be established using arguments analogous to those used above to establish the C^1-smoothness of h^*. It remains to show that the functions thus constructed are locally unique. Let us therefore assume for some pair (c, ϵ) the existence of a second quasi-front solution to (3.1). In view of the uniqueness claim in Lemma 5.10 and the uniqueness of solutions to fixed point equations, it suffices to show that the first two estimates in Definition 2.2(iv) hold with respect to an exponentially weighted norm and not merely with respect to the supremum norm. This can be established as in the proof of [43, Claim 3.7]. ■

5.4 Construction of quasi-backs

We now set out to proof Proposition 3.6. The ideas and techniques are very similar to those used previously to establish Proposition 3.4, so we will focus mainly on the differences needed here. Since the quasi-backs need to follow both $\mathcal{M}_R$ and $\mathcal{M}_L$, we will split the real line into four separate parts instead of three. In particular, we will study (3.1) on the four intervals $(-\infty, -\xi_0]$, $[-\xi_0, 0]$, $[0, \xi_0]$ and $[\xi_0, \infty)$.

To accommodate this, we introduce the family of function spaces

$$\begin{array}{rcl} BC_\alpha^\ominus & = & \{(v, \theta) \in C((-\infty, -\xi_0 + 1], \mathbb{R}) \times C((-\infty, -\xi_0 + 1], \mathbb{R}) \text{ for which} \\ & & \quad \|(v, \theta)\|_{BC_\alpha^\ominus} := \sup_{\xi \le -\xi_0 + 1} \mathrm{e}^{-\alpha|\xi + \xi_0|}(|v(\xi)| + |\theta(\xi)|) < \infty\}, \\ BC^{\odot -} & = & \{(v, \theta) \in C([-\xi_0 - 1, 1], \mathbb{R}) \times C([-\xi_0, 0], \mathbb{R}) \text{ for which} \\ & & \quad \|(v, \theta)\|_{BC^{\odot -}} := \sup_{-\xi_0 - 1 \le \xi \le 1} |v(\xi)| + \sup_{-\xi_0 \le \xi \le 0} |\theta(\xi)| < \infty\}, \\ BC^{\odot +} & = & \{(v, \theta) \in C([-1, \xi_0 + 1], \mathbb{R}) \times C([0, \xi_0], \mathbb{R}) \text{ for which} \\ & & \quad \|(v, \theta)\|_{BC^{\odot +}} := \sup_{-1 \le \xi \le \xi_0 + 1} |v(\xi)| + \sup_{0 \le \xi \le \xi_0} |\theta(\xi)| < \infty\}, \\ BC_\alpha^\oplus & = & \{(v, \theta) \in C([\xi_0 - 1, \infty), \mathbb{R}) \times C([\xi_0 - 1, \infty), \mathbb{R}) \text{ for which} \\ & & \quad \|(v, \theta)\|_{BC_\alpha^\oplus} := \sup_{\xi \ge \xi_0 - 1} \mathrm{e}^{-\alpha|\xi - \xi_0|}(|v(\xi)| + |\theta(\xi)|) < \infty\}, \end{array} \tag{5.96}$$

parametrized by $\alpha > 0$, together with the family

$$\begin{array}{lcl} BC^-_{\alpha} & = & \{g = (g_1, g_2) \in C((-\infty, -\xi_0], \mathbb{R}) \times C((-\infty, -\xi_0], \mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC^-_{\alpha}} := \sup_{\xi \le 0} \mathrm{e}^{-\alpha|\xi + \xi_0|} \big(|g_1(\xi)| + |g_2(\xi)| \big) < \infty \}, \\ BC^{\diamond -} & = & \{(g_1, g_2) \in C([-\xi_0, 0], \mathbb{R}) \times C([-\xi_0, 0], \mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC^{\diamond -}} := \sup_{-\xi_0 \le \xi \le 0} |g_1(\xi)| + \xi_0 \, |g_2(\xi)| < \infty \}, \\ BC^{\diamond +} & = & \{(g_1, g_2) \in C([0, \xi_0], \mathbb{R}) \times C([0, \xi_0], \mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC^{\diamond +}} := \sup_{0 \le \xi \le \xi_0} |g_1(\xi)| + \xi_0 \, |g_2(\xi)| < \infty \}, \\ BC^+_{\alpha} & = & \{(g_1, g_2) \in C([\xi_0, \infty), \mathbb{R}) \times C([\xi_0, \infty), \mathbb{R}) \text{ for which} \\ & & \quad \|g\|_{BC^+_{\alpha}} := \sup_{\xi \ge \xi_0} \mathrm{e}^{-\alpha|\xi - \xi_0|} \big(|g_1(\xi)| + |g_2(\xi)| \big) < \infty \}. \end{array} \tag{5.97}$$

As before, notice the additional factor ξ_0 that appears in the second component of the norms defined on $BC^{\diamond\pm}$.

Our goal is to find $\vartheta^-, \vartheta^+ \in [w_{\min}, w_{\max}]$ together with pairs $(v^-, \theta^-) \in BC^{\ominus}_{-\eta_*}$, $(v^{\diamond -}, \theta^{\diamond -}) \in BC^{\odot -}$, $(v^{\diamond +}, \theta^{\diamond +}) \in BC^{\odot +}$ and $(v^+, \theta^+) \in BC^{\oplus}_{-\eta_*}$, such that the choice

$$w_b(\xi) = \begin{cases} \Theta^{\mathrm{fs}}_R(\vartheta^-, c, \epsilon)(\xi) + \theta^-(\xi) & \text{for } \xi \le -\xi_0, \\ w_* + \theta^{\diamond -}(\xi) & \text{for } -\xi_0 \le \xi \le 0, \\ w_* + \theta^{\diamond +}(\xi) & \text{for } 0 \le \xi \le \xi_0, \\ \Theta^{\mathrm{fs}}_L(\vartheta^+, c, \epsilon)(\xi) + \theta^+(\xi) & \text{for } \xi \ge \xi_0, \end{cases} \tag{5.98}$$

in combination with

$$\begin{array}{lcl} u^-_b(\xi) & = & \begin{cases} s_R(w_b(\xi), c, \epsilon) + v^-(\xi) & \text{for } \xi \le -\xi_0, \\ q_b(\xi) + v^{\diamond -}(\xi) & \text{for } -\xi_0 \le \xi \le 1, \end{cases} \\ u^+_b(\xi) & = & \begin{cases} q_b(\xi) + v^{\diamond +}(\xi) & \text{for } -1 \le \xi \le \xi_0, \\ s_L(w_b(\xi), c, \epsilon) + v^+(\xi) & \text{for } \xi \ge \xi_0, \end{cases} \end{array} \tag{5.99}$$

satisfies the conditions of Proposition 3.6.

We will write the equations that arise after inserting the ansatz (5.98)-(5.99) into (3.1) in the following fashion,

$$\begin{array}{lcl} \Lambda^-(\vartheta^-, c, \epsilon)(v^-, \theta^-) & = & \mathcal{N}^-(v^-, \theta^-, \vartheta^-, c, \epsilon), \\ \Lambda^{\diamond -}(v^{\diamond -}, \theta^{\diamond -}) & = & (-\theta^{\diamond -}, 0) + \mathcal{N}^{\diamond -}(v^{\diamond -}, \theta^{\diamond -}, c, \epsilon), \\ \Lambda^{\diamond +}(v^{\diamond +}, \theta^{\diamond +}) & = & (-\theta^{\diamond +}, 0) + \mathcal{N}^{\diamond +}(v^{\diamond +}, \theta^{\diamond +}, c, \epsilon), \\ \Lambda^+(\vartheta^+, c, \epsilon,)(v^+, \theta^+) & = & \mathcal{N}^+(v^+, \theta^+, \vartheta^+, c, \epsilon). \end{array} \tag{5.100}$$

As before, we write

$$\begin{array}{lcl} \Lambda^{\pm}(\vartheta^{\pm}, c, \epsilon)(v^{\pm}, \theta^{\pm}) & = & \Big(\Lambda^{\pm}_1(\vartheta^{\pm}, c, \epsilon) v^{\pm}, \Lambda^{\pm}_2(c) \theta^{\pm} \Big), \\ \Lambda^{\diamond \pm}(v^{\diamond \pm}, \theta^{\diamond \pm}) & = & \Big(\Lambda^{\diamond \pm}_1 v^{\pm}, \Lambda^{\diamond \pm}_2 \theta^{\pm} \Big), \end{array} \tag{5.101}$$

now with $\Lambda^-_1(\vartheta^-, c, \epsilon) = \Lambda^{\mathrm{fb}}_R(\vartheta^-, c, \epsilon)$, $\Lambda^+_1(\vartheta^+, c, \epsilon) = \Lambda^{\mathrm{fb}}_L(\vartheta^+, c, \epsilon)$, $\Lambda^{\pm}_2(c) = cD$ and

$$\begin{array}{lcl} [\Lambda^{\diamond \pm}_1 v](\xi) & = & c_* v'(\xi) - L\big(q_b(\xi)\big) \mathrm{ev}_{\xi} v, \\ [\Lambda^{\diamond \pm}_2 \theta](\xi) & = & c_* \theta'(\xi). \end{array} \tag{5.102}$$

Up to obvious adjustments concerning the domain of definition, the nonlinearities $\mathcal{N}^{\pm}$ are given by $\mathcal{N}^- = \big(\mathcal{R}^{\mathrm{fb}}_R, \epsilon \mathcal{S}^{\mathrm{fb}}_R\big)$ and $\mathcal{N}^+ = \big(\mathcal{R}^{\mathrm{fb}}_L, \epsilon \mathcal{S}^{\mathrm{fb}}_L\big)$, while $\mathcal{N}^{\diamond \pm}$ are given by $\mathcal{N}^{\diamond \pm} = \big(\mathcal{R}_b, \epsilon \mathcal{S}_b\big)$.

The families of function spaces that are relevant for the construction of the quasi-backs are given by

$$\begin{array}{lcl} \mathcal{H}^{\circ}_{\zeta} & = & BC^{\ominus}_{-\eta_* + \zeta} \times BC^{\odot -} \times BC^{\odot +} \times BC^{\oplus}_{-\eta_* + \zeta}, \\ \mathcal{H}_{\zeta} & = & BC^-_{-\eta_* + \zeta} \times BC^{\diamond -} \times BC^{\diamond +} \times BC^+_{-\eta_* + \zeta}, \end{array} \tag{5.103}$$

both parametrized by $\zeta \geq 0$. As before, we employ the shorthands $\mathcal{H}^\circ = \mathcal{H}^\circ_0$ and $\mathcal{H} = \mathcal{H}_0$. For notational convenience, we now introduce the parameter vector $\widetilde{p} = (\vartheta^-, \vartheta^+, c, \epsilon)$, which we will take from the space

$$\widetilde{\mathcal{D}}_p(\delta_\vartheta, \delta_c, \delta_\epsilon) = [w_* - \delta_\vartheta, w_* + \delta_\vartheta]^2 \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_\epsilon]. \tag{5.104}$$

We also use the augmented parameter vector $p = (\vartheta^0, \widetilde{p}) = (\vartheta^0, \vartheta^-, \vartheta^+, c, \epsilon)$, which we will take from the space

$$\mathcal{D}_p(\delta_{\vartheta^0}, \delta_\vartheta, \delta_c, \delta_\epsilon) = [w_* - \delta_{\vartheta^0}, w_* + \delta_{\vartheta^0}] \times \widetilde{\mathcal{D}}_p(\delta_\vartheta, \delta_c, \delta_\epsilon). \tag{5.105}$$

The equivalent of Lemma 5.10 now reads as follows.

Lemma 5.11 *Fix a sufficiently large constant ξ_* and sufficiently small $\delta_\vartheta > 0$, $\delta_c > 0$ and $\delta_\epsilon > 0$. Choose any $\xi_0 \geq \xi_*$. Then for every $g = (g^-, g^{\diamond-}, g^{\diamond+}, g^+) \in \mathcal{H}$, any pair of boundary conditions $\phi_R, \phi_L \in X$, any $\vartheta^0 \in \mathbb{R}$ and any $\widetilde{p} = (\vartheta^-, \vartheta^+, c, \epsilon) \in \widetilde{\mathcal{D}}_p(\delta_\vartheta, \delta_c, \delta_\epsilon)$, there exists a unique*

$$h = (v^-, \theta^-, v^{\diamond-}, \theta^{\diamond-}, v^{\diamond+}, \theta^{\diamond+}, v^+, \theta^+) \in \mathcal{H}^\circ \tag{5.106}$$

that satisfies the following properties.

(i) The linear systems

$$\begin{array}{rcl} \Lambda^\pm(\widetilde{p})(v^\pm, \theta^\pm) & = & g^\pm, \\ \Lambda^{\diamond\pm}(v^{\diamond\pm}, \theta^{\diamond\pm}) & = & (-\theta^{\diamond\pm}, 0) + g^\pm \end{array} \tag{5.107}$$

are all satisfied.

(ii) The identity $\theta^{\diamond-}(0) = \theta^{\diamond+}(0) = \vartheta^0 - w_*$ holds.

(iii) We have the inclusions

$$\begin{array}{rcl} \mathrm{ev}_0 v^{\diamond-} & \in & \widehat{P}_b(0) \oplus \widehat{Q}_b(0) \oplus \Gamma_b, \\ \mathrm{ev}_0 v^{\diamond+} & \in & \widehat{P}_b(0) \oplus \widehat{Q}_b(0) \oplus \Gamma_b. \end{array} \tag{5.108}$$

(iv) The gap between $v^{\diamond-}$ and $v^{\diamond+}$ at zero satisfies $\mathrm{ev}_0[v^{\diamond-} - v^{\diamond+}] \in \Gamma_b$.

(v) Upon writing $\widetilde{v}_\theta^- = D_1 s_R(\Theta_R^{\mathrm{fs}}(\vartheta^-, c, \epsilon), c, \epsilon)\theta^-$ and $\widetilde{v}_\theta^+ = D_1 s_L(\Theta_L^{\mathrm{fs}}(\vartheta^+, c, \epsilon), c, \epsilon)\theta^+$, the following boundary conditions are satisfied,

$$\begin{array}{rcl} \mathrm{ev}_{-\xi_0}[v^{\diamond-} - v^- - \widetilde{v}_\theta^-] & = & \phi_R, \\ \mathrm{ev}_{\xi_0}[v^{\diamond+} - v^+ - \widetilde{v}_\theta^+] & = & \phi_L. \end{array} \tag{5.109}$$

The element $h \in \mathcal{H}^\circ$ given above will be denoted by

$$h = L_2(\widetilde{p})(\vartheta^0, g, \phi_R, \phi_L). \tag{5.110}$$

Recalling the integer r appearing in (H1), there exists a constant $N_1 > 0$ such that for any integer $0 \leq \ell \leq r + 2$ and any $\zeta > \ell\delta_\epsilon N_1$, the map $\widetilde{p} \mapsto L_2(\widetilde{p})$ is C^ℓ-smooth when considered as a map

$$L_2 : \widetilde{\mathcal{D}}_p \to \mathcal{L}\big(\mathbb{R} \times \mathcal{H} \times X \times X, \mathcal{H}^\circ_\zeta\big), \tag{5.111}$$

with derivatives that can be bounded independently of $\xi_0 \geq \xi_*$. Finally, consider any $d \in \mathbb{K}_b^*$. Then the following identity holds for the gap at zero,

$$\begin{array}{rcl} \langle \mathrm{ev}_0 d, \mathrm{ev}_0[v^{\diamond-} - v^{\diamond+}]\rangle & = & \int_{-\xi_0}^0 d(\xi') g_1^{\diamond-}(\xi')\,\mathrm{d}\xi' + \int_0^{\xi_0} d(\xi') g_1^{\diamond+}(\xi')\,\mathrm{d}\xi' \\ & & + \int_{-\xi_0}^0 d(\xi')[w_* - \vartheta^0 - \frac{1}{c_*}\int_0^{\xi''} g_2^{\diamond-}(\xi'')\,\mathrm{d}\xi'']\,\mathrm{d}\xi' \\ & & + \int_0^{\xi_0} d(\xi')[w_* - \vartheta^0 - \frac{1}{c_*}\int_0^{\xi'} g_2^{\diamond+}(\xi'')\,\mathrm{d}\xi'']\,\mathrm{d}\xi' \\ & & - \langle \mathrm{ev}_{\xi_0} d, \mathrm{ev}_{\xi_0} v^{\diamond+}\rangle + \langle \mathrm{ev}_{-\xi_0} d, \mathrm{ev}_{-\xi_0} v^{\diamond-}\rangle. \end{array} \tag{5.112}$$

Proof. As in the proof of Lemma 5.10, we can use Lemma 5.3 to define linear operators

$$\begin{array}{rcl} [\Lambda_1^-(\widetilde{p})]^{-1} & : & BC_{-\eta_*}((-\infty,\xi_0],\mathbb{R}) \to BC_{-\eta_*}((-\infty,\xi_0+1],\mathbb{R}), \\ [\Lambda_1^+(\widetilde{p})]^{-1} & : & BC_{-\eta_*}([\xi_0,\infty),\mathbb{R}) \to BC_{-\eta_*}([\xi_0-1,\infty),\mathbb{R}) \end{array} \tag{5.113}$$

and Lemma 5.7 to define

$$\begin{array}{rcl} [\Lambda_1^{\diamond -}]^{-1} & : & BC_0([-\xi_0,0],\mathbb{R}) \to BC_0([-\xi_0-1,1],\mathbb{R}), \\ [\Lambda_1^{\diamond +}]^{-1} & : & BC_0([0,\xi_0],\mathbb{R}) \to BC_0([-1,\xi_0+1],\mathbb{R}), \end{array} \tag{5.114}$$

such that the choice $h_0 = (v_0^-, \theta^-, v_0^{\diamond -}, \theta^{\diamond -}, v_0^{\diamond +}, \theta^{\diamond +}, v_0^+, \theta^+)$ with

$$\begin{array}{rclrcl} \theta^-(\xi) & = & \frac{1}{c}\int_{-\infty}^{\xi} g_2^-(\xi')\,\mathrm{d}\xi', & v_0^- & = & [\Lambda_1^-(\widetilde{p})]^{-1} g_1^-, \\ \theta^{\diamond -}(\xi) & = & \vartheta^0 - w_* + \frac{1}{c_*}\int_0^{\xi} g_2^{\diamond -}(\xi')\,\mathrm{d}\xi', & v_0^{\diamond -} & = & [\Lambda_1^{\diamond -}]^{-1}[g_1^{\diamond -} - \theta^{\diamond -}], \\ \theta^{\diamond +}(\xi) & = & \vartheta^0 - w_* + \frac{1}{c_*}\int_0^{\xi} g_2^{\diamond +}(\xi')\,\mathrm{d}\xi', & v_0^{\diamond +} & = & [\Lambda_1^{\diamond +}]^{-1}[g_1^{\diamond +} - \theta^{\diamond +}], \\ \theta^+(\xi) & = & \frac{1}{c}\int_{\infty}^{\xi} g_2^+(\xi')\,\mathrm{d}\xi', & v_0^+ & = & [\Lambda_1^+(\widetilde{p})]^{-1} g_1^+, \end{array} \tag{5.115}$$

satisfies items (i) and (ii).

As before, we will need to modify $v_0^{\pm}$ and $v_0^{\diamond\pm}$ to ensure that the remaining properties (iii) - (v) are satisfied. In particular, we will choose $\psi^{B\diamond\pm} \in B_b(0)$, $\psi^{\widehat{P}\diamond -} \in \widehat{P}_b(0)$, $\psi^{\widehat{Q}\diamond +} \in \widehat{Q}_b(0)$, $\psi^{R\diamond -} \in Q^{\mathrm{fb}}_{R,w_*}$, $\psi^{P-} \in P^{\mathrm{fb}}_{R,w_*}$, $\psi^{S\diamond +} \in P^{\mathrm{fb}}_{L,w_*}$, $\psi^{Q+} \in Q^{\mathrm{fb}}_{L,w_*}$ and write

$$\begin{array}{rcl} v^- & = & v_0^- + E^{P-}(\widetilde{p})\psi^{P-} \\ v^{\diamond -} & = & v_0^{\diamond -} + E^{P\diamond -}(\psi^{B\diamond -} + \psi^{P\diamond -}) + E^{R\diamond -}\psi^{R\diamond -}, \\ v^{\diamond +} & = & v_0^{\diamond +} + E^{Q\diamond +}(\psi^{B\diamond +} + \psi^{Q\diamond +}) + E^{S\diamond +}\psi^{S\diamond +}, \\ v^+ & = & v_0^+ + E^{Q+}(\widetilde{p})\psi^{Q+}, \end{array} \tag{5.116}$$

in which $E^{R\diamond -}$, $E^{P\diamond -}$, $E^{Q\diamond +}$ and $E^{S\diamond +}$ are relabelled versions of those defined in Lemma 5.9, while E^{P-} and E^{Q+} are constructed from the operators E^{fb}_{R,w_*} and E^{fb}_{L,w_*} appearing in Lemma 5.5. To complete the proof, a linear system analogous to (5.73)-(5.75) can be constructed and solved. ∎

Proof of Proposition 3.6 Similarly as before, one can choose the constants $\delta^0_{\vartheta} > 0$, $\delta_{\vartheta} > 0$, $\delta_c > 0$, $\delta_{\epsilon} > 0$ and $\delta_v > 0$ in such a way that the fixed point problem

$$h = L_2(\widetilde{p})\big(\vartheta^0, \mathcal{N}(h,p), \Phi_L(h,p), \Phi_R(h,p)\big) \tag{5.117}$$

has a solution $h = h^*(p)$ for each $p = (\vartheta^0, \widetilde{p}) \in \mathcal{D}_p$ that is unique in the set $\mathcal{B}^{\circ}_{\delta_v}$. Here the boundary operators Φ_L and Φ_R are defined in a fashion that is analogous to (5.77). In addition, one can define $\vartheta^{\pm}(\vartheta^0, c, \epsilon)$ in such a way that the function w_b defined in (5.98) is continuous.

Let us now fix $d \in \mathbb{K}_b^*$ and consider the map

$$M : p \mapsto \langle \mathrm{ev}_0 d, \mathrm{ev}_0 \pi_{v^{\diamond -}} h^*(p) - \mathrm{ev}_0 \pi_{v^{\diamond +}} h^*(p) \rangle. \tag{5.118}$$

Using (5.112) we may write

$$\begin{array}{rcl} M(p) & = & \int_{-\xi_0}^0 d(\xi')[w_* - \vartheta^0 + \mathcal{N}_1^{\diamond -}(h^*(p),p)(\xi')]\,\mathrm{d}\xi' + \int_0^{\xi_0} d(\xi')[w_* - \vartheta^0 + \mathcal{N}_1^{\diamond +}(h^*(p),p)(\xi')]\,\mathrm{d}\xi' \\ & & -\frac{1}{c_*}\int_{-\xi_0}^0 d(\xi')\int_0^{\xi'} \mathcal{N}_2^{\diamond -}(h^*(p),p)(\xi'')\,\mathrm{d}\xi''\,\mathrm{d}\xi' \\ & & -\frac{1}{c_*}\int_0^{\xi_0} d(\xi')\int_0^{\xi'} \mathcal{N}_2^{\diamond +}(h^*(p),p)(\xi'')d\xi''\,\mathrm{d}\xi' \\ & & -\langle \mathrm{ev}_{\xi_0} d, \mathrm{ev}_{\xi_0}\pi_{v^{\diamond +}} h^*(p)\rangle + \langle \mathrm{ev}_{-\xi_0} d, \mathrm{ev}_{-\xi_0}\pi_{v^{\diamond -}} h^*(p)\rangle. \end{array} \tag{5.119}$$

Let us now write $\widetilde{M} : [w_* - \delta_{\vartheta^0}, w_* + \delta_{\vartheta^0}] \times [c_* - \delta_c, c_* + \delta_c] \times [0, \delta_{\epsilon}] \to \mathbb{R}$ for the operator

$$\widetilde{M}(\vartheta^0, c, \epsilon) = M(\vartheta^0, \vartheta^-(\vartheta^0, c, \epsilon), \vartheta^+(\vartheta^0, c, \epsilon), c, \epsilon). \tag{5.120}$$

A short computation now yields

$$\begin{array}{lcl} D_{\vartheta^0}\widetilde{M}(w_*, c_*, 0) & = & -\int_{-\xi_0}^{\xi_0} d(\xi')\,\mathrm{d}\xi' + O(\mathrm{e}^{-\eta_*\xi_0}), \\ D_c\widetilde{M}(w_*, c_*, 0) & = & -\int_{-\xi_0}^{\xi_0} d(\xi')q_b'(\xi')\,\mathrm{d}\xi' + O(\mathrm{e}^{-\eta_*\xi_0}). \end{array} \tag{5.121}$$

The remaining part of the proof is identical to that of Proposition 3.4. ∎

6 The Exchange Lemma

In this section we set out to establish Proposition 3.8. First of all, we note that we will write $\widetilde{T}$ throughout this section to denote the variable T that appears in the statement of this proposition. After fixing a suitable large constant $\xi_0 > 0$, we will use the variable T here to denote the quantity $T = \frac{1}{2}\widetilde{T} - \xi_0$.

We will need to use the T-dependent families of function spaces

$$\begin{array}{lcl} BC^{\oplus}_{f,\alpha} & = & \{(v,\theta) \in C([\xi_0 - 1, \xi_0 + T + 1], \mathbb{R}) \times C([\xi_0, \xi_0 + T], \mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC^{\oplus}_{f,\alpha}} := \sup_{\xi_0 - 1 \le \xi \le \xi_0 + T + 1} \mathrm{e}^{-\alpha|\xi - \xi_0|}\,|v(\xi)| \\ & & \quad\quad + \sup_{\xi_0 \le \xi \le \xi_0 + T} \mathrm{e}^{-\alpha|\xi - \xi_0|}\,|\theta(\xi)| < \infty\}, \\ BC^{\ominus}_{b,\alpha} & = & \{(v,\theta) \in C([-\xi_0 - T - 1, -\xi_0 + 1], \mathbb{R}) \times C([-\xi_0 - T, -\xi_0], \mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC^{\ominus}_{b,\alpha}} := \sup_{-\xi_0 - T - 1 \le \xi \le -\xi_0 + 1} \mathrm{e}^{-\alpha|\xi + \xi_0|}\,|v(\xi)| \\ & & \quad\quad + \sup_{-\xi_0 - T \le \xi \le -\xi_0} \mathrm{e}^{-\alpha|\xi + \xi_0|}\,|\theta(\xi)| < \infty\}, \end{array} \tag{6.1}$$

together with the families

$$\begin{array}{lcl} BC^{+}_{f,\alpha} & = & \{(v,\theta) \in C([\xi_0, \xi_0 + T], \mathbb{R}) \times C([\xi_0, \xi_0 + T], \mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC^{+}_{f,\alpha}} := \sup_{\xi_0 \le \xi \le \xi_0 + T} \mathrm{e}^{-\alpha|\xi - \xi_0|}(|v(\xi)| + |\theta(\xi)|) < \infty\}, \\ BC^{-}_{b,\alpha} & = & \{(v,\theta) \in C([-\xi_0 - T, -\xi_0], \mathbb{R}) \times C([-\xi_0 - T, -\xi_0], \mathbb{R}) \text{ for which} \\ & & \quad \|(v,\theta)\|_{BC^{-}_{b,\alpha}} := \sup_{-\xi_0 - T \le \xi \le -\xi_0} \mathrm{e}^{-\alpha|\xi + \xi_0|}(|v(\xi)| + |\theta(\xi)|) < \infty\}, \end{array} \tag{6.2}$$

that are both parametrized by $\alpha \in \mathbb{R}$. In addition, we will reuse some of the function spaces introduced in §5. In particular, we recall the function spaces defined in (5.48)-(5.49) and write $BC^{\ominus}_{f,\alpha} = BC^{\ominus}_{\alpha}$, $BC^{-}_{f,\alpha} = BC^{-}_{\alpha}$, $BC^{\odot}_{f} = BC^{\odot}$ and $BC^{\diamond}_{f} = BC^{\diamond}$. Similarly, we recall the function spaces defined in (5.96)-(5.97) and write $BC^{\odot-}_{b} = BC^{\odot-}$, $BC^{\odot+}_{b} = BC^{\odot+}$, $BC^{\oplus}_{b,\alpha} = BC^{\oplus}_{\alpha}$, $BC^{\diamond-}_{b} = BC^{\diamond-}$, $BC^{\diamond+}_{b} = BC^{\diamond+}$ and $BC^{+}_{b,\alpha} = BC^{+}_{\alpha}$.

We also introduce the family of composite function spaces

$$\mathcal{H}^{\diamond}_{\zeta} = BC^{\ominus}_{f,-\eta_*} \times BC^{\odot}_{f} \times BC^{\oplus}_{f,\eta_*} \times BC^{\ominus}_{b,\eta_*} \times BC^{\odot-}_{b} \times BC^{\odot+}_{b} \times BC^{\oplus}_{b,-\eta_*+\zeta}, \tag{6.3}$$

together with the family

$$\mathcal{H}_{\zeta} = BC^{-}_{f,-\eta_*} \times BC^{\diamond}_{f} \times BC^{+}_{f,\eta_*} \times BC^{-}_{b,\eta_*} \times BC^{\diamond-}_{b} \times BC^{\diamond+}_{b} \times BC^{+}_{b,-\eta_*+\zeta}, \tag{6.4}$$

both parametrized by $\zeta \ge 0$. It is important to note that we are using positive weights in the function spaces that describe the passage near $\mathcal{M}_R$. This will allow us to establish the exponential estimates in (3.13).

We recall the slow time $T^{\mathrm{sl}} = \epsilon T$ and the set $\Omega = \Omega(\delta_c, \delta_{\mathrm{sl}}, T_*)$ consisting of triplets $\omega = (c, T^{\mathrm{sl}}, T)$ that was defined in (3.11). For any $\omega \in \Omega$, our goal in this section is to find $\vartheta^0 \in [w_{\min}, w_{\max}]$ and

$$h = (v_f^-, \theta_f^-, v_f^{\diamond}, \theta_f^{\diamond}, v_f^+, \theta_f^+, v_b^-, \theta_b^-, v_b^{\diamond-}, \theta_b^{\diamond-}, v_b^{\diamond+}, \theta_b^{\diamond+}, v_b^+, \theta_b^+) \in \mathcal{H}^{\diamond} \tag{6.5}$$

in such a way that the choice $\widetilde{T} = 2\xi_0 + 2T$ with

$$w(\xi) = \begin{cases} w_f(c,\epsilon)(\xi) + \theta_f^-(\xi) & \text{for } \xi \le 0, \\ w_f(c,\epsilon)(\xi) + \theta_f^\diamond(\xi) & \text{for } 0 \le \xi \le \xi_0, \\ w_f(c,\epsilon)(\xi) + \theta_f^+(\xi) & \text{for } \xi_0 \le \xi \le \xi_0 + T, \\ w_b(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + \theta_b^-(\xi - \widetilde{T}) & \text{for } \xi_0 + T \le \xi \le \xi_0 + 2T, \\ w_b(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + \theta_b^{\diamond -}(\xi - \widetilde{T}) & \text{for } \xi_0 + 2T \le \xi \le \widetilde{T}, \\ w_b(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + \theta_b^{\diamond +}(\xi - \widetilde{T}) & \text{for } \widetilde{T} \le \xi \le \widetilde{T} + \xi_0, \\ w_b(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + \theta_b^+(\xi - \widetilde{T}) & \text{for } \xi \ge \widetilde{T} + \xi_0, \end{cases} \tag{6.6}$$

in combination with $u_f = u_f^-(c,\epsilon) + v_f^-$ and

$$\begin{array}{rcl} u_b(\xi) & = & \begin{cases} u_b^+(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + v_b^{\diamond +}(\xi - \widetilde{T}) & \text{for } \widetilde{T} - 1 \le \xi \le \widetilde{T} + \xi_0, \\ u_b^+(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + v_b^+(\xi - \widetilde{T}) & \text{for } \xi \ge \widetilde{T} + \xi_0, \end{cases} \\ u_{\mathrm{xc}}(\xi) & = & \begin{cases} u_f^+(c,\epsilon)(\xi) + v_f^\diamond(\xi) & \text{for } -1 \le \xi \le \xi_0, \\ u_f^+(c,\epsilon)(\xi) + v_f^+(\xi) & \text{for } \xi_0 \le \xi \le \xi_0 + T, \\ u_b^-(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + v_b^-(\xi - \widetilde{T}) & \text{for } \xi_0 + T \le \xi \le \xi_0 + 2T, \\ u_b^-(\vartheta^0,c,\epsilon)(\xi - \widetilde{T}) + v_b^{\diamond -}(\xi - \widetilde{T}) & \text{for } \xi_0 + 2T \le \xi \le \widetilde{T} \end{cases} \end{array} \tag{6.7}$$

satisfies the conditions of Proposition 3.8.

We introduce the parameter vector $\widetilde{p} = (\vartheta^0, c, T^{\mathrm{sl}})$, which we will take from the space

$$\widetilde{\mathcal{D}}_p(\delta_{\vartheta^0}, \delta_c, \delta_{\mathrm{sl}}) = [w_* - \delta_{\vartheta^0}, w_* + \delta_{\vartheta^0}] \times [c_* - \delta_c, c_* + \delta_c] \times [T_*^{\mathrm{sl}} - \delta_{\mathrm{sl}}, T_*^{\mathrm{sl}} + \delta_{\mathrm{sl}}] \tag{6.8}$$

We also use the augmented parameter vector $p = (\widetilde{p}, T)$, which we take from the space

$$\mathcal{D}_p(\delta_{\vartheta^0}, \delta_c, \delta_{\mathrm{sl}}, T_*) = \widetilde{\mathcal{D}}_p(\delta_{\vartheta^0}, \delta_c, \delta_{\mathrm{sl}}) \times [T_*, \infty). \tag{6.9}$$

Substituting the ansatz (6.6)-(6.7) into (3.1), we arrive at a system of nonlinear equations that we write as

$$\begin{array}{rcl} \Lambda_f^-(c,\epsilon)(v_f^-, \theta_f^-) & = & (-\theta_f^-, 0) + \mathcal{M}_f^-(v_f^-, \theta_f^-, c, \epsilon), \\ \Lambda_f^\diamond(c,\epsilon)(v_f^\diamond, \theta_f^\diamond) & = & (-\theta_f^\diamond, 0) + \mathcal{M}_f^\diamond(v_f^\diamond, \theta_f^\diamond, c, \epsilon), \\ \Lambda_f^+(c,\epsilon)(v_f^+, \theta_f^+) & = & (-\theta_f^+, 0) + \mathcal{M}_f^+(v_f^+, \theta_f^+, c, \epsilon), \\ \Lambda_b^-(\vartheta^0,c,\epsilon)(v_b^-, \theta_b^-) & = & (-\theta_b^-, 0) + \mathcal{M}_b^-(v_b^-, \theta_b^-, \vartheta^0, c, \epsilon), \\ \Lambda_b^{\diamond -}(\vartheta^0,c,\epsilon)(v_b^{\diamond -}, \theta_b^{\diamond -}) & = & (-\theta_b^{\diamond -}, 0) + \mathcal{M}_b^{\diamond -}(v_b^{\diamond -}, \theta_b^{\diamond -}, \vartheta^0, c, \epsilon), \\ \Lambda_f^{\diamond +}(\vartheta^0,c,\epsilon)(v_b^{\diamond +}, \theta_b^{\diamond +}) & = & (-\theta_b^{\diamond +}, 0) + \mathcal{M}_b^{\diamond +}(v_b^{\diamond +}, \theta_b^{\diamond +}, \vartheta^0, c, \epsilon), \\ \Lambda_f^+(\vartheta^0,c,\epsilon)(v_b^+, \theta_b^+) & = & (-\theta_b^+, 0) + \mathcal{M}_b^+(v_b^+, \theta_b^+, \vartheta^0, c, \epsilon). \end{array} \tag{6.10}$$

Here we have

$$\Lambda_f^\#(c,\epsilon)(v_f^\#, \theta_f^\#) = (\Lambda_{f,1}^\#(c,\epsilon)v_f^\#, cD\theta_f^\#)$$

for $\# = -, \diamond, +$ and

$$\Lambda_b^\#(\vartheta^0,c,\epsilon)(v_b^\#, \theta_b^\#) = (\Lambda_{b,1}^\#(\vartheta^0,c,\epsilon)v_f^\#, cD\theta_b^\#)$$

for $\# = -, \diamond-, \diamond+, +$, in which

$$\begin{array}{rcl} [\Lambda_{f,1}^-(c,\epsilon)v](\xi) & = & cv'(\xi) - L\big(u_f^-(c,\epsilon)(\xi)\big)\mathrm{ev}_\xi v, \\ [\Lambda_{f,1}^\#(c,\epsilon)v](\xi) & = & cv'(\xi) - L\big(u_f^+(c,\epsilon)(\xi)\big)\mathrm{ev}_\xi v, \text{ for } \# = \diamond, +, \\ [\Lambda_{b,1}^\#(\vartheta^0,c,\epsilon)v](\xi) & = & cv'(\xi) - L\big(u_b^\pm(\vartheta^0,c,\epsilon)(\xi)\big)\mathrm{ev}_\xi v, \text{ for } \# = \pm, \diamond\pm\,. \end{array} \tag{6.11}$$

The nonlinearities can be written as $\mathcal{M}^{\#}_{f,2}(v,\theta) = \epsilon[v-\theta]$ for $\# = -, \diamond, +$ and $\mathcal{M}^{\#}_{b,2}(v,\theta) = \epsilon[v-\theta]$ for $\# = \pm, \diamond\pm$, together with

$$\begin{array}{lcl} \mathcal{M}^{-}_{f,1}(v,\theta,c,\epsilon)(\xi) & = & g\big(u_f^-(c,\epsilon)(\xi)+v(\xi)\big) - g'\big(u_f^-(c,\epsilon)(\xi)\big)v(\xi) \\ & & \quad -g\big(u_f^-(c,\epsilon)(\xi)\big), \\ \mathcal{M}^{\#}_{f,1}(v,\theta,c,\epsilon)(\xi) & = & g\big(u_f^+(c,\epsilon)(\xi)+v(\xi)\big) - g'\big(u_f^+(c,\epsilon)(\xi)\big)v(\xi) \\ & & \quad -g\big(u_f^+(c,\epsilon)(\xi)\big) \text{ for } \# = \diamond, +, \\ \mathcal{M}^{\#}_{b,1}(v,\theta,\vartheta^0,c,\epsilon)(\xi) & = & g\big(u_b^\pm(\vartheta^0,c,\epsilon)(\xi)+v(\xi)\big) - g'\big(u_b^\pm(\vartheta^0,c,\epsilon)(\xi)\big)v(\xi) \\ & & \quad -g\big(u_b^\pm(\vartheta^0,c,\epsilon)(\xi)\big) \text{ for } \# = \pm, \diamond\pm\,. \end{array} \tag{6.12}$$

We can combine these nonlinearities into a single entity

$$\mathcal{M} : \mathcal{H}^\circ \times \mathcal{D}_p \to \mathcal{H}, \tag{6.13}$$

with the warning that $\mathcal{H}^\circ$ and $\mathcal{H}$ both depend on T. For any $\delta > 0$, we write

$$\mathcal{B}^\circ_\delta = \{h \in \mathcal{H}^\circ \mid \|h\|_{\mathcal{H}^\circ} \le \delta\}. \tag{6.14}$$

We now pick a constant $\delta_v > 0$. For any $p \in \mathcal{D}_p$ and $h, h_1, h_2 \in \mathcal{B}^\circ_{\delta_v e^{-\eta_* T}}$, we have the bounds

$$\begin{array}{lcl} \|\mathcal{M}(h,p)\|_{\mathcal{H}} & \le_* & \epsilon\xi_0\delta_v e^{-\eta_* T} + \delta_v^2 e^{-\eta_* T}, \\ \|\mathcal{M}(h_1,p) - \mathcal{M}(h_2,p)\|_{\mathcal{H}} & \le_* & [\epsilon\xi_0 + \delta_v]\,\|h_1 - h_2\|_{\mathcal{H}^\circ}\,. \end{array} \tag{6.15}$$

In addition, let us pick an arbitrary small constant $\gamma > 0$ and recall the constant r appearing in (H1). By picking T_* to be sufficiently large, we can ensure that $\epsilon = T^{\mathrm{sl}}/T < \gamma$ for all $p = (\vartheta^0, c, T^{\mathrm{sl}}, T) \in \mathcal{D}_p$. For any integer $0 \le \ell \le r$ and $T \ge T_*$, we thus see that $(h, \widetilde{p}) \mapsto \mathcal{M}(h, \widetilde{p}, T)$ is C^ℓ-smooth when considered as a map from $\mathcal{H}^\circ \times \widetilde{\mathcal{D}}_p$ into $\mathcal{H}_\zeta$ for any $\zeta > \ell\gamma$. In addition, for any $h \in \mathcal{B}^\circ_{\delta_v e^{-\eta_* T}}$, $p \in \mathcal{D}_p$ and $\zeta > \gamma$, we have the bound

$$\|D_{\widetilde{p}}\mathcal{M}(h,p)\|_{\mathcal{H}_\zeta} \le_* e^{\gamma T}\delta_v^2 e^{-\eta_* T}. \tag{6.16}$$

The first step in the proof of Proposition 3.8 is to consider the linearized equations that the quasi-solutions must solve, imposing the constraints in Definition 3.7(iv)-(v) concerning the discontinuities that arise when passing from u_f to u_{xc} and from u_{xc} to u_b.

Lemma 6.1 *Fix a sufficiently large constant T_* and sufficiently small constants $\delta_{\vartheta^0} > 0$, $\delta_c > 0$ and $\delta_{\mathrm{sl}} > 0$. Choose any $T \ge T_*$. Then for every*

$$g = (g_f^-, g_f^\diamond, g_f^+, g_b^-, g_b^{\diamond-}, g_b^{\diamond+}, g_b^+) \in \mathcal{H}, \tag{6.17}$$

any boundary condition $\phi_{\mathrm{hw}} \in X$ and any

$$\widetilde{p} = (\vartheta^0, c, T^{\mathrm{sl}}) \in \widetilde{\mathcal{D}}_p = \widetilde{\mathcal{D}}_p(\delta_{\vartheta^0}, \delta_c, \delta_{\mathrm{sl}}), \tag{6.18}$$

there exists a unique

$$h = (v_f^-, \theta_f^-, v_f^\diamond, \theta_f^\diamond, v_f^+, \theta_f^+, v_b^-, \theta_b^-, v_b^{\diamond-}, \theta_b^{\diamond-}, v_b^{\diamond+}, \theta_b^{\diamond+}, v_b^+, \theta_b^+) \in \mathcal{H}^\circ \tag{6.19}$$

that satisfies the following properties.

(i) The linear system

$$\Lambda_f^{\#}(p)(v_f^{\#}, \theta_f^{\#}) = (-\theta_f^{\#}, 0) + g_f^{\#} \tag{6.20}$$

is satisfied for $\# = -, \diamond, +$. In addition, the linear system

$$\Lambda_b^{\#}(p)(v_b^{\#}, \theta_b^{\#}) = (-\theta_b^{\#}, 0) + g_b^{\#} \tag{6.21}$$

is satisfied for $\# = -, \diamond-, \diamond+, +$.

(ii) *We have the continuity conditions* $\theta_f^\diamond(0) = \theta_f^-(0)$, $\theta_f^\diamond(\xi_0) = \theta_f^+(\xi_0)$, *together with* $\theta_b^{\diamond-}(-\xi_0) = \theta_b^-(-\xi_0)$, $\theta_b^{\diamond+}(0) = \theta_b^{\diamond-}(0)$ *and* $\theta_b^+(\xi_0) = \theta_b^{\diamond+}(\xi_0)$.

(iii) *The following continuity conditions all hold,*

$$\begin{array}{rcl} \mathrm{ev}_{\xi_0} v_f^\diamond & = & \mathrm{ev}_{\xi_0} v_f^+, \\ \mathrm{ev}_{-\xi_0} v_b^{\diamond-} & = & \mathrm{ev}_{-\xi_0} v_b^-, \\ \mathrm{ev}_{\xi_0} v_b^{\diamond+} & = & \mathrm{ev}_{\xi_0} v_b^+. \end{array} \tag{6.22}$$

(iv) *We have the inclusions*

$$\begin{array}{rcl} \mathrm{ev}_0 v_f^-, \mathrm{ev}_0 v_f^\diamond & \in & \widehat{P}_f(0) \oplus \widehat{Q}_f(0) \oplus \Gamma_f, \\ \mathrm{ev}_0 v_b^{\diamond-}, \mathrm{ev}_0 v_b^{\diamond+} & \in & \widehat{P}_b(0) \oplus \widehat{Q}_b(0) \oplus \Gamma_b, \end{array} \tag{6.23}$$

(v) *The gap between* v_f^- *and* $v_f^\diamond$ *at zero satisfies* $\mathrm{ev}_0[v_f^- - v_f^\diamond] \in \Gamma_f$. *In addition, the corresponding gap between* $v^{\diamond-}$ *and* $v^{\diamond+}$ *satisfies* $\mathrm{ev}_0[v_b^{\diamond-} - v_b^{\diamond+}] \in \Gamma_b$.

(vi) *The following boundary condition holds,*

$$\mathrm{ev}_{-\xi_0 - T} v_b^- - \mathrm{ev}_{\xi_0 + T} v_f^+ \quad = \quad \phi_{\mathrm{hw}}. \tag{6.24}$$

The element $h \in \mathcal{H}^\circ$ *described above will be denoted by*

$$h = L_4(\widetilde{p}, T)(g, \phi_{\mathrm{hw}}), \tag{6.25}$$

and we have the estimate

$$\|h\|_{\mathcal{H}^\circ} \leq_* \|g\|_{\mathcal{H}} + \mathrm{e}^{-\eta_* T} \|\phi_{\mathrm{hw}}\|. \tag{6.26}$$

Recalling the integer r *appearing in (H1), there exists a small constant* γ *that tends to zero as* $T_* \to \infty$, *such that for any integer* $0 \leq \ell \leq r$ *and any* $\zeta > \ell\gamma$, *the map*

$$\widetilde{p} \mapsto L_4(\widetilde{p}, T) \in \mathcal{L}(\mathcal{H} \times X, \mathcal{H}^\circ_\zeta) \tag{6.27}$$

is C^ℓ*-smooth with norm*

$$\left\|D^\ell_{\widetilde{p}} L_4(p)\right\| \leq_* \mathrm{e}^{\ell\gamma T}[\|g\|_{\mathcal{H}} + \mathrm{e}^{-\eta_* T} \|\phi_{\mathrm{hw}}\|]. \tag{6.28}$$

Proof. First of all, let us write $p = (\widetilde{p}, T)$ and define

$$\begin{array}{rclrcl} \theta_f^-(\xi) & = & \frac{1}{c}\int_{-\infty}^{\xi} g_{f,2}^-(\xi')\,\mathrm{d}\xi' & v_{f,0}^- & = & [\Lambda_{f,1}^-(p)]^{-1}[g_{f,1}^- - \theta_f^-], \\ \theta_f^\diamond(\xi) & = & \theta_f^-(0) + \frac{1}{c}\int_0^{\xi} g_{f,2}^\diamond(\xi')\,\mathrm{d}\xi' & v_{f,0}^\diamond & = & [\Lambda_{f,1}^\diamond(p)]^{-1}[g_{f,1}^\diamond - \theta_f^\diamond], \\ \theta_f^+(\xi) & = & \theta_f^\diamond(\xi_0) + \frac{1}{c}\int_{\xi_0}^{\xi} g_{f,2}^+(\xi')\,\mathrm{d}\xi' & v_{f,0}^+ & = & [\Lambda_{f,1}^+(p)]^{-1}[g_{f,1}^+ - \theta_f^+], \\ \theta_b^+(\xi) & = & \frac{1}{c}\int_{-\infty}^{\xi} g_{b,2}^+(\xi')\,\mathrm{d}\xi' & v_{b,0}^+ & = & [\Lambda_{b,1}^+(p)]^{-1}[g_{b,1}^+ - \theta_b^+], \\ \theta_b^{\diamond+}(\xi) & = & \theta_b^+(\xi_0) + \frac{1}{c}\int_{\xi_0}^{\xi} g_{b,2}^{\diamond+}(\xi')\,\mathrm{d}\xi' & v_{b,0}^{\diamond+} & = & [\Lambda_{b,1}^{\diamond+}(p)]^{-1}[g_{b,1}^{\diamond+} - \theta_b^{\diamond+}], \\ \theta_b^{\diamond-}(\xi) & = & \theta_b^{\diamond+}(0) + \frac{1}{c}\int_0^{\xi} g_{b,2}^{\diamond-}(\xi')\,\mathrm{d}\xi' & v_{b,0}^{\diamond-} & = & [\Lambda_{b,1}^{\diamond-}(p)]^{-1}[g_{b,1}^{\diamond-} - \theta_b^{\diamond-}], \\ \theta_b^-(\xi) & = & \theta_b^{\diamond-}(-\xi_0) + \frac{1}{c}\int_{-\xi_0}^{\xi} g_{b,2}^-(\xi')\,\mathrm{d}\xi' & v_{b,0}^- & = & [\Lambda_{b,1}^-(p)]^{-1}[g_{b,1}^- - \theta_b^-], \end{array} \tag{6.29}$$

in which we have used Lemmas 4.1 and 5.7 to construct the various inverses used above.

We recall the slow time T_*^{sl} defined in (3.10) and write

$$P_{\mathrm{hw}} = P^{\mathrm{fb}}_{R, w_{\mathrm{hw}}}, \qquad Q_{\mathrm{hw}} = Q^{\mathrm{fb}}_{R, w_{\mathrm{hw}}}, \tag{6.30}$$

in which the half-way point w_{hw} is defined as $w_{\mathrm{hw}} = \Theta^{\mathrm{sl}}_R(0, c_*, 0)(\frac{1}{2}T^{\mathrm{sl}}_*)$. We now choose variables $\psi_f^{B-} \in B_f(0)$, $\psi_f^{\widehat{P}-} \in \widehat{P}_f(0)$, $\psi_f^{B\diamond} \in B_f(0)$, $\psi_f^{\widehat{Q}\diamond} \in \widehat{Q}_f(0)$, $\psi_f^{S\diamond} \in P^{\mathrm{fb}}_{R,0}$, $\psi_f^{Q+} \in Q^{\mathrm{fb}}_{R,0}$ and $\psi_f^{S+} \in P_{\mathrm{hw}}$, together with $\psi_b^{Q-} \in Q_{\mathrm{hw}}$, $\psi_b^{P-} \in P^{\mathrm{fb}}_{R,w_*}$, $\psi_b^{R\diamond-} \in Q^{\mathrm{fb}}_{R,w_*}$, $\psi_b^{B\diamond-} \in B_b(0)$, $\psi_b^{\widehat{P}\diamond-} \in \widehat{P}_b(0)$, $\psi_b^{B\diamond+} \in B_b(0)$, $\psi_b^{\widehat{Q}\diamond+} \in \widehat{Q}_b(0)$, $\psi_b^{S\diamond+} \in P^{\mathrm{fb}}_{L,w_*}$, $\psi_b^{Q+} \in Q^{\mathrm{fb}}_{L,w_*}$ and write

$$\begin{array}{lcl} v_f^- & = & v_{f,0}^- + E_f^{P-}(p)(\psi_f^{B-} + \psi_f^{\widehat{P}-}), \\ v_f^\diamond & = & v_{f,0}^\diamond + E_f^{Q\diamond}(p)(\psi_f^{B\diamond} + \psi_f^{\widehat{Q}\diamond}) + E_f^{S\diamond}(p)\psi_f^{S\diamond}, \\ v_f^+ & = & v_{f,0}^+ + E_f^{Q+}(p)\psi_f^{Q+} + E_f^{S+}(p)\psi_f^{S+}, \\ v_b^- & = & v_{b,0}^- + E_b^{P-}(p)\psi_B^{P-} + E_b^{R-}(p)\psi_b^{R-}, \\ v_b^{\diamond-} & = & v_{b,0}^{\diamond-} + E_b^{R\diamond-}(p)\psi_b^{R\diamond-} + E_b^{P\diamond-}(p)(\psi_b^{B\diamond-} + \psi_b^{\widehat{P}\diamond-}), \\ v_b^{\diamond+} & = & v_{b,0}^{\diamond+} + E_b^{S\diamond+}(p)\psi_b^{S\diamond+} + E_b^{Q\diamond+}(p)(\psi_b^{B\diamond+} + \psi_b^{\widehat{Q}\diamond+}), \\ v_b^+ & = & v_{b,0}^+ + E_b^{Q+}(p)\psi_b^{Q+}, \end{array} \tag{6.31}$$

in which the various extension operators $E^{\#}_{f,b}$ can be constructed from Lemmas 5.5 and 5.9 much as before. In view of the desired exponential estimate (6.26), it will turn out to be fruitful to choose a constant $\delta_\psi > 0$ and work with the rescaled variables

$$\begin{array}{lcl} \psi_f^{S+} & = & \delta_\psi e^{\eta_* T}\widetilde{\psi}_f^{S+}, \\ \psi_b^{R-} & = & \delta_\psi e^{\eta_* T}\widetilde{\psi}_b^{R-}. \end{array} \tag{6.32}$$

In order to satisfy the boundary condition (vi), we must have

$$\begin{array}{lcl} -\phi^P_{\mathrm{hw}} & = & \delta_\psi e^{\eta_* T}\widetilde{\psi}_f^{S+} - \delta_\psi e^{\eta_* T}\Pi_{P_{\mathrm{hw}}}\mathrm{ev}_{-\xi_0 - T}E_b^{R-}(p)\widetilde{\psi}_b^{R-} \\ & & +\Pi_{P_{\mathrm{hw}}}\mathrm{ev}_{\xi_0+T}E_f^{Q+}(p)\psi_f^{Q+} - \Pi_{P_{\mathrm{hw}}}\mathrm{ev}_{-\xi_0-T}E_b^{P-}(p)\psi_b^{P-}, \\ \phi^Q_{\mathrm{hw}} & = & \delta_\psi e^{\eta_* T}\widetilde{\psi}_b^{R-} - \delta_\psi e^{\eta_* T}\Pi_{Q_{\mathrm{hw}}}\mathrm{ev}_{\xi_0+T}E_f^{S+}(p)\widetilde{\psi}_f^{S+} \\ & & +\Pi_{Q_{\mathrm{hw}}}\mathrm{ev}_{-\xi_0-T}E_b^{P-}\psi_b^{P-} - \Pi_{Q_{\mathrm{hw}}}\mathrm{ev}_{\xi_0+T}E_f^{Q+}\psi_f^{Q+}, \end{array} \tag{6.33}$$

in which

$$\begin{array}{lcl} \phi^P_{\mathrm{hw}} & = & \Pi_{P_{\mathrm{hw}}}[\phi_{\mathrm{hw}} - \mathrm{ev}_{-\xi_0-T}v_{b,0}^- + \mathrm{ev}_{\xi_0+T}v_{f,0}^+], \\ \phi^Q_{\mathrm{hw}} & = & \Pi_{Q_{\mathrm{hw}}}[\phi_{\mathrm{hw}} - \mathrm{ev}_{-\xi_0-T}v_{b,0}^- + \mathrm{ev}_{\xi_0+T}v_{f,0}^+]. \end{array} \tag{6.34}$$

The inclusions in (iv) yield the conditions

$$\begin{array}{lcl} -\Pi_{B_f(0)}\mathrm{ev}_0 v_{f,0}^- & = & \psi_f^{B-}, \\ -\Pi_{B_f(0)}\mathrm{ev}_0 v_{f,0}^\diamond & = & \psi_f^{B\diamond} + \Pi_{B_f(0)}\mathrm{ev}_0 E_f^{S\diamond}(p)\psi_f^{S\diamond}, \\ -\Pi_{B_b(0)}\mathrm{ev}_0 v_{b,0}^{\diamond-} & = & \psi_b^{B\diamond-} + \Pi_{B_b(0)}\mathrm{ev}_0 E_b^{R\diamond-}(p)\psi_b^{R\diamond-}, \\ -\Pi_{B_b(0)}\mathrm{ev}_0 v_{b,0}^{\diamond+} & = & \psi_b^{B\diamond+} + \Pi_{B_b(0)}\mathrm{ev}_0 E_b^{S\diamond+}(p)\psi_b^{S\diamond+}, \end{array} \tag{6.35}$$

while the jump condition (v) yields

$$\begin{array}{lcl} -\Pi_{\widehat{P}_f(0)}\mathrm{ev}_0[v_{f,0}^- - v_{f,0}^\diamond] & = & \psi_f^{\widehat{P}-} - \Pi_{\widehat{P}_f(0)}\mathrm{ev}_0[E_f^{Q\diamond}(p)(\psi_f^{B\diamond} + \psi_f^{\widehat{Q}\diamond}) + E_f^{S\diamond}(p)\psi_f^{S\diamond}], \\ \Pi_{\widehat{Q}_f(0)}\mathrm{ev}_0[v_{f,0}^- - v_{f,0}^\diamond] & = & \psi_f^{\widehat{Q}\diamond} - \Pi_{\widehat{Q}_f(0)}\mathrm{ev}_0[E_f^{P-}(p)(\psi_f^{B-} + \psi_f^{\widehat{P}-}) - E_f^{S\diamond}(p)\psi_f^{S\diamond}], \\ -\Pi_{\widehat{P}_b(0)}\mathrm{ev}_0[v_{b,0}^{\diamond-} - v_{b,0}^{\diamond+}] & = & \psi_b^{\widehat{P}\diamond-} - \Pi_{\widehat{P}_b(0)}\mathrm{ev}_0\big[E_b^{Q\diamond+}(p)(\psi_b^{B\diamond+} + \psi_b^{\widehat{Q}\diamond+}) \\ & & \quad +E_b^{S\diamond+}(p)\psi_b^{S\diamond+} - E_b^{R\diamond-}(p)\psi_b^{R\diamond-}\big], \\ \Pi_{\widehat{Q}_b(0)}\mathrm{ev}_0[v_{b,0}^{\diamond-} - v_{b,0}^{\diamond+}] & = & \psi_b^{\widehat{Q}\diamond+} - \Pi_{\widehat{Q}_b(0)}\mathrm{ev}_0\big[E_b^{P\diamond-}(p)(\psi_b^{B\diamond-} + \psi_b^{\widehat{P}\diamond-}) \\ & & \quad -E_b^{S\diamond+}(p)\psi_b^{S\diamond+} + E_b^{R\diamond-}(p)\psi_b^{R\diamond-}\big]. \end{array} \tag{6.36}$$

The continuity conditions in (iii) for the variables associated with the front can be translated as

$$\begin{array}{rcl}-\Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[v^{\diamond}_{f,0}-v^{+}_{f,0}] & = & \psi^{S\diamond}_f+\Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}E^{Q\diamond}_f(p)(\psi^{B\diamond}_f+\psi^{\widehat{Q}\diamond}_f)\\ & & -\Pi_{P^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[E^{Q+}_f(p)\psi^{Q+}_f+\delta_\psi \mathrm{e}^{\eta_* T}E^{S+}_f(p)\widetilde{\psi}^{S+}_f],\\ \Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[v^{\diamond}_{f,0}-v^{+}_{f,0}] & = & \psi^{Q+}_f-\Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}E^{Q\diamond}_f(p)(\psi^{B\diamond}_f+\psi^{\widehat{Q}\diamond}_f)\\ & & +\Pi_{Q^{\mathrm{fb}}_{R,0}}\mathrm{ev}_{\xi_0}[\delta_\psi \mathrm{e}^{\eta_* T}E^{S+}_f(p)\widetilde{\psi}^{S+}_f-E^{S\diamond}_f(p)\psi^{S\diamond}_f],\end{array} \quad (6.37)$$

while the conditions in (iii) for the variables associated with the back yield

$$\begin{array}{rcl}-\Pi_{P^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}[v^{\diamond+}_{b,0}-v^{+}_{b,0}] & = & \psi^{S\diamond+}_b+\Pi_{P^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}E^{Q\diamond+}_b(p)(\psi^{B\diamond+}_b+\psi^{\widehat{Q}\diamond+}_b)\\ & & -\Pi_{P^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}E^{Q+}_b(p)\psi^{Q+}_b,\\ \Pi_{Q^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}[v^{\diamond+}_{b,0}-v^{+}_{b,0}] & = & \psi^{Q+}_b-\Pi_{Q^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}E^{Q\diamond+}_b(p)(\psi^{B\diamond+}_b+\psi^{\widehat{Q}\diamond+}_b)\\ & & -\Pi_{Q^{\mathrm{fb}}_{L,w_*}}\mathrm{ev}_{\xi_0}E^{S\diamond+}_b(p)\psi^{S\diamond+}_b,\\ -\Pi_{Q^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}[v^{\diamond-}_{b,0}-v^{-}_{b,0}] & = & \psi^{R\diamond-}_b+\Pi_{Q^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}E^{P\diamond-}_b(p)(\psi^{B\diamond-}_b+\psi^{\widehat{P}\diamond-}_b)\\ & & -\Pi_{Q^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}[E^{P-}_b(p)\psi^{P-}_b+\delta_\psi \mathrm{e}^{\eta_* T}E^{R-}_b(p)\widetilde{\psi}^{R-}_b],\\ \Pi_{P^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}[v^{\diamond-}_{b,0}-v^{-}_{b,0}] & = & \psi^{P-}_b-\Pi_{P^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}E^{P\diamond-}_b(p)(\psi^{B\diamond-}_b+\psi^{\widehat{P}\diamond-}_b)\\ & & +\Pi_{P^{\mathrm{fb}}_{R,w_*}}\mathrm{ev}_{-\xi_0}[\delta_\psi \mathrm{e}^{\eta_* T}E^{R-}_b(p)\widetilde{\psi}^{R-}_b-E^{R\diamond-}_b(p)\psi^{R\diamond-}_b].\end{array} \quad (6.38)$$

Putting these equations together, we arrive at a 16×16 linear system that can be solved for all $p \in \mathcal{D}_p$, provided that δ_ψ, δ_c, δ_{sl} and δ_{ϑ^0} are chosen to be sufficiently small and T_* is chosen to be sufficiently large. The bound (6.26) is a direct consequence of the scaling (6.32). The bound (6.28) on the derivatives can be obtained by using Lemma 4.1 and a modified version of Lemma 5.5 and noting that the embedding $BC^{\oplus}_{f,\alpha_1} \subset BC^{\oplus}_{f,\alpha_2}$ for $\alpha_1 > \alpha_2$ has norm $\mathrm{e}^{(\alpha_1-\alpha_2)T}$, together with a similar embedding for the family $BC^{\ominus}_b$. ∎

We are now ready to move on to the nonlinear system by considering the fixed point equation

$$h = \mathcal{F}_4(h, p, \phi_{\mathrm{hw}}) := L_4(p)(\mathcal{M}(h, p), \phi_{\mathrm{hw}}) \quad (6.39)$$

posed on the space $\mathcal{H}^\circ$, in which we take $p = (\widetilde{p}, T) \in \mathcal{D}_p$ and consider all sufficiently small ϕ_{hw}. Let us pick a constant $C_\phi > 0$ and write $\delta_\phi = \delta_v / C_\phi$ and

$$B_{\delta_\phi} = \{\phi_{\mathrm{hw}} \in X \mid \|\phi_{\mathrm{hw}}\| \le \delta_\phi\}. \quad (6.40)$$

In addition, consider for any $\phi_{\mathrm{hw}} \in B_{\delta_\phi}$ the space

$$\mathcal{B}_{\phi_{\mathrm{hw}}}(p) = \{h \in \mathcal{H}^\circ \mid \|h\|_{\mathcal{H}^\circ} \le C_\phi \|\phi_{\mathrm{hw}}\| \mathrm{e}^{-\eta_* T}\}. \quad (6.41)$$

For any $\phi_{\mathrm{hw}} \in B_{\delta_\phi}$, $p \in \mathcal{D}_p$ and $h \in \mathcal{B}_{\phi_{\mathrm{hw}}}(p)$, we have the estimate

$$\|\mathcal{F}_4(h, p, \phi_{\mathrm{hw}})\|_{\mathcal{H}^\circ} \le_* \epsilon \xi_0 C_\phi \|\phi_{\mathrm{hw}}\| \mathrm{e}^{-\eta_* T} + \|\phi_{\mathrm{hw}}\| \mathrm{e}^{-\eta_* T} + \mathrm{e}^{-\eta_* T} C_\phi^2 \|\phi_{\mathrm{hw}}\|^2 \quad (6.42)$$

Hence, after choosing C_ϕ to be sufficiently large and possibly decreasing the size of $\mathcal{D}_p$, we can ensure that $\mathcal{F}_4(h, p)$ maps $\mathcal{B}_{\phi_{\mathrm{hw}}}(p)$ into itself for all $p \in \mathcal{D}_p$ and $\phi_{\mathrm{hw}} \in B_{\delta_\phi}$. In addition, it is straightforward to show that $\mathcal{F}_4(\cdot, p, \phi_{\mathrm{hw}})$ is a contraction mapping on $\mathcal{B}_{\phi_{\mathrm{hw}}}(p)$ for all such p and ϕ_{hw}. We hence obtain a solution map

$$h^* : \mathcal{D}_p \times B_{\delta_\phi} \to \mathcal{B}_{\phi_{\mathrm{hw}}}(p) \subset \mathcal{H}^\circ \quad (6.43)$$

for the fixed point problem (6.39).

Differentiating (6.39), we find

$$D_{\phi_{\rm hw}} h^*(p, \phi_{\rm hw}) \quad = \quad L_4(p)\Big(D_h \mathcal{M}\big(h^*(p, \phi_{\rm hw})\big), 0\Big) D_{\phi_{\rm hw}} h^*(p, \phi_{\rm hw}) + L_4(p)(0, I). \tag{6.44}$$

Combining this with the bounds in Lemma 6.1, we conclude $\|D_{\phi_{\rm hw}} h^*(p, \phi_{\rm hw})\| \le_* e^{-\eta_* T}$. Substituting this back into (6.44), we obtain

$$D_{\phi_{\rm hw}} \pi_{\theta_f^+} \oplus \pi_{\theta_b^-} h^*(p, \phi_{\rm hw}) \le_* \epsilon e^{-\eta_* T}. \tag{6.45}$$

In addition, we may write

$$\begin{array}{rcl} D_{\widetilde{p}} h^*(p) & = & D_{\widetilde{p}} L_4(p)\big(\mathcal{M}(h^*(p), p), \phi_{\rm hw}\big) \\ & & + L_4(p)\big(D_h \mathcal{M}(h^*(p), p), 0\big) D_{\widetilde{p}} h^*(p) \\ & & + L_4(p)\big(D_{\widetilde{p}} \mathcal{M}(h^*(p), p), 0\big), \end{array} \tag{6.46}$$

which leads to the estimate

$$\|D_{\widetilde{p}} h^*(p, \phi_{\rm hw})\|_{\mathcal{H}^\circ_\zeta} \le_* e^{\gamma T} e^{-\eta_* T} \|\phi_{\rm hw}\| \tag{6.47}$$

for some $\zeta > \gamma$.

For ease of notation, we will need to reparametrize the variable ϑ^0. To this end, we will take variables

$$\overline{p} = (\vartheta^0_\Delta, \omega) = (\vartheta^0_\Delta, c, T^{\rm sl}, T) \tag{6.48}$$

from the new parameter space

$$\overline{\mathcal{D}}_p = \overline{\mathcal{D}}_p(\delta_{\vartheta^0}, \delta_c, \delta_{\rm sl}, T_*) = [-\delta_{\vartheta^0}, \delta_{\vartheta^0}] \times \Omega(\delta_c, \delta_{\rm sl}, T_*). \tag{6.49}$$

The relation between p and $\overline{p} = (\vartheta^0_\Delta, \omega)$ is given by

$$p = (\vartheta^0_\infty(\omega) + \vartheta^0_\Delta, \omega). \tag{6.50}$$

Let us write

$$\begin{array}{rcl} \Theta_{\rm hw}(\vartheta^0, c, \epsilon, T) & = & w_f^+(c, \epsilon)(\xi_0 + T) - w_b^-(\vartheta^0, c, \epsilon)(-\xi_0 - T), \\ \Phi_{\rm hw}(\vartheta^0, c, \epsilon, T) & = & {\rm ev}_{\xi_0 + T} u_f^+(c, \epsilon) - {\rm ev}_{-\xi_0 - T} u_b^-(\vartheta^0, c, \epsilon) \end{array} \tag{6.51}$$

for the gaps in the w and u variables that needs to be closed. Writing h_f^* for the solution of the fixed point problem (5.76) and h_b^* for the solution of (5.76), we note that we can represent $\Theta_{\rm hw}$ as follows,

$$\begin{array}{rcl} \Theta_{\rm hw}(\overline{p}) & = & \Theta^{\rm sl}_R\big(\vartheta_f^+(\omega), \omega\big)(T^{\rm sl}) - \Theta^{\rm sl}_R\big(\vartheta_b^-(\vartheta^0_\infty(\omega) + \vartheta^0_\Delta, \omega), \omega\big)(-T^{\rm sl}) \\ & & + \pi_{\theta^+} h_f^*\big(\vartheta_f^+(\omega), \omega\big)(\xi_0 + T) \\ & & - \pi_{\theta^-} h_b^*\big(\vartheta^0_\infty(\omega) + \vartheta^0_\Delta, \vartheta_b^+(\vartheta^0_\infty(\omega) + \vartheta^0_\Delta, \omega), \vartheta_b^-(\vartheta^0_\infty(\omega) + \vartheta^0_\Delta, \omega), \omega\big)(-\xi_0 - T). \end{array} \tag{6.52}$$

Using this representation and the estimates (5.86), it is not hard to see that for all $\overline{p} \in \overline{\mathcal{D}}_p$ we have

$$\begin{array}{rcl} \|\Theta_{\rm hw}(\overline{p})\| & \le_* & \left|\vartheta^0_\Delta\right| + e^{-\eta_* T}, \\ \|D_\omega \Theta_{\rm hw}(\overline{p})\| & \le_* & \left|\vartheta^0_\Delta\right| + e^{\gamma T} e^{-\eta_* T}, \\ \left\|D_{\vartheta^0_\Delta} \Theta_{\rm hw}(\overline{p})\right\| & \le_* & 1. \end{array} \tag{6.53}$$

We may obtain a similar representation for $\Phi_{\rm hw}$ as a function of $\overline{p}$, allowing us to obtain the bounds

$$\begin{array}{rcl} \|\Phi_{\rm hw}(\overline{p})\| & \le_* & \left|\vartheta^0_\Delta\right| + e^{-\eta_* T} \\ \|D_\omega \Phi_{\rm hw}(\overline{p})\| & \le_* & \left|\vartheta^0_\Delta\right| + e^{\gamma T} e^{-\eta_* T} \\ \left\|D_{\vartheta^0_\Delta} \Phi_{\rm hw}(\overline{p})\right\| & \le_* & 1. \end{array} \tag{6.54}$$

We note that the choice $\phi_{\rm hw} = \Phi_{\rm hw}(\overline{p})$ suffices to ensure that $u_{\rm xc}$ as defined in (6.6) is a continuous function. However, in order to make sure that the function w defined in (6.6) is also continuous, we must find an appropriate ϑ_Δ^0 for each $\omega = (c, T^{\rm sl}, T) \in \Omega$. In particular, we need solve the equation

$$\Theta_{\rm hw}(\vartheta_\Delta^0, \omega) = \theta_{\rm xc}(\vartheta_\Delta^0, \omega), \tag{6.55}$$

in which $\theta_{\rm xc}$ is given by

$$\theta_{\rm xc}(\overline{p}) = \pi_{\theta_b^-} h^*\big(\vartheta_\infty^0(\omega) + \vartheta_\Delta^0, \omega, \Phi_{\rm hw}(\overline{p})\big)(-\xi_0 - T) - \pi_{\theta_f^+} h^*\big(\vartheta_\infty^0(\omega) + \vartheta_\Delta^0, \omega, \Phi_{\rm hw}(\overline{p})\big)(\xi_0 + T). \tag{6.56}$$

The observation

$$\begin{array}{lcl} |\theta_{\rm xc}(\overline{p})| & \le_* & \big|\vartheta_\Delta^0\big| + {\rm e}^{-\eta_* T}, \\ \Big|D_{\vartheta_\Delta^0}\theta_{\rm xc}(\overline{p})\Big| & \le_* & {\rm e}^{\gamma T}\big[\big|\vartheta_\Delta^0\big| + {\rm e}^{-\eta_* T}\big] + \epsilon, \end{array} \tag{6.57}$$

allows (6.55) to be solved. In particular, we find a function $\vartheta_\Delta^0 : \Omega \to \mathbb{R}$ that admits the bounds

$$\begin{array}{lcl} \big|\vartheta_\Delta^0(\omega)\big| & \le_* & {\rm e}^{-\eta_* T}, \\ \big|D_c\vartheta_\Delta^0\big| + \big|D_{T^{\rm sl}}\vartheta_\Delta^0(\omega)\big| & \le_* & {\rm e}^{\gamma T}{\rm e}^{-\eta_* T}, \end{array} \tag{6.58}$$

for all $\omega \in \Omega$. Finally, writing

$$H^* : \omega \mapsto h^*\big(\vartheta_\infty^0(\omega) + \vartheta_\Delta^0(\omega), \omega, \Phi_{\rm hw}(\vartheta_\Delta^0(\omega), \omega)\big), \tag{6.59}$$

this means that

$$\|D_c H^*(\omega)\|_{\mathcal{H}_\zeta^\circ} + \|D_{T^{\rm sl}} H^*(\omega)\|_{\mathcal{H}_\zeta^\circ} \le_* {\rm e}^{\gamma T}{\rm e}^{-2\eta_* T}, \tag{6.60}$$

as desired.

The issue that now remains is smoothness with respect to the variable T. We proceed as in [27] and pick a $\overline{T} > T$. We reconsider the setting of Lemma 6.1 by considering $g \in \overline{\mathcal{H}}$ and looking for $h \in \overline{\mathcal{H}}^\circ$ instead of $g \in \mathcal{H}$ and $h \in \mathcal{H}^\circ$, in which $\overline{\mathcal{H}}^\circ$ and $\overline{\mathcal{H}}$ are defined as in (6.3) and (6.4), after replacing each occurrence of T by $\overline{T}$. We still require the properties (i) and (v) to hold in terms of the original T. The construction in the proof of this lemma remains valid if we adapt the operator E_f^{S+} to map into $C([\xi_0 - 1, \xi_0 + \overline{T} + 1], \mathbb{R})$, by providing an appropriate extension on the interval $[\xi_0 + T + 1, \xi_0 + \overline{T} + 1]$, together with a similar adaption for E_b^{R-}. For the purposes of solving the linear system (6.33) through (6.38), the parameter T can now be treated on the same footing as c. However, a complication arises when constructing v_f^+ and v_b^- according to (6.31). Indeed $E_f^{S+}(p)\psi_f^{S+}$ is only C^0-smooth on the interval $[\xi_0 + T, \xi_0 + T + 1]$, but calculating $D_T E_f^{S+}(p)$ involves taking a derivative on this interval. This issue can be resolved by studying the action of $\Pi_{P_{\rm hw}}$ and $\Pi_{Q_{\rm hw}}$ on the function space

$$X^{(1)} = \{\phi \in C([-1, 1], \mathbb{R}) \mid \phi_{|[-1,0]} \in C^1([-1, 0], \mathbb{R}) \text{ and } \phi_{|[0,1]} \in C^1([0, 1], \mathbb{R})\}. \tag{6.61}$$

It is not hard to see that Π_P and Π_Q map $X^{(1)}$ into $X^{(1)}$. This allows us to define

$$E_f^{S+}(p) : \Pi_{P_{\rm hw}}(X^{(1)}) \to W^{1,\infty}([\xi_0 - 1, \xi_0 + \overline{T} + 1], \mathbb{R}), \tag{6.62}$$

which now does allow taking a derivative with respect to ξ, with the remark that the resulting function may have a jump at $\xi = \xi_0 + T$. By noting that in fact $\Phi_{\rm hw}(p) \in X^{(1)}$ and taking $\psi_f^{S+} \in \Pi_{P_{\rm hw}}(X^{(1)})$ and $\psi_b^{R-} \in \Pi_{Q_{\rm hw}}(X^{(1)})$, we can argue that the gap functions ξ_f and ξ_b are also C^1-smooth with respect to T, and that the desired exponential bounds hold. Higher order smoothness with respect to T can be obtained in a similar fashion.

Proof of Proposition 3.8 We fix the parameter space Ω as above. For any $\omega \in \Omega$, we write $\vartheta^0(\omega) = \vartheta^0_\infty(\omega) + \vartheta^0_\Delta(\omega)$, together with

$$(v_f^-, \theta_f^-, v_f^\diamond, \theta_f^\diamond, v_f^+, \theta_f^+, v_b^-, \theta_b^-, v_b^{\diamond -}, \theta_b^{\diamond -}, v_b^{\diamond +}, \theta_b^{\diamond +}, v_b^+, \theta_b^+) = H^*(\omega) \tag{6.63}$$

and define w, u_{xc}, u_f and u_b according to (6.6) and (6.7). The properties required in Definition 3.7 follow immediately from the construction above. The exponents γ that appear in the estimates throughout this section can be eliminated by slightly decreasing η_*. The smoothness properties and bounds in (ii) and (iii) have been established for $\ell = 1$ and can be extended to $1 < \ell \leq r$ using similar arguments. As in the proof of Proposition 3.4, the local uniqueness properties can be established using the proof of [43, Claim 3.7]. ∎

7 Discussion

In this paper, we showed that the discrete FitzHugh–Nagumo equation

$$\begin{array}{rcl} \dot{u}_i(t) & = & \alpha[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] + u_i(t)(1 - u_i(t))(u_i(t) - a) - w_i(t), \\ \dot{w}_i(t) & = & \epsilon\big(u_i(t) - \gamma w_i(t)\big) \end{array} \tag{7.1}$$

with $i \in \mathbb{Z}$ supports travelling pulses

$$(u_i, w_i)(t) = (u_*, w_*)(i + ct) \tag{7.2}$$

for some wave speed $c > 0$ provided $0 < \epsilon \ll 1$ and $a \in (0, \frac{1}{2})$ is such that the discrete Nagumo equation supports travelling fronts. To establish this result, we extended various concepts from geometric singular perturbation theory from the more standard ODE setting to the functional differential equation

$$\begin{array}{rcl} cu_*'(\xi) & = & \alpha[u_*(\xi + 1) + u_*(\xi - 1) - 2u_*(\xi)] + g\big(u_*(\xi)\big) - w_*(\xi), \\ cw_*'(\xi) & = & \epsilon\big(u_*(\xi) - \gamma w_*(\xi)\big) \end{array} \tag{7.3}$$

that the travelling wave profiles $(u_*, w_*)(\xi)$ have to satisfy, in which $g(u) = u(1-u)(u-a)$. Specifically, we proved the persistence of slow manifolds and their stable and unstable foliations, and established an Exchange Lemma that can be used to track solutions that pass near hyperbolic slow manifolds. The main difficulty in proving these results is the fact the initial-value problem associated with the MFDE (7.3) is ill-posed: we overcame this difficulty by utilizing exponential dichotomies for linear MFDEs with slowly varying coefficients. In particular, we relied heavily on previous work by Sakamoto and various works by us and our coworkers. While the proofs given here are technically involved, they are inspired by the same geometric intuition, illustrated in Figure 1, that led to the ODE proofs of the existence of fast waves. As we already pointed out in the introduction, the techniques we developed here should be general enough to apply to a broader class of singularly perturbed functional differential equations.

We did not consider the stability of the travelling-pulse solutions (7.2) with respect to the underlying lattice dynamical system (7.1). The results in [3, 12] imply that spectral stability of the operator

$$\begin{array}{rcl} \mathcal{L}_* & : & L^2(\mathbb{R}, \mathbb{R}^2) \longrightarrow L^2(\mathbb{R}, \mathbb{R}^2), \\ & & \begin{pmatrix} u \\ w \end{pmatrix} \longmapsto \begin{pmatrix} cu'(\xi) - \alpha[u(\xi+1) + u(\xi-1) - 2u(\xi)] - g'\big(u_*(\xi)\big)u(\xi) - w(\xi) \\ cw'(\xi) - \epsilon(u(\xi) - \gamma w(\xi)) \end{pmatrix} \end{array}$$

with domain $H^1(\mathbb{R}, \mathbb{R}^2)$ implies nonlinear stability of the underlying pulse with respect to (7.1). The eigenvalue problem associated with the operator $\mathcal{L}_*$ is again singularly perturbed. It should therefore be possible to use the techniques outlined in this paper to study the spectrum of $\mathcal{L}_*$ and to assess the stability of the fast waves we constructed here: this is work in progress. In line with the results for the spatially continuous FitzHugh–Nagumo system in [30, 46] and with the numerical simulations of the spatially discrete FitzHugh–Nagumo system in [9], we expect that the fast pulses are stable.

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