Invariant Manifolds and Applications for Functional Differential-Algebraic Equations of Mixed Type

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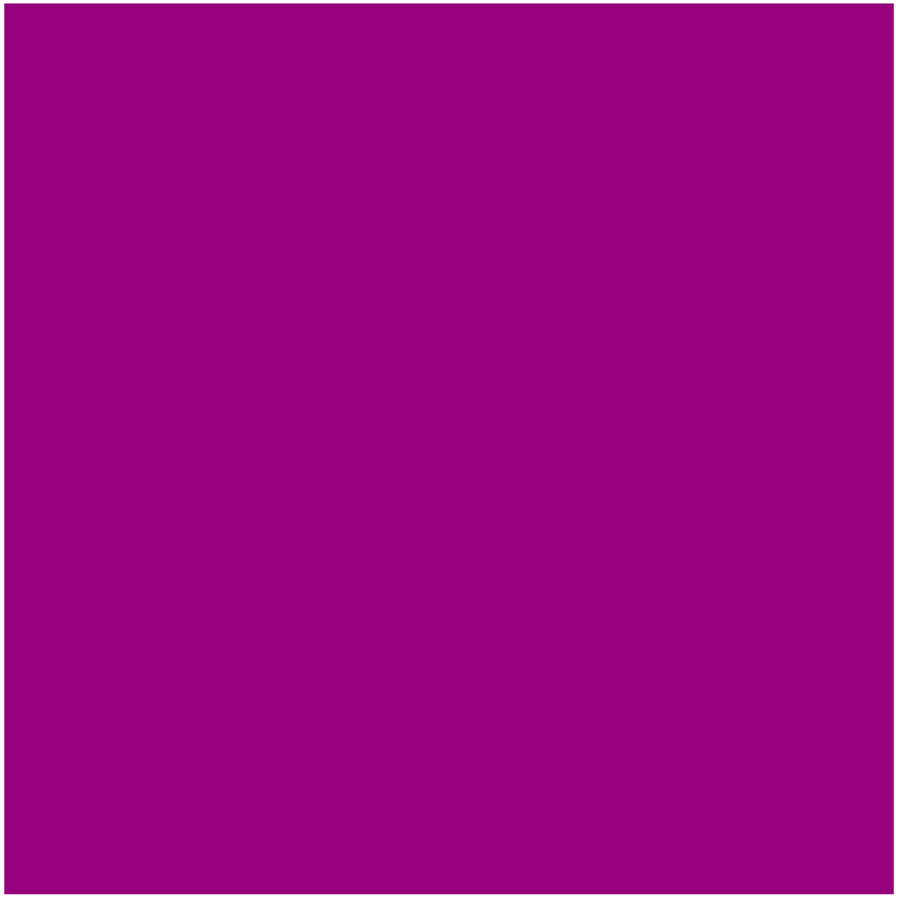
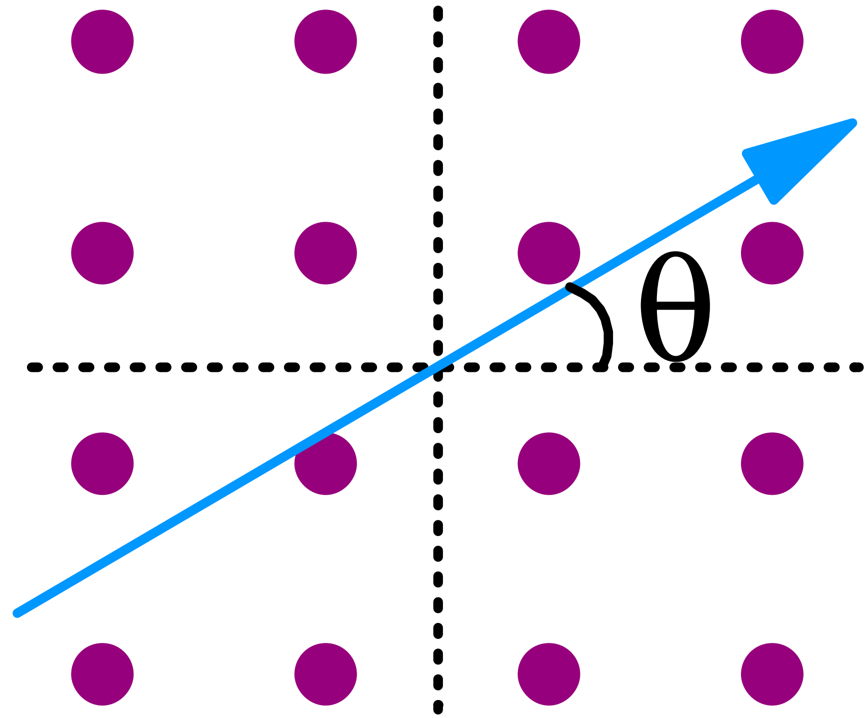
(Joint work with S.M. Verduyn Lunel)

Lattice equations



Continuous media (PDE)



Discrete media (Lattice Equation)

- Fruitful to include structure of underlying space into models.
- Differential equations on lattices (LDEs) are becoming increasingly popular.

Mixed Type Functional Differential Equations (MFDE)

Looking for travelling wave solutions for LDEs, one immediately encounters

$$\dot{x}(t) = G(x_t). \tag{1}$$

- x is a continuous function with $x(t) \in \mathbb{R}^n$.

- $x_t \in C([-1, 1])$ is the **state** of x at t, i.e.,

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-1, 1]\ .$$

- $G : C([-1, 1]) \to \mathbb{R}^n$ is sufficiently smooth.

Note that $\dot{x}(t)$ depends on both past and future values of x.

Eq. (1) is called a functional differential equation of mixed type (MFDE).

MFDE Applications

Lattice equations have arisen in many disciplines.

- Image processing

  Chua and Roska (1993): cellular networks for recognizing edges / outlines in pictures

- Material science (crystals)

  Bates and Chmaj (1999): Ising model for phase transitions.

- Biology

  Keener and Sneed (1998): signal propagation through nerves with discrete gaps.

Particular focus on direction dependence and propagation failure of travelling waves.

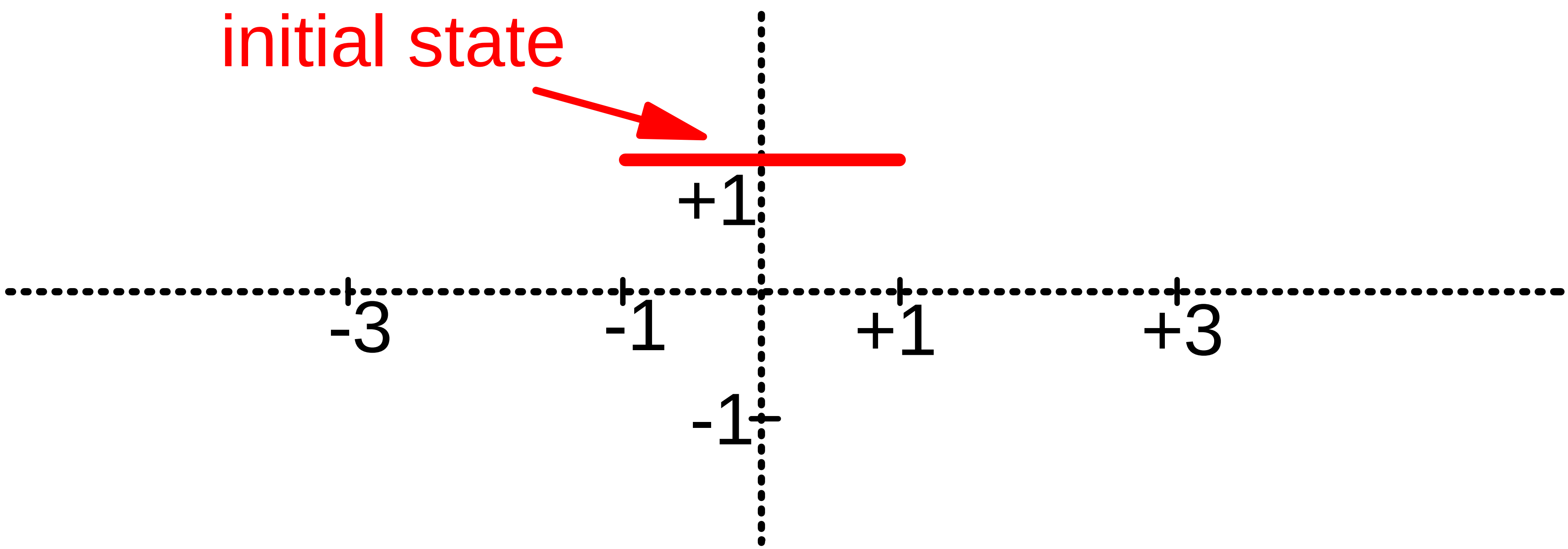
New numerical methods have been developed to study these features (E. Van Vleck, C. Elmer, S. Verduyn-Lunel, H.).

Propagation failure analyzed theoretically (J. Mallet Paret, A. Hoffman)

MFDEs vs Delay Equations

Consider the homogeneous MFDE

$$\dot{x}(t) = x(t-1) + x(t+1).$$

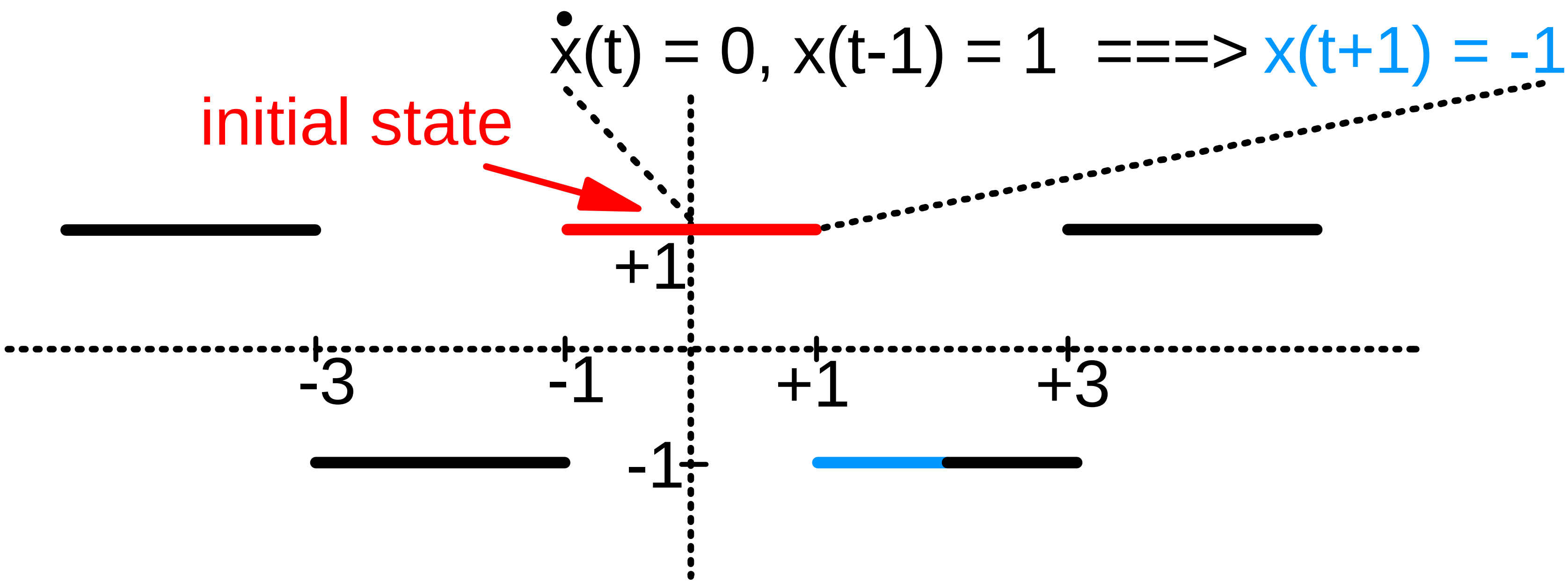


(Example due to Härterich, Sandstede, Scheel (2002))

MFDEs vs Delay Equations

Consider the MFDE

$$\dot{x}(t) = x(t-1) + x(t+1).$$



- Continuity lost $\Longrightarrow$ ill-defined as an initial value problem.

MFDE: Theory

- No semigroup techniques available; RFDE theory not directly applicable.
- Various authors have worked on theory for linear equation

$$\Lambda x = f, \qquad (\Lambda x)(t) = \dot{x}(t) - Lx_t.$$

  - Early work by Bellman and Cooke (1963), Rustichini (1989).
  - Mallet-Paret (1999) established Fredholm properties for Λ.
  - Mallet-Paret, Verduyn Lunel (2001) and independently Härterich et al. provided exponential dichotomies for Λ.

- Nonlinear theory until recently much less developed.
  - Global results by Mallet-Paret (1999) for bistable cubic systems.
  - Piecewise-linear cartoon nonlinearities analyzed using Fourier transform techniques, by C. Elmer, E. Van Vleck, J. Mallet-Paret, J. Cahn, S. Chow, W. Shen

MFDE: Center Manifolds

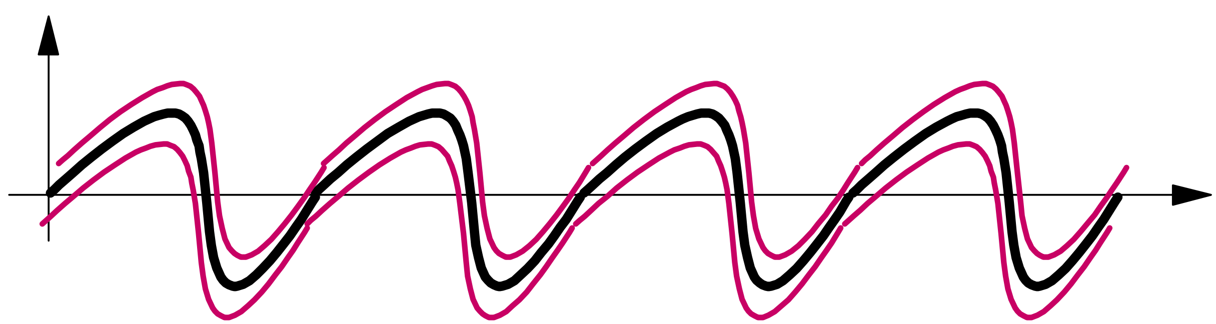
Interested in solutions to (2) near equilibria $\overline{x}$ or periodic solutions $x = p$.

$$\dot{x}(t) = G(x_t). \tag{2}$$

Flow cannot be defined for (2). Mielke and Kirchgässner faced with similar problem when considering elliptic PDEs, but still managed to construct a CM.

(2006) H. + VL: All solutions to (2) sufficiently close to equilibrium $\overline{x}$ lie on a finite dimensional center manifold. *J. Dyn. Diff. Eqns 19, 497-560.*

(2007) H. + VL: All solutions to (2) sufficiently close to periodic solution p lie on a finite dimensional center manifold (under discreteness condition on the Floquet spectrum). *MI report 2007-07; submitted.*



MFDE: Center Manifolds

Example: consider the variational equation around an equilibrium $\overline{x}$,

$$\dot{v}(t) = v(t-1) + v(t+1) - 2v(t) + v(t)^2.$$

Construction of CM based upon work by Mallet-Paret on linear operator Λ.

$$(\Lambda v)(t) = \dot{v}(t) - v(t-1) - v(t+1) + 2v(t).$$

Substitute $v(t) = \exp(zt)$ to obtain characteristic equation,

$$\Delta(z) = z - e^{-z} - e^{z} + 2.$$

We have $\Delta(0) = 0$ leading to eigenfunction $t \mapsto 1$ for eigenvalue $z = 0$.

Write $X_0 \subset C([-1,1])$ for the generalized eigenspace for all eigenvalues $\Delta(z) = 0$ on imaginary axis.

Dynamics on center manifold is described by an explicit ODE on X_0.

Opens up full finite dimensional toolbox in our ∞-dim setting.

MFDE Applications II

Lattice differential equations are not the only application of MFDEs.

- Solving optimal control problems with delays.

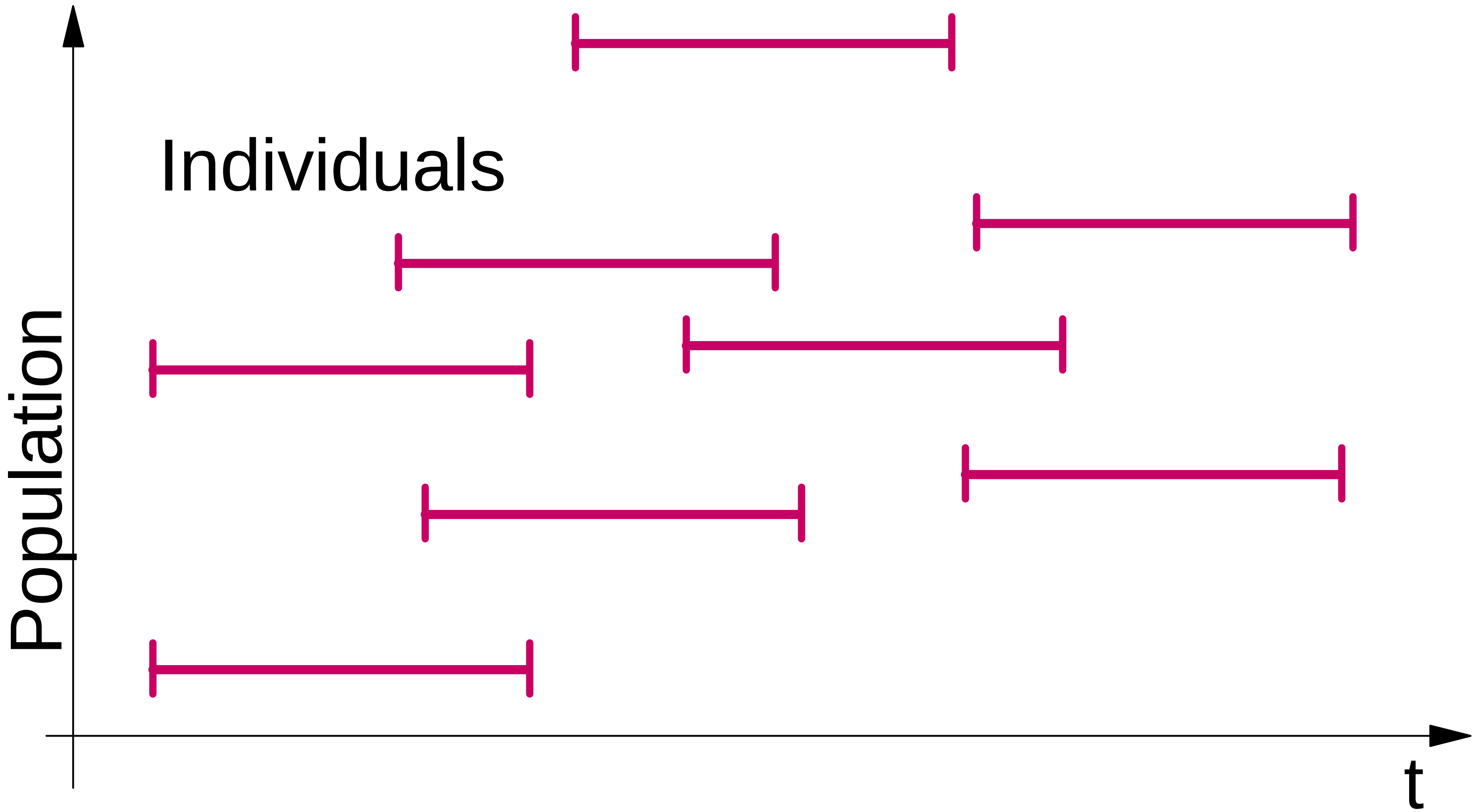
  Hughes (1968): Euler Lagrange equations for such problems are MFDEs.

  Benhabib & Nishimura (1979): introduced high dimensional economic growth optimal control model. Periodic orbits established.

  Rustichini (1989): Added delay into framework. E-L equations are MFDE. Even scalar model now yields the desired periodic orbits, using Hopf bifurcation theorem with the CM-reduction.

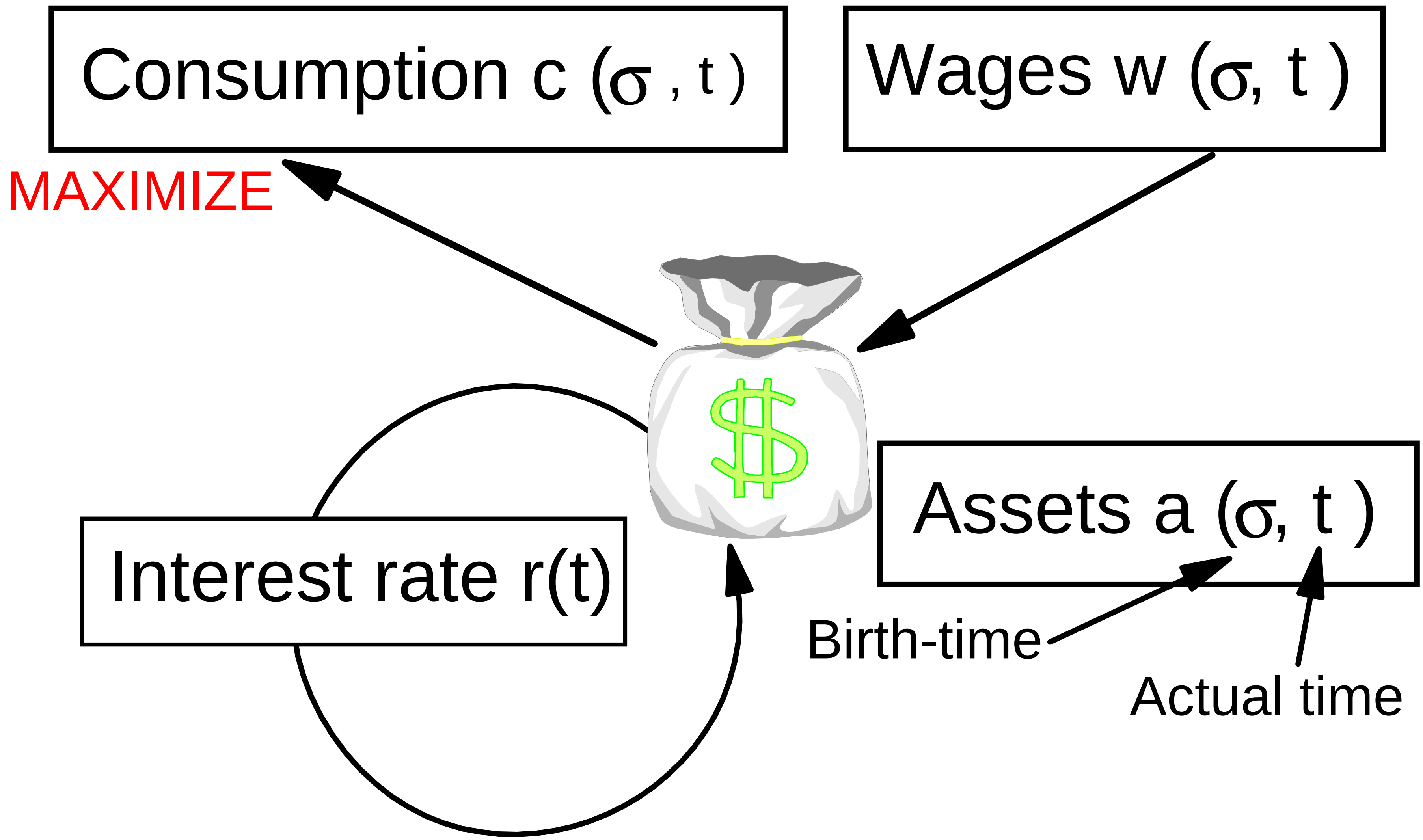
- Recent models in economic theory lead directly to MFDEs (H. d'Albis and E. Augeraud-Veron).

Overlapping Generations



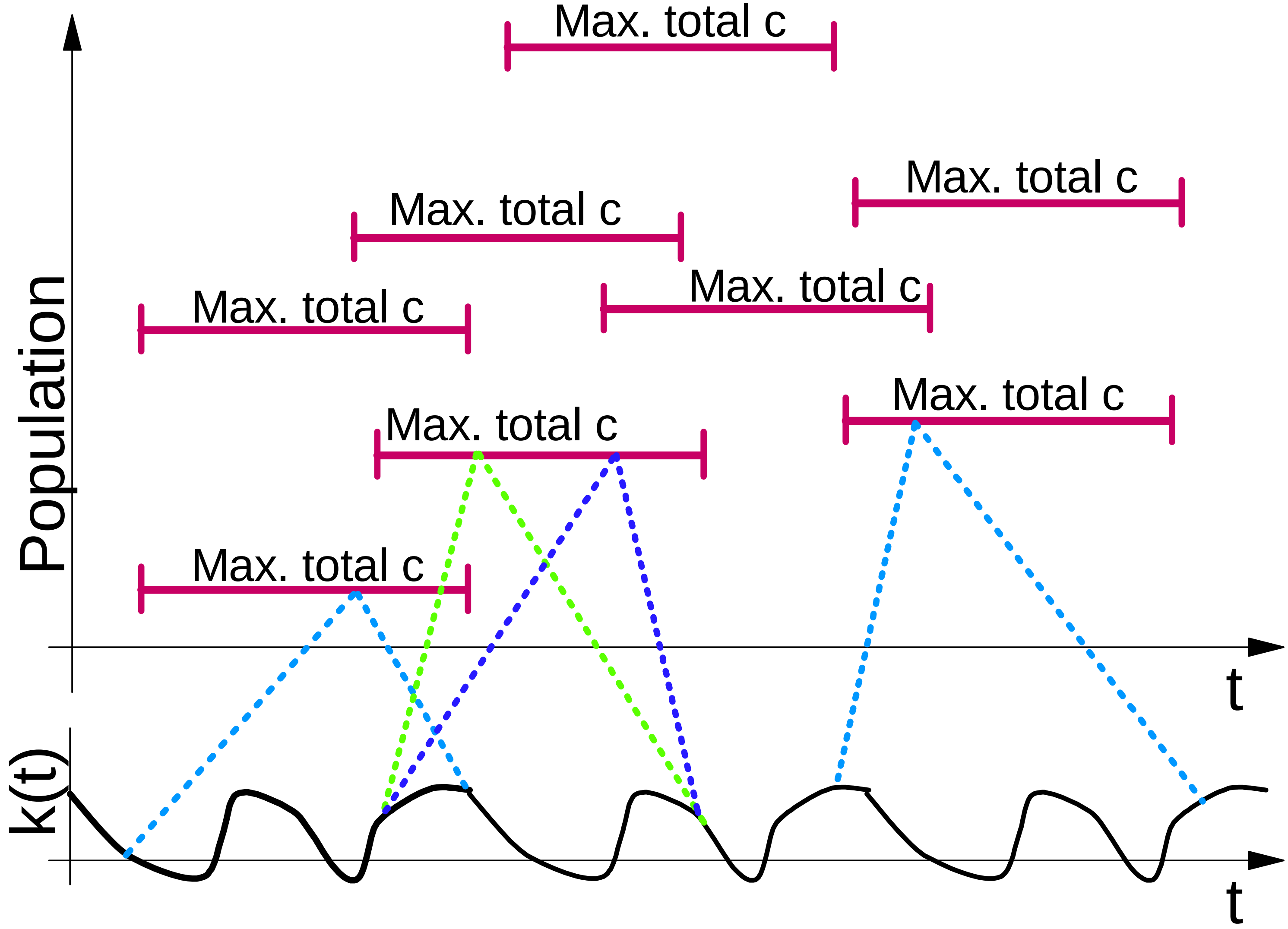
- Proposed by H. d'Albis + E. Augeraud-Véron (2006).
- Population consists of a continuum of individuals, that have fixed lifespan T_{life}.

Overlapping Generations: The Individual



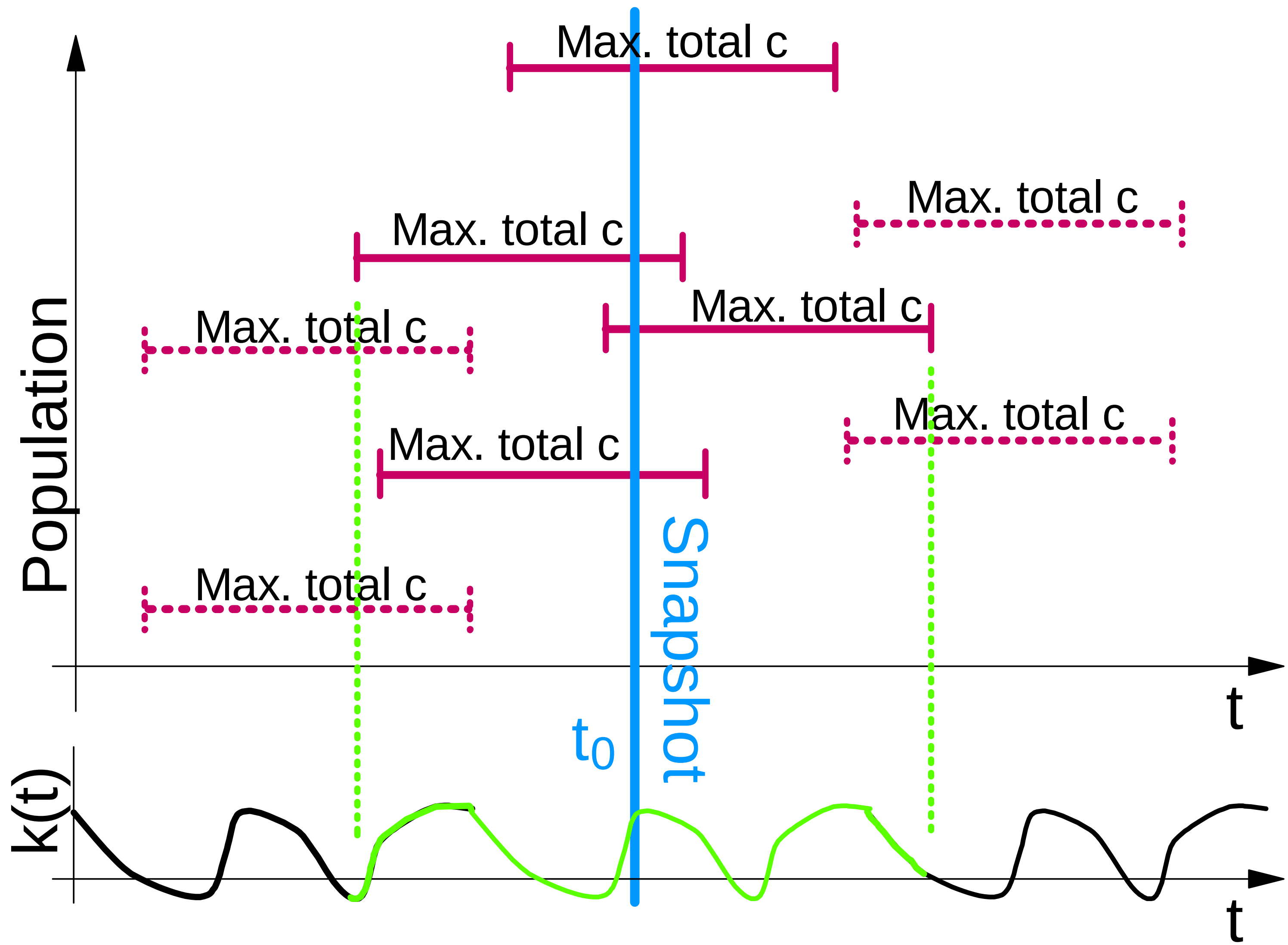
- Each individual wishes to maximize his lifetime consumption, but may not die in debt.

Overlapping Generations: The Collective



- Assumption: Consumers can make predictions for interest rate.

Overlapping Generations: The Dynamics



- Equilibrium at t_0 requires summation over all living indv (blue line).
- Condition for $k(t_0)$ involves all values $k(t_0 + \theta)$ for $\theta = -T_{\text{life}} \ldots T_{\text{life}}$.

Overlapping Generations: The Mathematics

We have $T_{\text{life}} = 1$.

Goal of every individual born at time s is to maximize his lifetime welfare, given by

$$\int_s^{s+1} \ln c(s,\tau) d\tau.$$

Solving this optimization problem shows that the optimal asset distribution $a^*(s,t)$ depends on the interest rates and wages during the lifetime of an individual, i.e.,

$$a^*(s,t) = F(r_{s+}, w_{s+}, t-s),$$

for some F. Here $r_{s+} \in C([0,1])$ is defined by $r_{s+}(\theta) = r(s+\theta)$.

The total amount of capital at any time t is given by

$$k(t) = \int_{t-1}^{t} a^*(\sigma,t) d\sigma,$$

namely the total amount of assets owned by living individuals.

Overlapping Generations: The Mathematics - II

The interest rate $r(t)$ and wages $w(t)$ can be related to the captial $k(t)$.

This allows one to derive an equilibrium condition for the amount of capital $k(t)$ in the market at time t.

$$\begin{aligned} k(t) &= f(k_t) = \alpha \int_{t-1}^{t+1} (s+1-t) \int_s^{s+1} k(v)^{2\alpha} \exp[-\alpha \int_t^v k(u)^{2\alpha-1} du] dv ds \\ &\quad -\alpha \int_{t-1}^{t} \int_t^{s+1} k(v)^{2\alpha} \exp[-\alpha \int_t^v k(u)^{2\alpha-1} du] dv ds. \end{aligned}$$

Due to all the integrals, a single differentiation leads to the MFDE

$$\dot{k}(t) = g(k_t).$$

We call this an index one problem.

Index 2 example

A slight variant of the model above involves retirement. Also proposed by H. d'Albis + E. Augeraud-Véron (2005). We now have $T_{\mathrm{life}} > 1$, while everybody retires at unit age. Every worker receives wages $w = 1$, while retired persons receive $w = 0$.

Welfare now given by

$$\int_s^{s+T_{\mathrm{life}}} \frac{c(s,\tau)^{1-\sigma^{-1}}}{1-\sigma^{-1}} d\tau.$$

Equilibrium condition for the interest rate given by

$$1 = \int_{t-T_{\mathrm{life}}}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+T_{\mathrm{life}}} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds.$$

We call this an index 2 problem, because 2 differentiations are needed to obtain MFDE.

Problems

Recall the equilibrium condition for the interest rate

$$1 = \int_{t-T_{\text{life}}}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+T_{\text{life}}} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds.$$

- Twofold differentiation leads to MFDE.
- Linearization of MFDE leads to characteristic equation (with $T = T_{\text{life}}$)

$$\Delta(z) \sim [-Te^z + (1-\sigma)e^{zT} + (Te^z - T + 1 - \sigma)e^{-zT} + (T - 2 + 2\sigma) + \sigma T^2 z^2].$$

- Double root at $z = 0$.
- Periodic orbits interesting from economic point of view.
- Would like to invoke Hopf bifurcation theorem.
- Double root adds annoying resonance.
- Unclear how to lift back solutions.

Differential-Algebraic Functional Equations

This gives motivation to study the differential-algebraic equation

$$A\dot{x}(t) = F(x_t). \tag{3}$$

- The matrix A is singular.
- We require that ℓ-fold differentiation turns (3) into a regular MFDE.
- This integer ℓ should be seen as the index of (3), in analogy with finite dimensional Differential-Algebraic equations (DAEs).
- Want to construct smooth center manifold directly for (3).
- Allows us to capture all solutions to (3) that stay in the neighbourhood of an equilibrium $\overline{x}$.

Overlapping Generations: Hopf bifurcations

Recall the equilibrium condition for the interest rate

$$1 = \int_{t-T_{\mathrm{life}}}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+T_{\mathrm{life}}} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds. \tag{4}$$

H + E. Augeraud-Veron + S. Verduyn-Lunel (2007): Center manifold can be constructed for smooth differential-algebraic equations like (4), *J. Diff. Eqns, to appear.*

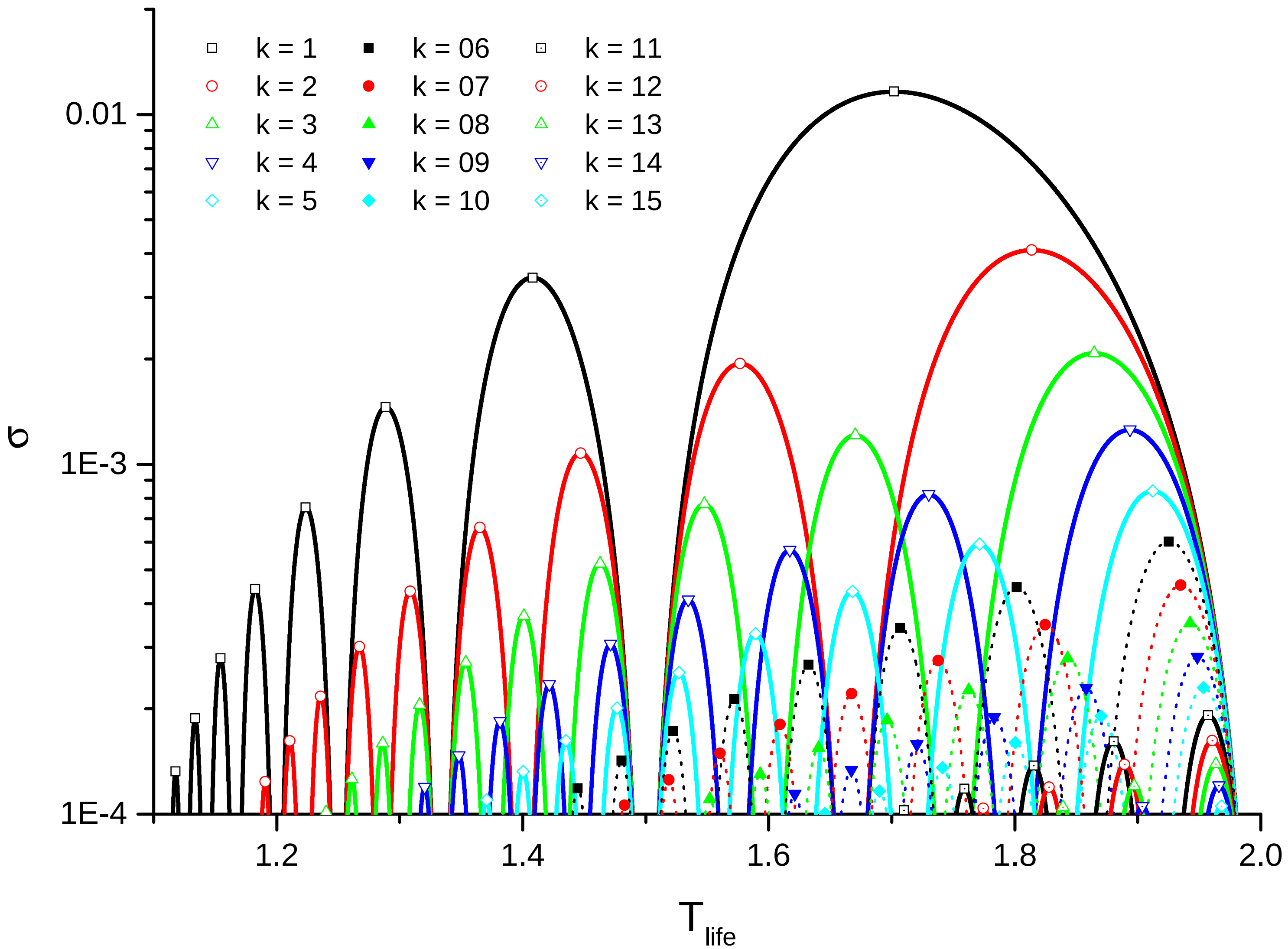
Flow on center manifold is described by explicit ODE.

Linear part of flow induced by characteristic equation for (4).

Double root at $z = 0$ thus no longer a problem!

Characteristic equation for (4) can be analyzed explicitly and leads to curves of Hopf bifurcations in $(T_{\mathrm{life}}, \sigma)$ plane.

Overlapping Generations: Hopf bifurcations



Optimizing of individual welfare leads to periodic cycles in interest rates. Many periods possible!

Smooth Differential-Algebraic Equations of Mixed Type

We set out to construct a center manifold for the smooth differential-algebraic equation

$$0 = Lx_t + R(x_t), \tag{5}$$

under the assumption that any solution x to (5) automatically satisfies an MFDE

$$\dot{x}(t) = Mx_t + S(x_t).$$

The nonlinearities $R : C([-1, 1]) \to \mathbb{R}$ and $S : C([-1, 1]) \to \mathbb{R}$ have

$$\begin{array}{rcccl} R(0) & = & S(0) & = & 0 \\ DR(0) & = & DS(0) & = & 0 \end{array}$$

and are sufficiently smooth.

Sketch of proof

- Analyze the linear homogeneous equation

$$0 = Lx_\xi$$

and determine set $\mathcal{N}_0$ of solutions that grow at most polynomially.

- Analyze the linear inhomogeneous equation

$$0 = Lx_\xi + f(\xi) \tag{6}$$

and find an "inverse" $\mathcal{K}$ such that $x = \mathcal{K}f$ solves (6) and $\mathcal{K}$ projects out $\mathcal{N}_0$.

- For $y \in \mathcal{N}_0$, construct solution $u^*(y)$ for fix point equation

$$u = y + \mathcal{K}R(u).$$

This fix point $u^*(y)$ is the desired solution of the main equation $Lu_t + R(u_t) = 0$.

Linear inhomogeneous equations

The important step is to analyze the linear inhomogeneous equation

$$0 = Lx_t + f(t), \tag{7}$$

for functions $f : \mathbb{R} \to \mathbb{R}$.

Due to our smoothness assumption, there exists an integer ℓ, such that (up to constants)

$$\Delta(z) = z^{-\ell}(z - M\exp(z\cdot)) = z^{-\ell}\Delta_M(z),$$

where Δ_M is characteristic matrix for the MFDE

$$\dot{x}(t) = Mx_t.$$

For $f \in W^{\ell,2}(\mathbb{R}, \mathbb{R}^n)$, we can hence solve (7) via Fourier transform, i.e.

$$\widehat{x}(\eta) = \Delta(i\eta)^{-1}\widehat{f}(\eta) = \Delta_M(i\eta)^{-1}\widehat{D^\ell f}(\eta).$$

Linear MFDE

Want to construct "inverse" for $\Lambda : BC_\eta(\mathbb{R}, \mathbb{R}^n) \to BC_\eta^\ell(\mathbb{R}, \mathbb{R}^n)$, i.e. for functions that grow as $e^{\eta|t|}$ at both $\pm\infty$.

Can be performed by splitting $f \in BC_\eta^\ell(\mathbb{R}, \mathbb{R}^n)$ into

$$f = \Phi_- f + \Phi_+ f + \Phi_\diamond f,$$

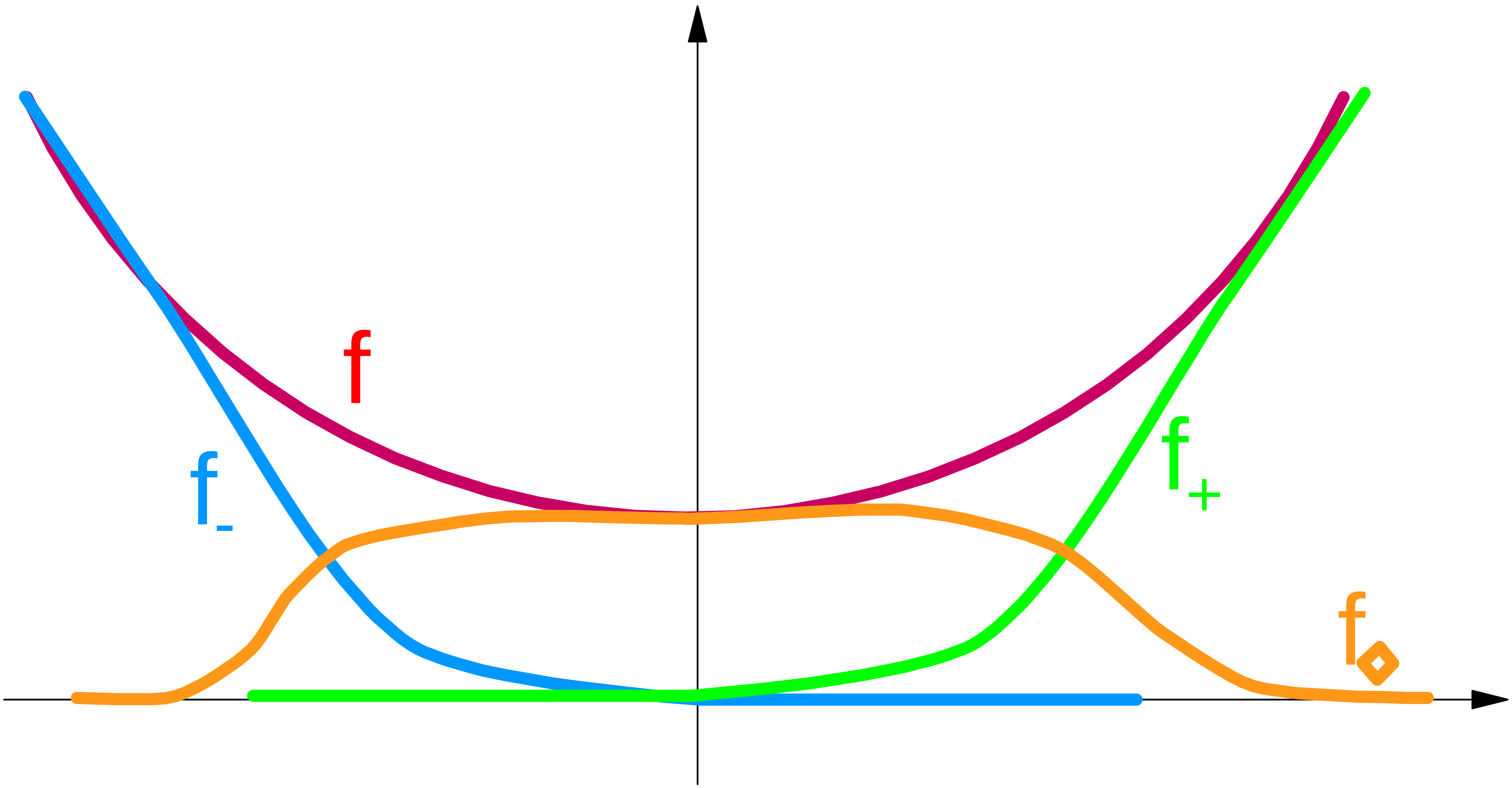
with

$$\begin{array}{rcl} \Phi_\pm f & \in & W^{\ell,2}_{\pm(\eta+\epsilon)}(\mathbb{R}, \mathbb{R}^n), \\ \Phi_\diamond f & \in & C_c^\infty \subset W^{\ell,2}_\eta(\mathbb{R}, \mathbb{R}^n). \end{array}$$

Care needs to be taken to ensure that components remain sufficiently smooth.

The operators $\Phi_\pm$ should be seen as restrictions to $[0, \infty)$ and $(-\infty, 0]$, while $\Phi_\diamond$ ensures that these restrictions can be taken smoothly.

Splitting f



Splitting $f = \Phi_- f + \Phi_+ f + \Phi_\diamond f = f_- + f_+ + f_\diamond$,

Cutoff function

Need to ensure that nonlinearity R is globally Lipschitz.

Normally this is done by choosing cutoff χ and writing, for $\phi \in C([-1,1])$,

$$R_\delta(\phi) \sim \chi(\|\phi\| / \delta) R(\phi)$$

However, this destroys our ability to take the ℓ-th derivative of R!

Need to embed our functions into bigger jet-space.

$$\begin{array}{rcl} BC^\ell_\eta(\mathbb{R}, \mathbb{R}^n) & \hookrightarrow & \big(BC_\eta(\mathbb{R}, \mathbb{R}^n)\big)^{\ell+1} := \underbrace{BC_\eta(\mathbb{R}, \mathbb{R}^n) \times \ldots \times BC_\eta(\mathbb{R}, \mathbb{R}^n)}_{\ell+1 \text{ copies}} \\ f & \mapsto & (f, Df, \ldots, D^\ell f) \end{array}$$

In the jet space the $\ell + 1$ components are independent.

Pseudo-inverse

The cut-off operators $\Phi_\pm$ and $\Phi_\diamond$ can be extended to act on functions $\mathbf{f}$ in the full jetspace $\big(BC_\eta(\mathbb{R},\mathbb{R}^n)\big)^{\ell+1}$.

In the definitions, each occurrence of $D^s f$ is simply replaced by $\mathbf{f}_s$.

The pseudo-inverse $\mathcal{K}_\eta$ is defined on the jet-space, namely

$$\begin{aligned}\mathcal{K}'_\eta\mathbf{f} &= \Lambda^{-1}_{(\eta)}\big(\tfrac{1}{2}\Phi_\diamond\mathbf{f} + \Phi_+\mathbf{f}\big) + \Lambda^{-1}_{(-\eta)}\big(\tfrac{1}{2}\Phi_\diamond\mathbf{f} + \Phi_-\mathbf{f}\big)\\ \mathcal{K}_\eta\mathbf{f} &= \big(I - P_{\mathcal{N}_0}\big)\mathcal{K}'\mathbf{f}.\end{aligned}$$

Here $P_{\mathcal{N}_0}$ is projection onto $\mathcal{N}_0$, the set of polynomially bounded solutions to homogeneous equations.

If in fact $f \in BC^\ell_\eta(\mathbb{R},\mathbb{R}^n)$, then indeed $\Lambda\mathcal{K}f = f$ and $P_{\mathcal{N}_0}\mathcal{K}f = 0$.

However, if $\mathbf{f} \notin BC^\ell_\eta(\mathbb{R},\mathbb{R}^n)$ is a true jet-function, then we have no interpretation of $\mathcal{K}\mathbf{f}$

Modification of Nonlinearity

Smoothness implies that for any $x \in C(\mathbb{R}, \mathbb{R}^n)$, $t \mapsto R(x_t)$ is C^ℓ-smooth.

Nonlinearity R induces substitution map $\widetilde{R} : BC_\eta \to BC^\ell_\eta$, given by

$$(\widetilde{R}x)(t) = R(x_t).$$

Let $R^{(s)} : C([-1,1]) \to \mathbb{R}^n$ for $0 \leq s \leq \ell$ be such that

$$D^s[t \mapsto R(x_t)] = R^{(s)}(x_t).$$

Modify each component separately with cut-off χ.

Leads to globally Lipschitz continuous substitution map $\widetilde{\mathbf{R}}_\delta : BC_\eta \to (BC_\eta)^{\ell+1}$, given by

$$(\widetilde{\mathbf{R}}_\delta x)_s = \chi(\|x\|_t / \delta) R^{(s)}(x_t).$$

Fix point equation

Fix point equation now given by

$$u = y + \mathcal{K}_\eta \widetilde{\mathbf{R}}_\delta(u),$$

with

- $y \in \mathcal{N}_0$ a polynomially bounded solution to $0 = Ly_t$.
- $u \in BC^1_\eta(\mathbb{R}, \mathbb{R}^n)$.

Can be treated mostly using standard techniques.

Special care needed to ensure that fix points are indeed solutions to our inital equation.

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