Propagation failure in the discrete Nagumo equation

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Abstract

We address the classical problem of propagation failure for monotonic fronts of the discrete Nagumo equation. For a special class of nonlinearities that support unpinned "translationally invariant" stationary monotonic fronts, we prove that propagation failure cannot occur. Properties of travelling fronts in the discrete Nagumo equation with such special nonlinear functions appear to be similar to those in the continuous Nagumo equation.

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1 Introduction

To illustrate the central topic of this paper, let us consider the discrete Nagumo equation

$$\dot{u}_j = \frac{1}{h^2}\big(u_{j+1} + u_{j-1} - 2u_j\big) + f(u_j\,;\,a), \qquad j \in \mathbb{Z}, \tag{1.1}$$

with the cubic nonlinearity

$$f(u\,;\,a) = 2(1-u^2)(u-a), \qquad -1 < a < 1. \tag{1.2}$$

This lattice differential equation (LDE) plays an important role when studying signal propagation through nerve fibres [7, 19] and has inspired a large volume of work on spatially discrete models in

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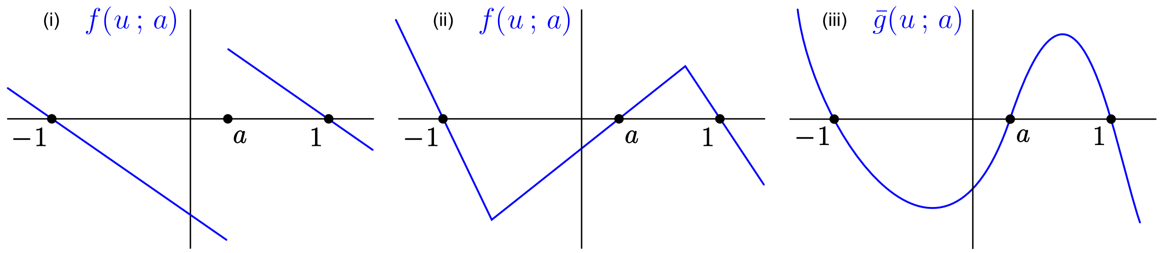

Fig. 1: *Panel (i) depicts the McKean caricature* $f(u\,;a) = \mathrm{sign}(u-a) - u$*, while the piecewise linear zigzag nonlinearity considered in [11] is shown in panel (ii). Panel (iii) illustrates the assumption (Hg2) on* $\bar{g}$*.*

many different scientific areas [4, 5, 9, 13, 22]. One may arrive at (1.1) by discretizing the Nagumo PDE

$$u_t = u_{xx} + f(u\,;a), \qquad x \in \mathbb{R}, \tag{1.3}$$

on a lattice with node distance h. It is well-known that for any $a \in (-1,1)$, the PDE (1.3) admits travelling front solutions $u(x,t) = \bar{u}(x - ct)$ with

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1. \tag{1.4}$$

The unique wave speed $c = c(a)$ satisfies $c(0) = 0$ and $\partial_a c(0) > 0$ and the wave profile has $\bar{u}'(\xi) > 0$ for all $\xi \in \mathbb{R}$.

Similarly, there exists a unique wave speed $c = c(a)$ for which the LDE (1.1) admits a travelling front $u_j(t) = \bar{u}(j + ct)$ that satisfies (1.4) and is nondecreasing with respect to j [23]. As above, the wave speed is nondecreasing with respect to a, but it no longer needs to be strictly increasing. In fact, writing $[a_-, a_+]$ for the maximal interval on which $c(a) = 0$, it may happen that $a_- < a_+$, in which case we say that (1.1) admits *propagation failure*.

Keener established that the Nagumo LDE (1.1) with the cubic nonlinearity (1.2) admits propagation failure for all sufficiently large h [20], but at present it is still unknown whether this holds for all $h > 0$. A significant step towards confirming this was made by Hoffman and Mallet-Paret [16]. These authors provided a generic condition on the nonlinearity f in (1.1) that guarantees propagation failure, but unfortunately this condition is hard to verify in practice.

We emphasize that these issues may depend subtly on the nonlinearity. For example, the explicit calculations in [6] show that (1.1) with the McKean caricature depicted in Figure 1(i) admits propagation failure for all $h > 0$. The theory developed in [24] shows that this also holds for smooth nonlinearities that are sufficiently close to this sawtooth. On the other hand, if f is given by a piecewise linear zigzag nonlinearity as in Figure 1(ii), one can exclude this phenomenon for all h in a countably infinite set [11]. Additional results and numerical studies can be found in [1, 12–14].

Stationary solutions: discrete families

In order to understand the mechanism that causes propagation failure, it is important to study the stationary solutions to (1.1). To this end, let us look for stationary solutions $u(t) = p$ to (1.1)-(1.2) by writing $r_j = p_{j+1}$ and solving the discrete planar system

$$\begin{array}{rcl} p_{j+1} & = & r_j, \\ r_{j+1} & = & -p_j + 2r_j - h^2 f(r_j\,;a). \end{array} \tag{1.5}$$

One may easily verify that the two equilibria $(\pm 1, \pm 1)$ are both saddles at $a = 0$. Following Qin and Xiao [25], dynamical system methods can now be used to show that (1.5) admits solutions $p^{(s)}$ and

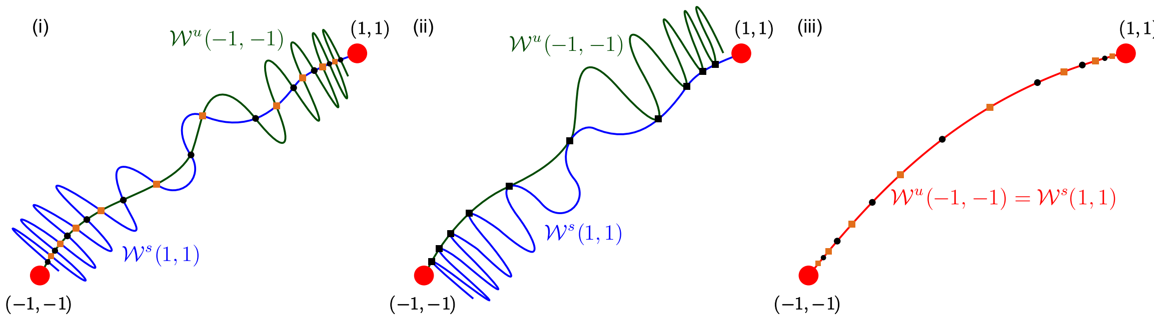

Fig. 2: *The stable and unstable manifolds $\mathcal{W}^s(1,1)$ and $\mathcal{W}^u(-1,-1)$ associated to the discrete system (1.5) can intersect in several ways. Black circles are used to represent the site-centered solutions $p^{(s)}$ and orange squares are used for the bond-centered solutions $p^{(b)}$. In Panel (i) the aforementioned manifolds intersect transversely, which means $p^{(s)}$ and $p^{(b)}$ will persist for $a \approx 0$. Panel (ii) illustrates the saddle-node bifurcation at $a = a_\pm$ that destroys these two branches of stationary solutions. Panel (iii) depicts the case covered by assumption (Hp). One can interpolate freely between $p^{(s)}$ and $p^{(b)}$ to find a one-parameter family of stationary solutions $p^{(\vartheta)}$ to (1.9) at $a = 0$.*

$p^{(b)}$ that satisfy

$$\lim_{j\to\pm\infty} p_j^{(s)} = \lim_{j\to\pm\infty} p_j^{(b)} = \pm 1, \qquad p_{-j}^{(s)} = -p_j^{(s)}, \quad p_{-j+1}^{(b)} = -p_j^{(b)}. \tag{1.6}$$

These solutions are hence referred to as site-centered and bond-centered solutions. For each $j \in \mathbb{Z}$, the pairs $\big(p_j^{(s)}, p_{j+1}^{(s)}\big)$ and $\big(p_j^{(b)}, p_{j+1}^{(b)}\big)$ lie in the intersection of the unstable manifold $W^u(-1,-1)$ and the stable manifold $W^s(1,1)$. If these intersections are transverse, both solutions $p^{(s)}$ and $p^{(b)}$ will persist for $a \approx 0$. One expects these branches to coalesce and terminate in a saddle–node bifurcation at $a = a_\pm$; see panels (i) and (ii) in Figure 2. The first nonrigorous analysis of this type for (1.1)-(1.2) was performed by Erneux and Nicolis [13].

Stationary solutions: continuous families

In this paper, we are interested in the degenerate situation that $p^{(s)}$ and $p^{(b)}$ are part of a smooth family of stationary solutions. As illustrated in Figure 2(iii), this means that $\mathcal{W}^s(1,1)$ and $\mathcal{W}^u(-1,-1)$ coincide at $a = 0$. In this case, it is not immediately clear if any intersections of these two manifolds survive for $a \neq 0$.

We do not expect this type of degeneracy to occur for (1.1) with the cubic nonlinearity (1.2), but there certainly are more general discretizations of the PDE (1.3) with (1.2) that do have this property. Consider for example the LDE

$$\dot{u} = \frac{1}{h^2}\big(u_{j-1} + u_{j+1} - 2u_j\big) + (1 - u_j^2)(u_{j+1} + u_{j-1} - 2a), \qquad j \in \mathbb{Z}. \tag{1.7}$$

One may directly verify that for any $a, \vartheta \in \mathbb{R}$, this equation is satisfied by

$$u_j(t) = \tanh\big(\operatorname{arcsinh}(h)(j - ct + \vartheta)\big), \quad c = \frac{2a}{\operatorname{arcsinh}(h)}. \tag{1.8}$$

We thus have a branch of stationary solutions at $a = 0$ that is parametrized by $\vartheta \in \mathbb{R}$. In particular, we are in the situation depicted in Figure 2(iii). In [3], Barashenkov and his coauthors constructed several additional discretizations of (1.3) that also have this degeneracy, generalizing earlier results in [10, 15, 21, 26].

In view of the explicit solutions (1.8), it is clear that $a_- = a_+ = 0$ holds for (1.7). In particular, this equation does not suffer from propagation failure. In terms of Figure 2(iii), the manifolds

$\mathcal{W}^u(-1,1)$ and $\mathcal{W}^s(1,1)$ separately completely as a moves away from zero and therefore none of the stationary solutions mentioned above survives this transition. The absence of propagation failure can be very useful in many situations, for example when using LDEs to spatially discretize a PDE.

Main Results

The main goal of this paper is to show that the situation described above for (1.7) extends to the broad class of bistable parameter-dependent LDEs that are commonly referred to as *normal families* [23]. In particular, we consider the LDE

$$\dot{u}_j = g\big(u_{j-1}, u_j, u_{j+1}\,;\, a\big), \qquad j \in \mathbb{Z}, \qquad u_j \in \mathbb{R} \tag{1.9}$$

and assume that g is monotonically increasing with respect to u_{j-1} and u_{j+1}, while $\bar{g}(u\,;\,a) := g(u,u,u\,;\,a)$ behaves much like the cubic $(u-a)(1-u^2)$; see Figure 1(iii). Furthermore, we assume that at some $a = a_*$, the system (1.9) admits a smooth one-parameter branch of stationary solutions that is 'translationally invariant' in the sense of Figure 2(iii). Our main results state that these conditions are sufficient to prevent (1.9) with $a \neq a_*$ from having any stationary solutions that increase monotonically with respect to $j \in \mathbb{Z}$.

The conditions (Hg1)-(Hg2) below give a more precise definition of the concept of a normal family. We remark that these assumptions are slightly stronger than their counterparts in [23], where less smoothness was imposed on the nonlinearity g.

(Hg1) The nonlinearity g is C^3-smooth, with $\partial_{u_1} g(u_1,u_2,u_3;a) > 0$ and $\partial_{u_3} g(u_1,u_2,u_3;a) > 0$ for all $(u_1,u_2,u_3) \in \mathbb{R}^3$ and $a \in (-1,1)$. In addition, we have

$$\partial_a g\big(u_1,u_2,u_3;a\big) < 0 \tag{1.10}$$

for all $a \in (-1,1)$ and all $(u_1,u_2,u_3) \in \mathbb{R}^3$ that have $-1 < u_1 < u_2 < u_3 < 1$.

(Hg2) Setting $\bar{g}(u;a) := g(u,u,u;a)$, we have

$$\begin{array}{ll} \bar{g}(\pm 1;a) = 0, & \bar{g}(a;a) = 0, \\ \bar{g}(u;a) < 0 \text{ for } u \in (-1,a) \cup (1,\infty), & \bar{g}(u;a) > 0 \text{ for } u \in (-\infty,-1) \cup (a,1), \end{array} \tag{1.11}$$

for every $-1 < a < 1$, together with

$$\begin{array}{ll} \partial_u \bar{g}(\pm 1;a) < 0, & \partial_u \bar{g}(a;a) > 0, \\ \partial_{ua} \bar{g}(-1;a) < 0, & \partial_{ua} \bar{g}(1;a) > 0. \end{array} \tag{1.12}$$

The degeneracy requirement that we need to impose on the stationary solutions to (1.9) at $a = a_*$ is given by the following.

(Hp) There exists a $\bar{p} \in BC^3(\mathbb{R},\mathbb{R})$[1] such that for any $\vartheta \in \mathbb{R}$, the constant function $u(t) = p^{(\vartheta)}$ given by

$$p_j^{(\vartheta)} = \bar{p}(j + \vartheta) \tag{1.13}$$

satisfies (1.9) with $a = a_*$ for some $a_* \in (-1,1)$. In addition, this function $\bar{p}$ has $\bar{p}'(\xi) > 0$ for all $\xi \in \mathbb{R}$ and satisfies the limits

$$\lim_{\xi \to \pm\infty} \bar{p}(\xi) = \pm 1. \tag{1.14}$$

[1] The notation BC^3 means that $\bar{p}$ and its first three derivatives are uniformly bounded on $\mathbb{R}$.

Theorem 1.1. *Consider the system (1.9) and suppose that (Hg1), (Hg2) and (Hp) are satisfied. Then for every* $a \in (-1, 1)$*, the system (1.9) admits a solution of the form*

$$u_j(t) = \bar{u}(j - ct) \tag{1.15}$$

for some wave speed $c \in \mathbb{R}$ *and wave profile* $\bar{u} \in C^1(\mathbb{R}, \mathbb{R})$ *that has* $\bar{u}'(\xi) > 0$ *for all* $\xi \in \mathbb{R}$ *and*

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1. \tag{1.16}$$

The wave speed $c = c(a)$ *depends* C^1*-smoothly on* a*, with* $c(a_*) = 0$ *and* $\partial_a c(a) > 0$ *for all* $a \in (-1, 1)$*.*

The next two results address the uniqueness of the travelling waves described above. The first one excludes two classes of stationary solutions for $a \neq a_*$, namely j-monotonic solutions and solutions that are close to p^{ϑ} for some $\vartheta \in \mathbb{R}$, but not necessarily j-monotonic. The second result states that the waves appearing in Theorem 1.1 are unique among all waves that connect ± 1 and travel with non-zero speed.[2] This actually follows directly from [23, Thm. 2.1], but we include it here for completeness.

Corollary 1.2. *Consider the system (1.9) and suppose that (Hg1), (Hg2) and (Hp) are satisfied. There exists a constant* $\delta > 0$ *such that the following holds true. Suppose that (1.9) admits a stationary solution*

$$u_j(t) = u_j \tag{1.17}$$

for some $a \in (-1, 1)$*. Suppose that* $u_{j_1} \leq u_{j_2}$ *holds for all* $j_1 \leq j_2$ *together with* $\lim_{j \to \pm\infty} u_j = \pm 1$*, or alternatively that* $|a - a_*| < \delta$ *and* $\left|u - p^{(\vartheta)}\right| < \delta$ *for some* $\vartheta \in \mathbb{R}$*. Then we must have* $a = a_*$*.*

Corollary 1.3 ([23, Thm. 2.1]). *Consider the system (1.9) and suppose that (Hg1), (Hg2) and (Hp) are satisfied. Pick any* $a \in (-1, 1)$ *and suppose that (1.9) admits a solution of the form*

$$u_j(t) = \bar{u}(j - ct) \tag{1.18}$$

for some $c \neq 0$ *and* $\bar{u} \in C^1(\mathbb{R}, \mathbb{R})$ *that satisfies the limits*

$$\lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1. \tag{1.19}$$

Then u *must be a temporal translate of the solution described in Theorem 1.1.*

The proof of Theorem 1.1 and Corollary 1.2 can be found in §2. We conclude the paper in §3 with some numerical examples and a brief discussion.

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2 Proof of Main Results

In this section we prove Theorem 1.1 and Corollary 1.2. We focus on the dynamics of the lattice system (1.9) for a near a_*. In particular, we show that the manifold of equilibria $\mathcal{M}(a_*) = \{p^{(\vartheta)}\}_{\vartheta \in \mathbb{R}}$ at $a = a_*$ persists as an invariant manifold $\mathcal{M}(a)$ for (1.9) with a near a_*. To aid us, we write $\mathcal{T}$ for the right-shift operator that acts as $(\mathcal{T}u)_j = u_{j-1}$ and note that

$$p^{(\vartheta)} = \mathcal{T} p^{(\vartheta+1)} \tag{2.1}$$

[2] Notice that this excludes the possibility of non-monotonic travelling waves that connect ± 1.

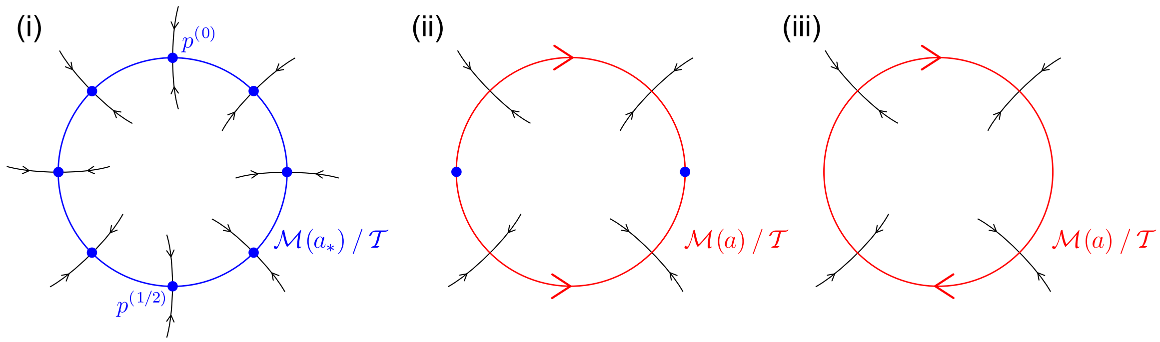

Fig. 3: *Panel (i) depicts the manifold $\mathcal{M}(a_*)$ after factoring out the symmetry $\mathcal{T}$. Each point on the circle represents an equivalence class $p^{(\vartheta)}$ with $\vartheta \in S^1$. We show that $\mathcal{M}(a_*)$ is normally hyperbolic and that it persists as $\mathcal{M}(a)$ for a near a_*. As illustrated in panel (ii), one or more of the equilibria can survive for $a \neq a_*$, in which case (1.9) suffers from propagation failure. However, if (1.9) is a normal family, the flow on $\mathcal{M}(a)$ must behave as shown in panel (iii), inducing travelling wave solutions to (1.9).*

holds for all $\vartheta \in \mathbb{R}$. After factoring out the symmetry $\mathcal{T}$, the manifold $\mathcal{M}(a_*)$ can hence be seen as a circle of equilibria; see Figure 3(i). In principle, some of these equilibria can survive for $a \neq a_*$, as illustrated in Figure 3(ii) and discussed in §3. However, by computing the flow on $\mathcal{M}(a)$ to leading order, we show that the situation described in Figure 3(iii) arises whenever the system (1.9) is a normal family. The travelling waves described in Theorem 1.1 can subsequently be read off from the shift-periodic solution to (1.9) that is induced by the flow on $\mathcal{M}(a)$.

For convenience, we rewrite the lattice system (1.9) as

$$\dot{u} = \mathcal{F}(u, a) \tag{2.2}$$

and look for solutions u that take values in the sequence space

$$\ell^\infty = \{u \in \mathbb{R}^{\mathbb{Z}} : |u|_{\ell^\infty} := \sup_{j\in\mathbb{Z}} |u_j| < \infty\}. \tag{2.3}$$

The nonlinearity $\mathcal{F} : \ell^\infty \times (-1, 1) \to \ell^\infty$ inherits the C^3-smoothness of the function g in (1.9). For $v, w \in \ell^\infty$, we introduce the pairing

$$\langle v, w\rangle = \sum_{j\in\mathbb{Z}} v_j w_j, \tag{2.4}$$

with the warning that additional constraints on v or w are needed to ensure that this sum is well-defined.

In order to show that $\mathcal{M}(a_*)$ persists as an invariant manifold, we need to show that it is normally hyperbolic. To this end, let us introduce the operator $L^{(\vartheta)} \in \mathcal{L}(\ell^\infty)$ that is given by

$$L^{(\vartheta)} = \partial_u \mathcal{F}\big(p^{(\vartheta)}, a_*\big). \tag{2.5}$$

Componentwise, we have

$$\begin{array}{rcl} (L^{(\vartheta)}v)_j & = & \partial_{u_1} g\big(\bar{p}(j+\vartheta-1), \bar{p}(j+\vartheta), \bar{p}(j+\vartheta+1)\,; a_*\big)v_{j-1} \\ & & +\partial_{u_2} g\big(\bar{p}(j+\vartheta-1), \bar{p}(j+\vartheta), \bar{p}(j+\vartheta+1)\,; a_*\big)v_j \\ & & +\partial_{u_3} g\big(\bar{p}(j+\vartheta-1), \bar{p}(j+\vartheta), \bar{p}(j+\vartheta+1)\,; a_*\big)v_{j+1} \end{array} \tag{2.6}$$

and it is easily verified that $L^{(\vartheta)}\bar{p}'(\vartheta + \cdot) = 0$. The formal adjoint of $L^{(\vartheta)}$ is given by

$$\begin{array}{rcl} (L^{(\vartheta)*}w)_j & = & \partial_{u_1} g\big(\bar{p}(j+\vartheta), \bar{p}(j+\vartheta+1), \bar{p}(j+\vartheta+2)\,; a_*\big)w_{j+1} \\ & & +\partial_{u_2} g\big(\bar{p}(j+\vartheta-1), \bar{p}(j+\vartheta), \bar{p}(j+\vartheta+1)\,; a_*\big)w_j \\ & & +\partial_{u_3} g\big(\bar{p}(j+\vartheta-2), \bar{p}(j+\vartheta-1), \bar{p}(j+\vartheta)\,; a_*\big)w_{j-1}. \end{array} \tag{2.7}$$

Our first result states that $L^{(\vartheta)}$ is a Fredholm operator.

Lemma 2.1. *For each $\vartheta \in \mathbb{R}$, $L^{(\vartheta)} : \ell^\infty \to \ell^\infty$ is a Fredholm operator with index zero. In addition, solutions $v = \{v_j\} \in \ell^\infty$ of $L^{(\vartheta)}v = 0$ or $L^{(\vartheta)*}v = 0$ decay exponentially as $j \to \pm\infty$. Finally, we have the characterization*

$$\operatorname{Range} L^{(\vartheta)} = \{x \in \ell^\infty \mid \langle w, x \rangle = 0 \textit{ for all } w \in \operatorname{Ker} L^{(\vartheta)*}\}. \tag{2.8}$$

Proof. We consider the characteristic functions associated to the operator $L^{(\vartheta)}$ in the limits $j \to \pm\infty$, which are given by

$$\Delta_\pm(z) = \partial_{u_1} g(\pm1, \pm1, \pm1; a_*)e^{-z} + \partial_{u_2} g(\pm1, \pm1, \pm1; a_*) + \partial_{u_3} g(\pm1, \pm1, \pm1; a_*)e^{z}. \tag{2.9}$$

Our assumption $\partial_u \bar{g}(\pm1, a_*) < 0$ implies that the equations $s\Delta_-(z) + (1-s)\Delta_+(z) = 0$ do not admit roots with $\operatorname{Re} z = 0$ for any $0 \le s \le 1$. The statement now follows directly from Corollary 2.6 and Theorems 3.2 and 4.3 in [2]. □

We now proceed to show that $\mathcal{M}$ normally hyperbolic. For any operator $L \in \mathcal{L}(\ell^\infty)$, we introduce the spectral sets[3]

$$\begin{aligned} \sigma_{\text{ess}}(L) &= \{\zeta \in \mathbb{C} : \lambda I - L \text{ is not a Fredholm operator with index zero}\}, \\ \sigma_p(L) &= \{\lambda \in \mathbb{C} \setminus \sigma_{\text{ess}}(L) : Lv = \lambda v \text{ for some non-zero } v \in \ell^\infty\}. \end{aligned} \tag{2.10}$$

Instead of studying the eigenvalue equation $L^{(\vartheta)}v = \lambda v$ directly, we adapt the comparison principle technique developed in [8, §8] to analyze the ODE

$$\dot{v} = L^{(\vartheta)}v \tag{2.11}$$

and determine the growth rate of solutions.

Lemma 2.2. *There exists $\delta > 0$ such that $\lambda I - L^{(\vartheta)}$ is invertible for each $\vartheta \in \mathbb{R}$ and each $\lambda \in \mathbb{C}\setminus\{0\}$ with $\operatorname{Re}\lambda \ge -\delta$. In addition, we have*

$$\operatorname{Ker} L^{(\vartheta)} = \operatorname{span}\{\bar{p}'(\vartheta + \cdot)\}, \qquad \operatorname{Ker} L^{(\vartheta)*} = \operatorname{span}\{\bar{q}(\vartheta + \cdot)\}, \tag{2.12}$$

in which $\bar{q} \in BC^2(\mathbb{R}, \mathbb{R})$ has $\bar{q}(\xi) > 0$ for all $\xi \in \mathbb{R}$. Finally, there is no $v \in \ell^\infty$ that satisfies $L^{(\vartheta)}v = \bar{p}'(\vartheta + \cdot)$.

Proof. Using the assumption $\partial_u \bar{g}(\pm1, a_*) < 0$, we may argue as in the proof of Lemma 2.1 to find a small $\delta > 0$ such that $\operatorname{Re}\lambda < -\delta$ holds for any $\lambda \in \sigma_{\text{ess}}(L^{(\vartheta)})$.

We study the point spectrum in an indirect fashion by looking at the ODE (2.11) posed on ℓ^∞, which admits the stationary solution $v_j(t) = \bar{p}'(\vartheta + j)$. Let us now pick $\beta > 0$ in such a way that $\partial_u \bar{g}(\pm1; a_*) < -\beta$. Since $\bar{p}' > 0$, there exists $K > 0$ such that

$$\beta K \bar{p}'(\vartheta + j) - \beta - (L^{(\vartheta)}\mathbf{1})_j \ge 0 \tag{2.13}$$

holds for all $j \in \mathbb{Z}$, which can be verified using the limits

$$\lim_{j \to \pm\infty} (L^{(\vartheta)}\mathbf{1})_j = \partial_u \bar{g}(\pm1; a_*) < -\beta. \tag{2.14}$$

The assumptions $\partial_{u_1} g > 0$ and $\partial_{u_3} g > 0$ imply that (2.11) admits a comparison principle. More precisely, any solution v to (2.11) satisfies

$$|v_j(t)| \le \Big(e^{-\beta t} + K(1 - e^{-\beta t})\bar{p}'(\vartheta + j)\Big)\, |v(0)|_{\ell^\infty} \tag{2.15}$$

[3]Where appropriate, ℓ^∞ should be interpreted as a subset of $\mathbb{C}^{\mathbb{Z}}$ instead of $\mathbb{R}^{\mathbb{Z}}$.

for all $t \geq 0$, which can be established as in [8, §8]. For any $T > 0$, we write Φ_T for the bounded linear map that sends $v(0)$ to $v(T)$ for solutions to (2.11). Following the construction in [8, §8], we conclude that

$$\left|\sigma_{\mathrm{ess}}\big(\Phi_T\big)\right| < 1, \qquad \sigma_p\big(\Phi_T\big) \setminus \{1\} \subset \{\zeta \in \mathbb{C} : |\zeta| < 1\}, \tag{2.16}$$

while $1 \in \sigma_p(\Phi_T)$ is a simple eigenvalue. Recall that eigenvalues outside the essential spectrum are isolated and $\sigma_p\big(L^{(\vartheta+1)}\big) = \sigma_p\big(L^{(\vartheta)}\big)$. All statements concerning $L^{(\vartheta)}$ now follow from these observations, exploiting the fact that $T > 0$ can be chosen arbitrarily.

For every $\vartheta \in \mathbb{R}$, Lemma 2.1 implies that there exists a non-trivial $q^{(\vartheta)} \in \mathrm{Ker}(L^{(\vartheta)*})$. Since $\bar{p}'(\vartheta + \cdot)$ depends C^2-smoothly on ϑ, one can ensure that the same holds for $q^{(\vartheta)}$, with $q_j^{(\vartheta+1)} = q_{j+1}^{(\vartheta)}$. One can now take $\bar{q}(\xi) = q_0^{(\xi)}$. Finally, one may imitate the proof of [23, Thm 4.1] to show that $\bar{q}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. □

We remark that [23, Thm. 2.2] implies that $\bar{p}$ approaches its limiting values at an exponential rate. In addition, since $\mathcal{F}$ is C^3-smooth, Lemma 2.1 implies that $\bar{p}'$, $\bar{p}''$, $\bar{p}'''$, $\bar{q}$, $\bar{q}'$ and $\bar{q}''$ all decay exponentially at $\pm\infty$. In particular, we may define a C^2-smooth family of projections $P(\vartheta) \in \mathcal{L}(\ell^\infty)$ by

$$P(\vartheta) = \bar{p}'(\vartheta + \cdot)Q(\vartheta), \qquad Q(\vartheta)v = \langle q^{(\vartheta)}, v\rangle, \tag{2.17}$$

after normalizing $q^{(\vartheta)}$ to ensure that $Q(\vartheta)\bar{p}'(\vartheta + \cdot) = 1$.

Since we are interested in the dynamics of (2.2) near the manifold $\mathcal{M} = \{p^{(\vartheta)}\}_{\vartheta\in\mathbb{R}}$, we look for solutions that can be written as

$$u(t) = p^{(\theta(t))} + v(t), \tag{2.18}$$

for some functions $\theta \in C^1(\mathbb{R}, \mathbb{R})$ and $v \in C^1(\mathbb{R}, \ell^\infty)$ that satisfy the normalization condition

$$Q\big(\theta(t)\big)v(t) = 0, \qquad t \in \mathbb{R}. \tag{2.19}$$

In terms of these new coordinates, we have $\mathcal{M}(a_*) = \{(\vartheta, 0)\}_{\vartheta\in\mathbb{R}}$. For a near a_*, this invariant manifold persists as $\mathcal{M}(a) = \{\vartheta, v_*(\vartheta, a)\}$, in which the function v_* is described by the following result.

Lemma 2.3. *Consider the LDE (2.2) and suppose that (Hg1),(Hg2) and (Hp) are all satisfied. Then there exists a constant $\delta_a > 0$ together with a C^1-smooth function $v_* : \mathbb{R} \times [a_* - \delta_a, a_* + \delta_a] \to \ell^\infty$ such that the following holds true.*

(i) For some constant $C > 0$ we have $|v_*(\vartheta, a)|_{\ell^\infty} \leq C\,|a - a_*|$ for all $\vartheta \in \mathbb{R}$. In addition, $v_*(\vartheta, a) = \mathcal{T} v_*(\vartheta + 1, a)$.

(ii) There exists a constant $\delta > 0$ such that any solution to (2.2) with $|a - a_*| < \delta_a$ of the form (2.18) - (2.19) that has $|v(t)|_{\ell^\infty} < \delta$ for all $t \in \mathbb{R}$, must have $v(t) = v_*\big(\theta(t), a\big)$ for all $t \in \mathbb{R}$.

(iii) Consider the C^2-smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(\vartheta) = Q(\vartheta)\partial_a\mathcal{F}(p^{(\vartheta)}, a_*). \tag{2.20}$$

There exists a C^1-smooth function $h_* : \mathbb{R} \times [a_* - \delta_a, a_* + \delta_a] \to \mathbb{R}$ with $h_*(\vartheta + 1, a) = h_*(\vartheta, a)$ and $h_*(\vartheta, a) = O(|a - a_*|^2)$, such that any solution to the ODE

$$\dot{\theta} = (a - a_*)\Psi(\theta) + h_*(\theta, a), \tag{2.21}$$

with $|a - a_*| < \delta_a$ yields a solution $u(t) = p^{(\theta(t))} + v_*(\theta(t), a)$ to (2.2).

Proof. Without loss of generality, we assume that $a_* = 0$. Choose a C^∞-smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(v) = v$ when $|v| < 1$ and $\chi(v) = 0$ when $|v| > 2$. For $\delta > 0$, write $\chi_\delta : \ell^\infty \to \ell^\infty$ for the C^∞-smooth function

$$\chi_\delta(v)_j = \chi(v_j/\delta), \tag{2.22}$$

Plugging the Ansatz (2.18) into (2.2), we find that any solution that has $|v(t)|_{\ell^\infty} < \delta$ for all $t \in \mathbb{R}$ will satisfy

$$\begin{array}{rcl} \dot{v} & = & L^{(\theta)}v + a\big(I - P(\theta)\big)\partial_a \mathcal{F}(p^{(\theta)}, 0) + \mathcal{N}_1(\theta, v, a) \\ \dot{\theta} & = & aQ(\theta)\partial_a \mathcal{F}(p^{(\theta)}, 0) + \mathcal{N}_2(\theta, v, a), \end{array} \tag{2.23}$$

in which

$$\begin{array}{rcl} \mathcal{N}_1(\vartheta, v, a) & = & \big(I - P(\vartheta)\big)\mathcal{N}(\vartheta, \chi_\delta(v), a) + P(\vartheta)\mathcal{S}(\vartheta, \chi_\delta(v), a), \\ \mathcal{N}_2(\vartheta, v, a) & = & Q(\vartheta)\mathcal{N}(\vartheta, \chi_\delta(v), a) - Q(\vartheta)\mathcal{S}(\vartheta, \chi_\delta(v), a), \end{array} \tag{2.24}$$

with

$$\begin{array}{rcl} \mathcal{N}(\vartheta, v, a) & = & \mathcal{F}(p^{(\vartheta)} + v, a) - L^{(\vartheta)}v - a\partial_a \mathcal{F}(p^{(\vartheta)}, 0), \\ \mathcal{S}(\vartheta, v, a) & = & \big[1 - (1 - Q'(\vartheta)v)^{-1}\big]\big[a\partial_a \mathcal{F}(p^{(\vartheta)}, 0) + \mathcal{N}(\vartheta, v, a)\big]. \end{array} \tag{2.25}$$

By construction, there exists $C > 0$ such that

$$\begin{array}{rcl} |\mathcal{N}_i(\vartheta, v, a)| & \leq & C(\delta + \delta_a)^2, \\ |\mathcal{N}_i(\vartheta_1, v_1, a) - \mathcal{N}_i(\vartheta_2, v_2, a)| & \leq & C(\delta + \delta_a)(|\vartheta_1 - \vartheta_2| + |v_1 - v_2|_{\ell^\infty}) \end{array} \tag{2.26}$$

hold for $i = 1, 2$ and $|a| < \delta_a$. One may now proceed as in [17, §6-7] to show that (2.23) admits a unique solution $(v(t), \theta(t))$ for any $|a| < \delta_a$ and any initial condition $\theta(0) = \vartheta_0$ with $Q(\vartheta_0)v(0) = 0$. The desired function v_* is now given by $v_*(\vartheta_0, a) = v(0)$. □

With Lemma 2.3 in hand, our main result can easily be established by analyzing the scalar ODE (2.21) that describes the flow on $\mathcal{M}(a)$.

Proof of Theorem 1.1. Without loss of generality, assume again that $a_* = 0$. The results in [23, §2] imply that $c(a)$ is C^1-smooth with $\partial_a c(a) > 0$ whenever $c \neq 0$.[4] We hence only need to consider a near zero. The inequality (1.10) implies that

$$\Psi(\vartheta) = Q(\vartheta)\partial_a \mathcal{F}(p^{(\vartheta)}, 0) < 0 \tag{2.27}$$

for all $\vartheta \in \mathbb{R}$. Introducing the rescaled time $\tau = at$, (2.21) becomes

$$\partial_\tau \theta(\tau) = \Psi\big(\theta(\tau)\big) + a^{-1}h_*(\theta(\tau), a). \tag{2.28}$$

In particular, there exists $T_* > 0$ and a C^1-smooth function $T(a) = O(a)$, defined for a near zero, such that any solution to (2.28) will have

$$\theta(\tau + T_* + T(a)) = \theta(\tau) - 1 \tag{2.29}$$

for all $\tau \in \mathbb{R}$. Picking $a \neq 0$ and writing $u(t)$ for the associated solution to (2.2), we see that

$$u\big(t + a^{-1}(T_* + T(a)\big) = \mathcal{T}u(t). \tag{2.30}$$

Choosing $\bar{u}(\xi) = u_0\big(-\xi a^{-1}(T_* + T(a))\big)$, one easily verifies that

$$u_j(t) = \bar{u}\Big(j - at/\big(T_* + T(a)\big)\Big), \tag{2.31}$$

which means that u is a travelling wave that moves to the right with speed $c(a) = a/(T_* + T(a))$. This formula remains valid for $a = 0$, which establishes the C^1-smoothness of $c(a)$. It remains to show that $\bar{u}$ satisfies the limits (1.16), but this follows directly from [23, Lemma 6.1]. □

[4] This is the only place where the condition $\pm\partial_{ua}\bar{g}(\pm 1, a) > 0$ in (Hg2) is needed.

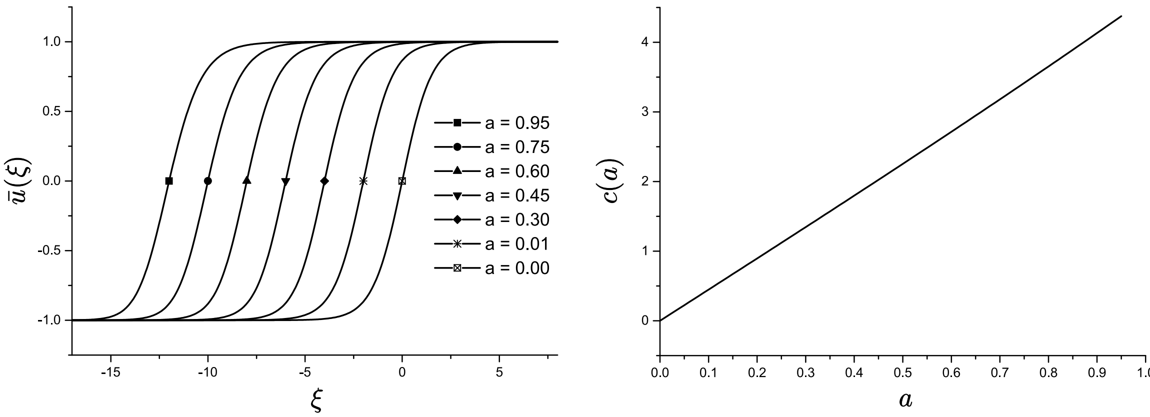

Fig. 4: *Selected wave profiles and wave speed plot for (3.1) with the nonlinearity $g = g_1$ defined in (3.7). No propagation failure occurs and the wave profiles $\bar{u}$ remain smooth as $a \to 0$.*

Proof of Corollary 1.2. If the stationary solution has $u_{j_1} \leq u_{j_2}$ whenever $j_1 \leq j_2$, the conclusion follows directly from [23, Thm. 2.1]. In the other case, the result follows directly from item (ii) in Lemma 2.3. □

3 Examples

In this section we illustrate the application range of our results, using the numerical method developed in [18]. In particular, we consider the lattice system

$$\dot{u} = g\big(u_{j-1}, u_j, u_{j+1}\,;\, a\big) \tag{3.1}$$

for three different families g. We search for wave solutions of the form

$$u_j(t) = \bar{u}(j - ct), \qquad \lim_{\xi \to \pm\infty} \bar{u}(\xi) = \pm 1, \tag{3.2}$$

by numerically solving the functional differential equation

$$-c\bar{u}'(\xi) = g\big(\bar{u}(\xi-1), \bar{u}(\xi), \bar{u}(\xi+1)\,;\, a\big). \tag{3.3}$$

All our examples admit a branch of stationary solutions $u(t) = p^{(\vartheta)} = \bar{p}(\vartheta + \cdot)$ at $a = 0$ that satisfy the condition (Hp). However, in our last two examples we consider two families g that violate the inequality (1.10) in the definition of a normal family. The presence of propagation failure now depends on whether or not the ODE

$$\dot{\theta} = (a - a_*)\Psi(\theta) + O(|a - a_*|^2) \tag{3.4}$$

with

$$\Psi(\vartheta) = \sum_{j \in \mathbb{Z}} \bar{q}(\vartheta + j)\partial_a g\big(\bar{p}(j + \vartheta - 1), \bar{p}(j + \vartheta), \bar{p}(j + \vartheta + 1)\,;\, a_*\big) \tag{3.5}$$

admits equilibria for $a \neq a_*$. To determine this, one needs to have detailed information on the function $\bar{q}$, which we compute numerically alongside with the wave profiles $\bar{u}$ by augmenting (3.3) at $a = 0$ with the appropriate equation[5] for $\bar{q}$.

[5] To avoid numerical issues that arise in the singular limit $c \to 0$, an extra term $-\gamma\bar{u}''(\xi)$ is added to the left hand side of (3.3), with $\gamma = 10^{-6}$. The actual equation used to determine $\bar{q}$ is given by

$$\begin{aligned} -\gamma\bar{q}''(\xi) &= (L^{(0)*}\bar{q})_\xi + \lambda(\xi)\bar{q}(\xi), \\ \lambda'(\xi) &= 0, \end{aligned} \tag{3.6}$$

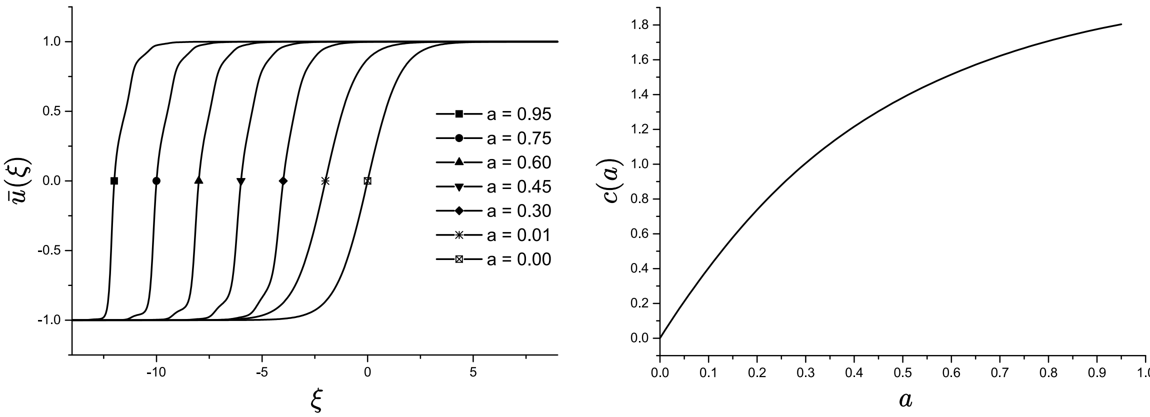

Fig. 5: *Selected wave profiles and wave speed plot for (3.1) with the nonlinearity $g = g_2$ defined in (3.9). As in Example 1, no propagation failure occurs and the wave profiles $\bar{u}$ remain smooth as $a \to 0$.*

Example 1: A Normal Family

Inspired by [3], we consider the lattice system (3.1) with the nonlinearity $g = g_1$ that is given by

$$g_1\big(u_{j-1}, u_j, u_{j+1}\,;\, a\big) = 4(u_{j-1} + u_{j+1} - 2u_j) + 2(u_j - a)(1 - u_{j-1}u_{j+1}). \tag{3.7}$$

This family satisfies the conditions (Hg1)-(Hg2) and is hence a normal family. When $a = 0$, one may easily verify that the branch[6]

$$u_j(t) = \tanh\big(\operatorname{arcsinh}(1/\sqrt{2})(j + \vartheta)\big), \qquad \vartheta \in \mathbb{R} \tag{3.8}$$

consists of stationary solutions to (3.1). In particular, (Hp) is also satisfied. Theorem 1.1 hence implies that (3.1) does not admit propagation failure, which is confirmed by the wave profiles and speeds depicted in Figure 4.

Example 2: A Non-Normal Family without Propagation Failure

Our second examples focusses on (3.1) with the nonlinearity $g = g_2$ that is given by

$$g_2\big(u_{j-1}, u_j, u_{j+1}\,;\, a\big) = 4(u_{j-1} + u_{j+1} - 2u_j) + 2(u_j - a)(1 - u_{j-1}u_{j+1}) + 5a\sin(\pi u_j). \tag{3.9}$$

Since $g_1(\cdot\,;\,0) = g_2(\cdot\,;\,0)$, condition (Hp) is also satisfied for this equation. However, the inequality (1.10) in the definition of a normal family is now violated. Nevertheless, the numerical results in Figure 5 indicate that (3.1) does not admit propagation failure. Indeed, one may verify numerically that $\Psi(\vartheta) < 0$ for all $\vartheta \in \mathbb{R}$, which shows that the ODE (3.4) admits no equilibria for sufficiently small $|a| > 0$.

Example 3: A Non-Normal Family with Propagation Failure

In our final example, we study (3.1) with the nonlinearity $g = g_3$ that is given by

$$\begin{aligned} g_3\big(u_{j-1}, u_j, u_{j+1}\,;\, a\big) \;&=\; u_{j-1} + u_{j+1} - 2u_j + (1 - u_j^2)(u_{j-1} + u_{j+1} - 2a) \\ &\qquad + 5a\sin(\pi u_j)(2 + \tfrac{4}{5}u_j). \end{aligned} \tag{3.10}$$

---

with boundary conditions $q(-L) = 0$, $q(L) = 0$ and $q(0) = 1$ for some large $L > 0$.

[6]Replacing the factor 4 in (3.7) by h^{-2}, the function $\xi \mapsto \tanh(\operatorname{arcsinh}(h/\sqrt{1-2h^2})\xi)$ generates stationary solutions to (3.1).

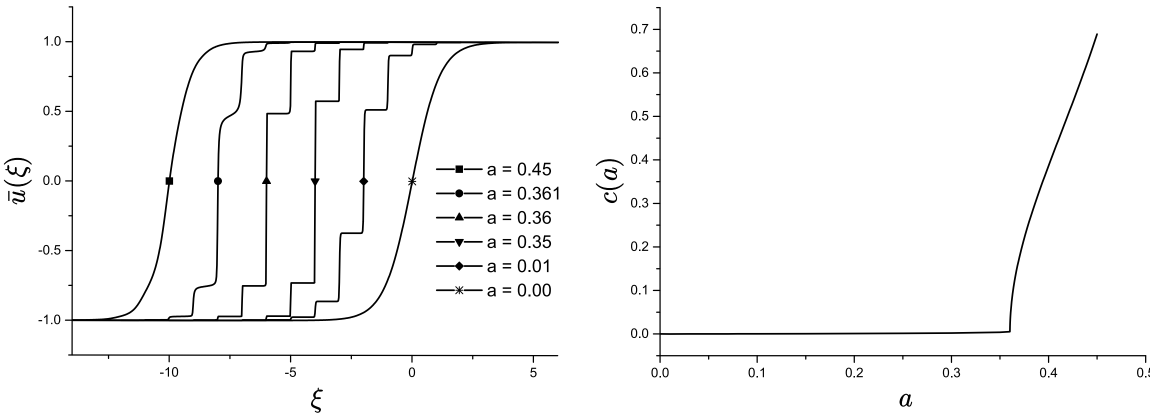

Fig. 6: *Selected wave profiles and wave speed plot for (3.1) with the nonlinearity* $g = g_3$ *defined in (3.10). Despite the smooth profile of the wave at* $a = 0$*, propagation failure occurs and the wave profiles are step functions for* $0 < a < a_+ \approx 0.36025$.

Recalling (1.8), we see that (3.1) with $a = 0$ has a branch of stationary solutions given by

$$u_j(t) = p_j^{(\vartheta)} = \bar{p}(j+\vartheta) = \tanh\big(\operatorname{arcsinh}(1)(j+\vartheta)\big), \tag{3.11}$$

which shows that (Hp) is satisfied. However, as in Example 2, the inequality (1.10) is violated.

We remark that the coefficients in g_3 were chosen in such a way that

$$\Psi(0) < 0 < \Psi(1/2). \tag{3.12}$$

The ODE (3.4) hence has at least two equilibria per unit interval whenever $|a|$ is sufficiently small. In particular, we expect (3.1) to admit propagation failure and this is confirmed in Figure 6.

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