Travelling Waves for Complete Discretizations of Reaction Diffusion Systems

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Abstract

In this paper we consider the impact that full spatial-temporal discretizations of reaction-diffusion systems have on the existence and uniqueness of travelling waves. In particular, we consider a standard second-difference spatial discretization of the Laplacian together with the six numerically stable backward differentiation formula (BDF) methods for the temporal discretization. For small temporal time-steps and a fixed spatial grid-size, we establish some useful Fredholm properties for the operator that arises after linearizing the system around a travelling wave. In particular, we perform a singular perturbation argument to lift these properties from the natural limiting operator. This limiting operator is associated to a lattice differential equation, where space has been discretized but time remains continuous.

For the backward-Euler temporal discretization, we also obtain travelling waves for arbitrary time-steps. In addition, we show that in the anti-continuum limit, in which the temporal time-step and the spatial grid-size are both very large, wave speeds are no longer unique. This is in contrast to the situation for the original continuous system and its spatial semi-discretization. This non-uniqueness is also explored numerically and discussed extensively away from the anti-continuum limit.

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1 Introduction

In this paper we study spatial-temporal discretizations of a class of bistable reaction-diffusion equations that includes the Nagumo PDE

$$u_t = u_{xx} + g_{\mathrm{cub}}(u; a), \qquad 0 < a < 1, \tag{1.1}$$

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which is also commonly referred to as the Allen-Cahn equation. The cubic nonlinearity is given by

$$g_{\mathrm{cub}}(u;a) = u(1-u)(u-a), \qquad 0 < a < 1. \tag{1.2}$$

Our goal is to understand the impact that such discretizations have on travelling front solutions of these systems. Such solutions have the form

$$u(x,t) = \Phi(x+ct), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1 \tag{1.3}$$

and play a fundamental role in the analysis of (1.1). Indeed, they are stable with a large domain of attraction, provide a mechanism by which the energetically favourable background state can invade the domain and serve as building-blocks for the construction of more complex patterns. Although explicit expressions for these fronts are available for (1.1), this is not the case in general and one frequently uses numerical approximations. It is hence rather desirable to understand the effects of the employed discretization scheme.

Discretization Schemes

The simplest spatial-temporal discretization scheme for (1.1) uses the forward-Euler method with time-step $\Delta t > 0$ for the temporal component, together with a second-difference stencil on a spatial grid with spacing $h > 0$. This provides approximants

$$u(hj, n\Delta t) \sim U_j(n\Delta t), \qquad (j,n) \in \mathbb{Z}^2 \tag{1.4}$$

that evolve as

$$\tfrac{1}{\Delta t}\big[U_j\big((n+1)\Delta t\big) - U_j(n\Delta t)\big] \quad = \quad \mathcal{F}_h\big(U_{j-1}(n\Delta t), U_j(n\Delta t), U_{j+1}(n\Delta t); a\big), \tag{1.5}$$

in which we have defined

$$\mathcal{F}_h\big(U_{j-1}, U_j, U_{j+1}; a\big) \quad = \quad \tfrac{1}{h^2}\big[U_{j-1} + U_{j+1} - 2U_j\big] + g_{\mathrm{cub}}\big(U_j; a\big). \tag{1.6}$$

Throughout most of the present paper we treat the spatial discretization as fixed. In this sense, one could alternatively state that we are interested in temporal discretizations of a class of bistable lattice differential equations (LDEs) that includes the Nagumo LDE

$$\dot{u}_j(t) = h^{-2}\big[u_{j-1}(t) + u_{j-1}(t) - 2u_j(t)\big] + g_{\mathrm{cub}}\big(u_j(t); a\big), \qquad 0 < a < 1. \tag{1.7}$$

Such LDEs arise naturally when modelling physical, chemical or biological systems that have an inherent discrete spatial structure, such as crystals [36], coupled chemical reactors [27] or myelinated nerve fibres [2]. The LDE (1.7) is by no means as well-studied as the PDE (1.1), but the literature concerning the former has expanded rapidly in recent decades.

Although intuitively appealing, the forward-Euler temporal discretization employed in (1.5) has a number of serious drawbacks. This can be seen by applying it to the test-problem $\dot{v} = \lambda v$ with $\lambda < 0$, which gives

$$v_{n+1} = v_n + \lambda \Delta t v_n = (1 + \lambda\Delta t)v_n. \tag{1.8}$$

In order to enforce $v_n \to 0$ we must hence demand $0 < \Delta t < 2\,|\lambda|^{-1}$, a restriction on the step-size that becomes increasingly severe as $\lambda \to -\infty$. This can be easily overcome by employing the backward-Euler discretization, which demands

$$v_{n+1} = v_n + \lambda \Delta t v_{n+1} \tag{1.9}$$

and hence yields

$$v_{n+1} = (1 - \lambda\Delta t)^{-1} v_n. \tag{1.10}$$

In this case we see $v_n \to 0$ for any time-step $\Delta t > 0$. We remark that a numerical scheme is called $A(\alpha)$-stable if this latter property holds for the entire wedge

$$\lambda \in \{z \in \mathbb{C} \setminus \{0\} : \arg(-z) < \alpha\}. \tag{1.11}$$

In particular, the backward-Euler discretization is $A(\frac{\pi}{2})$-stable, since

$$|1 - \lambda\Delta t| \geq |\mathrm{Re}\ 1 - \lambda\Delta t| \geq 1 - (\Delta t)\,\mathrm{Re}\,\lambda > 1 \tag{1.12}$$

holds whenever $\mathrm{Re}\,\lambda < 0$ and $\Delta t > 0$.

Replacing the forward-Euler temporal discretization in (1.5) by its backward-Euler counterpart, we obtain the evolution

$$\tfrac{1}{\Delta t}\big[U_j\big(n\Delta t\big) - U_j\big((n-1)\Delta t\big)\big] \;=\; \mathcal{F}_h\Big(U_{j-1}(n\Delta t), U_j(n\Delta t), U_{j+1}(n\Delta t); a\Big), \tag{1.13}$$

which plays a primary role in this paper. In fact, the backward-Euler discretization is the first member of a family of six discretization schemes commonly referred to as *backward differentiation formula* (BDF) methods. These methods are all $A(\alpha)$-stable with various coefficients $0 < \alpha \leq \frac{\pi}{2}$. The nature of their construction ensures that these schemes can be conveniently analyzed and they are commonly used in codes to solve parabolic problems. For these reasons, we have singled out this family of temporal discretization schemes for our analysis in this paper. We note however that there are other stiffly stable numerical methods, see for example [15].

In our case, the second BDF method takes the form

$$\tfrac{1}{2\Delta t}\big[3U_j\big(n\Delta t\big) - 4U_j\big((n-1)\Delta t\big) + U_j\big((n-1)\Delta t\big)\big] \;=\; \mathcal{F}_h\Big(U_{j-1}(n\Delta t), U_j(n\Delta t), U_{j+1}(n\Delta t); a\Big). \tag{1.14}$$

We take the opportunity here to point out an important difference between the backward-Euler evolution (1.13) and the two other fully-discretized systems (1.5) and (1.14) discussed above. In the former system, all terms that do not involve $U_j(n\Delta t)$ occur with coefficients of the same sign (after moving them to the same side of the equation). This is not the case for (1.5), (1.14) and the other four BDF methods considered in this paper. This powerful property allows us to embed (1.13) into a larger system that admits a comparison principle. In particular, we will be able to obtain results for (1.13) with arbitrary $\Delta t > 0$, while having to demand $\Delta t \approx 0$ for the other BDF discretizations.

Existence of travelling fronts

Continuous setting The front solutions (1.3) to the PDE (1.1) can be found explicitly by solving the planar ODE

$$c\Phi'(\xi) = \Phi''(\xi) + g_{\mathrm{cub}}\big(\Phi(\xi); a\big). \tag{1.15}$$

In particular, for each $a \in (0,1)$ there is a unique wave speed $c(a)$ for which a travelling front exists. The front profile itself is also unique up to translations. By symmetry, we have $c(\frac{1}{2}) = 0$. In addition, we have $\partial_a c(a) < 0$, which for some $c_0 > 0$ allows us to define a single-valued function $a(c)$ with $c \in (-c_0, c_0)$.

Semi-discrete setting By contrast, substitution of the travelling wave Ansatz

$$u_j(t) = \Phi(j + ct), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1 \tag{1.16}$$

into the LDE (1.7) with $h = 1$ leads to the mixed type functional differential equation (MFDE)

$$c\Phi'(\xi) \quad = \quad \Phi(\xi + 1) + \Phi(\xi - 1) - 2\Phi(\xi) + g_{\text{cub}}\big(\Phi(\xi); a\big). \tag{1.17}$$

A number of powerful tools have been developed in the past decades to analyze MFDEs, which present significant mathematical challenges [16, 28, 31, 34, 35].

As in the continuous case, there is a unique wave speed $c(a)$ that allows fronts to exists for each $a \in (0, 1)$ [29]. However, the inverse function $a(c)$ is typically multi-valued for $c = 0$, in which case wave profiles may become step-like and lose their uniqueness [11, 20]. This can be seen as a consequence of the broken translational invariance, which is manifested by the fact that the wave speed c appears in (1.17) in a singular fashion.

Fully discrete setting Our primary concern in this paper is to establish the existence of travelling fronts

$$U_j(n\Delta t) = \Phi(j + nc\Delta t), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1, \tag{1.18}$$

after temporally discretizing (1.7) using the BDF methods discussed above. For the backward-Euler discretization, such fronts must satisfy the system

$$\begin{aligned} \tfrac{1}{\Delta t}[\Phi(\xi) - \Phi(\xi - c\Delta t)] \quad &= \quad \mathcal{F}_h\big(\Phi(\xi - 1), \Phi(\xi), \Phi(\xi + 1); a\big) \\ &= \quad h^{-2}\big[\Phi(\xi - 1) + \Phi(\xi + 1) - 2\Phi(\xi)\big] + g_{\text{cub}}\big(\Phi(\xi); a\big). \end{aligned} \tag{1.19}$$

This is a difference equation for all $c \in \mathbb{R}$. In particular, it is natural to ask whether the $a(c)$ relation can be be multi-valued even for $c \neq 0$. We note that related phenomena have been observed in monostable KPP systems [32] in the presence of inhomogeneities. Investigating the $a(c)$ relationship is therefore our secondary concern in this paper. Our results cover three distinct regimes for the time-step $\Delta t > 0$, which we now briefly discuss.

The small time-step limit For $\Delta t \downarrow 0$, we set up a perturbation argument to construct solutions to (1.19) and its higher order counterparts that are close to solutions $(\overline{c}, \overline{\Phi})$ to (1.17). Fixing $h = 1$, the key technical ingredient here is the understanding of the fully discrete operator

$$[\mathcal{L}_{\text{fd}} v](\xi) = -\frac{1}{\Delta t}[v(\xi) - v(\xi - \overline{c}\Delta t)] + v(\xi - 1) + v(\xi + 1) - 2v(\xi) + g'_{\text{cub}}(\overline{\Phi}(\xi); a)v(\xi), \tag{1.20}$$

for ξ in an appropriate subset of $\mathbb{R}$. This operator is associated to the linearization of (1.19) around a solution of (1.17). The main question is in what sense this operator inherits properties from its semi-discrete counterpart

$$[\mathcal{L}_{\text{sd}} v](\xi) = -\overline{c}v'(\xi) + v(\xi - 1) + v(\xi + 1) - 2v(\xi) + g'_{\text{cub}}(\overline{\Phi}(\xi); a)v(\xi), \tag{1.21}$$

which by now is well-understood [21, 29]. The transition between $\mathcal{L}_{\text{sd}}$ and $\mathcal{L}_{\text{fd}}$ is highly singular, since an (unbounded) derivative is replaced by a (bounded) finite difference and the natural domain for ξ varies from the whole line to a subset of the line.

A related situation was encountered by Bates and coworkers when studying spatial discretizations of (1.1), although here the singular transition was between two differential equations of order two and one. Nevertheless, we are able to mimic the spirit of their approach in our situation to obtain a Fredholm-type result for $\mathcal{L}_{\text{fd}}$ in §3. This allows us to use a standard Liapunov-Schmidt perturbation argument to study the nonlinear problem (1.19).

At present our approach is limited to rational values of the combination $c\Delta t$. In such cases, we find that the natural domain for (1.19) is a discrete subset of the line. Since one can choose on which such subset the unperturbed wave $\overline{\Phi}$ is sampled, we in fact get a continuous branch of solutions to the perturbed problem (1.19). We believe that this is the mechanism by which the non-uniqueness of the $a(c)$ relation arises and we discuss this issue in considerable detail in §5.

When studying problems involving MFDEs, the need to distinguish between rationally and irrationally related shifts frequently arises. Indeed, when studying planar travelling wave solutions to LDEs posed on $\mathbb{Z}^2$, the rationality of the (tangent of the) direction of propagation has played an important role in the analysis of phenomena such as crystallographic pinning [4, 7, 20, 23, 30] and nonlinear wave stability [18, 19]. In §5 we discuss some potential connections between the results in [20] and the issues encountered here.

Fixed time-step Upon fixing the time-step $\Delta t > 0$, it is possible to analyze the backward-Euler travelling wave equation (1.19) (but not the other BDF methods) by embedding it into the MFDE

$$\nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] + h^{-2}\big[\Phi(\xi - 1) + \Phi(\xi + 1) - 2\Phi(\xi)\big] + g_{\mathrm{cub}}\big(\Phi(\xi); a\big) \qquad (1.22)$$

and looking for solutions with $\nu = 0$. This allows us to directly apply some important results obtained by Mallet-Paret in his landmark paper [29]. In particular, the (possibly) multi-valued $a(c)$ relation is non-empty for small $|c|$ and the non-uniqueness of this relation can be directly related to the phenomenon of propagation failure for solutions to bistable LDEs. Our results in this setting also work for $c\Delta t \notin \mathbb{Q}$ since we are able to exploit some powerful monotonicity properties of the auxiliary variable ν.

Anti-continuum limit Finally, for $\Delta t \to \infty$ and $h \to \infty$, which can be seen as the anti-continuum limit for (1.1), we can study (1.19) by adapting an elegant construction devised by Keener in his pioneering paper [25] for the Nagumo LDE (1.7). This allows us to show that the $a(c)$ relation is indeed multi-valued for all $c \in \mathbb{R}$, for choices of Δt and h that can be made explicit. The argument is essentially that a blocking region for Φ in (1.22) exists that prevents either of the two stable background states $\Phi \equiv 0$ and $\Phi \equiv 1$ from invading the domain. This forces $\nu = 0$ to hold for our auxiliary speed-like variable.

Motivation

Our primary motivation for this work is to contribute to the on-going systematic approach to understand the impact of discretization schemes on the dynamics that they are designed to capture. Of course, there is a tremendous amount of literature concerning the accuracy of numerical schemes, but these studies typically focus on *finite time* error bounds. Our concern is more related to the persistence of structures that exist for all time. An interesting discussion on this topic can be found in [13], which studies the impact of discretization on attractors for ODEs.

In some sense this work can be seen as a follow-up to the series [8–10], where ad-hoc techniques are developed to provide insight on the impact of spatial-, temporal- and spatial-temporal discetizations on the dynamics of travelling waves. These works include rigorous, formal, and first order results for smooth and piecewise linear bistable nonlinearities. Roughly speaking, it was found that spatial discretization schemes have a relatively high impact on slow waves, while temporal discretizations have more effect on fast waves. In addition, the non-uniqueness of the $a(c)$ relationship was established for fully discretized systems with a piecewise-linear nonlinearity; see [8, Fig. 3].

We note that complete discretizations have been analyzed by Chow, Mallet-Paret, and Shen [5] in their work on the stability of bistable lattice traveling waves. These authors obtain the existence of fully discretized travelling waves by looking directly at Poincare return-maps for the dynamics of (1.7), in contrast to our approach which focusses on the travelling wave equations (1.19) and the linear operator $\mathcal{L}_{\rm fd}$. The benefits of our linear analysis are that we are able to (partially) address questions concerning uniqueness and parameter dependence of these waves. In addition, based on our prior experience in [18, 19, 21], we believe that the full power of understanding $\mathcal{L}_{\rm fd}$ will come into play when addressing the stability of these waves under the fully discretized dynamics. Indeed, addressing this issue appears to be the natural next step in the broader program outlined above.

By now, there are many well-established codes such as DASSL [33], LSODE [17], and VODE [3] that are used by practitioners to solve parabolic problems such as those considered here. At their heart, these codes typically employ BDF discretizations to solve the underlying stiff problems. However, in order to increase accuracy and efficiency, non-uniform adaptive spatial discretizations are often considered together with temporal methods that involve variable orders and time-steps. Our hope is that the present work can be used as a starting point for rigorously understanding these more complicated algorithms.

Organization

This paper is organized as follows. In §2 we formulate the standard bistability assumptions we need to impose on our system and recall the k-step BDF methods for $k \in \{1, \ldots, 6\}$. We also state our main results for the three time-step regimes that were discussed above. Section 3 is focused on the analysis of the linearized operators $\mathcal{L}_{\rm fd}$ and their relation with a multi-component version of $\mathcal{L}_{\rm sd}$, inspired by the analysis of Bates and coworkers in [1]. We prove our main results in §4, exploiting the linear theory developed in §3 together with the work of Mallet-Paret [29] and Keener [25]. Finally, in §5 we discuss the significance of the results obtained, the complications arising in the case of irrational $c\Delta t$ and potential connections with work by other authors on asymptotic analysis [26] and crystallographic pinning [20].

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2 Main Results

Our main results concern the well-known Nagumo LDE

$$\dot{u}_j(t) = \kappa\big[u_{j+1} + u_{j-1} - 2u_j(t)\big] + g\big(u_j(t); a\big), \tag{2.1}$$

with $\kappa > 0$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$ and $u_j(t) \in \mathbb{R}$. We impose the following standard bistability conditions on the nonlinearity g, which in the terminology of [29] imply that (2.1) is a *normal family*.

(Hg) The nonlinearity $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^r-smooth for some integer $r \geq 2$, with $\partial_a g(u; a) < 0$ for all $a \in (0, 1)$ and $u \in (0, 1)$. In addition, we have the identities $g(0; a) = g(1; a) = g(a; a) = 0$ for all $a \in (0, 1)$ together with the inequalities $g(u; a) < 0$ for $u \in (0, a) \cup (1, \infty)$ and $g(u; a) > 0$ for $u \in (-\infty, 0) \cup (a, 1)$. Finally, the derivatives of g with respect to u satisfy the inequalities

$$\partial_u g(0; a) < 0, \qquad \partial_u g(a; a) > 0, \qquad \partial_u g(1; a) < 0 \tag{2.2}$$

together with

$$\partial_{au} g(0; a) < 0, \qquad \partial_{au} g(1; a) > 0. \tag{2.3}$$

The reader may wish to keep in mind the prototype cubic nonlinearity $g(u; a) = u(u - 1)(a - u)$, which may easily be verified to satisfy (Hg).

$\alpha_{n;k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	-1	$\frac{1}{3}$	$-\frac{2}{11}$	$\frac{3}{25}$	$-\frac{12}{137}$	$\frac{10}{147}$
$n=1$	1	$-\frac{4}{3}$	$\frac{9}{11}$	$-\frac{16}{25}$	$\frac{75}{137}$	$-\frac{72}{147}$
$n=2$		1	$-\frac{18}{11}$	$\frac{36}{25}$	$-\frac{200}{137}$	$\frac{225}{147}$
$n=3$			1	$-\frac{48}{25}$	$\frac{300}{137}$	$-\frac{400}{147}$
$n=4$				1	$-\frac{300}{137}$	$\frac{450}{147}$
$n=5$					1	$-\frac{360}{147}$
$n=6$						1
β_k	1	$\frac{2}{3}$	$\frac{6}{11}$	$\frac{12}{25}$	$\frac{60}{137}$	$\frac{60}{147}$

Table 1: *The coefficients $\alpha_{n;k}$ and β_k associated to the six BDF schemes as introduced in (2.5).*

In many situations one is unable or unwilling to solve (2.1) exactly for all time $t \geq 0$. Instead, the desire is to approximate the solution at discrete time intervals $t = n\Delta t$ by

$$u_j(n\Delta t) \sim U_j(n\Delta t), \qquad n \in \mathbb{Z}_{\geq 0}, \qquad j \in \mathbb{Z}. \tag{2.4}$$

In order to formulate an equation for the evolution of the approximant U, one needs to replace the temporal derivative appearing in (2.1) by an appropriate discretized version.

The BDF discretizations are a collection of six different methods to accomplish this task, utilizing interpolation polynomials of varying degree. In particular, the BDF method of order $k \in \{1, 2, \dots 6\}$ approximates $\dot{u}$ in (2.1) at $t = n\Delta t$ by constructing an interpolating polynomial of degree k through the $k+1$ values $\{U\big((n-n')\Delta t\big)\}_{n'=0}^{k}$ and computing the derivative of this polynomial at $U(n\Delta t)$.

In particular, for the BDF method of order k, the evolution of U is governed by

$$\begin{array}{rcl} \beta_k^{-1} \frac{1}{\Delta t} \sum_{n'=0}^{k} \alpha_{n';k} U_j\big(n\Delta t - (k-n')\Delta t\big) & = & \kappa \big[U_{j+1}(n\Delta t) + U_{j-1}(n\Delta t) - 2U_j(n\Delta t)\big] \\ & & + g\big(U_j(n\Delta t); a\big), \end{array} \tag{2.5}$$

in which the coefficients β_k and $\{\alpha_{n';k}\}$ are determined implicitly by the identities

$$\sum_{n'=0}^{k} \alpha_{n';k} U((n'-k)\Delta t) = \sum_{n''=1}^{k} [\partial^{n''} U](0), \qquad \beta_k = \sum_{n'=0}^{k} \alpha_{n';k}(n'-k), \tag{2.6}$$

where we have introduced the notation

$$[\partial U](n'\Delta t) = U(n'\Delta t) - U\big((n'-1)\Delta t\big). \tag{2.7}$$

This definition implies that $\sum_{n'=0}^{k} \alpha_{n';k} = 0$, which allows us to write

$$\beta_k = \sum_{n'=0}^{k} \alpha_{n';k}(n'-k) = \sum_{n'=1}^{k} \alpha_{n';k} n'. \tag{2.8}$$

We remark here that the BDF method of order $k=1$ is more commonly known as the backward Euler method. For convenience, the values of these coefficients can be found in Table 2. Naturally, the construction above can be repeated for arbitrary orders $k \geq 7$, but the resulting schemes are numerically unstable.

Our goal in this paper is to study travelling wave solutions to the fully discrete system (2.5). Such solutions have the special form

$$U_j(n\Delta t) = \Phi(j + nc\Delta t), \tag{2.9}$$

for some wave speed c and profile Φ that connects the two stable equilibria of the nonlinearity g. In particular, we demand

$$\Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1, \tag{2.10}$$

in a sense that we make precise below.

2.1 The small time-step limit $\Delta t \to 0$.

For notational convenience, we introduce the quantity $M = (c\Delta t)^{-1}$. Inserting the Ansatz (2.9) into (2.5), we find that the pair (c, Φ) must satisfy the system

$$c[\mathcal{D}_{k,M}\Phi](\xi) = \kappa\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big) \tag{2.11}$$

for all ξ that can be written as $\xi = n + jM^{-1}$ for $(j,n) \in \mathbb{Z}^2$. Here we have introduced the expressions

$$[\mathcal{D}_{k,M}\Phi](\xi) = \beta_k^{-1} M \sum_{n'=0}^{k} \alpha_{n';k} \Phi\big(\xi - (k-n')M^{-1}\big), \tag{2.12}$$

for $k \in \{1, 2, \ldots 6\}$. For instance, for $k = 1$ and $k = 2$ we have

$$\begin{aligned} [\mathcal{D}_{1,M}\Phi](\xi) &= M\big[\Phi\big(\xi\big) - \Phi\big(\xi - M^{-1}\big)\big], \\ [\mathcal{D}_{2,M}\Phi](\xi) &= \tfrac{3}{2}M\big[\Phi\big(\xi\big) - \tfrac{4}{3}\Phi\big(\xi - M^{-1}\big) + \tfrac{1}{3}\Phi\big(\xi - 2M^{-1}\big)\big]. \end{aligned} \tag{2.13}$$

The expressions $\mathcal{D}_{k,M}\Phi$ can be thought of as order k approximations of the derivative Φ'. Indeed, let us consider any function $\Phi \in C^{k+1}(\mathbb{R}, \mathbb{R})$ and suppose for concreteness that $M > 0$. For fixed ξ, one can then approximate the shifted terms in (2.12) by the k-th order Taylor polynomial centered at ξ, up to an error of order $M^{-(k+1)}\Phi^{(k+1)}(\xi + \vartheta)$ for some $\vartheta \in [-kM^{-1}, 0]$. The uniqueness of interpolating polynomials together with the defining property of the k-th order BDF method now imply the estimate

$$|[\mathcal{D}_{k,M}\Phi](\xi) - \Phi'(\xi)| \le C_k M^{-k} \sup_{-kM^{-1} \le \vartheta \le 0} \left\| \Phi^{(k+1)}(\xi + \vartheta) \right\|, \tag{2.14}$$

in which the constant $C_k \ge 1$ is independent of Φ and M.

Some of our results require a restriction on the values of M that are allowed. In particular, upon fixing an integer $q \ge 1$, we need to introduce the set

$$\mathcal{M}_q = \{\frac{p}{q} : p \in \mathbb{N} \text{ has } \gcd(p,q) = 1 \text{ and } p \ge q\}, \tag{2.15}$$

which contains all irreducible fractions larger than one that have q as their denominator. We often use the notation $M = \frac{p}{q} \in \mathcal{M}_q$, as an implicit definition for an integer $p = p(M) = qM$. We note that for $M = \frac{p}{q} \in \mathcal{M}_q$, the natural domain of definition for ξ in the discretized travelling wave equation (2.11) is the set $p^{-1}\mathbb{Z}$.

The fully discretized travelling wave system (2.11) should be contrasted to the travelling wave MFDE

$$c\Phi'(\xi) = \kappa\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big), \tag{2.16}$$

which arises after substituting $u_j(t) = \Phi(j + ct)$ into the LDE (2.1). Our first result constructs a branch of solutions to (2.11) for large M that bifurcates from a solution to (2.16) with non-zero wave speed. In particular, we need to impose the following condition, which is guaranteed [29] to hold for an open set of $\overline{a} \in (0, 1)$.

$(H\Phi)_{\overline{a}}$ The travelling wave MFDE (2.16) with $a = \overline{a}$ admits a solution $(c, \Phi) = (\overline{c}, \overline{\Phi})$ for which the wave speed has $\overline{c} \neq 0$ while the wave profile satisfies the limits

$$\lim_{\xi \to -\infty} \overline{\Phi}(\xi) = 0, \qquad \lim_{\xi \to +\infty} \overline{\Phi}(\xi) = 1. \tag{2.17}$$

The linearization of the MFDE (2.16) around a solution $(\overline{c}, \overline{\Phi})$ covered by $(H\Phi)_{\overline{a}}$ can be described by the operator $\overline{\mathcal{L}} : H^1(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R})$ that acts as

$$[\overline{\mathcal{L}} v](\xi) = -\overline{c} v'(\xi) + \kappa\big[v(\xi + 1) + v(\xi - 1) - 2v(\xi)\big] + g'\big(\overline{\Phi}(\xi); \overline{a}\big)v(\xi). \tag{2.18}$$

In [29, Thm. 4.1] it was established that $\overline{\mathcal{L}}$ is Fredholm with index zero, with a one dimensional kernel spanned by $\overline{\Phi}' > 0$. In addition, there is a strictly positive function $\overline{\Psi}$, normalized to have

$$\int \overline{\Psi}(\xi) \overline{\Phi}'(\xi) = 1, \tag{2.19}$$

so that the range of $\overline{\mathcal{L}}$ is given by

$$\mathrm{Range}\,(\overline{\mathcal{L}}) = \{w \in L^2(\mathbb{R}, \mathbb{R}) : \int \overline{\Psi}(\xi) w(\xi)\, d\xi = 0\}. \tag{2.20}$$

Theorem 2.1. *Fix $\kappa > 0$ and pick a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. Consider the LDE (2.1) and suppose that (Hg) is satisfied. Pick $\overline{a}$ in such a way that also $(H\Phi)_{\overline{a}}$ is satisfied. Then there exist constants $M_* \gg 1$ and $\delta_a > 0$ so that for any $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \geq M_*$, there are C^{r-1}-smooth functions*

$$c_M : \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta] \to \mathbb{R}, \qquad \Phi_M : \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta_a] \to \ell^\infty(p^{-1}\mathbb{Z}; \mathbb{R}) \tag{2.21}$$

that satisfy the following properties.

(i) For any $(\vartheta, a) \in \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta_a]$, the pair $c = c_M(\vartheta, a)$ and $\Phi = \Phi_M(\vartheta, a)$ satisfies the system

$$c[\mathcal{D}_{k,M}\Phi](\xi) = \kappa\big[\Phi(\xi + 1) + \Phi(\xi - 1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big), \qquad \xi \in p^{-1}\mathbb{Z}, \tag{2.22}$$

together with the boundary conditions

$$\lim_{\xi \to -\infty;\, \xi \in p^{-1}\mathbb{Z}} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty;\, \xi \in p^{-1}\mathbb{Z}} \Phi(\xi) = 1. \tag{2.23}$$

(ii) For any $(\vartheta, a) \in \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta_a]$, the function $\Phi = \Phi_M(\vartheta, a)$ admits the normalization

$$\sum_{\xi \in p^{-1}\mathbb{Z}} \overline{\Psi}(\xi + \vartheta)\big[\Phi(\xi) - \overline{\Phi}(\xi + \vartheta)\big] = 0. \tag{2.24}$$

(iii) For any $(\vartheta, a) \in \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta_a]$, we have the shift-periodicity

$$c_M(\vartheta + p^{-1}, a) = c_M(\vartheta, a), \qquad \Phi_M(\vartheta + p^{-1}, a)(\xi) = \Phi_M(\vartheta, a)(\xi + p^{-1}). \tag{2.25}$$

(iv) For any $(\vartheta, a) \in \mathbb{R} \times [\overline{a} - \delta_a, \overline{a} + \delta_a]$*, we have the inequality*

$$\partial_a c_M(\vartheta, a) < 0. \tag{2.26}$$

In addition, there exists $\delta > 0$ *such that the following holds true. Any triplet* $(c, \Phi, \vartheta) \in \mathbb{R} \times \ell^\infty(p^{-1}\mathbb{Z}, \mathbb{R}) \times \mathbb{R}$ *that satisfies (2.22) for some pair* $(a, M) \in \mathbb{R} \times \mathcal{M}_q$ *with*

$$|a - \overline{a}| < \delta, \qquad M = \frac{p}{q} > \delta^{-1} \geq M_* \tag{2.27}$$

and enjoys the estimate

$$p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} \Big[\big|\Phi(\xi) - \overline{\Phi}(\xi + \vartheta)\big|^2 + \big|[\mathcal{D}_{k,M}\Phi](\xi) - [\mathcal{D}_{k,M}\overline{\Phi}](\xi + \vartheta)\big|^2 \Big] < \delta^2, \tag{2.28}$$

must actually satisfy $\Phi = \Phi_M(\widetilde{\vartheta}, a)$ *and* $c = c_M(\widetilde{\vartheta}, a)$ *for some* $\widetilde{\vartheta} \in \mathbb{R}$*.*

The normalization factor p^{-1} appearing in (2.28) is required to compensate for the growing number of terms in the sum as $p \to \infty$, as we discuss more fully in §3. The final claim can hence be interpreted as a local uniqueness with respect to a ℓ^2-type norm. We also expect this uniqueness to hold for the supremum norm, but this would require some modifications to our arguments along the lines of [24, §4]. Notice however that there is no restriction of the type $c \approx \overline{c}$ on the wave speed appearing in this uniqueness claim.

Since $M = (c\Delta t)^{-1}$ remains fixed for the branches (c_M, Φ_M) obtained above, fluctuations in c automatically lead to fluctuations in Δt. Our main goal however is to understand the behaviour of (2.11) for fixed $\Delta t > 0$. To this end, we note that the inequality (2.26) implies that for each fixed $(\vartheta_0, a_0) \in \mathbb{R} \times (0, 1)$ with $|a_0 - \overline{a}| < \delta_a$, one can find a small constant $\delta_0 > 0$ together with a C^{r-1}-smooth function

$$a_* : (\vartheta_0 - \delta_0, \vartheta_0 + \delta_0) \to (0, 1), \tag{2.29}$$

with $a_*(\vartheta_0) = a_0$, so that

$$c_0 := c_M(\vartheta_0, a_0) = c_M\big(\vartheta, a_*(\vartheta)\big) \tag{2.30}$$

holds for all ϑ with $|\vartheta - \vartheta_0| < \delta_0$. This gives us a local one-parameter family of solutions to (2.11) that all share the same wave speed c_0 and time-step $\Delta t = (\Delta t)_0$, but with detuning parameters $a_*(\vartheta)$ that could potentially fluctuate.

Indeed, the implicit function theorem gives

$$\partial_\vartheta a_*(\vartheta_0) = -\partial_\vartheta c_M(\vartheta_0, a_0) / \partial_a c_M(\vartheta_0, a_0). \tag{2.31}$$

Unfortunately, the result above provides no information on $\partial_\vartheta c_M$, as we discuss in detail in §5. Nevertheless, if this quantity is non-zero, then there is a $\delta_* > 0$ so that the travelling wave problem (2.22) with boundary conditions 2.23 admits solutions with $\Delta t = (\Delta t)_0$ and $c = c_0$ for all detuning parameters $a \in (a_0 - \delta_*, a_0 + \delta_*)$. Stated more informally, the $a(c)$ relation is multi-valued at $c = c_0$.

2.2 The backward-Euler discretization

Let us now restrict ourselves to the BDF-method of order $k = 1$, also known as the backward-Euler discretization. In this case, substitution of the Ansatz (2.9) into the fully discretized system (2.5) yields the travelling wave equation

$$-\frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] = \kappa\big[\Phi(\xi + 1) + \Phi(\xi - 1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big). \tag{2.32}$$

Our goal is to study (2.32) by embedding it into the MFDE

$$\nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] + \kappa\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big). \tag{2.33}$$

This equation fits into the framework developed by Mallet-Paret in [29], since all terms with shifted arguments come with positive coefficients. As before, we impose the limiting behaviour

$$\lim_{\xi\to-\infty} \Phi(\xi) = 0, \qquad \lim_{\xi\to+\infty} \Phi(\xi) = 1. \tag{2.34}$$

The idea here is to fix $\Delta t > 0$ and $\kappa \geq 0$, consider $c \in \mathbb{R}$ and $a \in (0,1)$ as parameters and look for solutions (ν, Φ) to (2.33)-(2.34). We note that the limiting case $\kappa = 0$ is included here for technical reasons that will become apparent below.

The next result shows that ν is uniquely defined as a function of (c, a). We are specially interested in solutions for which $\nu(c, a) = 0$, since these are also solutions to the fully discrete travelling wave problem (2.32).

Theorem 2.2. *Consider the equation (2.33) with $\kappa \geq 0$, suppose that (Hg) is satisfied and fix a time step $\Delta t > 0$. Then there exists a continuous function $\nu : \mathbb{R} \times (0,1) \to \mathbb{R}$ that satisfies the following properties.*

(i) For every $c \in \mathbb{R}$ and $a \in (0,1)$, there exists a non-decreasing function $\Phi : \mathbb{R} \to \mathbb{R}$ that satisfies (2.33) with $\nu = \nu(c, a)$ together with the limits (2.34).

(ii) Suppose that (2.33) with $\nu = 0$ admits a non-decreasing solution Φ that satisfies the limits (2.34). Then we must have $\nu(c, a) = 0$.

(iii) Suppose that (2.33) with $\nu \neq 0$ admits a solution Φ that satisfies the limits (2.34) (but is not necessarily non-decreasing). Then $\nu = \nu(c, a)$ and Φ must be a translate of the solution described in (i).

(iv) The function ν depends C^r-smoothly on (c, a) wherever $\nu(c, a) \neq 0$, with the inequalities

$$\partial_c \nu(c, a) < 0, \qquad \partial_a \nu(c, a) < 0. \tag{2.35}$$

For explicitness, we write $\nu(c, a) = \nu(c, a; \kappa, \Delta t)$ for the function defined in the result above for (2.33). This allows us to introduce the quantities

$$\begin{array}{rcl} a^-(c;\kappa,\Delta t) & = & \sup\{a \in (0,1) : \nu(c,a;\kappa,\Delta t) > 0\} \in (0,1] \cup \{-\infty\}, \\ a^+(c;\kappa,\Delta t) & = & \inf\{a \in (0,1) : \nu(c,a;\kappa,\Delta t) < 0\} \in [0,1) \cup \{\infty\}. \end{array} \tag{2.36}$$

Exploiting the inequalities (2.35), we see that for any detuning parameter $a \in (0,1)$ that satisfies the inequalities

$$a^-(c;\kappa,\Delta t) \leq a \leq a^+(c;\kappa,\Delta t), \tag{2.37}$$

a solution exists for (2.32) with (2.34). We first state some basic properties of these functions $a^\pm$.

Corollary 2.3. *Consider (2.33) and suppose that (Hg) is satisfied. Fix $\kappa \geq 0$ and $\Delta t > 0$. Then the maps $c \mapsto a^\pm(c;\kappa,\Delta t)$ satisfy the following properties.*

(i) Both $c \mapsto a^\pm(c;\kappa,\Delta t)$ are non-increasing, while $c \mapsto a^+(c;\kappa,\Delta t)$ is left-continuous and $c \mapsto a^-(c;\kappa,\Delta t)$ is right-continuous.

(ii) There exists $\delta_c > 0$ so that for all $c \geq -\delta_c$ we have $a^+(c;\kappa,\Delta t) < 1$, while for all $c \leq \delta_c$ we have $a^-(c;\kappa,\Delta t) > 0$. In particular, for $|c| \leq \delta_c$ we have $0 < a^-(c;\kappa,\Delta t) \leq a^+(c;\kappa,\Delta t) < 1$.

Whenever the strict inequality

$$a^-(c;\kappa,\Delta t) < a^+(c;\kappa,\Delta t) \tag{2.38}$$

is satisfied, the discretized travelling wave problem (2.32) with (2.34) admits waves with the same wave speed c at multiple values of the detuning parameter a. The next result shows that in the anti-continuum limit, which can be thought of as a full discretization of the Nagumo PDE (1.1) with a large time-step $\Delta t \gg 1$ and a large spatial grid-spacing $h \gg 1$, this non-uniqueness of a indeed holds. In §5 we further discuss this question for different parameter regimes.

Corollary 2.4. *Consider (2.33) and suppose that (Hg) is satisfied. Fix $\overline{a} \in (0,1)$. Then there exists $\delta > 0$ so that for all $(\kappa, \Delta t)$ that have*

$$\Delta t > \delta^{-1}, \qquad 0 \le \kappa < \delta, \tag{2.39}$$

the strict inequalities

$$a^-(c;\kappa,\Delta t) < \overline{a} < a^+(c;\kappa,\Delta t) \tag{2.40}$$

hold for all $c \in \mathbb{R}$.

In order to state our final result, we introduce the quantities

$$a^{\pm}_{-\infty}(\Delta t) = a^{\pm}(-1;0,\Delta t), \qquad a^{\pm}_{+\infty}(\Delta t) = a^{\pm}(+1;0,\Delta t). \tag{2.41}$$

We note that the quantities $a^{\pm}_{-\infty}(\Delta t)$ are associated to the system

$$\nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi+\Delta t) - \Phi(\xi)] + g\big(\Phi(\xi);a\big), \tag{2.42}$$

while the quantities $a^{\pm}_{+\infty}(\Delta t)$ are associated to

$$\nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi-\Delta t) - \Phi(\xi)] + g\big(\Phi(\xi);a\big). \tag{2.43}$$

These systems can be interpreted in a suitable sense as the (rescaled) $c \to \pm\infty$ limits of (2.33), which no longer depend on the coefficient $\kappa \ge 0$. Our final result relates the quantities (2.41) to the $c \to \pm\infty$ limits of (2.36).

Corollary 2.5. *Consider (2.33) and suppose that (Hg) is satisfied. Fix $\Delta t > 0$ and $\kappa \ge 0$. We then have the identities*

$$a^-_{+\infty}(\Delta t) = -\infty, \qquad a^+_{-\infty}(\Delta t) = +\infty, \tag{2.44}$$

together with the limiting inequalities

$$\lim_{c\to\infty} a^+(c;\kappa,\Delta t) \le a^+_{+\infty}(\Delta t) < 1, \qquad \lim_{c\to-\infty} a^-(c;\kappa,\Delta t) \ge a^-_{-\infty}(\Delta t) > 0. \tag{2.45}$$

2.3 Numerical examples

In Fig. 1 plots can be found illustrating the functions $a^{\pm}(c)$ for the problems

$$-\frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] = \Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi) + g\big(\Phi(\xi);a\big), \tag{2.46}$$

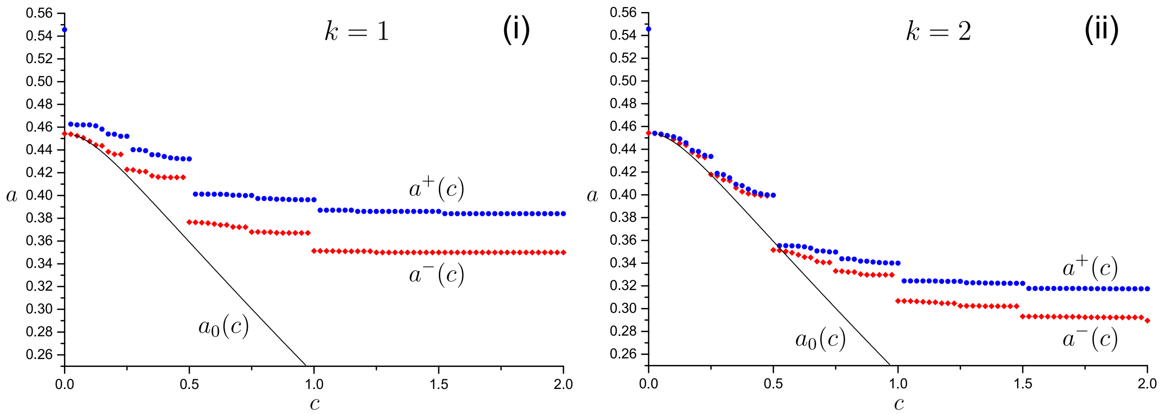

Fig. 1: *Numerical computation of the edges $a^-(c)$ and $a^+(c)$ of the interval of detuning parameters at which solutions to the fully discretized wave equations (2.46) (i) and (2.47) (ii) exist. Both plots also contain the function $a_0(c)$, which gives a as a function of c for the semi-discrete travelling wave MFDE (2.16), again with nonlinearity (2.48). The strict inequalities $a^-(c) < a^+(c)$ clearly hold in these examples. In panel (i) the temporal discretization causes a strict speed-up of the waves, while in panel (ii) this breaks down for $c \approx \frac{1}{2}$.*

and

$$-\frac{1}{2\Delta t}[-\Phi(\xi - 2c\Delta t) + 4\Phi(\xi - c\Delta t) - 3\Phi(\xi)] = \Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi) + g\big(\Phi(\xi); a\big), \quad (2.47)$$

both with $\Delta t = 2$ and nonlinearity

$$g(u;a) = \frac{121}{12}u(u-1)(a-u). \quad (2.48)$$

These two discretizations correspond to the BDF methods with order $k = 1$ and $k = 2$.

The plots were computed by repeatedly attempting to solve (2.46) and (2.47) on the finite interval $[-10, 10]$ for different values of $(c, a) \in \frac{1}{40}\mathbb{Z} \times (0, 1)$, recording at which parameter values solutions were successfully found. The accompanying boundary conditions are

$$\Phi(\xi) = 0 \text{ for } \xi \leq -10, \qquad \Phi(\xi) = 1 \text{ for } \xi \geq 10. \quad (2.49)$$

Although we have not defined the quantities $a^{\pm}$ for $k = 2$ in our discussion above, we simply define them here as the edges of the interval for a for which this recipe yields results.

These computations are rather delicate, since the success of the numerical solver depends heavily on the quality of the supplied initial conditions. Usually, a standard continuation approach can be applied to supply such high-quality initial conditions. In the current setup there however are two problems with such an approach that need to be addressed. The first problem is that the set of $\xi \in \mathbb{R}$ for which $\Phi(\xi)$ needs to be defined does not remain constant when varying the parameter c. For example, when $c = \frac{1}{2}$ one only requires $\Phi(\xi)$ for $\xi \in \{-10, 9, \ldots, 9, 10\}$, while for $c = \frac{1}{40}$ many additional values are needed. The second problem is that, even for fixed (c, a), solutions to (2.46) and (2.47) are not unique. In particular, when keeping c fixed and modifying a, one could be tracking a branch of solutions that terminates at some value of a that need not be a^+ or a^-.

In order to tackle these problems, we repeated the computations above for a large set of different initial conditions. In addition, to generate more data a second numerical procedure was followed to search directly for the branch termination points discussed above. In particular, after fixing $c \in \frac{1}{40}\mathbb{Z}$ but treating a as an unknown, we numerically solved the combined system that arises by supplementing (2.46) and (2.49) with the auxiliary problem

$$-\frac{1}{\Delta t}[v(\xi - c\Delta t) - v(\xi)] = v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\big(\Phi(\xi); a\big)v(\xi), \quad (2.50)$$

accompanied by the boundary conditions

$$v(-10) = 0 \text{ for } \xi \leq -10, \qquad v(0) = 1, \qquad v(10) = 0 \text{ for } \xi \geq 10. \tag{2.51}$$

As a final verification step, we found numerical solutions to the augmented system

$$-10^{-5}\Phi''(\xi) + \nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] + \Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi) + g\big(\Phi(\xi); a\big), \tag{2.52}$$

using the techniques developed in [7, 23].

This last step gives us a numerical approximation for the $\nu(c, a)$ relationship described in Theorem 2.2. As explained in detail in [7, 23], the small term involving Φ'' is required to handle the transition $c \to 0$, which in the absence of this smoothening term would be highly singular and thus hard to handle numerically. Although this extra small term prevents us from solving $\nu(c, a) = 0$ exactly, it does provide us with a visual means to reasonably verify that the data generated by our first two methods indeed finds the edges of the entire interval $[a^-(c), a^+(c)]$ at which solutions exist to (2.46). Naturally, an analogous approach was used to analyze (2.47).

3 Linear Theory for $\Delta t \to 0$

Throughout this section, we fix $\kappa = 1$ for notational convenience. Our goal is to study the linear operators that arise when linearizing the fully discrete travelling wave equation (2.11) around the semi-discrete travelling wave $(\overline{c}, \overline{\Phi})$ defined in $(H\overline{\Phi})_{\overline{a}}$. In particular, we define the linear expressions

$$[\mathcal{L}_{k,M} v](\xi) = -\overline{c}[\mathcal{D}_{k,M} v](\xi) + v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\big(\overline{\Phi}(\xi); \overline{a}\big)v(\xi) \tag{3.1}$$

and set out to study in what sense $\mathcal{L}_{k,M}$ inherits properties from the operator $\overline{\mathcal{L}}$ defined in (2.18).

In order to state our results, we need to introduce a number of function spaces. First of all, for any $\alpha \in \mathbb{R}$ we write

$$\begin{array}{rcl} BC_\alpha(\mathbb{R}, \mathbb{R}) & = & \{p \in C(\mathbb{R}, \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} e^{-\alpha|\xi|}\, |p(\xi)| < \infty\}, \\ BC^1_\alpha(\mathbb{R}, \mathbb{R}) & = & \{p \in C^1(\mathbb{R}, \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} e^{-\alpha|\xi|}[|p(\xi)| + |p'(\xi)|] < \infty\}. \end{array} \tag{3.2}$$

In addition, for any $\mu > 0$ and Hilbert space H, we introduce the sequence space

$$\ell^2_\mu(H) = \{v : \mu^{-1}\mathbb{Z} \to H \text{ with } \|v\|_{\ell^2_\mu(H)} := \langle v, v\rangle^{1/2}_{\ell^2_\mu(H)} < \infty\}, \tag{3.3}$$

in which the inner product is given by

$$\langle v, w\rangle_{\ell^2_\mu(H)} = \mu^{-1} \sum_{\xi \in \mu^{-1}\mathbb{Z}} \langle v(\xi), w(\xi)\rangle_H. \tag{3.4}$$

The role of the normalization factor μ^{-1} will become apparent in Lemma 3.1 below.

Let us now fix two integers $q \geq 1$ and $1 \leq k \leq 6$, together with a fraction $M = \frac{p}{q} \in \mathcal{M}_q$. In order to streamline our notation, we write $\mathcal{Y}_M$ to refer to the sequence space $\ell^2_p(\mathbb{R})$, i.e.,

$$\mathcal{Y}_M = \ell^2_p(\mathbb{R}), \qquad \langle v, w\rangle_{\mathcal{Y}_M} = \langle v, w\rangle_{\ell^2_p(\mathbb{R})}. \tag{3.5}$$

We also introduce the sequence space $\mathcal{Y}^1_{k,M}$, which differs from $\ell^2_p(\mathbb{R})$ only by the structure of its inner product. In particular, we write

$$\mathcal{Y}^1_{k,M} = \ell^2_p(\mathbb{R}), \qquad \langle v, w\rangle_{\mathcal{Y}^1_{k,M}} = \langle v, w\rangle_{\ell^2_p(\mathbb{R})} + \langle \mathcal{D}_{k,M} v, \mathcal{D}_{k,M} w\rangle_{\ell^2_p(\mathbb{R})}. \tag{3.6}$$

In addition, for any $f \in BC_{-\eta}(\mathbb{R},\mathbb{R})$ with $\eta > 0$, we write $\pi_{\mathcal{Y}_M} f \in \mathcal{Y}_M$ for the sequence

$$[\pi_{\mathcal{Y}_M} f](\xi) = f(\xi), \qquad \xi \in p^{-1}\mathbb{Z}. \tag{3.7}$$

If also $f \in BC^1_{-\eta}(\mathbb{R},\mathbb{R})$, we sometimes use the notation $\pi_{\mathcal{Y}^1_{k,M}} f$ to refer to the same function (3.7) if we wish to be explicit.

Lemma 3.1. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$, together with a constant $\eta > 0$. Then there exists $C \ge 1$ so that for all $M \in \mathcal{M}_q$, all functions $f \in BC_{-\eta}(\mathbb{R},\mathbb{R})$ and all functions $g \in BC^1_{-\eta}(\mathbb{R},\mathbb{R})$, we have the bounds*

$$\|\pi_{\mathcal{Y}_M} f\|_{\mathcal{Y}_M} \le C\,\|f\|_{BC_{-\eta}}, \qquad \left\|\pi_{\mathcal{Y}^1_{k,M}} g\right\|_{\mathcal{Y}^1_{k,M}} \le C\,\|g\|_{BC^1_{-\eta}}. \tag{3.8}$$

Proof. Observe first that for $M = \frac{p}{q} \in \mathcal{M}_q$ we have $p^{-1} \le M^{-1} \le 1$. We may hence compute

$$\begin{array}{lcl} \left\|e^{-\eta|\cdot|}\right\|^2_{\ell^2_p(\mathbb{R})} & = & p^{-1}\big[1 + \sum_{j>0} e^{-2\eta p^{-1} j} + \sum_{j<0} e^{2\eta p^{-1} j}\big] \\ & = & p^{-1}\frac{1+e^{-2\eta p^{-1}}}{1-e^{-2\eta p^{-1}}} \\ & \le & C_1'(1+p^{-1}) \\ & \le & 2C_1', \end{array} \tag{3.9}$$

for some constant $C_1' \ge 1$ that depends only on $\eta > 0$. The desired bounds (3.8) follow directly from this computation together with the estimate

$$|[\mathcal{D}_{k,M} f](\xi)| \le \sup_{-k \le -kM^{-1} \le \vartheta \le 0} \|f'(\xi+\vartheta)\|. \tag{3.10}$$

□

These preparations in hand, we can now consider the operators $\mathcal{L}_{k,M}$ appearing in (3.1) as bounded linear maps

$$\mathcal{L}_{k,M} : \mathcal{Y}^1_{k,M} \to \mathcal{Y}_M. \tag{3.11}$$

The remainder of this section is devoted to the proof of the following result, which shows in what sense the Fredholm structure of the operator $\overline{\mathcal{L}}$ described in §2 can be maintained under the transition from a continuous to a discrete setting. Indeed, for any $f \in L^2(\mathbb{R},\mathbb{R})$ one can find $v \in H^1(\mathbb{R},\mathbb{R})$ for which we have

$$\overline{\mathcal{L}} v = f - \overline{\Phi}' \int_{-\infty}^{\infty} \overline{\Psi}(\xi) f(\xi)\, d\xi. \tag{3.12}$$

In view of the normalization (2.19), one can subsequently arrange for the normalization condition

$$\int_{-\infty}^{\infty} \overline{\Psi}(\xi) v(\xi)\, d\xi = 0 \tag{3.13}$$

to hold by subtracting an appropriate multiple of $\overline{\Phi}'$ from v. Since $\overline{\mathcal{L}}\,\overline{\Phi}' = 0$, this does not affect the identity (3.12).

Proposition 3.2. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$, together with a constant $\eta > 0$. Consider the LDE (2.1) and suppose that (Hg) is satisfied. Pick $\overline{a}$ in such a way that also $(H\Phi)_{\overline{a}}$ is satisfied. Then there exists $M_* \ge 1$ together with a constant $C > 1$ so that for all $M = \frac{p}{q} \in \mathcal{M}_q$ with $M \ge M_*$, there exist linear maps*

$$\gamma^*_{k,M} : \mathcal{Y}_M \to \mathbb{R}, \qquad \mathcal{V}^*_{k,M} : \mathcal{Y}_M \to \mathcal{Y}^1_{k,M} \tag{3.14}$$

that satisfy the following properties for all such M.

(i) For all $f \in \mathcal{Y}_M$, we have the bounds

$$\left|\gamma^*_{k,M} f\right| + \left\|\mathcal{V}^*_{k,M} f\right\|_{\mathcal{Y}^1_{k,M}} \le C \left\|f\right\|_{\mathcal{Y}_M}. \tag{3.15}$$

(ii) For all $f \in \mathcal{Y}_M$, the pair

$$(\gamma, v) = \left(\gamma^*_{k,M} f, \mathcal{V}^*_{k,M} f\right) \in \mathbb{R} \times \mathcal{Y}^1_{k,M} \tag{3.16}$$

is the unique solution to the problem

$$\mathcal{L}_{k,M} v = f + \gamma \mathcal{D}_{k,M} \overline{\Phi} \tag{3.17}$$

that satisfies the normalization condition

$$\langle \pi_{\mathcal{Y}_M} \overline{\Psi}, v \rangle_{\mathcal{Y}_M} = 0. \tag{3.18}$$

(iii) For all $f \in BC^1_{-\eta}(\mathbb{R}, \mathbb{R})$, we have the bound

$$\left|\gamma + \langle \pi_{\mathcal{Y}_M} \overline{\Psi}, \pi_{\mathcal{Y}_M} f \rangle_{\mathcal{Y}_M}\right| \le C M^{-1} \left\|f\right\|_{BC^1_{-\eta}(\mathbb{R},\mathbb{R})}. \tag{3.19}$$

3.1 Reformulation

In this subsection we formulate our strategy towards proving Proposition 3.2, which is rather indirect. Indeed, with the exception of §3.3, our efforts will be focused on establishing the following technical result.

Proposition 3.3. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. Consider the LDE (2.1) and suppose that (Hg) is satisfied. Pick $\overline{a}$ in such a way that also $(H\Phi)_{\overline{a}}$ is satisfied. Then there exists $C_0 > 0$ together with a map $M_0 : (0,1) \to [1, \infty)$ so that the following holds true. For any $0 < \delta < 1$ and any $M \in \mathcal{M}_q$ for which $M \ge M_0(\delta)$, the operator $\mathcal{L}_{k,M} - \delta$ is invertible as a map from $\mathcal{Y}^1_{k,M}$ onto $\mathcal{Y}_M$, with the bound*

$$\left\|(\mathcal{L}_{k,M} - \delta)^{-1} w\right\|_{\mathcal{Y}^1_{k,M}} \le C_0 \Big[\left\|w\right\|_{\mathcal{Y}_M} + \delta^{-1} \left|\langle \pi_{\mathcal{Y}_M} \overline{\Psi}, w \rangle_{\mathcal{Y}_M}\right| \Big]. \tag{3.20}$$

This result can be seen as the analogue of [1, Thm. 4]. As a consequence, our strategy here will follow the same broad ideas as those developed in [1], but we will need to make significant modifications. We first state a preliminary result to aid the reader in interpreting the inner products appearing in (3.19) and (3.20).

Lemma 3.4. *Fix an integer $q \ge 1$. There exists $C > 1$ so that for all $M \in \mathcal{M}_q$ and all functions $f, g \in BC^1_{-\eta}(\mathbb{R}, \mathbb{R})$, we have the bound*

$$\left|\langle f, g \rangle_{L^2(\mathbb{R},\mathbb{R})} - \langle \pi_{\mathcal{Y}_M} f, \pi_{\mathcal{Y}_M} g \rangle_{\mathcal{Y}_M}\right| \le C M^{-1} \left\|f\right\|_{BC^1_{-\eta}} \left\|g\right\|_{BC^1_{-\eta}}. \tag{3.21}$$

Proof. Upon introducing the quantity

$$\mathcal{I}_* = \sum_{\xi \in p^{-1}\mathbb{Z}} \int_{\xi}^{\xi + p^{-1}} \left[f(\xi') g(\xi') - f(\xi) g(\xi) \right] d\xi', \tag{3.22}$$

we may compute

$$\begin{array}{rcl} \langle f, g \rangle_{L^2(\mathbb{R},\mathbb{R})} & = & \int_{\mathbb{R}} f(\xi') g(\xi') \, d\xi' \\ & = & \sum_{\xi \in p^{-1}\mathbb{Z}} \int_{\xi}^{\xi+p^{-1}} f(\xi') g(\xi') \, d\xi' \\ & = & \sum_{\xi \in p^{-1}\mathbb{Z}} \int_{\xi}^{\xi+p^{-1}} f(\xi) g(\xi) \, d\xi' + \mathcal{I}_* \\ & = & p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} [\pi_{\mathcal{Y}_M} f](\xi) [\pi_{\mathcal{Y}_M} g](\xi) + \mathcal{I}_* \\ & = & \langle \pi_{\mathcal{Y}_M} f, \pi_{\mathcal{Y}_M} g \rangle_{\mathcal{Y}_M} + \mathcal{I}_*. \end{array} \tag{3.23}$$

Whenever the pair (ξ, ξ') satisfies the inequality

$$\xi \le \xi' \le \xi + p^{-1}, \tag{3.24}$$

we may estimate

$$|f(\xi')| + |f'(\xi')| \le \|f\|_{BC^1_{-\eta}} e^{-\eta|\xi'|} \le e^{p^{-1}} \|f\|_{BC^1_{-\eta}} e^{-\eta|\xi|} \tag{3.25}$$

with the analogous estimate for g. In particular, assuming (3.24) and exploiting the product rule $(fg)' = f'g + fg'$, we have

$$|f(\xi')g(\xi') - f(\xi)g(\xi)| \le 2p^{-1}e^{\eta p^{-1}} \big[\|f\|_{BC^1_{-\eta}} \|g\|_{BC^1_{-\eta}} \big] e^{-2\eta|\xi|}. \tag{3.26}$$

This allows us to estimate

$$\begin{array}{lcl} |\mathcal{I}_*| & \le & 2p^{-1}e^{2\eta p^{-1}} \|f\|_{BC^1_{-\eta}} \|g\|_{BC^1_{-\eta}} p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} e^{-2\eta|\xi|} \\ & = & 2p^{-1}e^{2\eta p^{-1}} \|f\|_{BC^1_{-\eta}} \|g\|_{BC^1_{-\eta}} \big\|e^{-\eta|\cdot|}\big\|^2_{\ell^2_p(\mathbb{R})}. \end{array} \tag{3.27}$$

The estimate (3.9) can now be used to complete the proof. □

Our next task is to set up a series of additional sequence spaces that will allow us to pass to the $M \to \infty$ limit in a controlled fashion. The main idea is to construct H^1 interpolants for functions in $\mathcal{Y}^1_{k,M}$ and L^2 interpolants for functions in $\mathcal{Y}_M$, so that sequences in these spaces can be compared regardless of the precise value of M. The main issue is that for $M = \frac{p}{q}$ with $q > 1$, understanding $\mathcal{D}_{k,M}v$ for $v \in \mathcal{Y}^1_{k,M}$ gives insufficient control over differences of the form $v(\xi + p^{-1}) - v(\xi)$.

To compensate for this, we need to perform q separate interpolations, each bridging gaps of size $M^{-1} = \frac{q}{p}$. In particular, fixing an integer $q \ge 1$ and writing

$$\mathbb{Z}_q = \{0, 1, 2, \ldots q\}, \qquad \mathbb{Z}^\circ_q = \{1, 2, \ldots q-1\}, \tag{3.28}$$

we introduce the space

$$\ell^2_{q,\perp} = \{v : q^{-1}\mathbb{Z}_q \to \mathbb{R}\}, \tag{3.29}$$

equipped with the inner product

$$\langle v, w \rangle_{\ell^2_{q,\perp}} \quad = \quad q^{-1}\big[\tfrac{1}{2}v(0)w(0) + \tfrac{1}{2}v(1)w(1) + \textstyle\sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} v(\zeta)w(\zeta)\big]. \tag{3.30}$$

This allows us to define the space

$$\mathcal{H}_M = \{v \in \ell^2_M(\ell^2_{q,\perp}) : v(1,\xi) = v(0, \xi + M^{-1}) \text{ for all } \xi \in M^{-1}\mathbb{Z}\}, \tag{3.31}$$

equipped with the inner product

$$\langle v, w \rangle_{\mathcal{H}_M} = M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \langle v(\cdot, \xi), w(\cdot, \xi) \rangle_{\ell^2_{q,\perp}}. \tag{3.32}$$

Here we have introduced the notation $v(\zeta, \xi) = [v(\xi)](\zeta)$ for $v \in \mathcal{H}_M$, with $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$.

We extend the operators $\mathcal{D}_{k,M}$ defined in (2.12) to $\mathcal{H}_M$ by writing

$$[\mathcal{D}_{k,M}v](\zeta, \xi) = [\mathcal{D}_{k,M}v(\zeta, \cdot)](\xi), \tag{3.33}$$

which implies that these operators act only on the second component of v. This allows us to define our final space

$$\mathcal{H}^1_{k,M} = \mathcal{H}_M, \tag{3.34}$$

but now equipped with the inner product

$$\langle v, w\rangle_{\mathcal{H}^1_{k,M}} = \langle v, w\rangle_{\mathcal{H}_M} + \langle \mathcal{D}_{k,M} v, \mathcal{D}_{k,M} w\rangle_{\mathcal{H}_M}. \tag{3.35}$$

In order to relate these new spaces back to the spaces defined earlier, we introduce for $M = \frac{p}{q} \in \mathcal{M}_q$ the operators

$$\mathcal{J}_M : \mathcal{Y}_M \to \mathcal{H}_M, \qquad \mathcal{J}^1_{k,M} : \mathcal{Y}^1_{k,M} \to \mathcal{H}^1_{k,M} \tag{3.36}$$

that both act as

$$[\mathcal{J}_M v](\zeta, \xi) = v(\xi + M^{-1}\zeta), \qquad [\mathcal{J}^1_{k,M} v](\zeta, \xi) = v(\xi + M^{-1}\zeta), \tag{3.37}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$.

Lemma 3.5. *Fix an integer $q \geq 1$. For any $M = \frac{p}{q} \in \mathcal{M}_q$, the operators $\mathcal{J}_M$ and $\mathcal{J}^1_{k,M}$ defined in (3.36) are isometries.*

Proof. Since $\mathcal{J}_M \mathcal{D}_{k,M} = \mathcal{D}_{k,M} \mathcal{J}_M$, we only have to consider the statement for $\mathcal{J}_M$. The invertibility of $\mathcal{J}_M$ follows directly from the construction of the space $\mathcal{H}_M$. In addition, for any $v \in \mathcal{Y}_M$ we may write $w = \mathcal{J}_M v$ and compute

$$\begin{array}{lcl} \|w\|^2_{\mathcal{H}_M} & = & \frac{q}{p} \sum_{\xi \in M^{-1}\mathbb{Z}} |w(\cdot, \xi)|^2_{\ell^2_{q,\perp}} \\ & = & \frac{q}{p} \sum_{\xi \in M^{-1}\mathbb{Z}} \frac{1}{q} \Big[\frac{1}{2} w(0, \xi)^2 + \frac{1}{2} w(1, \xi)^2 + \sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} w(\zeta, \xi)^2 \Big] \\ & = & \frac{1}{p} \sum_{\xi \in M^{-1}\mathbb{Z}} \Big[\frac{1}{2} v(\xi)^2 + \frac{1}{2} v(\xi + M^{-1})^2 + \sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} v(\xi + \frac{q}{p}\zeta)^2 \Big] \\ & = & \frac{1}{p} \sum_{\xi \in p^{-1}\mathbb{Z}} v(\xi)^2 \\ & = & \|v\|^2_{\mathcal{Y}_M}. \end{array} \tag{3.38}$$

□

Let us again fix $\eta > 0$. For any $f \in BC_{-\eta}(\mathbb{R}, \mathbb{R})$, we now write $\pi_{\mathcal{H}_M} f \in \mathcal{H}_M$ for the function

$$[\pi_{\mathcal{H}_M} f](\zeta, \xi) = f(\xi + \zeta M^{-1}), \qquad \zeta \in q^{-1}\mathbb{Z}_q, \qquad \xi \in M^{-1}\mathbb{Z}, \tag{3.39}$$

so that $\pi_{\mathcal{H}_M} = \mathcal{J}_M \pi_{\mathcal{Y}_M}$.

Our task now is to understand the action of $\mathcal{L}_{k,M}$ interpreted as a map from $\mathcal{H}^1_{k,M}$ into $\mathcal{H}_M$. To this end, we pick $m \in \mathbb{Z}$ such that

$$1 = (m + \varrho) M^{-1}, \qquad 0 < \varrho \leq 1, \tag{3.40}$$

which with $M = \frac{p}{q} \in \mathcal{M}_q$ gives $\varrho = \frac{p - mq}{q}$ and so

$$m M^{-1} = 1 - \varrho M^{-1}, \qquad \varrho \in q^{-1}\mathbb{Z}_q \setminus \{0\}. \tag{3.41}$$

In fact, because $\gcd(p, q) = 1$ we also have $\gcd(q\varrho, q) = 1$.

We now write $\mathcal{K}_{k,M} : \mathcal{H}^1_{k,M} \to \mathcal{H}_M$ for the linear operator that acts as

$$\begin{array}{lcl} [\mathcal{K}_{k,M} v](\zeta, \xi) & = & -\overline{c}[\mathcal{D}_{k,M} v](\zeta, \xi) + v(\zeta + \varrho, \xi + 1 - \varrho M^{-1}) + v(\zeta - \varrho, \xi - 1 + \varrho M^{-1}) - 2v(\zeta, \xi) \\ & & + g'\big(\overline{\Phi}(\xi + \zeta M^{-1}); \overline{a}\big) v(\zeta, \xi), \end{array} \tag{3.42}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in M^{-1}\mathbb{Z}$, where we introduce the convention

$$v(\zeta \pm 1, \xi) = v(\zeta, \xi \pm M^{-1}). \tag{3.43}$$

The shift ϱ hence acts as a rotation number, connecting the different components of v in the ζ-direction.

For notational convenience, we introduce the twist operator $T_M : \mathcal{H}_M \to \mathcal{H}_M$ that acts as

$$[T_M v](\zeta, \xi) = v(\zeta + \varrho, \xi + mM^{-1}), \tag{3.44}$$

again with the convention (3.43). In addition, we introduce the notation

$$g'\big(\pi_{\mathcal{H}_M}\overline{\Phi}; \overline{a}\big) : \mathcal{H}_M \to \mathcal{H}_M \tag{3.45}$$

to refer to the multiplication operator

$$[g'\big(\pi_{\mathcal{H}_M}\overline{\Phi}; \overline{a}\big)v](\zeta, \xi) = g'\big(\overline{\Phi}(\xi + \zeta M^{-1}); \overline{a}\big)v(\zeta, \xi). \tag{3.46}$$

These conventions allow us to write

$$\mathcal{K}_{k,M}v = -\overline{c}\mathcal{D}_{k,M}v + T_M v + T_M^{-1}v - 2v + g'\big(\pi_{\mathcal{H}_M}\overline{\Phi}; \overline{a}\big)v \tag{3.47}$$

and one may easily verify that in fact

$$\mathcal{K}_{k,M}\mathcal{J}^1_{k,M}v = \mathcal{J}_M\mathcal{L}_{k,M}v, \tag{3.48}$$

showing that $\mathcal{K}_{k,M}$ and $\mathcal{L}_{k,M}$ are equivalent.

In order to study the formal adjoint of $\mathcal{K}_{k,M}$, we need to introduce the operator $\mathcal{D}^*_{k,M}$ that acts as

$$[\mathcal{D}^*_{k,M}v](\zeta, \xi) = \beta_k^{-1} M \sum_{n'=0}^{k} \alpha_{n';k} v\big(\xi + (k - n')M^{-1}\big). \tag{3.49}$$

This allows us to define $\mathcal{K}^*_{k,M} : \mathcal{H}^1_{k,M} \to \mathcal{H}_M$ by writing

$$\mathcal{K}^*_{k,M}v = -\overline{c}\mathcal{D}^*_{k,M}v + T_M v + T_M^{-1}v - 2v + g'\big(\pi_{\mathcal{H}_M}\overline{\Phi}; \overline{a}\big)v. \tag{3.50}$$

As a final preparation, we introduce the subspace

$$\ell^2_{q,\perp;\infty} = \{v \in \ell^2_{q,\perp} : v(1) = v(0)\}, \tag{3.51}$$

together with the notation

$$[\pi_\perp f](\zeta, \xi) = f(\xi), \qquad \zeta \in q^{-1}\mathbb{Z}_q, \qquad \xi \in \mathbb{R}, \tag{3.52}$$

which constructs a function $\pi_\perp f \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})$ from a function $f \in L^2(\mathbb{R}, \mathbb{R})$.

Taking the limit $M \to \infty$ while keeping ϱ and q fixed as in (3.40), we find that $\mathcal{K}_{k,M}$ and $\mathcal{K}^*_{k,M}$ formally approach the limiting operators

$$\overline{\mathcal{K}}_{q,\varrho} : H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}) \to L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}), \qquad \overline{\mathcal{K}}^*_{q,\rho} : H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}) \to L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}) \tag{3.53}$$

that act as

$$\begin{aligned}
[\overline{\mathcal{K}}_{q,\varrho}V](\zeta, \xi) &= -\overline{c}\partial_\xi V(\zeta, \xi) + V(\zeta + \rho, \xi + 1) + V(\zeta - \rho, \xi - 1) - 2V(\zeta, \xi) \\
&\quad + g'(\overline{\Phi}(\xi); \overline{a})V(\zeta, \xi), \\
[\overline{\mathcal{K}}^*_{q,\varrho}V](\zeta, \xi) &= +\overline{c}\partial_\xi V(\zeta, \xi) + V(\zeta - \rho, \xi - 1) + V(\zeta + \rho, \xi + 1) - 2V(\zeta, \xi) \\
&\quad + g'(\overline{\Phi}(\xi); \overline{a})V(\zeta, \xi),
\end{aligned} \tag{3.54}$$

both with $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in \mathbb{R}$. Here we have made the identification $V(\zeta + 1, \xi) = V(\zeta, \xi)$.

The result below states some basic properties of these limiting operators $\overline{\mathcal{K}}_{q,\varrho}$ and $\overline{\mathcal{K}}^*_{q,\varrho}$. The key ingredient for the proof is [22, Prop 8.2], which generalizes the important scalar result [29, Thm. 4.1] to the multi-component setting considered here. Indeed, the latter result states that $\overline{\mathcal{L}}$ is Fredholm with index zero and a one-dimensional kernel, while the former establishes this for $\overline{\mathcal{K}}_{q,\rho}$.

Lemma 3.6. *Fix an integer $q \geq 1$ together with a constant $\varrho \in q^{-1}\mathbb{Z}_q$ that has $\gcd(q\varrho, q) = 1$. Consider the LDE (2.1) and suppose that (Hg) is satisfied. Pick $\overline{a}$ in such a way that also $(H\Phi)_{\overline{a}}$ is satisfied. Then the operators $\overline{\mathcal{K}}_{q,\varrho}$ and $\overline{\mathcal{K}}^*_{q,\varrho}$ are both Fredholm with index zero, with*

$$\operatorname{Ker} \overline{\mathcal{K}}_{q,\varrho} = \operatorname{span}\{\pi_\perp \overline{\Phi}'\}, \qquad \operatorname{Ker} \overline{\mathcal{K}}^*_{q,\varrho} = \operatorname{span}\{\pi_\perp \overline{\Psi}\}. \tag{3.55}$$

In addition, for any $\delta > 0$ the operator $\overline{\mathcal{K}}_{q,\varrho} - \delta$ is invertible and there exists $C > 1$ so that

$$\left\| [\overline{\mathcal{K}}_{q,\varrho} - \delta]^{-1} f - \frac{1}{\delta} \pi_\perp \overline{\Phi}' \langle \pi_\perp \overline{\Psi}, f \rangle_{L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})} \right\|_{H^1(\mathbb{R}, \ell^2_{q,\perp;\infty})} \leq C \, \|f\|_{L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})} \tag{3.56}$$

holds for any $\delta > 0$ and $f \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty})$.

Proof. Consider the problem

$$-\overline{c}\partial_\xi P(\zeta, \xi) = P(\zeta + \rho, \xi + 1) + P(\zeta - \rho, \xi - 1) - 2P(\zeta, \xi) + g\big(P(\zeta, \xi); \overline{a}\big), \tag{3.57}$$

for $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in \mathbb{R}$, with the identification $P(\zeta + 1, \xi) = P(\zeta, \xi)$. This problem clearly has a solution $P(\zeta, \xi) = \overline{\Phi}(\xi)$. In addition, the condition on ϱ ensures that (3.57) satisfies the conditions (HA), (HS1)-(HS2) and (Hf1)-(Hf3) formulated in [22, §2]. This allows us to apply [22, Prop 8.2], which directly gives the Fredholm properties stated above.

To see that $\overline{\mathcal{K}}_{q,\varrho} - \delta$ is invertible for $\delta > 0$ and that the conditions (S1)-(S3) in [21, §2] hold, one can use a comparison principle argument analogous to [19, Lem. 6.2] and [5, Lem. 8.3]. The bound (3.56) now follows from [21, Eq. (2.44)]. □

We now introduce the quantities

$$\begin{aligned} \mathcal{E}_{k,M}(\delta) &= \inf_{\|v\|_{\mathcal{H}^1_{k,M}} = 1} \Big\{ \|\mathcal{K}_{k,M} v - \delta v\|_{\mathcal{H}_M} + \delta^{-1} \big| \langle \pi_{\mathcal{H}_M} \overline{\Psi}, \mathcal{K}_{k,M} v - \delta v \rangle_{\mathcal{H}_M} \big| \Big\}, \\ \mathcal{E}^*_{k,M}(\delta) &= \inf_{\|v\|_{\mathcal{H}^1_{k,M}} = 1} \Big\{ \left\|\mathcal{K}^*_{k,M} v - \delta v\right\|_{\mathcal{H}_M} + \delta^{-1} \left| \langle \pi_{\mathcal{H}_M} \overline{\Phi}', \mathcal{K}^*_{k,M} v - \delta v \rangle_{\mathcal{H}_M} \right| \Big\}. \end{aligned} \tag{3.58}$$

Our next result provides a lower bound on these quantities, analogous to [1, Lem. 6]. The proof is postponed to §3.2, but we already use it here to establish Proposition 3.3 by making some minor adjustments to the proof of [1, Thm. 4].

Proposition 3.7. *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. Consider the LDE (2.1) and suppose that (Hg) is satisfied. Pick $\overline{a}$ in such a way that also $(H\Phi)_{\overline{a}}$ is satisfied. Then there exists $\kappa > 0$ such that for every $0 < \delta < 1$ we have*

$$\begin{aligned} \kappa(\delta) &:= \operatorname{liminf}_{M \to \infty,\ M \in \mathcal{M}_q} \mathcal{E}_{k,M}(\delta) \geq \kappa, \\ \kappa^*(\delta) &:= \operatorname{liminf}_{M \to \infty,\ M \in \mathcal{M}_q} \mathcal{E}^*_{k,M}(\delta) \geq \kappa. \end{aligned} \tag{3.59}$$

Proof of Proposition 3.3. Fix $0 < \delta < 1$ and $M \in \mathcal{M}_q$ sufficiently large. By Proposition 3.7 and the equivalence (3.48), $\mathcal{L}_{k,M} - \delta$ is an homeomorphism from $\mathcal{Y}^1_{k,M}$ onto its range

$$\mathcal{R} = (\mathcal{L}_{k,M} - \delta)\big(\mathcal{Y}^1_{k,M}\big) \subset \mathcal{Y}_M, \tag{3.60}$$

with a bounded inverse $\mathcal{I} : \mathcal{R} \to \mathcal{Y}^1_{k,M}$. The latter fact shows that $\mathcal{R}$ is a closed subset of $\mathcal{Y}_M$. If $\mathcal{R} \neq \mathcal{Y}_M$, there exists a non-zero $w \in \mathcal{Y}_M$ so that $\langle w, \mathcal{R} \rangle_{\mathcal{Y}_M} = 0$, i.e.,

$$\big\langle w, (\mathcal{L}_{k,M} - \delta) v \big\rangle_{\mathcal{Y}_M} = 0 \text{ for all } v \in \mathcal{Y}^1_{k,M}. \tag{3.61}$$

Since also $w \in \mathcal{Y}^1_{k,M}$, this implies

$$\big\langle (\mathcal{L}^*_{k,M} - \delta) w, v \big\rangle_{\mathcal{Y}_M} = 0 \text{ for all } v \in \mathcal{Y}^1_{k,M}. \tag{3.62}$$

Since $\mathcal{Y}^1_{k,M}$ and $\mathcal{Y}_M$ are equal as sets, this shows that in fact $\mathcal{L}^*_{k,M} w = 0$. Applying Proposition 3.7 once more and possibly increasing the lower bound for M, this gives the contradiction $w = 0$ and establishes that $\mathcal{R} = \mathcal{Y}_M$. The bound (3.20) with the δ-independent constant $C_0 > 1$ now follows directly from the definition (3.58) of the quantities $\mathcal{E}_{k,M}(\delta)$ and the uniform lower bound (3.59). □

3.2 Proof of Proposition 3.7

Our first task is to understand some basic properties concerning the discrete derivatives $\mathcal{D}_{k,M}$. Recalling the coefficients (2.6) appearing in the definition (2.12) for $\mathcal{D}_{k,M}$, we implicitly define the polynomial ϱ_k by writing

$$\varrho_k(z)(z-1) = \sum_{j=0}^{k} \alpha_{j;k} z^j. \tag{3.63}$$

Introducing the operator $S_M : \mathcal{H}_M \to \mathcal{H}_M$ that acts as

$$[S_M v](\zeta, \xi) = v\big(\zeta, \xi + M^{-1}\big), \tag{3.64}$$

we may compute

$$\begin{array}{lcl} \mathcal{D}_{k,M} & = & \beta_k^{-1} M \sum_{j=0}^{k} \alpha_{j;k} S_M^{j-k} \\ & = & \beta_k^{-1} M S_M^{-k} \sum_{j=0}^{k} \alpha_{j;k} S_M^{j} \\ & = & \beta_k^{-1} M S_M^{-k} \varrho_k(S_M)(S_M - I) \\ & = & \beta_k^{-1} M S_M^{-k} \varrho_k(S_M) S_M (I - S_M^{-1}) \\ & = & \beta_k^{-1} S_M^{-k} \varrho_k(S_M) S_M \mathcal{D}_{1,M} \\ & = & \beta_k^{-1} S_M^{-(k-1)} \varrho_k(S_M) \mathcal{D}_{1,M}. \end{array} \tag{3.65}$$

In view of this factorization, the following result allow us to recover information concerning $\mathcal{D}_{1,M} v$ from $\mathcal{D}_{k,M} v$ for $k \neq 1$.

Lemma 3.8. *For all integers $1 \le k \le 6$, the $k-1$ roots of the equation $\varrho_k(z) = 0$ all lie inside the unit circle.*

Proof. See [14, Ex 4; Sec III.3]. □

Corollary 3.9. *Fix two integers $q \ge 1$ and $1 \le k \le 6$. Then there exists constants $\kappa_{\min} > 0$ and $\kappa_{\max} > 0$ such that for any $M \in \mathcal{M}_q$ and any $v \in \mathcal{H}^1_{k,M}$, we have the inequalities*

$$\kappa_{\min} \left\| \mathcal{D}_{k,M} v \right\|_{\mathcal{H}_M} \le \left\| \mathcal{D}_{1,M} v \right\|_{\mathcal{H}_M} \le \kappa_{\max} \left\| \mathcal{D}_{k,M} v \right\|_{\mathcal{H}_M}. \tag{3.66}$$

Proof. On account of Lemma 3.8, the operator $\varrho_k(S_M)$ is invertible, which in view of the factorization (3.65) shows that $\mathcal{D}_{k,M}$ and $\mathcal{D}_{1,M}$ are equivalent. □

We are now ready to turn to our interpolation procedure. For any $\xi \in \mathbb{R}$, we define two quantities $\xi_M^{\pm}(\xi) \in M^{-1}\mathbb{Z}$ in such a way that

$$\xi_M^{-}(\xi) \le \xi < \xi_M^{+}(\xi), \qquad \xi_M^{+}(\xi) - \xi_M^{-}(\xi) = M^{-1}. \tag{3.67}$$

This allows to introduce two interpolation operators

$$\begin{array}{lcccl} \mathcal{I}_M^0 & : & \mathcal{H}_M & \to & L^2\big(\mathbb{R}, \ell^2_{q,\perp}\big), \\ \mathcal{I}_{k,M}^1 & : & \mathcal{H}^1_{k,M} & \to & H^1\big(\mathbb{R}, \ell^2_{q,\perp}\big) \end{array} \tag{3.68}$$

that act as

$$\begin{array}{lcl} [\mathcal{I}_M^0 v](\zeta, \xi) & = & v\big(\zeta, \xi_M^{-}(\xi)\big), \\ [\mathcal{I}_{k,M}^1 v](\zeta, \xi) & = & M\big[(\xi - \xi_M^{-}(\xi)) v(\zeta, \xi_M^{+}(\xi)) + (\xi_M^{+}(\xi) - \xi) v\big(\zeta, \xi_M^{-}(\xi)\big)\big], \end{array} \tag{3.69}$$

for all $\zeta \in q^{-1}\mathbb{Z}_q$ and $\xi \in \mathbb{R}$. These can be seen as interpolations of order zero respectively one, acting on the second coordinate of v. The next three results show that these operators are well-defined and establish some useful bounds.

Lemma 3.10. *Fix a pair of integers $q \ge 1$ and $1 \le k \le 6$. For any $M \in \mathcal{M}_q$ and $v \in \mathcal{H}_M$ we have*

$$\left\| \mathcal{I}^0_M v \right\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} = \|v\|_{\mathcal{H}_M} . \tag{3.70}$$

In addition, there exist constants $\kappa_{\min} > 0$ and $\kappa_{\max} > 0$ so that for any $M \in \mathcal{M}_q$ and $v \in \mathcal{H}^1_{k,M}$ we have

$$\begin{array}{rcccl} \kappa_{\min} \|v\|_{\mathcal{H}_M} & \le & \left\| \mathcal{I}^1_{k,M} v \right\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} & \le & \kappa_{\max} \|v\|_{\mathcal{H}_M} , \\ \kappa_{\min} \|v\|_{\mathcal{H}^1_{k,M}} & \le & \left\| \mathcal{I}^1_{k,M} v \right\|_{H^1(\mathbb{R},\ell^2_{q,\perp})} & \le & \kappa_{\max} \|v\|_{\mathcal{H}^1_{k,M}} . \end{array} \tag{3.71}$$

Proof. Picking $v \in \mathcal{H}_M$, we write $V_0 = \mathcal{I}^0_M v$ and compute

$$\begin{array}{rcl} \|V_0\|^2_{L^2(\mathbb{R},\ell^2_{q,\perp})} & = & \int_{-\infty}^{\infty} |V_0(\cdot,\xi')|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi+M^{-1}} |v(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} |v(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \\ & = & \|v\|^2_{\mathcal{H}_M} . \end{array} \tag{3.72}$$

In addition, picking $v \in \mathcal{H}^1_{k,M}$ and writing $V_1 = \mathcal{I}^1_{k,M} v$, we compute

$$\begin{array}{rcl} \|V_1\|^2_{L^2(\mathbb{R},\ell^2_{q,\perp})} & = & \int_{-\infty}^{\infty} |V_1(\cdot,\xi')|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi+M^{-1}} M^2 \big|(\xi' - \xi) v(\cdot, \xi + M^{-1}) + (\xi + M^{-1} - \xi') v\big(\cdot,\xi\big)\big|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \frac{1}{3} M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \Big[\big|v(\cdot,\xi + M^{-1})\big|^2_{\ell^2_{q,\perp}} + \big|v\big(\cdot,\xi\big)\big|^2_{\ell^2_{q,\perp}} \\ & & \qquad + \langle v(\cdot, \xi + M^{-1}), v\big(\cdot,\xi\big) \rangle_{\ell^2_{q,\perp}} \Big] \\ & = & \frac{1}{3} \Big[2 \|v\|^2_{\mathcal{H}_M} + \langle v, S_M v \rangle_{\mathcal{H}_M} \Big]. \end{array} \tag{3.73}$$

The first line in (3.71) now follows from the bound

$$|\langle v, S_M v \rangle_{\mathcal{H}_M}| \le \|v\|^2_{\mathcal{H}_M} . \tag{3.74}$$

On the other hand, we can compute

$$\begin{array}{rcl} \|V_1'\|^2_{L^2(\mathbb{R},\ell^2_{q,\perp})} & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi+M^{-1}} |V_1'(\cdot,\xi')|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi+M^{-1}} M^2 \left|v(\cdot,\xi + M^{-1}) - v(\cdot,\xi)\right|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi+M^{-1}} M^2 \left|[(S_M - I)v](\cdot,\xi)\right|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} M^2 \left|[S_M(I - S_M^{-1})v](\cdot,\xi)\right|^2_{\ell^2_{q,\perp}} \\ & = & M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \left|[S_M \mathcal{D}_{1,M} v](\cdot,\xi)\right|^2_{\ell^2_{q,\perp}} \\ & = & \|S_M \mathcal{D}_{1,M} v\|_{\mathcal{H}_M} \\ & = & \|\mathcal{D}_{1,M} v\|_{\mathcal{H}_M} . \end{array} \tag{3.75}$$

The second line of (3.71) now follows from the inequalities (3.66). □

Lemma 3.11. *Fix a pair of integers $q \geq 1$ and $1 \leq k \leq 6$. For any $M \in \mathcal{M}_q$ and $v \in \mathcal{H}^1_{k,M}$, we have the identity*

$$\left\| \mathcal{I}^0_M v - \mathcal{I}^1_{k,M} v \right\|_{L^2(\mathbb{R}, \ell^2_{q,\perp})} = \frac{1}{3}\sqrt{3} M^{-1} \left\| \mathcal{D}_{1,M} v \right\|_{\mathcal{H}_M}. \tag{3.76}$$

Proof. Writing $V_0 = \mathcal{I}^0_M v$ and $V_1 = \mathcal{I}^1_{k,M} v$, we compute

$$\begin{array}{lcl} \|V_0 - V_1\|^2_{L^2(\mathbb{R}, \ell^2_{q,\perp})} & = & \int_{-\infty}^{\infty} |V_0(\cdot, \xi') - V_1(\cdot, \xi')|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi + M^{-1}} |V_1(\cdot, \xi') - V_0(\cdot, \xi')|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi + M^{-1}} M^2 (\xi' - \xi)^2 \left| v(\cdot, \xi + M^{-1}) - v(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \frac{1}{3} M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \left| v(\cdot, \xi + M^{-1}) - v(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} \\ & = & \frac{1}{3} M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} |[(S_M - I)v](\cdot, \xi)|^2_{\ell^2_{q,\perp}} \\ & = & \frac{1}{3} M^{-3} \sum_{\xi \in M^{-1}\mathbb{Z}} \left| [M S_M (I - S_M^{-1}) v](\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} \\ & = & \frac{1}{3} M^{-2} \left\| S_M \mathcal{D}_{1,M} v \right\|^2_{\mathcal{H}_M} \\ & = & \frac{1}{3} M^{-2} \left\| \mathcal{D}_{1,M} v \right\|^2_{\mathcal{H}_M}. \end{array} \tag{3.77}$$

□

Lemma 3.12. *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$, together with a constant $\eta > 0$. Then there exists a constant $C > 1$ such that for any function $f \in BC^1_{-\eta}(\mathbb{R}, \mathbb{R})$, any $M \in \mathcal{M}_q$ and any $v \in \mathcal{H}_M$, we have*

$$\left| \langle \pi_\perp f, \mathcal{I}^0_M v \rangle_{L^2(\mathbb{R}, \ell^2_{q,\perp})} - \langle \pi_{\mathcal{H}_M} f, v \rangle_{\mathcal{H}_M} \right| \leq C M^{-1} \left\| f \right\|_{BC^1_{-\eta}} \left\| v \right\|_{\mathcal{H}_M}. \tag{3.78}$$

Proof. Upon introducing the quantity

$$\mathcal{I}_* = \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi + M^{-1}} \left\langle [\pi_\perp f](\cdot, \xi') - [\pi_{\mathcal{H}_M} f](\cdot, \xi), v(\cdot, \xi) \right\rangle_{\ell^2_{q,\perp}} d\xi', \tag{3.79}$$

we may compute

$$\begin{array}{lcl} \langle \pi_\perp f, \mathcal{I}^0_M v \rangle_{L^2(\mathbb{R}, \ell^2_{q,\perp})} & = & \int_{-\infty}^{\infty} \langle [\pi_\perp f_\perp](\cdot, \xi'), [\mathcal{I}^0_M v](\cdot, \xi') \rangle_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi + M^{-1}} \langle [\pi_\perp f](\cdot, \xi'), v(\cdot, \xi) \rangle_{\ell^2_{q,\perp}} \, d\xi' \\ & = & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_{\xi}^{\xi + M^{-1}} \langle [\pi_{\mathcal{H}_M} f](\cdot, \xi), v(\cdot, \xi) \rangle_{\ell^2_{q,\perp}} \, d\xi' + \mathcal{I}_* \\ & = & M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \langle [\pi_{\mathcal{H}_M} f](\cdot, \xi), v(\cdot, \xi) \rangle_{\ell^2_{q,\perp}} + \mathcal{I}_* \\ & = & \langle \pi_{\mathcal{H}_M} f, v \rangle_{\mathcal{H}_M} + \mathcal{I}_*. \end{array} \tag{3.80}$$

Whenever the pair (ξ, ξ') satisfies the inequality

$$\xi \leq \xi' \leq \xi + M^{-1}, \tag{3.81}$$

we may estimate

$$\|Df(\xi')\| \leq \|f\|_{BC^1_{-\eta}} e^{-\eta|\xi'|} \leq e^{\eta M^{-1}} \|f\|_{BC^1_{-\eta}} e^{-\eta|\xi|}. \tag{3.82}$$

In particular, assuming (3.81) we have

$$|f(\xi') - f(\xi)| \le M^{-1} e^{\eta M^{-1}} \, \|f\|_{BC^1_{-\eta}} \, e^{-\eta|\xi|}, \tag{3.83}$$

which under the additional assumption $\xi \in M^{-1}\mathbb{Z}$ gives

$$\begin{array}{rcl} q\Big|[\pi_\perp f](\cdot,\xi') - [\pi_{\mathcal{H}_M} f](\cdot,\xi)\Big|^2_{\ell^2_{q,\perp}} & = & \frac{1}{2}\Big|[\pi_\perp f](0,\xi') - [\pi_{\mathcal{H}_M} f](0,\xi)\Big|^2 + \frac{1}{2}\Big|[\pi_\perp f](1,\xi') - [\pi_{\mathcal{H}_M} f](1,\xi)\Big|^2 \\ & & \quad + \sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} \Big|[\pi_\perp f](\zeta,\xi') - [\pi_{\mathcal{H}_M} f](\zeta,\xi)\Big|^2 \\ & = & \frac{1}{2}\,|f(\xi') - f(\xi)|^2 + \frac{1}{2}\,\big|f(\xi') - f(\xi + M^{-1})\big|^2 \\ & & \quad + \sum_{\zeta \in q^{-1}\mathbb{Z}^\circ_q} \big|f(\xi') - f(\xi + \zeta M^{-1})\big|^2 \\ & \le & qM^{-2} e^{2\eta M^{-1}} \, \|f\|^2_{BC^1_{-\eta}} \, e^{-2\eta|\xi|}. \end{array} \tag{3.84}$$

This allows us to estimate

$$\begin{array}{rcl} |\mathcal{I}_*| & \le & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_\xi^{\xi+M^{-1}} \Big|[\pi_\perp f](\cdot,\xi') - [\pi_{\mathcal{H}_M} f](\cdot,\xi)\Big|_{\ell^2_{q,\perp}} \big|v(\cdot,\xi)\big|_{\ell^2_{q,\perp}} \, d\xi' \\ & \le & \sum_{\xi \in M^{-1}\mathbb{Z}} \int_\xi^{\xi+M^{-1}} M^{-1} e^{\eta M^{-1}} \, \|f\|_{BC^1_{-\eta}} \, e^{-\eta|\xi|} \big|v(\cdot,\xi)\big|_{\ell^2_{q,\perp}} \, d\xi' \\ & = & M^{-1} e^{\eta M^{-1}} \, \|f\|_{BC^1_{-\eta}} \, M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} e^{-\eta|\xi|} \big|v(\cdot,\xi)\big|_{\ell^2_{q,\perp}} \\ & \le & M^{-1} e^{\eta M^{-1}} \, \|f\|_{BC^1_{-\eta}} \, \big\|e^{-\eta|\cdot|}\big\|_{\ell^2_M(\mathbb{R})} \, \|v\|_{\ell^2_M(\ell^2_{q,\perp})} \, . \end{array} \tag{3.85}$$

The proof can now be completed exactly as in the final part of the proof of Lemma 3.4. □

A key ingredient in the proof of Proposition 3.7 is that certain inner products involving terms appearing in $\mathcal{K}_{k,M}$ have a well-defined sign or vanish in the limit $M \to \infty$. This issue is explored in the following set of results.

Lemma 3.13. *Fix an integer $q \ge 1$. For any $M \in \mathcal{M}_q$ and $v \in \mathcal{H}_M$ we have the inequality*

$$\langle v, [T_M + T_M^{-1} - 2]v \rangle_{\mathcal{H}_M} \le 0. \tag{3.86}$$

Proof. In view of the fact that T_M is an isometry, the inequality follows directly from Cauchy-Schwartz. □

Fix an integer $q \ge 1$ and pick $M \in \mathcal{M}_q$. For $v \in \mathcal{H}_M \subset \ell^2_M(\ell^2_{q,\perp})$, we define the Fourier transform

$$\widehat{v}(\zeta,\omega) = M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} e^{-i\xi\omega} v(\zeta,\xi), \qquad \omega \in [-M\pi, M\pi], \qquad \zeta \in q^{-1}\mathbb{Z}_q \tag{3.87}$$

and recall the accompanying inversion formula

$$v(\zeta,\xi) = \frac{1}{2\pi} \int_{-M\pi}^{M\pi} e^{i\xi\omega} \widehat{v}(\zeta,\omega) d\omega. \tag{3.88}$$

For $v \in \mathcal{H}_M$ and $w_{\mathcal{H}_M}$, Parseval's identity can be written as

$$\langle v, w \rangle_{\mathcal{H}_M} = M^{-1} \sum_{\xi \in M^{-1}\mathbb{Z}} \langle v(\cdot,\xi), w(\cdot,\xi) \rangle_{\ell^2_{q,\perp}} = \frac{1}{2\pi} \int_{-M\pi}^{M\pi} \langle \widehat{v}(\cdot,\omega), \widehat{w}(\cdot,\omega) \rangle_{\ell^2_{q,\perp}} \, d\omega. \tag{3.89}$$

For any $v \in \mathcal{H}^1_{k,M}$, computing the Fourier transform of $w = \mathcal{D}_{k,M} v \in \mathcal{H}_M$ yields

$$\widehat{w}(\zeta, \omega) = \beta_k^{-1} M \sum_{j=0}^{k} \alpha_{j;k} e^{i\omega(j'-k)M^{-1}} \widehat{v}(\zeta, \omega). \tag{3.90}$$

This motivates the definition

$$\widehat{\mathcal{D}}_{k,M}(\omega) = \beta_k^{-1} M \sum_{j=0}^{k} \alpha_{j;k} e^{i\omega(j'-k)M^{-1}} \in \mathbb{C} \tag{3.91}$$

for $\omega \in [-M\pi, M\pi]$ and $k \in \{1, \ldots, 6\}$.

Lemma 3.14. *There exists a constant $K > 1$ so that we have the bound*

$$\left| \operatorname{Re} \widehat{\mathcal{D}}_{k,M}(\omega) \right| \leq K M^{-1} \left| \widehat{\mathcal{D}}_{k,M}(\omega) \right|^2 \tag{3.92}$$

for all $k \in \{1, \ldots 6\}$, all $M > 0$ and all $\omega \in [-M\pi, M\pi]$.

Proof. In view of the scaling

$$\widehat{\mathcal{D}}_{k,M}(\omega) = M \widehat{\mathcal{D}}_{k,1}(\omega M^{-1}), \tag{3.93}$$

it suffices to show that for some $K > 1$ we have

$$\left| \operatorname{Re} \widehat{\mathcal{D}}_{k,1}(\omega) \right| \leq K \left| \widehat{\mathcal{D}}_{k,1}(\omega) \right|^2, \qquad \omega \in [-\pi, \pi], \qquad k \in \{1, \ldots, 6\}. \tag{3.94}$$

The identity

$$\widehat{\mathcal{D}}_{k,1}(\omega) = \beta_k^{-1} e^{-i\omega k} \varrho_k(e^{i\omega})(e^{i\omega} - 1) \tag{3.95}$$

together with Lemma 3.8 implies that $\mathcal{D}_{k,1}(\omega) \neq 0$ for all $\omega \notin 2\pi\mathbb{Z}$. In particular, it suffices to establish (3.94) for ω in some small neighbourhood of $\omega = 0$. To this end, we note that

$$\begin{aligned} \operatorname{Re} \widehat{\mathcal{D}}_{k,1}(\omega) &= \beta_k^{-1} \textstyle\sum_{j=0}^{k-1} \alpha_{j;k} [\cos\big(\omega(j-k)\big) - 1], \\ \operatorname{Im} \widehat{\mathcal{D}}_{k,1}(\omega) &= \beta_k^{-1} \textstyle\sum_{j=0}^{k-1} \alpha_{j;k} \sin\big(\omega(j-k)\big), \end{aligned} \tag{3.96}$$

which using (2.8) gives

$$\begin{aligned} [\tfrac{d}{d\omega} \operatorname{Re} \widehat{\mathcal{D}}_{k,1}(\omega)]_{\omega=0} &= 0, \\ [\tfrac{d}{d\omega} \operatorname{Im} \widehat{\mathcal{D}}_{k,1}(\omega)]_{\omega=0} &= \beta_k^{-1} \textstyle\sum_{j=0}^{k-1} \alpha_{j;k}(j-k) \\ &= 1. \end{aligned} \tag{3.97}$$

We hence see that

$$\begin{aligned} \left| \operatorname{Re} \widehat{\mathcal{D}}_{k,1}(\omega) \right| &= O(\omega^2), \\ \left| \widehat{\mathcal{D}}_{k,1}(\omega) \right|^2 &= \omega^2 + O(\omega^4), \end{aligned} \tag{3.98}$$

as $\omega \to 0$, which completes the proof. $\square$

Corollary 3.15. *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. There exists a constant $K > 1$ so that for all $M \in \mathcal{M}_q$ and all $v \in \mathcal{H}^1_{k,M}$ we have the bound*

$$\left| \langle v, \mathcal{D}_{k,M} v \rangle_{\mathcal{H}_M} \right| \leq K M^{-1} \left\| \mathcal{D}_{k,M} v \right\|^2_{\mathcal{H}_M}. \tag{3.99}$$

Proof. Using Parseval's identity (3.89) and applying Lemma 3.14, we may estimate

$$\begin{array}{lll} \left|\langle v, \mathcal{D}_{k,M} v\rangle_{\mathcal{H}_M}\right| & = & \left|\frac{1}{2\pi}\operatorname{Re}\int_{-M\pi}^{M\pi}\widehat{\mathcal{D}}_{k,M}(\omega)\,|\widehat{v}(\cdot,\omega)|^2_{\ell^2_{q,\perp}}\,d\omega\right| \\ & \leq & \frac{1}{2\pi}\int_{-M\pi}^{M\pi}\left|\operatorname{Re}\widehat{\mathcal{D}}_{k,M}(\omega)\right|\,|\widehat{v}(\cdot,\omega)|^2_{\ell^2_{q,\perp}}\,d\omega \\ & \leq & KM^{-1}\frac{1}{2\pi}\int_{-M\pi}^{M\pi}\left|\widehat{\mathcal{D}}_{k,M}(\omega)\right|^2|\widehat{v}(\cdot,\omega)|^2_{\ell^2_{q,\perp}}\,d\omega \\ & = & KM^{-1}\langle\mathcal{D}_{k,M}v, \mathcal{D}_{k,M}v\rangle_{\mathcal{H}_M}. \end{array} \tag{3.100}$$

□

We are now ready to establish a lower bound for the quantities $\mathcal{E}_{k,M}(\delta)$ defined in (3.58), noting that $\mathcal{E}^*_{k,M}(\delta)$ can be treated in a similar fashion. As a first step, we show that the limiting value $\kappa(\delta)$ can be approached via a sequence of realizations that allow us to take weak and strong limits in suitable function spaces. It is here that our need to work in the Hilbert spaces $L^2(\mathbb{R}, \ell^2_{q,\perp})$ and $H^1(\mathbb{R}, \ell^2_{q,\perp})$ becomes apparent, as we exploit the fact that bounded subsets of these spaces are weakly compact.

Lemma 3.16. *Consider the setting of Proposition 3.7 and fix $0 < \delta < 1$. Then there exist two functions*

$$V_* \in H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}) \subset H^1(\mathbb{R}, \ell^2_{q,\perp}), \qquad W_* \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}) \subset L^2(\mathbb{R}, \ell^2_{q,\perp}), \tag{3.101}$$

together with three sequences

$$\{M_j\}_{j\in\mathbb{N}} \subset \mathcal{M}_q, \qquad \{v_j\}_{j\in\mathbb{N}} \subset \mathcal{H}^1_{k,M_j}, \qquad \{w_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_{M_j} \tag{3.102}$$

that satisfy the following properties.

(i) We have $\lim_{j\to\infty} M_j = \infty$.

(ii) For any $j \in \mathbb{N}$, we have $\|v_j\|_{\mathcal{H}^1_{k,M_j}} = 1$ together with

$$w_j = \mathcal{K}_{k,M_j} v_j - \delta v_j. \tag{3.103}$$

(iii) Recalling the constant $\kappa(\delta)$ defined in (3.59), we have the limit

$$\kappa(\delta) = \lim_{j\to\infty}\{\|w_j\|_{\mathcal{H}_{M_j}} + \delta^{-1}\left|\langle \pi_{\mathcal{H}_{M_j}}\overline{\Psi}, w_j\rangle_{\mathcal{H}_{M_j}}\right|\}. \tag{3.104}$$

(iv) As $j \to \infty$, we have the weak convergences

$$\mathcal{I}^1_{k,M_j} v_j \rightharpoonup V_* \in H^1(\mathbb{R}, \ell^2_{q,\perp}), \qquad \mathcal{I}^0_{M_j} w_j \rightharpoonup W_* \in L^2(\mathbb{R}, \ell^2_{q,\perp}). \tag{3.105}$$

(v) For any compact interval $\mathcal{I} \subset \mathbb{R}$, we have the strong convergences

$$\mathcal{I}^0_{M_j} v_j \to V_* \in L^2(\mathcal{I}, \ell^2_{q,\perp}), \qquad \mathcal{I}^1_{k,M_j} v_j \to V_* \in L^2(\mathcal{I}, \ell^2_{q,\perp}) \tag{3.106}$$

as $j \to \infty$.

Proof. The existence of the sequences (3.102) that satisfy (i) through (iii) follows directly from the definition of $\kappa(\delta)$. Notice that (3.104) implies that $\|w_j\|_{\mathcal{H}_{M_j}}$ can be bounded uniformly for $j \in \mathbb{N}$. Upon introducing the functions

$$V_j = \mathcal{I}^1_{k,M_j} v_j \in H^1(\mathbb{R}, \ell^2_{q,\perp}), \qquad W_j = \mathcal{I}^0_{M_j} w_j \in L^2(\mathbb{R}, \ell^2_{q,\perp}), \tag{3.107}$$

Lemma 3.10 hence yields the bounds

$$\|V_j\|_{H^1(\mathbb{R},\ell^2_{q,\perp})} + \|W_j\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} \leq C_1' \tag{3.108}$$

for some $C_1' > 0$.

Since $L^2(\mathbb{R}, \ell^2_{q,\perp})$ and $H^1(\mathbb{R}, \ell^2_{q,\perp})$ are weakly compact, we can take a subsequence to obtain the weak convergence

$$V_j \rightharpoonup V_* \in H^1(\mathbb{R}, \ell^2_{q,\perp}), \qquad W_j \rightharpoonup W_* \in L^2(\mathbb{R}, \ell^2_{q,\perp}). \tag{3.109}$$

In addition, for any compact interval $\mathcal{I} \subset \mathbb{R}$, the compact embedding $H^1(\mathcal{I}, \ell^2_{q,\perp}) \subset L^2(\mathcal{I}, \ell^2_{q,\perp})$ yields the strong convergence $V_j \to V_* \in L^2(\mathcal{I}, \ell^2_{q,\perp})$. On account of Lemma 3.11 we also have the strong convergence

$$\mathcal{I}^0_{M_j} v_j \to V_* \in L^2(\mathcal{I}, \ell^2_{q,\perp}). \tag{3.110}$$

Finally, on account of the strong continuity of the shift-semigroup [12, Example I.5.4], we may in fact conclude

$$V_* \in H^1(\mathbb{R}, \ell^2_{q,\perp;\infty}), \qquad W_* \in L^2(\mathbb{R}, \ell^2_{q,\perp;\infty}). \tag{3.111}$$

□

In the next step we study the relation between the limiting functions V_* and W_*. By integrating against smooth test functions ζ, which naturally satisfy $\mathcal{D}_{k,M_j}\zeta \to \zeta'$, we are able to show that $W_* = (\overline{\mathcal{K}}_{q,\rho} - \delta)V_*$ for some appropriate ρ. This allows us to obtain an upper bound on the H^1 norm of V_*.

Lemma 3.17. *There exists a constant $K_1 > 0$ so that for any $0 < \delta < 1$, the function V_* defined in Lemma 3.16 satisfies the bound*

$$\|V_*\|_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \leq K_1 \kappa(\delta). \tag{3.112}$$

Proof. Again writing $W_j = \mathcal{I}^0_{M_j} w_j$, the weak lower semi-continuity of the L^2 norm implies that

$$\|W_*\|_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} \leq \liminf_{j\to\infty} \|W_j\|_{L^2(\mathbb{R},\ell^2_{q,\perp})} = \liminf_{j\to\infty} \|w_j\|_{\mathcal{H}_{M_j}}, \tag{3.113}$$

where the last identity follows from (3.70). In addition, we have the identities

$$\langle \pi_\perp \overline{\Psi}, W_* \rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} = \langle \pi_\perp \overline{\Psi}, W_* \rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})} = \lim_{j\to\infty} \langle \pi_\perp \overline{\Psi}, W_j \rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})} = \lim_{j\to\infty} \langle \pi_{\mathcal{H}_M} \overline{\Psi}, w_j \rangle_{\mathcal{H}_{M_j}}, \tag{3.114}$$

in which the second equality follows from the weak converge $W_j \rightharpoonup W_*$ and the third equality follows from Lemma 3.12, using the fact that $\overline{\Psi} \in BC^1_{-\eta}(\mathbb{R}, \mathbb{R})$ for all sufficiently small $\eta > 0$. In particular, we see that

$$\|W_*\|_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} + \delta^{-1} \big| \langle \pi_\perp \overline{\Psi}, W_* \rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} \big| \leq \kappa(\delta). \tag{3.115}$$

Let us fix $M \in \mathcal{M}_q$ for the moment. Observe that we have the commutation relations

$$\mathcal{I}^0_M T_M = T_M \mathcal{I}^0_M, \qquad \mathcal{I}^0_M S_M = S_M \mathcal{I}^0_M \tag{3.116}$$

for the twist operator T_M defined in (3.44) and the shift operator S_M defined in (3.64), both naturally extended to $L^2(\mathbb{R}, \ell^2_{q,\perp})$. This immediately also gives

$$\mathcal{I}^0_M \mathcal{D}_{k,M} = \mathcal{D}_{k,M} \mathcal{I}^0_M, \tag{3.117}$$

again extending $\mathcal{D}_{k,M}$ to act on $L^2(\mathbb{R},\ell^2_{q,\perp})$. In addition, for any $v \in \mathcal{H}_M$ we have

$$\mathcal{I}^0_M\Big[g'\big(\pi_{\mathcal{H}_M}\overline{\Phi};\overline{a}\big)v\Big] = [\mathcal{I}^0_M g'\big(\pi_{\mathcal{H}_M}\overline{\Phi};\overline{a}\big)]\mathcal{I}^0_M v, \tag{3.118}$$

where the right hand part is a multiplication of functions in $L^2(\mathbb{R},\ell^2_{q,\perp})$.

In view of these considerations, we introduce the operators

$$\mathcal{K}_{k,M;\mathcal{I}^0} : L^2(\mathbb{R},\ell^2_{q,\perp}) \to L^2(\mathbb{R},\ell^2_{q,\perp}), \qquad \mathcal{K}^*_{k,M;\mathcal{I}^0} : L^2(\mathbb{R},\ell^2_{q,\perp}) \to L^2(\mathbb{R},\ell^2_{q,\perp}) \tag{3.119}$$

that act as

$$\begin{array}{rcl} \mathcal{K}_{k,M;\mathcal{I}^0}V & = & -\overline{c}\mathcal{D}_{k,M}V + T_M V + T_M^{-1}V - 2V + [\mathcal{I}^0_M g'\big(\pi_{\mathcal{H}_M}\overline{\Phi};\overline{a}\big)]V, \\ \mathcal{K}^*_{k,M;\mathcal{I}^0}V & = & +\overline{c}\mathcal{D}^*_{k,M}V + T_M V + T_M^{-1}V - 2V + [\mathcal{I}^0_M g'\big(\pi_{\mathcal{H}_M}\overline{\Phi};\overline{a}\big)]V. \end{array} \tag{3.120}$$

For any $v \in \mathcal{H}^1_{k,M}$, we now have

$$\mathcal{I}^0_M \mathcal{K}_{k,M} v = \mathcal{K}_{k,M;\mathcal{I}^0}\mathcal{I}^0_M v. \tag{3.121}$$

For any test-function $\zeta \in C^\infty_0(\mathbb{R};\ell^2_{q,\perp;\infty}) \subset C^\infty_0(\mathbb{R},\ell^2_{q,\perp})$, we may compute

$$\begin{array}{rcl} \langle \zeta, \mathcal{I}^0_{M_j} w_j\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})} & = & \langle \zeta, \mathcal{I}^0_{M_j}[\mathcal{K}_{k,M_j} - \delta]v_j\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})} \\ & = & \langle \zeta, [\mathcal{K}_{k,M_j;\mathcal{I}^0} - \delta]\mathcal{I}^0_{M_j} v_j\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})} \\ & = & \langle [\mathcal{K}^*_{k,M_j;\mathcal{I}^0} - \delta]\zeta, \mathcal{I}^0_{M_j} v_j\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp})}. \end{array} \tag{3.122}$$

Since ζ has compact support, we can pick $m > 0$ so that $\mathrm{supp}(\zeta) \subset [-m+1, m-1]$ and hence

$$\mathrm{supp}[\mathcal{K}^*_{k,M_j;\mathcal{I}^0} - \delta]\zeta \in [-m,m]. \tag{3.123}$$

Without loss of generality, we assume that we can pick ρ from the finite set $q^{-1}\mathbb{Z}_q \setminus \{0\}$ in such a way that $\gcd(q\rho, q) = 1$ and $\rho(M_j) = \rho$ holds for all $j \in \mathbb{N}$. Here we use the notation $\rho(M_j)$ to refer to the value of ϱ in (3.40) with $M = M_j$.

The smoothness of ζ now implies that

$$[\mathcal{K}^*_{k,M_j;\mathcal{I}^0} - \delta]\zeta \to [\overline{\mathcal{K}}^*_{q,\rho} - \delta]\zeta \in L^2([-m,m],\ell^2_{q,\perp;\infty}). \tag{3.124}$$

Together with the strong limit

$$\mathcal{I}^0_{M_j} v_j \to V_* \in L^2([-m,m],\ell^2_{q,\perp}) \tag{3.125}$$

and (3.111), this allows us to conclude

$$\langle \zeta, W_*\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} = \langle [\overline{\mathcal{K}}^*_{q,\rho} - \delta]\zeta, V_*\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})}. \tag{3.126}$$

Since ζ was arbitrary, we see that $W_* = (\overline{\mathcal{K}}_{q,\rho} - \delta)V_*$ in the sense of distributions, which in view of Lemma 3.6 and (3.111) implies that

$$\begin{array}{rcl} \|V_*\|_{H^1(\mathbb{R},\ell^2_{q,\perp})} & = & \|V_*\|_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \\ & \leq & K_1\Big[\,\|W_*\|_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})} + \delta^{-1}\Big|\langle \pi_\perp\overline{\Psi}, W_*\rangle_{L^2(\mathbb{R},\ell^2_{q,\perp;\infty})}\Big|\,\Big] \\ & \leq & K_1\kappa(\delta) \end{array} \tag{3.127}$$

for some $K_1 > 1$. □

In the final step we obtain a lower bound on the H^1 norm of V_*. It is here that we exploit the specific structure of the terms in $\mathcal{K}_{k,M_j}$ and the bistable nature of the nonlinearity g. In particular, the expression $g'(\overline{\Phi}(\xi);\overline{a})$ is only positive on a bounded set for ξ, allowing us to exploit the strong convergence of $\mathcal{I}^0_{M_j} v_j$ to V_* on such sets.

Lemma 3.18. *There exist constants $K_2 > 1$ and $K_3 > 1$ so that for any $0 < \delta < 1$, the function V_* defined in Lemma 3.16 satisfies the bound*

$$\|V_*\|^2_{H^1(\mathbb{R},\ell^2_{q,\perp;\infty})} \geq K_2 - K_3 \kappa(\delta)^2. \tag{3.128}$$

Proof. For definiteness, we will assume $\overline{c} > 0$. In view of the identity

$$\begin{aligned} w_j &= \mathcal{K}_{k,M_j} v_j - \delta v_j \\ &= -\overline{c}\mathcal{D}_{k,M_j} v_j + [T_{M_j} + T^{-1}_{M_j} - 2] v_j + g'(\pi_{\mathcal{H}_{M_j}} \overline{\Phi}; \overline{a}) v_j - \delta v_j, \end{aligned} \tag{3.129}$$

we may write

$$\begin{aligned} \langle w_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_M} + \overline{c} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} &= \langle g'(\pi_{\mathcal{H}_{M_j}} \overline{\Phi}; \overline{a}) v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} \\ &\quad + \langle [T_{M_j} + T^{-1}_{M_j} - 2] v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} \\ &\quad - \delta \langle v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}}. \end{aligned} \tag{3.130}$$

Writing $C_1' = \|g'\|_\infty + 6 > 0$, remembering that $0 < \delta < 1$ and invoking Cauchy-Schwartz, we obtain

$$\begin{aligned} C_1' \|v_j\|_{\mathcal{H}_{M_j}} \left\| \mathcal{D}_{k,M_j} v_j \right\|_{\mathcal{H}_{M_j}} &\geq \langle g'(\pi_{\mathcal{H}_{M_j}} \overline{\Phi}; \overline{a}) v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} \\ &\quad + \langle [T_{M_j} + T^{-1}_{M_j} - 2] v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} \\ &\quad - \delta \langle v_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} \\ &= \langle w_j, \mathcal{D}_{k,M_j} v_j \rangle_{\mathcal{H}_{M_j}} + \overline{c} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} \\ &\geq \overline{c} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} - \left\| \mathcal{D}_{k,M_j} v_j \right\|_{\mathcal{H}_{M_j}} \|w_j\|_{\mathcal{H}_{M_j}}. \end{aligned} \tag{3.131}$$

This yields

$$\|w_j\|_{\mathcal{H}_{M_j}} + C_1' \|v_j\|_{\mathcal{H}_{M_j}} \geq \overline{c} \|\mathcal{D}_{k,M} v_j\|_{\mathcal{H}_{M_j}}, \tag{3.132}$$

which can be squared to give

$$\begin{aligned} 2\|w_j\|^2_{\mathcal{H}_{M_j}} + 2C_1'^2 \|v_j\|^2_{\mathcal{H}_{M_j}} &\geq \|w_j\|^2_{\mathcal{H}_{M_j}} + C_1'^2 \|v_j\|^2_{\mathcal{H}_{M_j}} + 2C_1' \|w_j\|_{\mathcal{H}_{M_j}} \|v_j\|_{\mathcal{H}_{M_j}} \\ &\geq \overline{c}^2 \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}}, \end{aligned} \tag{3.133}$$

which is reminiscent of [1, Eq. (3.9)].

Let us now pick a constant $m > 1$ in such a way that

$$0 < \alpha := \frac{1}{2} \min\{-g'(0;\overline{a}), -g'(1;\overline{a})\} = \min_{|\xi| \geq m-1} \{-g'(\overline{\Phi}(\xi);\overline{a})\}. \tag{3.134}$$

This allows us to estimate

$$
\begin{array}{rcl}
\langle w_j, v_j \rangle_{\mathcal{H}_{M_j}} & = & \big\langle [\mathcal{K}_{k,M_j} - \delta] v_j, v_j \big\rangle_{\mathcal{H}_{M_j}} \\
& = & -\overline{c} \langle \mathcal{D}_{k,M} v_j, v_j \rangle_{\mathcal{H}_{M_j}} \\
& & \quad + \big\langle [T_{M_j} + T_{M_j}^{-1} - 2] v_j, v_j \big\rangle_{\mathcal{H}_{M_j}} \\
& & \quad + \big\langle g'(\pi_{\mathcal{H}_{M_j}} \overline{\Phi}; \overline{a}) v_j, v_j \big\rangle_{\mathcal{H}_{M_j}} - \delta \langle v_j, v_j \rangle_{\mathcal{H}_{M_j}} \\
& \leq & \big\langle g'(\pi_{\mathcal{H}_{M_j}} \overline{\Phi}; \overline{a}) v_j, v_j \big\rangle_{\mathcal{H}_{M_j}} - \overline{c} \langle \mathcal{D}_{k,M} v_j, v_j \rangle_{\mathcal{H}_{M_j}} \\
& \leq & -\alpha \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} + \big(\left\| g' \right\|_\infty + \alpha \big) M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z} : |\xi| \leq m} \left| v_j(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} \\
& & \quad + C_2' M_j^{-1} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} ,
\end{array}
\tag{3.135}
$$

for some $C_2' > 1$, where we used (3.134) and Corollary 3.15 for the last bound. Using the basic inequality

$$
xy = (\sqrt{\alpha} x)(y / \sqrt{\alpha}) \leq \frac{\alpha}{2} x^2 + \frac{1}{2\alpha} y^2,
\tag{3.136}
$$

we find

$$
\begin{array}{rcl}
\big(\left\| g' \right\|_\infty + \alpha \big) M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z} : |\xi| \leq m} \left| v_j(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} & \geq & \alpha \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} + \langle w_j, v_j \rangle_{\mathcal{H}_{M_j}} \\
& & \quad - C_2' M_j^{-1} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} \\
& \geq & \alpha \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} - \left\| w_j \right\|_{\mathcal{H}_{M_j}} \left\| v_j \right\|_{\mathcal{H}_{M_j}} \\
& & \quad - C_2' M_j^{-1} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} \\
& \geq & \frac{\alpha}{2} \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} - \frac{1}{2\alpha} \left\| w_j \right\|^2_{\mathcal{H}_{M_j}} \\
& & \quad - C_2' M_j^{-1} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} .
\end{array}
\tag{3.137}
$$

Rescaling (3.133) yields

$$
0 \geq \frac{\alpha}{2(\overline{c}^2 + 2C_1'^2)} \Big[\overline{c}^2 \left\| \mathcal{D}_{k,M} v_j \right\|^2_{\mathcal{H}_{M_j}} - 2C_1'^2 \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} - 2 \left\| w_j \right\|^2_{\mathcal{H}_{M_j}} \Big],
\tag{3.138}
$$

which can be added to (3.137) to obtain

$$
\begin{array}{rcl}
\big(\left\| g' \right\|_\infty + \alpha \big) M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z} : |\xi| \leq m} \left| v_j(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} & \geq & \frac{\overline{c}^2 \alpha}{2(\overline{c}^2 + 2C_1'^2)} \big[\left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} + \left\| v_j \right\|^2_{\mathcal{H}_{M_j}} \big] \\
& & \quad - \big[\frac{1}{2\alpha} + \frac{\alpha}{\overline{c}^2 + 2C_1'^2} \big] \left\| w_j \right\|^2_{\mathcal{H}_{M_j}} \\
& & \quad - C_2' M_j^{-1} \left\| \mathcal{D}_{k,M_j} v_j \right\|^2_{\mathcal{H}_{M_j}} .
\end{array}
\tag{3.139}
$$

Remembering that $\left\| v_j \right\|_{\mathcal{H}^1_{k,M_j}} = 1$, we find that there exist constants $K_2 > 0$ and $K_3 > 0$, which both are independent of $0 < \delta < 1$, such that

$$
M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z} : |\xi| \leq m} \left| v_j(\cdot, \xi) \right|^2_{\ell^2_{q,\perp}} \geq K_2 - K_3 \left\| w_j \right\|^2_{\mathcal{H}_{M_j}} - C_2' M_j^{-1}.
\tag{3.140}
$$

The strong convergence $\mathcal{I}^0_{M_j} v_j \to V_* \in L^2([-m-1, m+1], \ell^2_{q,\perp})$ now implies that

$$\begin{array}{lcl} M_j^{-1} \sum_{\xi \in M_j^{-1}\mathbb{Z}: |\xi| \le m} |v_j(\cdot,\xi)|^2_{\ell^2_{q,\perp}} & = & \int_{-m}^{m+M_j^{-1}} \Big| [\mathcal{I}^0_{M_j} v_j](\cdot,\xi) \Big|^2_{\ell^2_{q,\perp}} \, d\xi \\ & \le & \int_{-m}^{m+1} \Big| [\mathcal{I}^0_{M_j} v_j](\cdot,\xi) \Big|^2_{\ell^2_{q,\perp}} \, d\xi \\ & \to & \int_{-m}^{m+1} |V_*(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \, d\xi, \end{array} \tag{3.141}$$

which in view of the bound $\limsup_{j\to\infty} \|w_j\|^2_{\mathcal{H}_{M_j}} \le \kappa(\delta)^2$ gives

$$\|V_*\|^2_{H^1(\mathbb{R},\ell^2_{q,\perp})} \ge \int_{-m}^{m+1} |V_*(\cdot,\xi)|^2_{\ell^2_{q,\perp}} \, d\xi \ge K_2 - K_3\kappa(\delta)^2. \tag{3.142}$$

□

Proof of Proposition 3.7. For any $0 < \delta < 1$, Lemma's 3.17 and 3.18 show that the function V_* defined in Lemma 3.16 satisfies

$$K_1^2 \kappa(\delta)^2 \ge \|V_*\|^2_{H^1(\mathbb{R},\ell^2_{q,\perp})} \ge K_2 - K_3\kappa(\delta)^2, \tag{3.143}$$

which gives $\big(K_1^2 + K_3\big)\kappa(\delta)^2 \ge K_2 > 0$, as desired. □

3.3 Proof of Proposition 3.2

We are now ready to turn to the proof of this section's main result. The basic strategy is to exploit the fact that we already know that $\mathcal{L}_{k,M} - \delta$ is invertible to study the difference between $(\mathcal{L}_{k,M} - \delta)^{-1}$ and $(\overline{\mathcal{L}} - \delta)^{-1}$.

Lemma 3.19. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$, together with a sufficiently small constant $\eta > 0$. Recall the function $\delta \mapsto M_0(\delta)$ defined in the statement of Proposition 3.3. Then there exists a constant $K > 1$ so that for any $0 < \delta < 1$, any $f \in BC^1_{-\eta}(\mathbb{R},\mathbb{R})$ and any $M \in \mathcal{M}_q$ with $M \ge M_0(\delta)$, we have the bound*

$$\Big\| (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} f - \pi_{\mathcal{Y}^1_{k,M}} (\overline{\mathcal{L}} - \delta)^{-1} f \Big\|_{\mathcal{Y}^1_{k,M}} \le K\delta^{-2} M^{-1} \|f\|_{BC^1_{-\eta}} . \tag{3.144}$$

Proof. Consider the functions $v_{k,M} \in \mathcal{Y}^1_{k,M}$ and $\overline{v} \in BC^2_{-\eta}(\mathbb{R},\mathbb{R})$ defined by

$$\begin{array}{lcl} v_{k,M} & = & (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} f, \\ \overline{v} & = & (\overline{\mathcal{L}} - \delta)^{-1} f. \end{array} \tag{3.145}$$

Again applying [21, Eq. (2.44)], we find the bound

$$\|\overline{v}\|_{BC^2_{-\eta}} \le C_1' \delta^{-1} \|f\|_{BC^1_{-\eta}} , \tag{3.146}$$

for some $C_1' > 1$ that does not depend on δ and f.

Writing $x = v_{k,M} - \pi_{\mathcal{Y}^1_{k,M}} \overline{v} \in \mathcal{Y}^1_{k,M}$, we may compute

$$(\mathcal{L}_{k,M} - \delta)x = \overline{c}\pi_{\mathcal{Y}_M} \big[\mathcal{D}_{k,M} \overline{v} - \overline{v}' \big]. \tag{3.147}$$

The estimate (2.14) now implies that the $\mathcal{Y}_M$ norm of the right-hand side can be bounded by $C_2' M^{-1} \|\overline{v}\|_{BC^2_{-\eta}}$ for some $C_2' > 1$ that does not depend on $\overline{v}$. The desired bound now follows from an application of Proposition 3.3. □

Proof of Proposition 3.2. All constants introduced below are independent of $0 < \delta < 1$ and $M \in \mathcal{M}_q$ with $M \geq \max\{\delta^{-2}, M_0(\delta)\}$, together with f and v where applicable. For convenience, we introduce the set

$$\mathcal{Z}^1_{k,M} = \{v \in \mathcal{Y}^1_{k,M} : \langle \pi_{\mathcal{Y}_M} \overline{\Psi}, v \rangle_{\mathcal{Y}_M} = 0\}. \tag{3.148}$$

Our goal is to find, for any $f \in \mathcal{Y}_M$, a solution $(\gamma, v) \in \mathbb{R} \times \mathcal{Z}^1_{k,M}$ to the problem

$$v = \mathcal{V}_{k,M;\delta}(f, v, \gamma) := (\mathcal{L}_{k,M} - \delta)^{-1}\big[f + \gamma \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M} \overline{\Phi} - \delta v\big]. \tag{3.149}$$

In order to ensure that the linear operator $\mathcal{V}_{k,M;\delta}$ indeed maps into $\mathcal{Z}^1_{k,M}$, it suffices to choose γ in such a way that

$$\gamma \langle \pi_{\mathcal{Y}_M} \overline{\Psi}, (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M} \overline{\Phi} \rangle_{\mathcal{Y}_M} = -\langle \pi_{\mathcal{Y}_M} \overline{\Psi}, (\mathcal{L}_{k,M} - \delta)^{-1}(f - \delta v) \rangle_{\mathcal{Y}_M}. \tag{3.150}$$

It is easy to verify that

$$(\overline{\mathcal{L}} - \delta)^{-1} \Phi' = -\delta^{-1} \Phi', \tag{3.151}$$

which using Lemma 3.19 together with (2.14) and remembering $\delta^{-2} > \delta^{-1}$ gives

$$\left\| (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M} \Phi + \delta^{-1} \pi_{\mathcal{Y}^1_{k,M}} \Phi' \right\|_{\mathcal{Y}^1_{k,M}} \leq C_1' \delta^{-2} M^{-1} \tag{3.152}$$

for some $C_1' > 1$. Applying Lemma 3.4, we hence see that

$$\left| \langle \pi_{\mathcal{Y}_M} \Psi, (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M} \Phi \rangle_{\mathcal{Y}_M} + \delta^{-1} \right| \leq C_2' \delta^{-2} M^{-1} \tag{3.153}$$

for some $C_2' > 1$. In particular, using $\frac{1}{x} + \frac{1}{\alpha} = \frac{\alpha + x}{\alpha x}$ and $|x| \geq \alpha - |x + \alpha|$, we see that there exists $C_3' > 1$ for which

$$\left| \langle \pi_{\mathcal{Y}_M} \Psi, (\mathcal{L}_{k,M} - \delta)^{-1} \pi_{\mathcal{Y}_M} \mathcal{D}_{k,M} \Phi \rangle^{-1}_{\mathcal{Y}_M} + \delta \right| \leq C_3' M^{-1} \tag{3.154}$$

holds for all sufficiently large $M \gg \delta^{-2}$. For such pairs (δ, M), one can hence find a unique solution $\gamma = \gamma_{k,M;\delta}(f, v)$ to (3.150) for every $v \in \mathcal{Z}^1_{k,M}$ and $f \in \mathcal{Y}_M$. Since we may estimate

$$\left| \langle \pi_{\mathcal{Y}_M} \overline{\Psi}, (\mathcal{L}_{k,M} - \delta)^{-1}(f - \delta v) \rangle_{\mathcal{Y}_M} \right| \leq C_4' [\delta^{-1} \|f\|_{\mathcal{Y}_M} + \delta \|v\|_{\mathcal{Y}_M}], \tag{3.155}$$

we see that

$$|\gamma_{k,M;\delta}(f, v)| \leq C_5' \big[\|f\|_{\mathcal{Y}_M} + \delta^2 \|v\|_{\mathcal{Y}_M} \big] \tag{3.156}$$

for some $C_4' > 1$ and $C_5' > 1$. We emphasize that it is a consequence of $v \in \mathcal{Z}^1_{k,M}$ that we have gained an extra factor δ in front of v here.

We now find

$$\|\mathcal{V}_{k,M;\delta}(f, v, \gamma_{k,M;\delta}(f, v))\|_{\mathcal{Y}^1_{k,M}} \leq C_6' \big[\delta^{-1} \|f\|_{\mathcal{H}_M} + \delta \|v\|_{\mathcal{Y}_M} \big] \tag{3.157}$$

for some $C_6' > 1$. By choosing $\delta > 0$ to be sufficiently small, we hence see that the linear fixed point problem

$$v = \mathcal{V}_{k,M;\delta}\big(f, v, \gamma_{k,M;\delta}(f, v)\big) \tag{3.158}$$

posed on $\mathcal{Z}^1_{k,M}$ has a unique solution for all $f \in \mathcal{Y}_M$. Writing $v = \mathcal{V}^*_{k,M;\delta} f$ for this solution together with

$$\gamma^*_{k,M;\delta} f = \gamma_{k,M;\delta}\big(f, \mathcal{V}^*_{k,M;\delta} f\big), \tag{3.159}$$

we obtain the estimates

$$\left\|\mathcal{V}^*_{k,M;\delta} f\right\|_{\mathcal{Y}^1_{k,M}} \le C_7' \delta^{-1} \left\| f \right\|_{\mathcal{Y}_M}, \qquad \left|\gamma^*_{k,M;\delta} f\right| \le C_7' \left\| f \right\|_{\mathcal{Y}_M} \tag{3.160}$$

for some $C_7' \ge 1$. Inspection of (3.149) shows that $\mathcal{V}^*_{k,M;\delta}$ and $\gamma^*_{k,M;\delta}$ are actually independent of δ, which allows us to fix a suitably small $\delta > 0$ and obtain the desired bounds (3.15).

Now turning to the bound (3.19), we note that for every sufficiently large M and every $f \in BC^1_{-\eta}(\mathbb{R}, \mathbb{R})$, we can find $\overline{v}_M \in BC^2_{-\eta}(\mathbb{R}, \mathbb{R})$ so that

$$\overline{\mathcal{L}}\, \overline{v}_M = f - \Phi' \langle \overline{\Psi}, f \rangle_{L^2}, \qquad \langle \pi_{\mathcal{Y}_M} \overline{\Psi}, \pi_{\mathcal{Y}_M} \overline{v}_M \rangle_{\mathcal{Y}_M} = 0, \tag{3.161}$$

with the estimate

$$\left\| \overline{v}_M \right\|_{BC^2_{-\eta}} \le C_8' \left\| f \right\|_{BC^1_{-\eta}} \tag{3.162}$$

for some $C_8' > 1$. In particular, upon writing

$$\mathcal{V}^*_{k,M} f = \overline{v}_M + v_{k,M}, \qquad \gamma^*_{k,M} f = -\langle \overline{\Psi}, f \rangle_{L^2} + \gamma_{k,M}, \tag{3.163}$$

we find that

$$\mathcal{L}_{k,M} v_{k,M} = g_{k,M} + \gamma_{k,M} \mathcal{D}_{k,M} \overline{\Phi}. \tag{3.164}$$

Here we have introduced the sequence

$$g_{k,M} = \overline{c} \pi_{\mathcal{Y}_{k,M}} [\mathcal{D}_{k,M} \overline{v}_M - \overline{v}_M'] - \langle \overline{\Psi}, f \rangle_{L^2} \pi_{\mathcal{Y}_{k,M}} [\mathcal{D}_{k,M} \overline{\Phi} - \overline{\Phi}'], \tag{3.165}$$

which implies

$$v_{k,M} = \mathcal{V}^*_{k,M} g_{k,M}, \qquad \gamma_{k,M} = \gamma^*_{k,M} g_{k,M}. \tag{3.166}$$

Using (2.14), we obtain the estimate

$$\left\| g_{k,M} \right\|_{\mathcal{Y}_M} \le C_9' M^{-1} \left\| f \right\|_{BC^1_{-\eta}}, \tag{3.167}$$

which in view of Lemma 3.4 gives the desired bound (3.19). □

4 Proof of Main Results

In this section we set out to prove the results stated in §2. In §4.1 we study the limit $\Delta t \to 0$, exploiting the linear theory developed in §3 to set up a fixed point argument and prove Theorem 2.1. The backward-Euler discretization is analyzed in §4.2, where we primarily exploit the work of Mallet-Paret [29] to prove Theorem 2.2 and Corollaries 2.3 and 2.5. Finally, in §4.3 we prove Corollary 2.4, which concerns the anti-continuum limit of the PDE (1.1). This part is heavily based on the pioneering work of Keener [25].

4.1 The small time-step limit $\Delta t \to 0$

Let us fix an integer $q \ge 1$ and a constant $M = \frac{p}{q} \in \mathcal{M}_q$. We seek a solution to the nonlinear problem

$$c[\mathcal{D}_{k,M} \Phi](\xi) = \Phi(\xi + 1) + \Phi(\xi - 1) - 2\Phi(\xi) + g\big(\Phi(\xi); a\big), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.1}$$

that has the form

$$\Phi(\xi) = \overline{\Phi}(\xi + \vartheta) + v(\xi), \qquad \xi \in p^{-1}\mathbb{Z}, \tag{4.2}$$

for some $\vartheta \in \mathbb{R}$ and $v \in \mathcal{Y}_M$. Note that this automatically ensures that Φ satisfies the boundary conditions

$$\lim_{\xi \to -\infty;\, \xi \in p^{-1}\mathbb{Z}} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty;\, \xi \in p^{-1}\mathbb{Z}} \Phi(\xi) = 1. \tag{4.3}$$

In addition, the normalization condition (2.24) is satisfied provided that

$$\langle \overline{\Psi}(\cdot + \vartheta), v \rangle_{\mathcal{Y}_M} = 0. \tag{4.4}$$

For convenience, we introduce the shorthands

$$\overline{\Phi}_\vartheta(\xi) = \overline{\Phi}(\xi + \vartheta), \qquad \overline{\Psi}_\vartheta(\xi) = \overline{\Psi}(\xi + \vartheta). \tag{4.5}$$

In addition, we introduce the linear operators

$$\mathcal{L}_{k,M;\vartheta} : \mathcal{Y}^1_{k,M} \to \mathcal{Y}_M \tag{4.6}$$

that act as

$$[\mathcal{L}_{k,M;\vartheta} v](\xi) = -\overline{c}[\mathcal{D}_{k,M} v](\xi) + v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'\big(\overline{\Phi}_\vartheta(\xi); \overline{a}\big)v(\xi), \qquad \xi \in p^{-1}\mathbb{Z}. \tag{4.7}$$

Naturally, these operators satisfy the properties described in Proposition 3.2 provided all occurrences of $\overline{\Phi}$ and $\overline{\Psi}$ are replaced by $\overline{\Phi}_\vartheta$ respectively $\overline{\Psi}_\vartheta$. We write

$$\gamma^*_{k,M;\vartheta} : \mathcal{Y}_M \to \mathbb{R}, \qquad \mathcal{V}^*_{k,M;\vartheta} : \mathcal{Y}_M \to \mathcal{Y}^1_{k,M} \tag{4.8}$$

for the maps appearing in that result. The properties (Hg) imply that the map

$$\vartheta \mapsto \mathcal{L}_{k,M;\vartheta} \in \mathcal{L}\big(\mathcal{Y}^1_{k,M}, \mathcal{Y}_M\big) \tag{4.9}$$

is C^{r-1}-smooth. The same hence holds for the maps

$$\vartheta \mapsto \gamma^*_{k,M;\vartheta} \in \mathcal{L}(\mathcal{Y}_M, \mathbb{R}), \qquad \vartheta \mapsto \mathcal{V}^*_{k,M;\vartheta} \in \mathcal{L}(\mathcal{Y}_M, \mathcal{Y}^1_{k,M}), \tag{4.10}$$

with derivatives that can be uniformly bounded for large M.

Plugging the Ansatz (4.2) into (4.1), we arrive at

$$\begin{array}{rcl} c[\mathcal{D}_{k,M}\overline{\Phi}_\vartheta](\xi) + c[\mathcal{D}_{k,M} v](\xi) & = & \overline{\Phi}_\vartheta(\xi+1) + \overline{\Phi}_\vartheta(\xi-1) - 2\overline{\Phi}_\vartheta(\xi) \\ & & +v(\xi+1) + v(\xi-1) - 2v(\xi) \\ & & +g\Big(\overline{\Phi}_\vartheta(\xi) + v(\xi); a\Big). \end{array} \tag{4.11}$$

For any $v \in \mathbb{R}$ and $(\xi, \vartheta, a) \in \mathbb{R}^2 \times (0,1)$ we introduce the expression

$$\mathcal{N}(v; \xi, \vartheta, a) = g\Big(\overline{\Phi}(\xi+\vartheta) + v; a\Big) - g\Big(\overline{\Phi}(\xi+\vartheta); a\Big) - g'\Big(\overline{\Phi}(\xi+\vartheta); a\Big)v, \tag{4.12}$$

which allows us to rephrase as (4.11) as

$$\begin{array}{rcl} c[\mathcal{D}_{k,M}\overline{\Phi}_\vartheta](\xi) + c[\mathcal{D}_{k,M} v](\xi) & = & \overline{\Phi}_\vartheta(\xi+1) + \overline{\Phi}_\vartheta(\xi-1) - 2\overline{\Phi}_\vartheta(\xi) \\ & & +v(\xi+1) + v(\xi-1) - 2v(\xi) + g'\Big(\overline{\Phi}_\vartheta(\xi); \overline{a}\Big)v(\xi) \\ & & +g\Big(\overline{\Phi}_\vartheta(\xi); a\Big) + \mathcal{N}(v(\xi); \xi, \vartheta, a) \\ & & +g'\Big(\overline{\Phi}_\vartheta(\xi); a\Big)v(\xi) - g'\Big(\overline{\Phi}_\vartheta(\xi); \overline{a}\Big)v(\xi). \end{array} \tag{4.13}$$

Exploiting the identity

$$\overline{c}\overline{\Phi}_\vartheta' = \overline{\Phi}_\vartheta(\xi+1) + \overline{\Phi}_\vartheta(\xi-1) - 2\overline{\Phi}_\vartheta(\xi) + g\big(\overline{\Phi}_\vartheta(\xi);\overline{a}\big), \tag{4.14}$$

we find that the pair (c, v) must satisfy

$$\begin{array}{rcl} -[\mathcal{L}_{k,M,\vartheta} v](\xi) & = & (\overline{c}-c)[\mathcal{D}_{k,M}\overline{\Phi}_\vartheta](\xi) + [\mathcal{R}_A(c,\mathcal{D}_{k,M}v)](\xi) + [\mathcal{R}_B(v;\vartheta,a)](\xi) \\ & & \quad +[\mathcal{R}_C(\vartheta,M)](\xi), \end{array} \tag{4.15}$$

in which we have introduced the expressions

$$\begin{array}{rcl} [\mathcal{R}_A(c,\mathcal{D}_{k,M}v)](\xi) & = & (\overline{c}-c)[\mathcal{D}_{k,M}v](\xi) \\ {[\mathcal{R}_B(v;\vartheta,a)](\xi)} & = & g'\Big(\overline{\Phi}_\vartheta(\xi);a\Big)v(\xi) - g'\Big(\overline{\Phi}_\vartheta(\xi);\overline{a}\Big)v(\xi) \\ & & \quad +g\Big(\overline{\Phi}_\vartheta(\xi);a\Big) - g\Big(\overline{\Phi}_\vartheta(\xi);\overline{a}\Big) \\ & & \quad +\mathcal{N}(v(\xi);\xi,\vartheta,a) \\ & = & g\Big(\overline{\Phi}_\vartheta(\xi)+v(\xi);a\Big) - g\Big(\overline{\Phi}_\vartheta(\xi)+v(\xi);\overline{a}\Big) + \mathcal{N}\Big(v(\xi);\xi,\vartheta,\overline{a}\Big), \end{array} \tag{4.16}$$

together with

$$[\mathcal{R}_C(\vartheta,M)](\xi) \quad = \quad -\overline{c}[\mathcal{D}_{k,M}\overline{\Phi}_\vartheta](\xi) + \overline{\Phi}_\vartheta(\xi+1) + \overline{\Phi}_\vartheta(\xi-1) - 2\overline{\Phi}_\vartheta(\xi) + g\Big(\overline{\Phi}_\vartheta(\xi);\overline{a}\Big), \tag{4.17}$$

which can be simplified to

$$[\mathcal{R}_C(\vartheta,M)](\xi) \quad = \quad \overline{c}\Big[\overline{\Phi}_\vartheta' - \mathcal{D}_{k,M}\overline{\Phi}_\vartheta\Big](\xi). \tag{4.18}$$

The motivation for this split is that the $\mathcal{R}_C$ term incorporates the entire effect of moving from the pure derivative to the sampled derivative, while $\mathcal{R}_B$ describes the effects caused by varying the parameters in our equation.

Proposition 3.2 shows that solutions to (4.15) must satisfy the fixed point problem

$$\begin{array}{rcl} c-\overline{c} & = & \gamma^*_{k,M;\vartheta}\big[\mathcal{R}_A(c,\mathcal{D}_{k,M}v) + \mathcal{R}_B(v;\vartheta,a) + \mathcal{R}_C(\vartheta,M)\big], \\ -v & = & \mathcal{V}^*_{k,M;\vartheta}\big[\mathcal{R}_A(c,\mathcal{D}_{k,M}v) + \mathcal{R}_B(v;\vartheta,a) + \mathcal{R}_C(\vartheta,M)\big]. \end{array} \tag{4.19}$$

In order to construct solutions to (4.19) that depend smoothly on the parameters $(\vartheta, a) \in \mathbb{R}\times(0,1)$, we need to obtain appropriate bounds and smoothness conditions on the nonlinear terms. This is addressed in the following series of results.

Lemma 4.1. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. There exists $C > 1$ so that for all $M = \frac{p}{q} \in \mathcal{M}_q$ and $v \in \mathcal{Y}^1_{k,M}$ we have*

$$\|v\|_\infty := \sup_{\xi\in p^{-1}\mathbb{Z}} |v(\xi)| \le C\,\|v\|_{\mathcal{Y}^1_{k,M}}. \tag{4.20}$$

Proof. This follows from the bounded embedding $H^1(\mathbb{R},\mathbb{R}^{q+1}) \subset L^\infty(\mathbb{R},\mathbb{R}^{q+1})$, the interpolation estimate (3.71) and the isometries described in Lemma 3.5. □

Lemma 4.2. *Fix a pair of integers $1 \le k \le 6$ and $q \ge 1$. There exists $C > 1$ so that for all $M \in \mathcal{M}_q$, all $(c,\vartheta,a) \in \mathbb{R}^2\times(0,1)$ and all $v \in \mathcal{Y}^1_{k,M}$ with $\|v\|_{\mathcal{Y}^1_{k,M}} \le 1$, we have the estimates*

$$\begin{array}{rcl} \|\mathcal{R}_A(c,\mathcal{D}_{k,M}v)\|_{\mathcal{Y}_M} & \le & |c-\overline{c}|\,\|\mathcal{D}_{k,M}v\|_{\mathcal{Y}_M}, \\ \|\mathcal{R}_B(v;\vartheta,a)\|_{\mathcal{Y}_M} & \le & C\,|a-\overline{a}| + C\,\|v\|_{\mathcal{Y}^1_{k,M}}\,\|v\|_{\mathcal{Y}_M}, \\ \|\mathcal{R}_C(\vartheta,M)\|_{\mathcal{Y}_M} & \le & CM^{-1}. \end{array} \tag{4.21}$$

Proof. The bound for $\mathcal{R}_A$ is immediate. The restriction on v together with Lemma 4.1 implies that $\|v\|_\infty \leq C_1'$ for some $C_1' > 1$, which allows us to obtain

$$\left|\mathcal{N}(v(\xi);\xi,\vartheta,a)\right| \leq C_2' \left|v(\xi)\right|^2 \tag{4.22}$$

for some $C_2' > 1$. This allows us to estimate

$$\begin{array}{rcl} \|\mathcal{N}(v(\cdot);\cdot,\vartheta,a)\|^2_{\mathcal{Y}_M} & = & p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}}\left|\mathcal{N}(v(\xi);\xi,\vartheta,a)\right|^2 \\ & \leq & [C_2']^2 p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}} v(\xi)^4 \\ & \leq & [C_2']^2\,\|v\|^2_\infty\, p^{-1}\sum_{\xi\in M^{-1}\mathbb{Z}} v(\xi)^2 \\ & \leq & C_3'\,\|v\|^2_{\mathcal{Y}^1_{k,M}}\,\|v\|^2_{\mathcal{Y}_M} \end{array} \tag{4.23}$$

for some $C_3' > 1$. Observe that $\partial_{au} g(u;a)$ is uniformly bounded for $a \in (0,1)$ and $u \in [0,1]$, while also $\partial_a g(1;a) = \partial_a g(0;a) = 0$. This yields the estimate

$$\left|g(\overline{\Phi}(\xi);a) - g(\overline{\Phi}(\xi);\overline{a})\right| \leq C_4'\left|a-\overline{a}\right|\min\{\left|\overline{\Phi}\right|,\left|1-\overline{\Phi}\right|\} \tag{4.24}$$

for some $C_4' > 1$, which due to the exponential decay of $\overline{\Phi}$ to its limiting values $\overline{\Phi}(-\infty) = 0$ and $\overline{\Phi}(\infty) = 1$ shows that

$$\left\|g(\overline{\Phi}(\cdot);a) - g(\overline{\Phi}(\cdot);\overline{a})\right\|_{\mathcal{Y}_M} \leq C_5'\left|a-\overline{a}\right| \tag{4.25}$$

for some $C_5' > 1$. The stated bound for $\mathcal{R}_B$ now follows from the elementary estimate

$$\left\|\left[g'(\overline{\Phi}(\cdot);a) - g'(\overline{\Phi}(\cdot);\overline{a})\right]v(\cdot)\right\|_{\mathcal{Y}_M} \leq C_6'\left|a-\overline{a}\right|\|v\|_{\mathcal{Y}_M} \tag{4.26}$$

for some $C_6' > 1$.

Turning finally to $\mathcal{R}_C$, we note that the desired estimate follows from (2.14) and the exponential decay of $\overline{\Phi}''$, which guarantees that

$$\xi \mapsto \sup_{-kM^{-1}\leq\tau\leq 0}\left|\overline{\Phi}''(\xi+\tau)\right| \tag{4.27}$$

is an element of $BC_{-\eta}(\mathbb{R},\mathbb{R})$ and hence of $\mathcal{Y}_M$. □

Lemma 4.3. *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$, together with two constants $\delta_c > 0$ and $0 < \delta_v < 1$. Then there exists $C > 1$ so that for any set*

$$(v_1,v_2,c_1,c_2,a,\vartheta) \in \mathcal{Y}^1_M \times \mathcal{Y}^1_M \times \mathbb{R}\times\mathbb{R}\times(0,1)\times\mathbb{R} \tag{4.28}$$

with

$$\|v_1\|_{\mathcal{Y}^1_{k,M}} + \|v_2\|_{\mathcal{Y}^1_{k,M}} \leq \delta_v, \qquad |c_1-\overline{c}| + |c_2-\overline{c}| \leq \delta_c, \tag{4.29}$$

we have the estimates

$$\begin{array}{rcl} \|\mathcal{R}_A(c_1,\mathcal{D}_{k,M}v_1) - \mathcal{R}_A(c_2,\mathcal{D}_{k,M}v_2)\|_{\mathcal{Y}_M} & \leq & \delta_v\,|c_1-c_2| + \delta_c\,\|\mathcal{D}_{k,M}[v_1-v_2]\|_{\mathcal{Y}_M}, \\ \|\mathcal{R}_B(v_1;\vartheta,a) - \mathcal{R}_B(v_2;\vartheta,a)\|_{\mathcal{Y}_M} & \leq & C\,|a-\overline{a}|\,\|v_1-v_2\|_{\mathcal{Y}_M} + C\delta_v\,\|v_1-v_2\|_{\mathcal{Y}_M}. \end{array} \tag{4.30}$$

Proof. The estimate for $\mathcal{R}_A$ is immediate. Lemma 4.1 again implies $\|v_1\|_\infty + \|v_2\|_\infty \leq C_1'\delta_v$ for some $C_1' > 0$, which shows that

$$\left|\mathcal{N}(v_1(\xi);\xi,\vartheta,a) - \mathcal{N}(v_2(\xi);\xi,\vartheta,a)\right| \leq C_2'\delta_v\left|v_1(\xi)-v_2(\xi)\right| \tag{4.31}$$

for some $C_2' > 1$. This allows us to compute

$$\begin{array}{rcl} \|\mathcal{N}(v_1(\cdot);\cdot,\vartheta,a) - \mathcal{N}(v_2(\cdot);\cdot,\vartheta,a)\|_{\mathcal{Y}_M}^2 & = & p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}} \left|\mathcal{N}(v_1(\xi);\xi,\vartheta,a) - \mathcal{N}(v_2(\xi);\xi,\vartheta,a)\right|^2 \\ & \leq & [C_2']^2\delta_v^2 p^{-1}\sum_{\xi\in p^{-1}\mathbb{Z}} |v_1(\xi) - v_2(\xi)|^2 \\ & = & [C_2']^2\delta_v^2 \|v_1 - v_2\|_{\mathcal{Y}_M}^2 . \end{array} \tag{4.32}$$

Together with

$$\left\| \left[g'(\overline{\Phi}(\cdot);a) - g'(\overline{\Phi}(\cdot);\overline{a})\right]\left[v_1(\cdot) - v_2(\cdot)\right]\right\|_{\mathcal{Y}_M} \leq C_3' |a - \overline{a}| \, \|v_1 - v_2\|_{\mathcal{Y}_M} \tag{4.33}$$

for some $C_3' > 1$, one obtains the stated estimate for $\mathcal{R}_B$. □

Lemma 4.4. *Fix a pair of integers $1 \leq k \leq 6$ and $q \geq 1$. For all $M = \frac{p}{q} \in \mathcal{M}_q$, the function*

$$\widetilde{\mathcal{N}} : \mathcal{Y}_{k,M}^1 \times \mathbb{R} \times (0,1) \to \mathcal{Y}_{k,M} \tag{4.34}$$

defined by

$$[\widetilde{\mathcal{N}}(v;\vartheta,a)](\xi) = \mathcal{N}(v(\xi);\xi,\vartheta,a), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.35}$$

is C^{r-1}-smooth. The derivatives can be bounded uniformly for $M \in \mathcal{M}_q$, $\vartheta \in \mathbb{R}$, $a \in (0,1)$ and v in bounded subsets of $\mathcal{Y}_{k,M}^1$.

Proof. In view of the estimate $\|v\|_\infty \leq C\,\|v\|_{\mathcal{Y}_{k,M}^1}$ and the C^r-smoothness of the nonlinearity g, the smoothness can be obtained as in the proof of item (iv) of [6, Lem. App.IV.1.1]. □

Proof of Theorem 2.1. Without loss of generality, we fix $\kappa = 1$. On account of the estimates in Lemma's 4.2 and 4.3, the fixed point problem (4.19) posed on the space

$$\mathcal{Z}_{\delta_v,\delta_c} = \{(c,v) \in \mathbb{R}\times\mathcal{Y}_{k,M}^1 : |c - \overline{c}| \leq \delta_c \text{ and } \|v\|_{\mathcal{Y}_{k,M}^1} \leq \delta_v\} \tag{4.36}$$

has a unique solution $c_M^*(\vartheta,a), v_M^*(\vartheta,a))$, provided that $\delta_v > 0$, $\delta_c > 0$ and $|a - \overline{a}|$ are chosen to be sufficiently small and $M \in \mathcal{M}_q$ is chosen to be sufficiently large. The solution to this fixed point problem depends C^{r-1}-smoothly on the parameters (ϑ,a) on account of Lemma 4.4 and the observations above concerning the C^{r-1} smoothness of $\vartheta \mapsto \mathcal{V}_{k,M;\vartheta}^*$ and $\vartheta \mapsto \gamma_{k,M;\vartheta}^*$.

The shift-periodicity stated in (iii) follows from the uniqueness of solutions to (4.19). The inequality (iv) can be seen by inspecting the nonlinear terms appearing in (4.19) and observing that the leading order dependence on a arises in the $\mathcal{R}_B$ term. In particular, applying Proposition 3.2, we find that

$$\partial_a c_M(\vartheta,\overline{a}) \quad = \quad \langle \pi_{\mathcal{Y}_M}\overline{\Psi}_\vartheta, \pi_{\mathcal{Y}_M}\partial_a g(\overline{\Phi}_\vartheta;\overline{a})\rangle_{\mathcal{Y}_M} + O(M^{-1}). \tag{4.37}$$

Since $\partial_a g(u;a) < 0$ for all $(u,a) \in (0,1)\times(0,1)$, the desired inequality follows from Lemma 3.4 for all sufficiently large $M \in \mathcal{M}_q$.

We now turn to the uniqueness claim in the statement. First, we note that any $\Phi \in \ell^\infty(p^{-1}\mathbb{Z};\mathbb{R})$ that satisfies (2.28) for sufficiently small $\delta > 0$, can be decomposed as

$$\Phi = \overline{\Phi}_{\widetilde{\vartheta}} + v \tag{4.38}$$

for some $v \in \mathcal{Y}_{k,M}^1$ with $\langle \pi_{\mathcal{Y}_M}\overline{\Psi}_{\widetilde{\vartheta}}, v\rangle = 0$. This is a consequence of the inequality $\langle \pi_{\mathcal{Y}_M}\overline{\Psi}, \pi_{\mathcal{Y}_M}\overline{\Phi}'\rangle > 0$, which holds for all sufficiently large $M \in \mathcal{M}_q$.

Inspection of the first line of the fixed point problem (4.19) shows that for fixed $v \in \mathcal{Y}^1_{k,M}$ with $\|v\|_{\mathcal{Y}^1_{k,M}} < \delta$, the remaining problem for c is linear and uniquely solvable as $c = c(v)$ provided that $\delta > 0$ is sufficiently small. In addition, we see that $|c(v) - \overline{c}| \le C_1' \, \|v\|_{\mathcal{Y}^1_{k,M}}$ holds for the solution of this problem, for some $C_1' > 0$. In particular, possibly after further decreasing $\delta > 0$, we see that $|c - \overline{c}| \le \delta_c$ holds for the wave speed c associated to any profile Φ satisfying (2.28). The desired uniqueness hence again follows from the uniqueness of solutions to the full fixed point problem (4.19). □

4.2 The backward-Euler discretization

Fix two constants $\Delta t > 0$ and $\kappa \ge 0$. In this subsection we study the problem

$$\nu\Phi'(\xi) = \frac{1}{\Delta t}[\Phi(\xi - c\Delta t) - \Phi(\xi)] + \kappa\big[\Phi(\xi+1) + \Phi(\xi-1) - 2\Phi(\xi)\big] + g\big(\Phi(\xi); a\big). \tag{4.39}$$

The conditions (Hg) imply that this system satisfies the conditions (i)-(v) in [29, §2]. We may therefore directly apply [29, Thm. 2.1] to obtain the existence of a function $\nu : \mathbb{R} \times (0,1) \to \mathbb{R}$ so that (4.39) with $c \in \mathbb{R}$ and $a \in (0,1)$ admits a non-decreasing solution Φ with the limits

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi(\xi) = 1, \tag{4.40}$$

if and only if $\nu = \nu(c,a)$. This theorem also shows that ν depends continuously on a, but does not cover variations in c. In addition, the conditions (vi)-(x) are also satisfied, allowing us to apply [29, Cor. 2.5] to conclude that $a \mapsto \nu(c,a)$ is a non-decreasing function, with $\partial_a \nu(c,a) < 0$ whenever $\nu(c,a) \neq 0$.

The main task for our proof of Theorem 2.2 is hence to understand the dependence of ν on c. As a first step, we establish the equivalent of [29, Prop 7.2], which shows that we only need to be concerned about the continuity of $(c,a) \mapsto \nu(c,a)$ in the regime where $\nu(c,a) \neq 0$.

Lemma 4.5. *Consider the problem (4.39) and suppose that (Hg) is satisfied. Consider a sequence* $\{(c_j, a_j)\}_{j \in \mathbb{N}} \subset \mathbb{R} \times (0,1)$ *for which we have the convergence*

$$\lim_{j \to \infty} (c_j, a_j) = (c_*, a_*) \in \mathbb{R} \times (0,1). \tag{4.41}$$

Suppose furthermore that $\nu(c_j, a_j) \neq 0$ *for all* $j \in \mathbb{N}$ *but* $\nu(c_*, a_*) = 0$*. Then we have the limit*

$$\lim_{j \to \infty} \nu(c_j, a_j) = 0. \tag{4.42}$$

Proof. Without loss, we assume that $\nu_j := \nu(c_j, a_j) > 0$ and that $\nu_j \to \nu_*$ as $j \to \infty$, with $0 \le \nu_* \le \infty$. We write Φ_j for the wave profiles associated with (c_j, a_j).

Consider first the case $\nu_* = \infty$. Upon introducing the new functions $x_j(\xi) = \Phi_j(\nu_j \xi)$, one sees that

$$x_j'(\xi) = \frac{1}{T}\big[x_j(\xi - \nu_j^{-1} cT) - x_j(\xi)\big] + x_j(\xi - \nu_j^{-1}) + x_j(\xi + \nu_j^{-1}) - 2x_j(\xi) + g\big(x_j(\xi); a_j\big). \tag{4.43}$$

On account of the equicontinuity of the families x_j and x_j', one can pass to a subsequence for which one can take the limits $x_j(\xi) \to x_*(\xi)$ and $x_j'(\xi) \to x_*'(\xi)$, uniformly on compact intervals of ξ. The limiting function x_* satisfies

$$x_*'(\xi) = g\big(x_*(\xi); a_*\big). \tag{4.44}$$

One can now proceed as in [29, Prop 7.2] to obtain a contradiction.

For the remaining case $0 < \nu_* < \infty$, we note that the families Φ_j and Φ_j' are equicontinuous. After passing to a subsequence, we obtain the convergence $\Phi_j(\xi) \to \Phi_*(\xi)$, $\Phi_j'(\xi) \to \Phi_*'(\xi)$, uniformly on compact intervals of ξ. In fact, the equicontinuity also gives $\Phi_j(\xi - c_j T) \to \Phi_*(\xi - c_* T)$, which allows us to conclude

$$\nu_* \Phi_*' = \frac{1}{T}\big[\Phi_*(\xi - c_* T) - \Phi_*(\xi)\big] + \Phi_*(\xi - 1) + \Phi_*(\xi + 1) - 2\Phi_*(\xi) + g\big(\Phi_*(\xi); a_*\big). \tag{4.45}$$

One can now again proceed as in [29, Prop 7.2] to obtain a contradiction. $\square$

For $\nu(c,a) \neq 0$, we can set up a modified implicit function argument in order to study the impact of variations in (c,a). In particular, let us suppose that $\overline{\nu} = \nu(\overline{c}, \overline{a}) \neq 0$ for some $(\overline{c}, \overline{a}) \in \mathbb{R} \times (0,1)$. Write $\overline{\Phi}$ for the associated wave profile and $\overline{\Psi}$ for the associated strictly positive adjoint eigenfunction; see [29, Eq. (4.6)]. We now write

$$\overline{\mathcal{L}}_{\mathrm{be}} : H^1(\mathbb{R},\mathbb{R}) \to L^2(\mathbb{R},\mathbb{R}) \tag{4.46}$$

for the operator associated to the linearization of the backward-Euler wave equation (4.39), which acts as

$$\begin{aligned} [\overline{\mathcal{L}}_{\mathrm{be}} w](\xi) \quad = \quad & -\overline{\nu} w'(\xi) + \tfrac{1}{\Delta t}\big[w(\xi - \overline{c}T) - w(\xi)\big] + \kappa\big[w(\xi+1) + w(\xi-1) - 2w(\xi)\big] \\ & + g'(\overline{\Phi}(\xi); \overline{a}) w(\xi). \end{aligned} \tag{4.47}$$

For normalization purposes, let us write

$$\mathcal{Z}^s = \{ w \in H^s(\mathbb{R},\mathbb{R}) : \langle \overline{\Psi}, w \rangle_{L^2} = 0 \} \tag{4.48}$$

for any integer $s \geq 1$. Looking for a solution to (4.39) of the form $\Phi(\xi) = \overline{\Phi}(\xi) + w(\xi)$ with $w \in \mathcal{Z}^1$ is equivalent to looking for zeroes of the function

$$\mathcal{F} : \mathcal{Z}^1 \times \mathbb{R} \times \mathbb{R} \times (0,1) \to L^2(\mathbb{R},\mathbb{R}) \tag{4.49}$$

that acts as

$$\mathcal{F}(w, \nu, c, a) \quad = \quad \overline{\mathcal{L}}_{\mathrm{be}} w + (\overline{\nu} - \nu)\overline{\Phi}' + \mathcal{S}_A(w', \nu) + \mathcal{S}_B(w, a) + \mathcal{S}_C(c) + \mathcal{S}_D(w, c). \tag{4.50}$$

Here we have introduced the nonlinear expressions

$$\begin{aligned} [\mathcal{S}_A(w', \nu)](\xi) \quad &= \quad (\overline{\nu} - \nu) w'(\xi), \\ [\mathcal{S}_B(w, a)](\xi) \quad &= \quad g(\overline{\Phi}(\xi) + w(\xi); a) - g(\overline{\Phi}(\xi) + w(\xi); \overline{a}) + \mathcal{N}(w(\xi); \xi, \overline{a}), \\ [\mathcal{S}_C(c)](\xi) \quad &= \quad \tfrac{1}{\Delta t}[\overline{\Phi}(\xi - c\Delta t) - \overline{\Phi}(\xi - \overline{c}\Delta t)], \\ [\mathcal{S}_D(w, c)](\xi) \quad &= \quad \tfrac{1}{\Delta t}[w(\xi - c\Delta t) - w(\xi - \overline{c}\Delta t)]. \end{aligned} \tag{4.51}$$

Inspection of these definitions immediately shows that $\mathcal{S}_A$ and $\mathcal{S}_B$ share the estimates obtained in §4.1 for $\mathcal{R}_A$ and $\mathcal{R}_B$, provided one makes the replacements

$$c \mapsto \nu, \qquad v \mapsto w, \qquad \mathcal{Y}^1_{k,M} \mapsto H^1, \qquad \mathcal{Y}_M \mapsto L^2. \tag{4.52}$$

Lemma 4.6. *For any set*

$$(w, w_1, w_2, c) \in H^1(\mathbb{R},\mathbb{R})^3 \times \mathbb{R}, \tag{4.53}$$

we have the estimates

$$\begin{aligned} \|\mathcal{S}_C(c)\|_{L^2} \quad &\leq \quad |c - \overline{c}| \left\| \overline{\Phi}' \right\|_{L^2} \\ \|\mathcal{S}_D(w, c)\|_{L^2} \quad &\leq \quad |c - \overline{c}| \, \|w'\|_{L^2}, \\ \|\mathcal{S}_D(w_1, c) - \mathcal{S}_D(w_2, c)\| \quad &\leq \quad |c - \overline{c}| \, \|w_1' - w_2'\|_{L^2}. \end{aligned} \tag{4.54}$$

Proof. Using Jensen's inequality, we compute

$$\begin{array}{rcl} |\mathcal{S}_D(w,c)(\xi)|^2 & = & \frac{1}{(\Delta t)^2} \left| \int_{u=0}^{(\overline{c}-c)\Delta t} w'(\xi - \overline{c}\Delta t + u)\, du \right|^2 \\ & \le & \frac{1}{\Delta t} |c - \overline{c}| \int_{u=0}^{(\overline{c}-c)\Delta t} |w'(\xi - \overline{c}\Delta t + u)|^2 \, du, \end{array} \tag{4.55}$$

which gives

$$\begin{array}{rcl} \|\mathcal{S}_D(w,c)\|_{L^2}^2 & \le & \frac{1}{\Delta t} |c - \overline{c}| \int_{-\infty}^{\infty} \int_{u=0}^{(\overline{c}-c)\Delta t} |w'(\xi - \overline{c}T + u)|^2 \, du\, d\xi \\ & = & |c - \overline{c}|^2 \int_{-\infty}^{\infty} |w'(\xi)|^2 \, d\xi. \end{array} \tag{4.56}$$

The other estimates follow analogously. □

Writing

$$\Omega_\delta = \{(c,a) \in \mathbb{R} \times (0,1) : |c - \overline{c}| + |a - \overline{a}| < \delta\}, \tag{4.57}$$

we may proceed precisely as in §4.1 to find solutions

$$w = w^*(c,a) \in H^1(\mathbb{R},\mathbb{R}), \qquad \nu = \nu^*(c,a) \in \mathbb{R} \tag{4.58}$$

to the problem $\mathcal{F}(w,\nu,c,a) = 0$ whenever $(c,a) \in \Omega_\delta$ for some sufficiently small $\delta > 0$. However, since the nonlinear term $c \mapsto \mathcal{S}_D(w,c)$ is not of class C^r, special care needs to be taken when studying the smoothness of ν^* and w^*.

Lemma 4.7. *Fix a sufficiently small $\delta > 0$. The map $\nu^* : \Omega_\delta \to \mathbb{R}$ is C^r-smooth. In addition, for each integer $0 \le l \le r+1$, the map $w^* : \Omega_\delta \to\in H^{r+1-l}$ is C^l-smooth.*

Proof. Consider $\mathcal{F}$ as a map from $\mathcal{Z}^r \times \mathbb{R}^2 \times (0,1)$ into H^{r-1}, which is C^1-smooth. We note that

$$D_{(w,\nu)}\mathcal{F}(0,\overline{\nu},\overline{c},\overline{a}) = \big(\overline{\mathcal{L}}_{\mathrm{be}}, -\overline{\Phi}'\big) \in \mathcal{L}\big(\mathcal{Z}^r \times \mathbb{R}, H^{r-1}\big). \tag{4.59}$$

Since this linear operator is invertible, the implicit function theorem gives us a C^1-smooth branch of solutions $w^*(c,a) \in \mathcal{Z}^r$ and $\nu^*(c,a) \in \mathbb{R}$ for $(c,a) \in \Omega_\delta$, after possibly decreasing $\delta > 0$. Differentiating (4.39) with respect to ξ subsequently shows that $(c,a) \mapsto w^*(c,a) \in \mathcal{Z}^{r+1}$ is C^0-smooth. In addition, upon writing

$$\mathcal{S}_*(w,\nu,c,a) = \mathcal{S}_A(w',\nu) + \mathcal{S}_B(w,a) + \mathcal{S}_C(c) + \mathcal{S}_D(w,c) \tag{4.60}$$

and introducing the operator $\mathcal{F}^{(c)} : \Omega \times H^r \times \mathbb{R} \to H^{r-1}$ that acts as

$$\begin{array}{rcl} \mathcal{F}^{(c)}(c,a,\tilde{w},\tilde{\nu}) & = & \overline{\mathcal{L}}_{\mathrm{be}}\tilde{w} - \tilde{\nu}\overline{\Phi}' + D_{(w,\nu)}\mathcal{S}_*(w^*(c,a),\nu^*(c,a),c,a)[\tilde{w},\tilde{\nu}] \\ & & + D_c\mathcal{S}_*(w^*(c,a),\nu^*(c,a),c,a), \end{array} \tag{4.61}$$

we see that

$$\mathcal{F}^{(c)}\big(c,a,\partial_c w^*(c,a),\partial_c \nu^*(c,a)\big) = 0 \tag{4.62}$$

for all $(c,a) \in \Omega_\delta$. Unfortunately, $\mathcal{F}^{(c)}$ does not depend C^1-smoothly on the variable c, on account of the term $g'(\overline{\Phi}(\xi) + w^*(c,a)(\xi);\overline{a})$ appearing in $D_w\mathcal{N}(w^*(c,a)(\xi),\xi,\overline{a})$. Indeed, one cannot take $(r-1)$ derivatives with respect to ξ followed by one derivative with respect to c, since g is only of class C^r. However, $\mathcal{F}^{(c)}$ is in fact C^1-smooth when interpreted as a map from $\Omega \times H^{r-1} \times \mathbb{R}$ into H^{r-2}. Arguing as above, one may now apply the implicit function theorem to the problem

$$\mathcal{F}^{(c)}(c,a,\tilde{w},\tilde{\nu}) = 0, \tag{4.63}$$

establishing that the solution branches $(c,a) \mapsto \partial_c w^*(c,a) \in H^{r-1}$ coupled with $(c,a) \mapsto \partial_c \nu^*(c,a) \in \mathbb{R}$ are C^1-smooth. The desired smoothness can now be obtained by repeating this argument a sufficient number of times. □

Proof of Theorem 2.2. In view of the discussion above, it only remains to establish the inequalities (2.35). Inspecting (4.61) and exploiting the identities

$$w^*(\overline{c}, \overline{a}) = 0, \qquad \nu^*(\overline{c}, \overline{a}) = \overline{\nu} \tag{4.64}$$

shows that

$$-\overline{\mathcal{L}}_{\mathrm{be}} \partial_c w^*(\overline{c}, \overline{a}) + \partial_c \nu^*(\overline{c}, \overline{a}) \overline{\Phi}' = D_c \mathcal{S}_C(\overline{c}) = -\Phi'(\xi - \overline{c}\Delta t), \tag{4.65}$$

which yields

$$\partial_c \nu^*(\overline{c}, \overline{a}) = -\int_{-\infty}^{\infty} \overline{\Psi}(\xi) \overline{\Phi}'(\xi - \overline{c}\Delta t)\, d\xi < 0. \tag{4.66}$$

Proceeding in a similar fashion and exploiting (Hg), we also find

$$\partial_a \nu^*(\overline{c}, \overline{a}) = \int_{-\infty}^{\infty} \overline{\Psi}(\xi) \partial_a g(\overline{\Phi}(\xi); \overline{a})\, d\xi < 0, \tag{4.67}$$

as desired. □

Proof of Corollary 2.3. Item (i) is a direct consequence of the continuity of the map $(c, a) \mapsto \nu(c, a)$, the definitions of $a^{\pm}(c)$ and the inequalities (2.35). Item (ii) follows from [29, Thm. 2.6] and the fact that (4.39) with $\kappa > 0$ and $c = 0$ is *weakly coercive* in the terminology of [29]. In particular, this result states that $0 < a^-(0) \le a^+(0) < 1$ and the desired inequalities now follow from continuity of the map $(c, a) \mapsto \nu(c, a)$ □

Proof of Corollary 2.5. Let us first consider the inequalities

$$\lim_{c \to \infty} a^+(c) \le a^+_{+\infty}, \qquad \lim_{c \to -\infty} a^-(c) \ge a^-_{-\infty}. \tag{4.68}$$

We only establish the first inequality here, as the second one follows in a similar fashion. In order to relate the system (4.39) with $\kappa = 1$ to $\kappa = 0$, we assume $c > 0$ and perform the rescaling

$$\zeta = c^{-1}\xi, \qquad \mu = \frac{\nu}{c}, \qquad \epsilon = c^{-1}, \tag{4.69}$$

which transforms (4.39) with $\kappa = 1$ to

$$\mu \Phi'(\zeta) = \frac{1}{\Delta t}[\Phi(\zeta - \Delta t) - \Phi(\zeta)] + \Phi(\zeta - \epsilon) + \Phi(\zeta + \epsilon) - 2\Phi(\zeta) + g(\Phi(\zeta); a). \tag{4.70}$$

We write $\mu(\epsilon, a)$ for the unique value of μ for which (4.70) admits a non-decreasing solution Φ with

$$\lim_{\zeta \to -\infty} \Phi(\zeta) = 0, \qquad \lim_{\zeta \to +\infty} \Phi(\zeta) = 1. \tag{4.71}$$

By the same arguments as developed in this section, the map $(\epsilon, a) \to \mu(\epsilon, a)$ is continuous for all $\epsilon \in \mathbb{R}$ and $a \in (0, 1)$. Since μ has the same sign as ν, we obtain the first inequality of (4.68) by taking the limit $\epsilon \downarrow 0$.

The statements

$$a^+_{+\infty} < 1, \qquad a^-_{-\infty} > 0 \tag{4.72}$$

again follow from [29, Thm. 2.6], as (4.39) with $\kappa = 0$ is *coercive* at $\Phi = +1$ or $\Phi = 0$ when $c > 0$ respectively $c < 0$. Finally, the identities

$$a^-_{+\infty} = -\infty, \qquad a^+_{-\infty} = +\infty \tag{4.73}$$

follow directly from [29, Thm. 2.2]. □

4.3 The anti-continuum limit

In this subsection we continue our discussion from (4.2). We first consider the special case that $(c\Delta t)^{-1} = M = \frac{p}{q} \in \mathbb{Q}$, with $\gcd(p, q) = 1$. In this case, solutions to (4.39) generate solutions to the LDE

$$\frac{d}{dt}u(t,\xi) = \frac{1}{\Delta t}[u(t, \xi - M^{-1}) - u(t,\xi)] + \kappa\big[u(t,\xi+1) + u(t,\xi-1) - 2u(t,\xi)\big] + g\big(u(t,\xi); a\big), \tag{4.74}$$

posed on the lattice $\xi \in p^{-1}\mathbb{Z}$, via the correspondence

$$u(t,\xi) = \Phi(\nu t + \xi). \tag{4.75}$$

We note that (4.74) can be embedded into the more general system

$$\frac{d}{dt}u(t,\xi) = \sum_{j=1}^{N} d_j^- [u(t, \xi - jp^{-1}) - u(t,\xi)] + d_j^+ \sum_{j=1}^{N}[u(t, \xi + jp^{-1}) - u(t,\xi)] + g\big(u(t,\xi); a\big) \tag{4.76}$$

for some integer $N \geq 1$ and coefficients $d_j^{\pm} \geq 0$. Following [22, Prop. 4.1], we see that the LDE (4.76) admits a comparison principle. In particular, any two solutions u_1 and u_2 to (4.76) that have

$$u_1(t_0, \xi) \leq u_2(t_0, \xi), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.77}$$

for some $t_0 \in \mathbb{R}$, will in fact have

$$u_1(t, \xi) \leq u_2(t, \xi), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.78}$$

for all $t \geq t_0$.

As a consequence, if a solution u to (4.76) has the weak monotonicity property

$$u(t_0, \xi) \leq u(t_0, \xi + p^{-1}), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.79}$$

for some $t_0 \in \mathbb{R}$, then we also have

$$u(t, \xi) \leq u(t, \xi + p^{-1}), \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.80}$$

for all $t \geq t_0$. This is useful in conjunction with the following two results, which are closely related to [25, Thm. 2.8].

Lemma 4.8. *Consider the LDE (4.76) and suppose that (Hg) is satisfied. Fix $a \in (0, 1)$ and introduce the quantity*

$$d_*^+ = \sum_{j=1}^{N} d_j^+. \tag{4.81}$$

Suppose that there exist $0 \leq u_l < u_r \leq a$ so that for all $u \in (u_l, u_r)$ we have

$$g(u; a) < d_*^+(u - 1), \tag{4.82}$$

which is the case whenever $d_*^+ \geq 0$ is sufficiently small. Consider any solution to (4.76) that has the property

$$0 \leq u(t_0, \xi) \leq u(t_0, \xi + p^{-1}) \leq 1, \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.83}$$

for some $t_0 \in \mathbb{R}$. Then for all pairs $(t_*, \xi_*) \in \mathbb{R} \times p^{-1}\mathbb{Z}$ for which

$$t_* \geq t_0, \qquad u(t_*, \xi_*) \in (u_l, u_r), \tag{4.84}$$

we have the inequality

$$\frac{d}{dt}u(t_*, \xi_*) < 0. \tag{4.85}$$

Proof. The remarks above imply that also

$$0 \le u(t_*, \xi) \le u(t_*, \xi + p^{-1}) \le 1, \qquad \xi \in p^{-1}\mathbb{Z}. \tag{4.86}$$

In particular, we may estimate

$$\begin{array}{rcl} \frac{d}{dt} u(t_*, \xi_*) & = & \sum_{j=1}^N d_j^- [u(t_*, \xi_* - jp^{-1}) - u(t_*, \xi_*)] \\ & & \quad + \sum_{j=1}^N d_j^+ [u(t_*, \xi + jp^{-1}) - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & \le & \sum_{j=1}^N d_j^+ [u(t_*, \xi + jp^{-1}) - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & \le & \sum_{j=1}^N d_j^+ [1 - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & = & d_*^+ [1 - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & < & 0. \end{array} \tag{4.87}$$

□

Lemma 4.9. *Consider the LDE (4.76) and suppose that (Hg) is satisfied. Fix $a \in (0, 1)$ and introduce the quantity*

$$d_*^- = \sum_{j=1}^N d_j^-. \tag{4.88}$$

Suppose that there exist $a \le u_l < u_r \le 1$ so that for all $u \in (u_l, u_r)$ we have

$$g(u; a) > d_*^- u, \tag{4.89}$$

which is the case whenever $d_*^- \ge 0$ is sufficiently small. Consider any solution to (4.76) that has the property

$$0 \le u(t_0, \xi) \le u(t_0, \xi + p^{-1}) \le 1, \qquad \xi \in p^{-1}\mathbb{Z} \tag{4.90}$$

for some $t_0 \in \mathbb{R}$. Then for all pairs $(t, \xi) \in \mathbb{R} \times p^{-1}\mathbb{Z}$ for which

$$t \ge t_0, \qquad u(t, \xi) \in (u_l, u_r), \tag{4.91}$$

we have the inequality

$$\frac{d}{dt} u(t, \xi) > 0. \tag{4.92}$$

Proof. Proceeding as in the proof of the Lemma above, we estimate

$$\begin{array}{rcl} \frac{d}{dt} u(t_*, \xi_*) & = & \sum_{j=1}^N d_j^- [u(t_*, \xi_* - jp^{-1}) - u(t_*, \xi_*)] \\ & & \quad + \sum_{j=1}^N d_j^+ [u(t_*, \xi + jp^{-1}) - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & \ge & \sum_{j=1}^N d_j^- [u(t_*, \xi_* - jp^{-1}) - u(t_*, \xi_*)] + g\big(u(t_*, \xi_*); a\big) \\ & \ge & -\sum_{j=1}^N d_j^- u(t_*, \xi_*) + g\big(u(t_*, \xi_*); a\big) \\ & = & -d_*^- u(t_*, \xi_*) + g\big(u(t_*, \xi_*); a\big) \\ & > & 0. \end{array} \tag{4.93}$$

□

Proof of Corollary 2.4. For $c > 0$, we have $d_*^+ = \kappa$ and $d_*^- = \kappa + \frac{1}{\Delta t}$, while for $c < 0$, we have $d_*^+ = \kappa + \frac{1}{\Delta t}$ and $d_*^- = \kappa$. Since g depends continuously on a, it is possible to restrict the size of $d_*^\pm > 0$ in such a way that both conditions (4.82) and (4.89) are satisfied for all a that have $|a - \overline{a}| < \delta_a$ for some small $\delta_a > 0$. For all such a we necessarily have

$$\nu(c, a; \kappa, \Delta t) = 0, \tag{4.94}$$

since (4.85) and (4.92) together preclude any non-decreasing wave profile Φ that satisfies the limits (4.92) from actually travelling. We thus obtain

$$a^-(c; \kappa, \Delta t) < \overline{a} - \frac{1}{2}\delta_a < \overline{a} + \frac{1}{2}\delta_a < a^+(c; \kappa, \Delta t) \tag{4.95}$$

for all c for which $c\Delta t \in \mathbb{Q}$. This last rationality restriction can be removed by using the fact that the maps $c \mapsto a^\pm(c; \kappa, \Delta t)$ are non-increasing. □

5 Discussion

In this paper we studied the existence of travelling wave solutions to fully discretized scalar reaction-diffusion systems in one spatial dimension. We considered a family of discretization schemes commonly referred to as the BDF methods, which include the well-known backward-Euler discretization. We constructed travelling waves in a variety of different limits by employing several distinct techniques. In addition, we were able to prove the non-uniqueness of the $a(c)$ relationship in the anti-continuum regime. In this final section we discuss various issues that we encountered during the preparation of this paper.

Irrational values of $M = (c\Delta t)^{-1}$

At present there is still a large disconnect between the results of Theorems 2.1 and 2.2. Indeed, the latter result is independent of the (ir)rationality of M, while this distinction plays a major role in the former result. In addition, the existence results obtained in [5] for fully discretized travelling waves also cover both rational and irrational M. Those results were however obtained using completely different techniques that do not involve the operators $\mathcal{L}_{k,M}$ and do not address questions such as uniqueness and parameter dependence.

The technical obstruction in our approach is the interpolating procedure used in §3. In particular, one would like to perform a second interpolation procedure in the transverse direction and build functions in $H^1(\mathbb{R}, L^2([0,1]))$ from elements of $\mathcal{H}^1_{k,M}$. This would allow functions defined on different subsets of $\mathbb{R}$ to be compared to each other, which is a natural first step towards taking the limit $q \to \infty$.

The problem however is that $\ell^2_{q,\perp}$ is compact, while $L^2([0,1], \mathbb{R})$ is not. In particular, one does not have any control from below on the $L^2([0,1], \mathbb{R})$ norm of the components of the limiting functions $V_*(\cdot, \xi)$ and $W_*(\cdot, \xi)$ in the proof of Proposition 3.7. This means that the inequalities (3.141) and (3.143) fail. The difference with respect to the first interpolation in the ξ-direction is that the travelling wave equations provide a natural bound on the differences $\mathcal{D}_{1,M_j} v_j$, while there is no similar control over terms of the form $p_j^{-1}[v_j(\xi + p_j^{-1}) - v_j(\xi)]$.

Exponentially small effects

Arguing in the fashion of the proof of Theorem 2.1, one finds that the leading order dependence on ϑ occurs in the term $\mathcal{R}_C$. This suggests writing

$$\partial_\vartheta c_M(\vartheta, \overline{a}) \quad = \quad \overline{c}\partial_\vartheta \langle \pi_{\mathcal{Y}_M} \overline{\Psi}_\vartheta, \pi_{\mathcal{Y}_M}[\overline{\Phi}_\vartheta - \mathcal{D}_{k,M}\overline{\Phi}_\vartheta]\rangle_{\mathcal{Y}_M} + O(M^{-1}). \tag{5.1}$$

Unfortunately, this expression does not appear to be very useful, as we now discuss in some detail.

Following [37], let us consider a function $f : \mathbb{R} \to \mathbb{R}$ that decays as

$$f(\xi) = O\big(\,|\xi|^{-1-\epsilon}\,\big), \qquad \xi \to \pm\infty, \tag{5.2}$$

for some $\epsilon > 0$. Let us also assume that f can be extended to an analytic function

$$f : \{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1\} \to \mathbb{C} \tag{5.3}$$

for which $\xi \mapsto f(i\eta + \xi) \in L^1(\mathbb{R}, \mathbb{R})$ for each $\eta \in (-1, 1)$. Upon writing

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi}\, d\xi \tag{5.4}$$

for the Fourier transform of f, our assumptions imply the decay rate

$$\widehat{f}(\omega) = O\big(e^{-\alpha|\omega|}\big), \qquad \omega \to \pm\infty \tag{5.5}$$

for every $\alpha < 1$.

Let us now introduce, for any integer $p \geq 1$ and $\vartheta \in \mathbb{R}$, the quantity

$$T_p(\vartheta) = p^{-1} \sum_{\xi \in p^{-1}\mathbb{Z}} f(\xi + \vartheta), \tag{5.6}$$

which is well-defined on account of our assumptions above. The well-known Poisson summation formula states that

$$T_p(\vartheta) - \int_{-\infty}^{\infty} f(\xi)\, d\xi = \sum_{j=1}^{\infty} \big[\widehat{f}(2\pi pj) e^{2\pi i\vartheta pj} + \widehat{f}(-2\pi pj) e^{-2\pi i\vartheta pj}\big], \tag{5.7}$$

which gives

$$\partial_\vartheta T_p(\vartheta) = \sum_{j=1}^{\infty} (2\pi ipj) \big[\widehat{f}(2\pi pj) e^{2\pi i\vartheta pj} - \widehat{f}(-2\pi pj) e^{-2\pi i\vartheta pj}\big]. \tag{5.8}$$

In particular, we find that for any $\alpha < 1$ we have

$$\partial_\vartheta T_p(\vartheta) = O(e^{-2\pi\alpha p}), \qquad p \to \infty. \tag{5.9}$$

As an example, for the function

$$f(\xi) = \frac{1}{1 + \xi^2} \tag{5.10}$$

we may explicitly compute

$$\begin{aligned} T_p(\vartheta) &= p^{-1} \textstyle\sum_{j\in\mathbb{Z}} \frac{1}{1+(p^{-1}j+\vartheta)^2} \\ &= \pi \frac{\cosh(\pi p)\sinh(\pi M)}{\cosh^2(\pi p) - \cos^2(\pi p\vartheta)} \\ &= \pi \tanh(\pi p)^{-1} \big[1 + \tfrac{1-\cos(\pi p\vartheta)^2}{\sinh^2(\pi p)}\big]^{-1}, \end{aligned} \tag{5.11}$$

which gives

$$\begin{aligned} \partial_\vartheta T_p(\vartheta) &= -\sinh^{-2}(\pi p)\pi \tanh(\pi p)^{-1} \big[1 + \tfrac{1-\cos(\pi p\vartheta)^2}{\sinh^2(\pi p)}\big]^{-2} (\sin(2\pi p\vartheta)\pi p) \\ &= -4\pi^2 p \sin(2\pi p\vartheta) e^{-2\pi p} + o(pe^{-2\pi p}) \end{aligned} \tag{5.12}$$

as $p \to \infty$. Such terms are hence exponentially small as $p \to \infty$, which means that they will not show up at any order when performing a Taylor expansion in the variable p^{-1} near zero. In particular, we do not expect to be able to analyze the term (5.1) and its propagation through the fixed point argument outlined in §4 by using only standard Melnikov methods.

A way around this could be a formal asymptotics-beyond-all-orders method such as the one outlined by King and Chapman [26], which could potentially be used to study systems of the form

$$c\epsilon^{-2}[\Phi(\xi - \epsilon^2) - \Phi(\xi)] = \epsilon^{-2}[\Phi(\xi + \epsilon) + \Phi(\xi - \epsilon) - 2\Phi(\xi)] + g(u; a) \tag{5.13}$$

for small $\epsilon > 0$, again with the usual limits

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi(\xi) = 1. \tag{5.14}$$

If $u \mapsto g(u, \frac{1}{2})$ is anti-symmetric around $u = \frac{1}{2}$, one can follow the cut-off procedure that was developed in [26] for the spatial discretization

$$c\Phi'(\xi) = \epsilon^{-2}[\Phi(\xi + \epsilon) + \Phi(\xi - \epsilon) - 2\Phi(\xi)] + g(u; a), \tag{5.15}$$

to formally find an interval $\mathcal{I}_a = [\frac{1}{2} - \frac{1}{2}\delta_a, \frac{1}{2} + \frac{1}{2}\delta_a]$ of width $\delta_a \sim e^{-\alpha/\epsilon}$ for some $\alpha > 0$ so that (5.13) admits only solutions with $c = 0$ whenever $a \in \mathcal{I}_a$. For a slightly outside $\mathcal{I}_a$, we believe a second cut-off procedure could be used to uncover the difference between (5.13) and (5.15). In particular, we believe that the expected exponentially small fluctuations in c could be uncovered by appropriately sampling the remainder equations on different subsets of the line.

Another way could be to understand how poles of the functions $\overline{\Phi}$ and $\overline{\Psi}$ behave under the fixed point iteration procedure described here. This would require understanding the solutions to MFDEs of the form

$$-cv'(z) + v(z+1) + v(z-1) - 2v(z) + g'\big(\overline{\Phi}(z); \overline{a}\big)v(z) = \frac{1}{(z-\beta)^n} \tag{5.16}$$

with a complex variable z. Compared to meromorphic ODEs, the difficulty here is that one now expects poles to occur at more locations that just $z = \beta$ and the poles of $\overline{\Phi}$. In particular, one can no longer perform local expansions as in [26].

Discontinuities of $a^{\pm}$.

For the purpose of this discussion, let us fix $\kappa > 0$ and $\Delta t > 0$ and introduce the shorthands $a^{\pm}(c) = a^{\pm}(c; \kappa, \Delta t)$. In addition, let us fix c_0 in such a way that $c_0 \Delta t \in \mathbb{Q}$. The left-continuity of a^+ and right-continuity of a^- stated in Corollary 2.3 imply that

$$\lim_{c \uparrow c_0} a^+(c) = a^+(c_0), \qquad \lim_{c \downarrow c_0} a^-(c) = a^-(c_0). \tag{5.17}$$

In particular, whenever the strict inequalities

$$\lim_{c \uparrow c_0} a^-(c) > a^-(c_0), \qquad \lim_{c \downarrow c_0} a^+(c) < a^+(c_0) \tag{5.18}$$

hold, we may conclude that

$$a^-(c_0) < a^+(c_0), \tag{5.19}$$

implying that solutions to (2.32) with the limits (2.34) exist with the same wave speed $c = c_0$ at multiple values of a. The numerical results in Figure 1 indicate that the strict inequalities (5.18) can indeed be expected to hold for $c_0 \Delta t \in \mathbb{Q}$, with the size of the jumps roughly increasing with the strength of the resonance. This is reminiscent of the crystallographic pinning phenomenon described in [7, 20, 23, 30], which concerns the interval of detuning parameters a for which planar wave solutions to the LDE

$$\dot{u}_{ij} = u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} + g\big(u_{ij}; a\big), \qquad (i,j) \in \mathbb{Z}^2, \tag{5.20}$$

fail to propagate. Such solutions can be written as

$$u_{ij}(t) = \Phi(i\cos\theta + j\sin\theta + \nu t), \tag{5.21}$$

again with the limits

$$\lim_{\xi \to -\infty} \Phi(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Phi(\xi) = 1. \tag{5.22}$$

The wave speed $\nu = \nu(\theta, a)$ depends uniquely on the angle of propagation θ and the detuning parameter $a \in (0,1)$, allowing us define the quantities

$$\begin{array}{rcl} a^-_{\rm tw}(\theta) & = & \sup\{a \in (0,1) : \nu(\theta, a) > 0\} \in (0,1), \\ a^+_{\rm tw}(\theta) & = & \inf\{a \in (0,1) : \nu(\theta, a) < 0\} \in (0,1). \end{array} \tag{5.23}$$

The conjecture now is that

$$\lim_{\theta \to \theta_0} a^+_{\rm tw}(\theta) < a^+_{\rm tw}(\theta_0), \qquad \lim_{\theta \to \theta_0} a^-_{\rm tw}(\theta) > a^-_{\rm tw}(\theta_0) \tag{5.24}$$

whenever $\tan\theta_0 \in \mathbb{Q}$. In [20], the authors provide a proof for these inequalities for the horizontal and vertical directions $\theta_0 \in \frac{\pi}{2}\mathbb{Z}$, provided a generic Melnikov condition is satisfied.

Although the quantities $a^{\pm}_{\rm tw}$ do not depend in a monotonic fashion upon the angle θ, we believe that the root mechanisms leading to the jumps (5.24) and (5.18) are closely related. Indeed, we expect that the general spirit of the center manifold approach developed in [20] should also be applicable towards establishing (5.18). However, significant hurdles still remain to be overcome. In particular, the dimension of the systems that need to be analyzed can become large as the height of the fraction $c\Delta t$ increases. In addition, in order to prove the strictness of the inequalities (2.45), which correspond to the limiting case $c_0 = \infty$, one would need to overcome the lack of monotonicity of the eigenfunctions described in [20, Prop. 1.3]. This is a consequence of the fact that the limiting system (4.70) is a pure delayed or advanced equation.

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