Travelling Waves for Adaptive Grid Discretizations of Reaction Diffusion Systems

H. J. Hupkes [a,*], E. S. Van Vleck [b]

[a] *Mathematisch Instituut - Universiteit Leiden*
P.O. Box 9512; 2300 RA Leiden; The Netherlands
Email: hhupkes@math.leidenuniv.nl

[b] *Department of Mathematics - University of Kansas*
1460 Jayhawk Blvd; Lawrence, KS 66045; USA
Email: erikvv@ku.edu

Abstract

In this paper we consider a spatial discretization scheme with an adaptive grid for the Nagumo PDE and establish the existence of travelling waves. In particular, we consider the MMPDE5 grid update method that aims to equidistribute the arclength of the solution under consideration. We assume that this equidistribution is strictly enforced, which leads to a non-local problem with infinite range interactions.

For small spatial grid-sizes, we establish some useful Fredholm properties for the operator that arises after linearizing our system around the travelling wave solutions to the original Nagumo PDE. In particular, we perform a singular perturbation argument to lift these properties from the natural limiting operator. This limiting operator is a spatially stretched and twisted version of the standard second order differential operator that is associated to the PDE waves.

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1 Introduction

In this paper we consider adaptive discretization schemes for a class of scalar reaction-diffusion equations that includes the Nagumo PDE

$$u_t = u_{xx} + g_{\mathrm{cub}}(u; a), \tag{1.1}$$

with the bistable cubic nonlinearity

$$g_{\mathrm{cub}}(u) = u(1-u)(u-a), \qquad 0 < a < 1. \tag{1.2}$$

In particular, we discretize (1.1) on a time-dependent spatial grid and add an extra equation that aims to distribute the gridpoints in such a way that the arclength of the solution is equal between any two consecutive gridpoints.

*Corresponding author.

Our main contribution in this paper is to show that the resulting coupled semi-discrete system admits solutions that can be interpreted as travelling waves. In particular, our results here are part of an ongoing program that aims to systematically explore the impact of commonly used spatial, temporal and full discretization schemes on the dynamical behaviour of dissipative PDEs.

Reaction-diffusion systems Reaction-diffusion PDEs have been extensively studied in the past decades. Indeed, their rich pattern-forming properties allow many intriguing localized structures that can be observed in nature to be reproduced analytically and numerically. For example, the classical paper by Aronson and Weinberger [1] shows how (1.1) and its higher-dimensional counterparts can be used to model the spreading of a dominant biological species throughout a spatial domain. Upon adding a slowly-varying second component to (1.1) by writing

$$\begin{aligned} u_t &= u_{xx} + g_{\mathrm{cub}}(u;a) - v, \\ v_t &= \epsilon\big(u - v\big), \end{aligned} \tag{1.3}$$

Fitzhugh [27, 28] was able to effectively describe the propagation of signal spikes through nerve fibres. Sparked by his interest in morphogenesis, Turing [62] described the famous bifurcation through which equilibria of general two-component reaction-diffusion systems can destabilize and generate spatially periodic structures such as spots and stripes.

These early results led to the development of many important technical tools that today are indispensable to the field of dynamical systems. For example, comparison principle techniques have been used to study the global dynamics of (1.1) in one [26] and two [9] spatial dimensions. The rigorous construction of the pulses observed by FitzHugh for (1.3) led to the birth of geometric singular perturbation theory [14, 32, 47]. The development of Evans function [44] and semigroup theory [59] was heavily influenced by the desire to analyze the stability of many of these localized structures.

The systems (1.1) and (1.3) are both still under active investigation. For example, the behaviour of perturbed spherical [56] or planar [50] fronts has been investigated for higher-dimensional versions of (1.1). In addition, in [15, 16] the authors consider (1.3) in the $a \downarrow 0$ limit and describe the birth of pulse solutions with oscillating tails.

Travelling waves It is well-known that travelling waves play an important role in the global dynamics of (1.1). Such solutions have the form $u(x,t) = \Phi(x+ct)$, which implies that the waveprofile Φ and wavespeed c must satisfy the travelling wave ODE

$$c\Phi' = \Phi'' + g_{\mathrm{cub}}(\Phi;a). \tag{1.4}$$

Using a now standard phase-plane analysis [26], one readily shows that (1.4) coupled with the boundary conditions

$$\Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1 \tag{1.5}$$

admits a unique solution pair $(\Phi, c) = \big(\Phi(a), c(a)\big)$, with

$$\mathrm{sign}\big(c(a)\big) = \mathrm{sign}(\frac{1}{2} - a). \tag{1.6}$$

Such solutions provide a mechanism through which the fitter biological species (corresponding to the deepest well of the potential $-\int g_{\mathrm{cub}}$) can become dominant throughout a spatial domain. For this reason they are sometimes referred to as *invasion waves*.

Using a squeezing technique based on the comparison principle, one can show that these waves are nonlinearly stable under a large class of perturbations [26]. This can be generalized to planar travelling wave solutions to

$$u_t = u_{xx} + u_{yy} + g_{\mathrm{cub}}(u;a) \tag{1.7}$$

by carefully constructing appropriate sub- and supersolutions [9].

Travelling waves have been used extensively as building blocks to construct general time dependent solutions of reaction-diffusion systems. For example, (1.7) supports travelling corners [10, 30], expansion waves [56], and scattering waves [9]. Changing the nonlinearity g, (1.1) supports modulated waves [21] that connect periodic travelling waves of nearby frequencies.

Uniform spatial discretizations Introducing the approximants $U_j(t) \sim u(jh, t)$ and applying a standard discretization to the second derivative in (1.1), one obtains

$$\dot{U}_j(t) = \frac{1}{h^2}[U_{j-1}(t) + U_{j+1}(t) - 2U_j(t)] + g_{\mathrm{cub}}(u_j). \tag{1.8}$$

This lattice differential equation (LDE) can be seen as the nearest-neighbour uniform spatial discretization of the PDE (1.1) on the grid $h\mathbb{Z}$.

Mathematically speaking, the transition from (1.1) to (1.8) breaks the continuous translational symmetry of the underlying space. Indeed, (1.8) merely admits the discrete group of symmetries $j \mapsto j + k$ with $k \in \mathbb{Z}$. As a consequence, travelling wave solutions

$$U_j(t) = \Phi(jh + ct) \tag{1.9}$$

can no longer be seen as equilibria in an appropriate comoving frame. Instead, they must be treated as periodic solutions modulo the discrete shift symmetry discussed above. The resulting challenges occur frequently in similar discrete settings and general techniques have been developed to overcome them [8, 17, 29].

Naturally, the $0 < h \ll 1$ regime is the most interesting from the perspective of numerical analysis. However, we remark here that many physical and biological systems have a discrete spatial structure for which it is natural to take $h \sim 1$. Indeed, genuinely discrete phenomena such as phase transitions in Ising models [5], crystal growth in materials [13], propagation of action potentials in myelinated nerve fibers [7] and phase mixing in martensitic structures [63] have all been modelled using equations similar to (1.8). The list of applications will undoubtedly expand over time as the mathematical tools for analyzing LDEs are improved.

Substituting the Ansatz (1.9) into (1.8), we obtain the travelling wave equation

$$c\Phi'(\xi) = \frac{1}{h^2}[\Phi(\xi - h) + \Phi(\xi + h) - 2\Phi(\xi)] + g_{\mathrm{cub}}(\Phi(\xi); a). \tag{1.10}$$

Due to the presence of the shifted arguments such equations are known as functional differential equations of mixed type (MFDEs). Mathematically speaking, the unbounded second derivative operator in (1.4) has been replaced by a bounded second-difference operator. In addition, the transition $c \to 0$ is now singular since it changes the structure of the equation. As a consequence, there is a fundamental difference between standing and moving wave solutions to (1.8).

In the anti-continuum regime $h \gg 1$, the second-difference operator can be treated as a small perturbation to the remaining ODE. An elegant construction pioneered by Keener [45] allows one to construct standing waves for $a \neq \frac{1}{2}$ that satisfy the boundary conditions (1.5) and block the two stable background states $\Phi \equiv 0$ and $\Phi \equiv 1$ from invading the domain. In particular, the simple geometric condition (1.6) is violated in this setting. This phenomenon is often referred to as *pinning* or *propagation failure* and has attracted a considerable amount of attention [3, 20, 22, 23, 34, 39].

In the intermediate $h \sim 1$ regime the shifted terms cannot be handled so easily and one needs to understand the full MFDE. Such equations are ill-posed as initial value problems and hence must be handled delicately. Several important tools have been developed to accomplish this, such as Fredholm theory [48] and exponential dichotomies [31, 49, 57, 58].

Using a global homotopy argument together with the comparison principle, Mallet-Paret constructed a branch of solutions $\big(\Phi(a), c(a)\big)$ to (1.10) with (1.5), in which $c(a)$ is unique and $\Phi(a)$

is unique up to translation when $c(a) \neq 0$. For the uniform spatial discretization of the FitzHugh-Nagumo PDE (1.3), a generalization of Lin's method can be used to establish a version of the exchange Lemma for MFDEs and construct stable travelling pulses [40, 41].

In the continuum regime $0 < h \ll 1$, a natural first step is to construct spatially-discrete waves as small perturbations from the PDE waves (Φ_*, c_*). As explained above however, the transition between (1.4) and (1.10) is highly singular. Nevertheless, Johann [43] developed a version of the implicit function theorem that can achieve this in some settings. Our inspiration for the present paper however comes from the spectral convergence approach developed by Bates and his coauthors in [4].

A key role in this approach is reserved for the linear operator

$$[\mathcal{L}_{h;\mathrm{unif}} v](\xi) = -cv'(\xi) + \frac{1}{h^2}[v(\xi + h) + v(\xi - h) - 2v(\xi)] + g'_{\mathrm{cub}}(\Phi_*(\xi); a)v(\xi), \tag{1.11}$$

which can be seen as the linearization of (1.10) around the PDE wave Φ_*. This operator is a singularly perturbed version of the PDE linearization

$$[\mathcal{L}_{\mathrm{tw}} v](\xi) = -cv'(\xi) + v''(\xi) + g'_{\mathrm{cub}}(\Phi_*(\xi); a)v(\xi). \tag{1.12}$$

The main contribution in [4] is that Fredholm properties of $\mathcal{L}_{\mathrm{tw}}$ are transferred to $\mathcal{L}_{h;\mathrm{unif}}$. The latter operator can then be used in a standard fashion to close a fixed-point argument and construct a solution to (1.10) that is close to (Φ_*, c_*).

Stated more precisely, the authors fix a constant $\delta > 0$ and use the invertibility of $\mathcal{L}_{\mathrm{tw}} + \delta$ to show that also $\mathcal{L}_{h;\mathrm{unif}} + \delta$ is invertible for small $h > 0$. In particular, they consider bounded weakly-converging sequences $\{v_j\} \subset H^1$ and $\{w_j\} \subset L^2$ with $(\mathcal{L}_{h;\mathrm{unif}} + \delta)v_j = w_j$ and set out to find a lower bound for w_j that is uniform in δ and h. This can be achieved by picking a large compact interval K and extracting a subsequence of $\{v_j\}$ that converges strongly in $L^2(K)$. Special care must therefore be taken to rule out the limitless transfer of energy into oscillatory or tail modes, which are not visible in this strong limit. Spectral properties of the (discrete) Laplacian together with the bistable structure of the nonlinearity g provide the control on $\{v_j\}$ that is necessary for this.

The results in [4] are actually strong enough to handle discretizations of the Laplacian that have infinite range interactions. In addition, this approach was recently generalized [60] for use in multi-component reaction-diffusion problems such as the FitzHugh-Nagumo system (1.3). We emphasize that this generalization also allows one to establish the *stability* of the constructed waves, which is an important reason for us to pursue this line of thought in the present paper.

Uniform spatial-temporal discretizations A natural next step is to study the impact of temporal discretization schemes. In order to set the stage, we apply the backward-Euler discretization with time-step Δt to the temporal derivative in (1.8), which leads to the fully discrete system

$$\begin{aligned} \tfrac{1}{\Delta t}\big[U_j\big(n\Delta t\big) - U_j\big((n-1)\Delta t\big)\big] \quad = \quad & \tfrac{1}{h^2}\big[U_{j-1}(n\Delta t) + U_{j+1}(n\Delta t) - 2U_j(n\Delta t)\big] \\ & + g_{\mathrm{cub}}\big(U_j(n\Delta t); a\big). \end{aligned} \tag{1.13}$$

This type of system is commonly referred to as a *coupled map lattice* (CML). Such systems are used as stand-alone models across a wide range of disciplines, from the construction of hash functions [65] to the study of population dynamics [19].

The backward-Euler discretization is part of a family of six so-called backward differentiation formula (BDF) schemes for discretizing the temporal derivative. These are well-known multistep methods that are appropriate for parabolic PDEs due to their numerical stability properties. In [42] we analyzed these BDF methods and constructed fully discretized travelling wave solutions

$$U_j(n\Delta t) = \Phi(j + nc\Delta t), \qquad \Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1 \tag{1.14}$$

for (1.13) and its five higher order counterparts. This continued the program that was initiated in [23–25] to study the impact of temporal and full discretization schemes on various reaction-diffusion systems. Indeed, these papers studied versions of (1.1) with various smooth and piecewise linear bistable nonlinearities. The authors used adhoc techniques to obtain rigorous, formal and first order information concerning the change in the dynamics of traveling wave solutions. In addition, in [17] the authors considered the forward-Euler counterpart of (1.13) and used Poincaré return-maps and topological arguments to obtain the existence of fully-discretized waves.

We note that the fully discrete front solutions (1.14) to (1.13) must satisfy the difference equation

$$\tfrac{1}{\Delta t}[\Phi(\xi) - \Phi(\xi - c\Delta t)] \quad = \quad \tfrac{1}{h^2}\big[\Phi(\xi - h) + \Phi(\xi + h) - 2\Phi(\xi)\big] + g_{\mathrm{cub}}\big(\Phi(\xi); a\big). \tag{1.15}$$

In view of the discussion above it is natural to ask whether the $c(a)$ relation can become multi-valued. This question is answered affirmatively by the numerical and theoretical results in [42]. Related phenomena have been observed in monostable KPP systems [51] in the presence of inhomogeneities.

The key technical ingredient in our construction of the front solutions (1.14) is the understanding of the fully discrete operator

$$\begin{array}{rcl}[\mathcal{L}_{h,\Delta t} v](\xi) & = & -\tfrac{1}{\Delta t}[v(\xi) - v(\xi - c\Delta t)] + \tfrac{1}{h^2}\big[v(\xi - 1) + v(\xi + 1) - 2v(\xi)\big] \\ & & +g'_{\mathrm{cub}}(\Phi(\xi); a)v(\xi),\end{array} \tag{1.16}$$

in which (Φ, c) is the spatially-discrete travelling wave (1.10). The main contribution in [42] is that we modified the approach of [4] that was discussed above in such a way that Fredholm properties can be transferred from the spatially-discrete operators $\mathcal{L}_{h;\mathrm{unif}}$ to the fully-discrete operators $\mathcal{L}_{h,\Delta t}$. In our view this presents a further reason for focussing on this spectral convergence approach here.

Arclength equidistribution Most efficient modern solvers do not use fixed spatial grids but concentrate their meshpoints in areas where the solution under construction fluctuates the most. In particular, let us write $\{x_j(t)\}$ for the positions of the grid points. Introducing the approximants

$$U_j(t) \approx u(x_j(t), t), \tag{1.17}$$

we may use (1.1) to compute

$$\begin{array}{rcl}\tfrac{d}{dt}U_j(t) & = & u_x\big(x_j(t), t\big)\dot{x}_j(t) + u_t\big(x_j(t), t\big) \\ & = & u_x\big(x_j(t), t\big)\dot{x}_j(t) + u_{xx}\big(x_j(t), t\big) + g_{\mathrm{cub}}\Big(u\big(x_j(t), t\big); a\Big)\end{array} \tag{1.18}$$

in the special case that the approximation (1.17) is exact. Using central differences to discretize the spatial derivatives in (1.18) on the grid $x_j(t)$, we obtain the LDE

$$\dot{U}_j \quad = \quad \Big[\tfrac{U_{j+1} - U_{j-1}}{x_{j+1} - x_{j-1}}\Big]\dot{x}_j + \tfrac{2}{x_{j+1} - x_{j-1}}\big[\tfrac{U_{j-1} - U_j}{x_j - x_{j-1}} + \tfrac{U_{j+1} - U_j}{x_{j+1} - x_j}\big] + g_{\mathrm{cub}}(U_j; a). \tag{1.19}$$

This system should be compared to [38, Eqs. (1.12)-(1.13)] where a similar procedure was applied to Burgers' equation.

In order to close the system, we need to describe the behaviour of the gridpoints. For illustrative purposes, let us consider the so-called MMPDE5 method that was originally developed by Huang, Ren, and Russell [36, 37, 55]. This method is efficient and relatively easy to formulate for our problem. In particular, inspecting [38, Eqs. (2.52), (2.53), (2.57)], the gridpoint behaviour can be described by

$$\tau\dot{x}_j \quad = \quad \sqrt{(x_{j+1} - x_j)^2 + (U_{j+1} - U_j)^2} - \sqrt{(x_{j-1} - x_j)^2 + (U_{j-1} - U_j)^2}, \tag{1.20}$$

in which $\tau > 0$ is a tunable speed parameter. In the terminology of [38], we are using the arclength monitor function

$$\rho(x, t) = \sqrt{1 + u_x^2}. \tag{1.21}$$

Indeed, the update rule (1.20) acts to equalize the arclength of the solution profile between gridpoints.

Adaptive meshing Numerical techniques involving non-constant grids have attracted tremendous attention in the search for accurate and efficient approximation procedures for differential equations. The first method of this type that is based upon an equidistribution principle was described by de Boor [18]. The method was developed to efficiently solve boundary value problems for ordinary differential equations. After each step in the numerical iteration scheme, the error is computed in a pointwise fashion. One can subsequently choose new gridpoints in such a way that this error is equally distributed over each subinterval in the new mesh. This technique turned out to be very effective and has also been used for time dependent (parabolic) PDEs in one space dimension.

The MMPDE5 method described above is an r-adaptive refinement scheme in the terminology of the finite element community, since the mesh is continuously relocating as it adapts to the solution of the PDE being solved. The equations that determine the movement of the mesh are generally independent of the PDE being solved, but are dependent on the solution of the underlying physical PDE. Several approaches have been developed that are relatively simple to program and robust with respect to the choice of adjustable parameters. The recent book [38] contains a comprehensive treatment of the most important moving mesh methods, including the MMPDE5 scheme described above. Further references can be found in the review articles [12] and [33].

The literature concerning convergence results for moving mesh methods is somewhat limited. Results have been obtained [6, 53, 54] for finite difference methods applied to singularly perturbed two-point boundary value problems and reaction-diffusion equations. However, these require a-priori knowledge of the mesh behaviour and explicitly use the singular part of the exact solution. Results that do not require such a-priori knowledge are available for linear one-dimensional elliptic equations [2] and one-dimensional quasi-linear convection-diffusion problems [46]. For combustion PDEs that feature blow-up behaviour, one can use scaling invariance and moving mesh methods to recreate the scaling laws inherent in the exact blow-up solutions [11]. Finally, the behavior of moving mesh schemes for travelling wave solutions of the Fisher equation, which is the monostable counterpart of (1.1), was investigated in [52].

Results and broader goals Inspection of the coupled system (1.19)-(1.20) shows that one loses the comparison principle, even if x is treated as a known function. Such drastic structural changes are a common feature when applying discretization schemes and we refer to [61] for an interesting discussion. For our purposes here, this means that we will have to consider perturbative techniques to analyze (1.19)-(1.20), viewing the speed parameter τ and the average arclength between gridpoints as small parameters.

In this paper we focus on the singular case $\tau = 0$, which allows us to rewrite (1.20) as

$$h = \sqrt{(x_{j+1} - x_j)^2 + (U_{j+1} - U_j)^2} = \sqrt{(x_{j-1} - x_j)^2 + (U_{j-1} - U_j)^2} \tag{1.22}$$

for some constant $h > 0$ that we take to be small. In particular, we obtain

$$x_{j+1} - x_j = \sqrt{h^2 - (U_{j+1} - U_j)^2} \tag{1.23}$$

for all $j \in \mathbb{Z}$. In order to fix the absolute positions of the gridpoints, we impose the boundary condition

$$\lim_{j \to -\infty} x_j(t) - jh = 0 \tag{1.24}$$

at each time $t \geq 0$. Our main results state that the resulting system is well-posed and admits travelling wave solutions

$$U_j(t) = \Phi(x_j(t) + ct) \tag{1.25}$$

that satisfy the boundary conditions (1.5). These travelling waves (Φ, c) are small perturbations of the PDE waves (Φ_*, c_*).

We view our work here as a first step towards understanding the impact of adaptive discretization schemes on travelling waves and other patterns that exist for all time. In particular, we believe that the waves constructed here can be seen as a slow manifold for the dynamics of the full system (1.19)-(1.20). Using the Fredholm theory that we develop in this paper one should be able to effectively track the fast grid-dynamics in the $0 < \tau \ll 1$ regime. A further step in the program would be to also handle temporal discretizations, inspired by the approach developed in [42] that we described above. Finally, we feel that it is important to understand the stability of the discretized waves under the full dynamics of the numerical scheme.

We are specially interested here in the pinning phenomenon. Indeed, numerical observations indicate that the set of detuning parameters a for which $c(a) = 0$ shrinks dramatically when using adaptive discretizations. Understanding this in a rigorous fashion would give considerable insight into the *theoretical* benefits of adaptive grids compared to the *practical* benefits of increased performance. Preliminary results in this direction can be found in [35].

Let us emphasize that the application range of our techniques does not appear to be restricted to the scalar problem (1.1) or the specific grid-update scheme (1.20). Indeed, using the framework developed in [60], it should be possible to perform a similar analysis for the FitzHugh-Nagumo equation PDE (1.3) and other multi-component reaction-diffusion problems. In addition, any numerical scheme based on the arclength monitor function will share (1.23) as the instantaneous equidistribution limit.

Reduction procedure The first step in our program to construct the travelling waves (1.25) is to eliminate the variable x from (1.19). In view of the boundary condition (1.24), we can repeatedly apply (1.23) to obtain

$$\begin{aligned} x_k - kh &= \textstyle\sum_{j=-\infty}^{k-1} \left(\sqrt{h^2 - (U_{j+1} - U_j)^2} - h\right) \\ &= -\textstyle\sum_{j=-\infty}^{k-1} \frac{(U_{j+1}-U_j)^2}{\sqrt{h^2-(U_{j+1}-U_j)^2}+h}. \end{aligned} \tag{1.26}$$

Upon introducing the discrete derivative

$$[\partial_h^+ U]_j = h^{-1}[U_{j+1} - U_j], \tag{1.27}$$

we note that this expression is well-defined if we impose the conditions

$$\left\|\partial_h^+ U\right\|_\infty < 1, \qquad \partial_h^+ U \in \ell^2. \tag{1.28}$$

A direct differentiation yields

$$\dot{x}_k = -\sum_{j=-\infty}^{k-1} \frac{U_{j+1} - U_j}{\sqrt{h^2 - (U_{j+1} - U_j)^2}} \big(\dot{U}_{j+1} - \dot{U}_j\big), \tag{1.29}$$

which is well-defined if also $\partial_h^+ \dot{U} \in \ell^2$. Using (1.19) to eliminate $\partial_h^+ \dot{U}$, we obtain the implicit expression

$$\dot{x}_k = \sum_{j=-\infty}^{k-1} \mathcal{F}\Big(U_{j-1}, U_{j+1}, U_j, U_{j+2}, \dot{x}_{j+1}, \dot{x}_j\Big). \tag{1.30}$$

This equation has a unique solution that can be written as

$$\dot{x}_k = \mathcal{Y}_k\Big(\{U_j\}_{j\le k}, \{\partial_h^+ U_j\}_{j\le k}, \{\partial_h^+\partial_h^+ U_j\}_{j\le k-1}, \{\partial_h^+\partial_h^+\partial_h^+ U_j\}_{j\le k-2}\Big) \tag{1.31}$$

for some function $\mathcal{Y}$ that we compute explicitly in §8. Using (1.23) to eliminate the remaining terms involving x from (1.19), this allows us to write

$$\dot{U}_k = \mathcal{G}_k\Big(\{U_j\}_{j\leq k}, \{\partial_h^+ U_j\}_{j\leq k}, \{\partial_h^+\partial_h^+ U_j\}_{j\leq k-1}, \{\partial_h^+\partial_h^+\partial_h^+ U_j\}_{j\leq k-2}\Big) \tag{1.32}$$

for some function $\mathcal{G}$ that we describe explicitly in §9. We note that the partial derivatives of $\mathcal{G}$ can be controlled uniformly for small h, so the representation (1.32) isolates all the terms that have the potential to blow up as $h \downarrow 0$. By choosing an appropriate space for the sequences U, we show in §12 that (1.32) can be seen as a well-posed initial value problem.

The discrete third derivative in (1.32) arises directly from (1.29), which forces us to take a discrete derivative of our second-order original system. Fortunately, one can use a discrete summation-by-parts technique to eliminate this derivative. The price that needs to be paid is that the right-hand-side of (1.32) becomes rather convoluted, containing terms of the form $(\partial_h^+\partial_h^+ U)^2$. Using PDE terminology, the equation becomes fully nonlinear rather than semi-linear and this requires considerable care.

We are aided by the special structure of $\mathcal{G}$, which is a product of two sums. More precisely, taking a discrete derivative of $\mathcal{G}$ does **not** involve fourth-order discrete derivatives of U. In fact, taking a discrete derivative of (1.32) leads to a semi-linear third-order equation that plays a major role in our construction. The main purpose of §6 and §10-§13 is to build a framework that allows us to control the convoluted expressions that arise from this procedure.

Computational frame Based on the discussion above, it appears to be much more natural to construct wave-like solutions to the scalar LDE (1.32) in the computational coordinate $\tau = jh + ct$ rather than the physical coordinate $\xi = x_j(t) + ct$. Indeed, attempting to use ξ will lead to an equation for the waveprofile Φ with shifts that depend on the waveprofile Φ itself. In particular, the resulting wave equation is a state-dependent MFDE with infinite range interactions. At the moment, even state-dependent delay equations with a finite number of shifts are technically very challenging to analyze, requiring special care in the linearization procedure [64]. Indeed, linearizations typically involve higher order (continuous) derivatives, making it very hard to close fixed-point arguments.

It turns out that the two points of view described above are closely related. In order to see this, let us assume for the moment that we have found a triplet (Φ, c, x) for which x and the function U defined in (1.25) satisfy (1.19) together with (1.23)-(1.24). Let us also assume that for each $\vartheta \in \mathbb{R}$ there is a unique increasing sequence $y_{j;\vartheta}$ with $y_{0;\vartheta} = \vartheta$ for which

$$\big(\Phi(y_{j+1;\vartheta}) - \Phi(y_{j;\vartheta})\big)^2 + (y_{j+1;\vartheta} - y_{j;\vartheta})^2 = h^2 \tag{1.33}$$

holds for all $j \in \mathbb{Z}$. This can be arranged by imposing a-priori Lipschitz bounds on Φ and Φ' and picking $h > 0$ to be sufficiently small. Finally, let us assume for definiteness that $c > 0$ and that the wave outruns the grid in the sense that $\dot{x}_0(t) + c > \epsilon > 0$.

A direct consequence of this inequality is that

$$x_0(T) + cT = x_1(0) \tag{1.34}$$

for some $T > 0$, which implies

$$U_0(T) = U_1(0) = \Phi\big(x_1(0)\big). \tag{1.35}$$

The uniqueness property discussed above hence implies that

$$U_j(T) = \Phi(y_{j;x_1(0)}) = \Phi\big(x_{j+1}(0)\big) \tag{1.36}$$

for all $j \in \mathbb{Z}$. Since

$$\begin{array}{rcl} \big(x_{j+1}(T) - x_j(T)\big)^2 & = & h^2 - \big(U_{j+1}(T) - U_j(T)\big)^2 \\ & = & h^2 - \Big(\Phi\big(x_{j+2}(0)\big) - \Phi\big(x_{j+1}(0)\big)\Big)^2 \\ & = & \big(x_{j+2}(0) - x_{j+1}(0)\big)^2, \end{array} \tag{1.37}$$

we see that in fact

$$x_j(T) + cT = x_{j+1}(0) \tag{1.38}$$

for all $j \in \mathbb{Z}$. Taking the limit $j \to -\infty$, the boundary conditions (1.24) imply that $cT = h$. Exploiting the well-posedness of our dynamics in forward and backward time, we conclude that

$$x_j(t) = x_0(jT + t) + jh \tag{1.39}$$

holds for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Writing $\Psi_x(\vartheta) = x_0(\vartheta/c)$, we hence find

$$x_j(t) - jh = \Psi_x(jh + ct), \tag{1.40}$$

which implies that

$$U_j(t) = \Phi(x_j(t) + ct) = \Phi\big(jh + \Psi_x(jh + ct) + ct\big) \tag{1.41}$$

for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Upon introducing the function

$$\Psi_U(\tau) = \Phi\big(\tau + \Psi_x(\tau)\big), \tag{1.42}$$

this allows us to obtain the representation

$$\big(U_j(t), x_j(t) - jh\big) = \big(\Psi_U(jh + ct), \Psi_x(jh + ct)\big). \tag{1.43}$$

Motivated by these considerations, the main focus of this paper is to construct the waveprofiles Ψ_U directly in the computational coordinates. We treat these profiles as small perturbations from the function Ψ_* that is defined as the arclength reparametrization of the PDE waveprofile Φ_*. In fact, we show that for arbitrary solutions U to (1.32) for which $U(t_0)$ is close to $\Psi_*(h\mathbb{Z} + \vartheta)$, we indeed have the pointwise inequalities $|\dot{x}(t_0)| < |c|$ whenever c is sufficiently close to c_*. This can be used to show that the coordinate transformation (1.42) can be inverted, allowing us to reconstruct the profile $\Phi(\xi)$ from $\Psi_U(\tau)$.

In §3 we show that the stretched profile Ψ_* satisfies the ODE

$$c_*\big[1 - \Psi_*'(\tau)^2\big]^{-1/2}\Psi_*'(\tau) = \big[1 - \Psi_*'(\tau)^2\big]^{-2}\Psi_*''(\tau) + g_{\mathrm{cub}}\big(\Psi_*(\tau); a\big). \tag{1.44}$$

Linearizing this equation around Ψ_*, we obtain the operator

$$\begin{array}{rcl} [\mathcal{L}_{\mathrm{cmp}} v](\tau) & = & -c_*\big[1 - \Psi_*'(\tau)^2\big]^{-3/2} v'(\tau) + \big[1 - \Psi_*'(\tau)^2\big]^{-2} v''(\tau) \\ & & +4\big[1 - \Psi_*'(\tau)^2\big]^{-3}\Psi_*'(\tau)\Psi_*''(\tau)v'(\tau) + g_{\mathrm{cub}}'\big(\Psi_*(\tau); a\big)v(\tau). \end{array} \tag{1.45}$$

In §3.2 we analyze this operator in some detail and recast it back into the original physical coordinates. In fact, we show that it is not equivalent to the standard linearization $\mathcal{L}_{\mathrm{tw}}$ introduced in (1.12). It contains an extra term related to the stretching procedure that vanishes when applied to $\partial_\xi \Phi_*$. On the other hand, in the limits $\tau \to \pm\infty$ the differences between $\mathcal{L}_{\mathrm{cmp}}$ and $\mathcal{L}_{\mathrm{tw}}$ disappear. The essential spectrum hence remains unchanged. In addition, we explicitly show that the kernel of $\mathcal{L}_{\mathrm{cmp}}$ is also one-dimensional.

The singular perturbation The travelling wave equation in the computational coordinates can be written as

$$c\Psi' = \mathcal{G}\Big(\Psi, \partial_h^+\Psi, \partial_h^+\partial_h^+\Psi, \partial_h^+\partial_h^+\partial_h^+\Psi\Big), \tag{1.46}$$

in which the discrete derivatives now act on functions instead of sequences. Linearizing around the stretched wave Ψ_*, we obtain operators $\mathcal{L}_h$ that in the formal $h \downarrow 0$ limit reduce to

$$[\mathcal{L}_* v](\tau) \quad = \quad \big[1 - \Psi_*'(\tau)^2\big]\big[\mathcal{L}_{\mathrm{cmp}} v\big](\tau) + \Psi_*'(\tau)\int_{-\infty}^{\tau}\Psi_*''(\tau')[\mathcal{L}_{\mathrm{cmp}} v](\tau')\, d\tau'. \tag{1.47}$$

The twisted structure is a direct consequence of the procedure that we used to eliminate the $\dot{x}$ terms from the LDE (1.19). In §3.3 we study the integral transform present in (1.47), which allows us to transfer key properties of the operator $\mathcal{L}_{\mathrm{cmp}}$ to $\mathcal{L}_*$.

In §12 we compute the precise expression for $\mathcal{L}_h$, which is too convoluted to present here. We remark however that it is a version of $\mathcal{L}_*$ where the integral has been replaced by a sum and all derivatives except $-c_* v'$ have been replaced by their discrete counterparts.

A crucial step in our program is to establish Fredholm properties for the operator $\mathcal{L}_h$. In particular, we generalize the spectral convergence approach described above to understand the singular transition from $\mathcal{L}_*$ to $\mathcal{L}_h$. This is a delicate task, since the structure of the operators $\mathcal{L}_h$ is significantly more complicated than that of $\mathcal{L}_{h;\mathrm{unif}}$. In particular, the integral transform and the non-autonomous coefficients generate several new terms that were not present in [4].

Our approach hinges on the fact that the new terms can all be shown to be localized in an appropriate sense. Nevertheless, recalling the sequences $\{v_j\} \subset H^1$ and $\{w_j\} \subset L^2$ with $(\mathcal{L}_h + \delta)v_j = w_j$, we need to extract subsequences for which the discrete derivatives of v_j also converge strongly on compact intervals. We accomplish this by carefully controlling the size of the second-order discrete derivatives. This requires frequent use of a discrete summation-by-parts procedure to isolate this derivative from the convoluted expressions.

We believe that our understanding of the operators $\mathcal{L}_h$ will turn out to be very helpful when carrying out the broader program discussed above. Indeed, following the approach in [60], we can obtain information on the linearization around the actual adaptive travelling waves constructed in this paper. Such information is crucial in order to understand the stability properties of the waves and could be very helpful towards understanding the $0 < \tau \ll 1$ regime for the full coupled system (1.19)-(1.20).

Overview This paper is organized as follows. Our main results are formulated in §2. In §3 we discuss the impact on the PDE wave (Φ_*, c_*) caused by the transition from the physical coordinates to the computational coordinates. We develop some basic tools that link discrete and continuous calculus in §4-§5. We continue in §6-§7 by building the framework that we use to obtain our estimates on $\mathcal{G}$ and discussing the properties of several important error functions.

The behaviour of the gridpoints is discussed in §8, where we derive an equation for the nonlinearity $\mathcal{Y}$ that describes $\dot{x}$. We use this expression in §9 to analyze the function $\mathcal{G}$ that appears in the reduced scalar LDE (1.32). In particular, we perform an initial summation by parts procedure to eliminate the third discrete derivative. In §10-11 we obtain estimates on all the nonlinear functions that appear as factors in the product structure of $\mathcal{G}$. These estimates are used in §12-§13 to compute tractable expressions for the linearization of $\mathcal{G}$ around Ψ_* and obtain errors on the residuals.

In §14 we analyze the structure of the linearizations $\mathcal{L}_h$ and generalize the spectral convergence method to establish Fredholm properties for these operators. Finally, in §15 we combine all these ingredients and establish our main results. In particular, we construct the desired travelling waves by setting up an appropriate fixed-point argument.

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2 Main results

The main results of this paper concern adaptive-grid discretizations of the scalar PDE

$$u_t = u_{xx} + g(u). \tag{2.1}$$

In particular, we fix $h > 0$ and consider a sequence of gridpoints that we index somewhat unconventionally by $h\mathbb{Z}$, in order to highlight the scale of their spatial distribution and prevent cumbersome coordinate transformations.

For any $j \in \mathbb{Z}$, we write $x_{jh}(t)$ for the time-dependent location of the relevant gridpoint and $U_{jh}(t)$ for the associated function value, which ideally should be a close approximation for $u\big(x_{jh}(t), t\big)$. The adaptive scheme that we study here can be formulated as

$$\begin{array}{lcl} \dot{U}_{jh}(t) & = & \Big[\frac{U_{(j+1)h}(t) - U_{(j-1)h}(t)}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big]\dot{x}_{jh}(t) \\ & & \quad + \frac{2}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big[\frac{U_{(j-1)h}(t) - U_{jh}(t)}{x_{jh}(t) - x_{(j-1)h}(t)} + \frac{U_{(j+1)h}(t) - U_{jh}(t)}{x_{(j+1)h}(t) - x_{jh}(t)}\Big] + g\big(U_{jh}(t)\big), \end{array} \tag{2.2}$$

in which $x(t)$ is defined implicitly by demanding that

$$\big(x_{(j+1)h}(t) - x_{jh}(t)\big)^2 + \big(U_{(j+1)h}(t) - U_{jh}(t)\big)^2 \quad = \quad h^2 \tag{2.3}$$

and imposing the boundary constraint

$$\lim_{j \to -\infty} [x_{jh}(t) - jh] = 0. \tag{2.4}$$

Throughout the paper, we assume that the nonlinearity g satisfies the following standard bistability condition.

(Hg) The nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is C^3-smooth and has a bistable structure, in the sense that there exists a constant $0 < a < 1$ such that we have

$$g(0) = g(a) = g(1) = 0, \qquad g'(0) < 0, \qquad g'(1) < 0, \tag{2.5}$$

together with

$$g(u) < 0 \text{ for } u \in (0, a) \cup (1, \infty), \qquad g(u) > 0 \text{ for } u \in (-\infty, -1) \cup (a, 1). \tag{2.6}$$

In §2.1 we introduce a scalar lattice differential equation for U that is equivalent to (2.2)-(2.4) in an appropriate sense, but much more suitable for analysis. In §2.2 we exploit this reduced equation to describe a bifurcation result that allows us to obtain travelling wave solutions to (2.2) for $0 < h \ll 1$.

2.1 The reduced system

Our main results in this first part show how the implicit requirements (2.3)-(2.4) can be made explicit. In particular, we introduce the equilibrium grid

$$[x_{\mathrm{eq};h}]_{jh} = jh \tag{2.7}$$

together with the sequence space

$$\mathcal{X}_h \quad = \quad \{x : h\mathbb{Z} \to \mathbb{R} \text{ for which } \|x\|_{\mathcal{X}_h} := \|x - x_{\mathrm{eq};h}\|_\infty = \sup_{j \in \mathbb{Z}} |x_{jh} - jh| < \infty\} \tag{2.8}$$

and write

$$x(t) = \{x_{jh}(t)\}_{j \in \mathbb{Z}} \in \mathcal{X}_h. \tag{2.9}$$

Our goal is to formulate a well-posed equation for the dynamics of

$$U(t) = \{U_{jh}(t)\}_{j \in \mathbb{Z}} \in \ell^\infty(h\mathbb{Z}; \mathbb{R}) \tag{2.10}$$

from which the dependence on x and $\dot{x}$ has been eliminated.

As a preparation, for any $U \in \ell^\infty(h\mathbb{Z}; \mathbb{R})$ we introduce the notation

$$\partial^+ U \in \ell^\infty(h\mathbb{Z}; \mathbb{R}), \qquad \partial^- U \in \ell^\infty(h\mathbb{Z}; \mathbb{R}), \qquad \partial^0 U \in \ell^\infty(h\mathbb{Z}; \mathbb{R}) \tag{2.11}$$

for the sequences

$$\begin{array}{rcl} [\partial^+ U]_{jh} & = & h^{-1}\big[U_{(j+1)h} - U_{jh}\big], \\ [\partial^- U]_{jh} & = & h^{-1}\big[U_{jh} - U_{(j-1)h}\big], \\ [\partial^0 U]_{jh} & = & (2h)^{-1}\big[U_{(j+1)h} - U_{(j-1)h}\big]. \end{array} \tag{2.12}$$

In addition, for any $U \in \ell^\infty(h\mathbb{Z}; \mathbb{R})$ for which $\|\partial^+ U\|_\infty < 1$, we define the sequences

$$\mathcal{F}^{\diamond_\pm}(U) \in \ell^\infty(h\mathbb{Z}; \mathbb{R}), \qquad \mathcal{F}^{\diamond_0}(U) \in \ell^\infty(h\mathbb{Z}; \mathbb{R}), \qquad \mathcal{F}^{\diamond\diamond_0}(U) \in \ell^\infty(h\mathbb{Z}; \mathbb{R}) \tag{2.13}$$

by means of the pointwise identities

$$\begin{array}{rcl} \mathcal{F}^{\diamond_-}(U) & = & \frac{\partial^- U}{\sqrt{1-(\partial^- U)^2}}, \\ \mathcal{F}^{\diamond_+}(U) & = & \frac{\partial^+ U}{\sqrt{1-(\partial^+ U)^2}}, \\ \mathcal{F}^{\diamond_0}(U) & = & \frac{2\partial^0 U}{\sqrt{1-(\partial^+ U)^2} + \sqrt{1-(\partial^- U)^2}}, \\ \mathcal{F}^{\diamond\diamond_0}(U) & = & \frac{1}{h} \frac{\mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_-}(U)}{\sqrt{1-(\partial^+ U)^2} + \sqrt{1-(\partial^- U)^2}}. \end{array} \tag{2.14}$$

We also introduce the Heaviside sequence $H \in \ell^\infty(h\mathbb{Z}; \mathbb{R})$ that has

$$H_{jh} = \begin{cases} 1 & \text{for } j \geq 0 \\ 0 & \text{for } j < 0 \end{cases} \tag{2.15}$$

Finally, we introduce the formal expression

$$\mathcal{Q}_{jh}(U) = \sum_{j' < j} \Big[\ln\big[1 + \mathcal{F}^{\diamond_+}_{j'h}(U) \mathcal{F}^{\diamond_0}_{(j'+1)h}(U)\big] - \ln\big[1 + \mathcal{F}^{\diamond_+}_{j'h}(U) \mathcal{F}^{\diamond_0}_{j'h}(U)\big] \Big], \tag{2.16}$$

together with

$$\mathcal{Y}_{jh}(U) = -\exp[-\mathcal{Q}_{jh}(U)] h \sum_{j' < j} \frac{\mathcal{F}^{\diamond_+}_{j'h}(U) \exp[\mathcal{Q}_{j'h}(U)]}{1 + \mathcal{F}^{\diamond_+}_{j'h}(U) \mathcal{F}^{\diamond_0}_{j'h}(U)} \partial^+ \big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]_{j'h}. \tag{2.17}$$

Upon imposing a summability condition on U it is possible to show that these expressions are well-defined.

Lemma 2.1 (see §8). *Suppose that (Hg) is satisfied, fix $h > 0$ and consider any $U \in \ell^\infty(h\mathbb{Z}; \mathbb{R})$ for which $U - H \in \ell^2(h\mathbb{Z}; \mathbb{R})$ and $\|\partial^+ U\|_\infty < 1$. Then the sequences*

$$\mathcal{Q}(U) = \{\mathcal{Q}_{jh}\}_{j \in \mathbb{Z}}, \qquad \mathcal{Y}(U) = \{\mathcal{Y}_{jh}\}_{j \in \mathbb{Z}} \tag{2.18}$$

are both well-defined and we have

$$\mathcal{Q}(U) \in \ell^\infty(h\mathbb{Z}; \mathbb{R}), \qquad \mathcal{Y}(U) \in \ell^\infty(h\mathbb{Z}; \mathbb{R}). \tag{2.19}$$

Our first main result shows that the expression $\mathcal{Y}(U)$ can be used to replace the $\dot{x}$ term appearing in (2.2). The remaining terms involving x can be eliminated using the implicit relation (2.3).

Proposition 2.2 (see §8). *Suppose that (Hg) is satisfied and fix $h > 0$ together with $T > 0$. Consider two functions*

$$x : [0,T] \mapsto \mathcal{X}_h, \qquad U : [0,T] \mapsto \ell^\infty(h\mathbb{Z};\mathbb{R}) \tag{2.20}$$

that satisfy the following properties.

(a) We have the inclusions

$$\begin{array}{lcl} t \mapsto U(t) - H & \in & C^1\big([0,T];\ell^2(h\mathbb{Z};\mathbb{R})\big), \\ t \mapsto x(t) - x_{\mathrm{eq};h} & \in & C^1\big([0,T];\ell^\infty(h\mathbb{Z};\mathbb{R})\big). \end{array} \tag{2.21}$$

(b) For every $j \in \mathbb{Z}$ and $0 \le t \le T$ we have the identity

$$x_{(j+1)h}(t) - x_{jh}(t) \quad = \quad \sqrt{h^2 - \big(U_{(j+1)h}(t) - U_{jh}(t)\big)^2}. \tag{2.22}$$

(c) For every $0 \le t \le T$ we have the limit

$$\lim_{j \to -\infty} \big[x_{jh}(t) - jh\big] = 0. \tag{2.23}$$

(d) For every $0 \le t \le T$ we have the strict inequality

$$\inf_{j \in \mathbb{Z}} \big[x_{(j+1)h}(t) - x_{jh}(t)\big] > 0. \tag{2.24}$$

(e) For every $0 \le t \le T$ and $j \in \mathbb{Z}$ we have the identity

$$\begin{array}{lcl} \dot{U}_{jh}(t) & = & \Big[\frac{U_{(j+1)h}(t) - U_{(j-1)h}(t)}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big]\dot{x}_{jh}(t) \\ & & + \frac{2}{x_{(j+1)h}(t) - x_{(j-1)h}(t)}\Big[\frac{U_{(j-1)h}(t) - U_{jh}(t)}{x_{jh}(t) - x_{(j-1)h}(t)} + \frac{U_{(j+1)h}(t) - U_{jh}(t)}{x_{(j+1)h}(t) - x_{jh}(t)}\Big] + g\big(U_{jh}(t)\big). \end{array} \tag{2.25}$$

Then the function U satisfies the system

$$\dot{U}(t) = \mathcal{F}^{\diamond_0}\big(U(t)\big)\mathcal{Y}\big(U(t)\big) + 2\mathcal{F}^{\diamond\diamond_0}\big(U(t)\big) + g\big(U(t)\big) \tag{2.26}$$

for all $0 \le t \le T$.

Conversely, once a solution to (2.26) has been obtained, it is possible to construct a solution to the full problem (2.2). Indeed, the following result shows how the position of the gridpoints can be recovered from $U(t)$.

Proposition 2.3 (see §8). *Suppose that (Hg) is satisfied and fix $h > 0$ together with $T > 0$. Consider a function $U : [0,T] \to \ell^\infty(h\mathbb{Z};\mathbb{R})$ that satisfies the following properties.*

(a') We have the inclusion

$$t \mapsto U(t) - H \in C^1\big([0,T];\ell^2(h\mathbb{Z};\mathbb{R})\big). \tag{2.27}$$

(b') The strict inequality

$$\left\|\partial^+ U(t)\right\|_\infty < 1 \tag{2.28}$$

holds for every $t \in [0,T]$.

(c') For every $t \in [0,T]$ *the identity*

$$\dot U(t) = \mathcal{F}^{\diamond_0}(U(t))\mathcal{Y}\big(U(t)\big) + 2\mathcal{F}^{\diamond\diamond_0}\big(U(t)\big) + g\big(U(t)\big) \tag{2.29}$$

is satisfied.

Then upon writing

$$x_{jh}(t) \quad = \quad jh - \sum_{j'<j} \frac{\big(U_{(j'+1)h}(t) - U_{j'h}(t)\big)^2}{\sqrt{h^2 - (U_{(j'+1)h}(t) - U_{j'h}(t))^2} + h} \tag{2.30}$$

the properties (a), (b), (c), (d) and (e) from Proposition 2.2 are all satisfied.

We conclude our general analysis of the full problem (2.2) by showing that the reduced system (2.29) is well-posed in an appropriate sense. Indeed, we establish the following short-term existence result for a class of summable initial conditions. We remark that the restriction (2.31) on the initial condition is a natural and unavoidable consequence of the requirement (2.3).

Proposition 2.4 (see §12). *Suppose that (Hg) is satisfied and fix* $h > 0$. *Consider any* $U^0 \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ *for which* $U^0 - H \in \ell^2(h\mathbb{Z};\mathbb{R})$ *and for which*

$$\left\|\partial^+ U^0\right\|_\infty < 1. \tag{2.31}$$

Then there exists $\delta_T > 0$ *and a function* $U : [0,\delta_T] \to \ell^\infty(h\mathbb{Z};\mathbb{R})$ *that has* $U(0) = U^0$ *and that satisfies the properties (a'), (b') and (c') from Proposition 2.3 with* $T = \delta_T$.

2.2 Travelling waves

It is well-known that the PDE (2.1) admits a travelling wave solution that connects the two stable equilibria of g [26]. The key requirement in our next assumption is that this wave is not stationary, which can be arranged by demanding $\int_0^1 g(u)\,du \neq 0$.

(HΦ_*) There exists a wave speed $c_* \neq 0$ and a profile $\Phi_* \in C^5(\mathbb{R},\mathbb{R})$ that satisfies the limits

$$\lim_{\xi\to-\infty} \Phi_*(\xi) = 0, \qquad \lim_{\xi\to+\infty} \Phi_*(\xi) = 1 \tag{2.32}$$

and yields a solution to the PDE (2.1) upon writing

$$u(x,t) = \Phi_*(x + c_* t). \tag{2.33}$$

The physical wave coordinate $\xi = x + c_* t$ appearing in (HΦ_*) is not well-suited for our purposes here, since the reduced equation (2.29) is formulated in terms of the grid-coordinates $h\mathbb{Z}$. In order to compensate for this, we introduce the arclength

$$\mathcal{A}(\xi) = \int_0^\xi \sqrt{1 + [\partial_{\xi'}\Phi_*(\xi')]^2}\, d\xi'. \tag{2.34}$$

Lemma 2.5. *For every* $\tau \in \mathbb{R}$, *there is a unique* $\xi_*(\tau)$ *for which*

$$\mathcal{A}\big(\xi_*(\tau)\big) = \tau. \tag{2.35}$$

Proof. The existence of the right-inverse ξ_* for $\mathcal{A}$ follows from

$$\partial_\xi \mathcal{A}(\xi) = \sqrt{1 + [\partial_\xi \Phi_*(\xi)]^2} \geq 1. \tag{2.36}$$

□

We are now in a position to introduce the stretched waveprofile $\Psi_* : \mathbb{R} \to \mathbb{R}$ that is given by

$$\Psi_*(\tau) = \Phi_*\big(\xi_*(\tau)\big). \tag{2.37}$$

This profile Ψ_* can be seen as the arclength parametrization of the graph of the physical wave Φ_*.

The main result of this paper states that for sufficiently small $h > 0$, the reduced problem (2.29) admits a travelling wave solution

$$U_{jh}(t) = \Psi_h(jh + c_h t) \tag{2.38}$$

with $(\Psi_h, c_h) \approx (\Psi_*, c_*)$ in an appropriate sense. These waves are locally unique up to translation. We note that items (iv) and (v) use the notation $\partial_h^+ v = h^{-1}[v(\cdot + h) - v(\cdot)]$ for functions v. In addition, we use the shorthands $L^2 = L^2(\mathbb{R};\mathbb{R})$ and $H^1 = H^1(\mathbb{R};\mathbb{R})$.

Theorem 2.6 (see §15). *Suppose that (Hg) and (HΦ_*) are satisfied. Then there exists a constant $\delta_h > 0$ together with pairs*

$$(\Psi_h, c_h) \in C^1(\mathbb{R};\mathbb{R}) \times \mathbb{R}, \tag{2.39}$$

defined for $0 < h \leq \delta_h$, such that the following properties are satisfied.

(i) For every $0 < h \leq \delta_h$ we have the limits

$$\lim_{\xi \to -\infty} \Psi_h(\xi) = 0, \qquad \lim_{\xi \to +\infty} \Psi_h(\xi) = 1. \tag{2.40}$$

(ii) For every $0 < h \leq \delta_h$ we have the strict inequality

$$\sup_{\tau \in \mathbb{R}} |\Psi_h(\tau + h) - \Psi_h(\tau)| < h. \tag{2.41}$$

(iii) For every $0 < h \leq \delta_h$, the function $U : \mathbb{R} \to \ell^\infty(h\mathbb{Z};\mathbb{R})$ defined by

$$U_{jh}(t) = \Psi_h(jh + c_h t) \tag{2.42}$$

satisfies the inclusion

$$t \mapsto U(t) - H \in C^1\big(\mathbb{R}; \ell^2(h\mathbb{Z};\mathbb{R})\big). \tag{2.43}$$

In addition, the identity (2.29) and the strict inequality $\|\partial^+ U(t)\|_\infty < 1$ both hold for all $t \in \mathbb{R}$.

(iv) We have $\Psi_h - \Psi_* \in H^1$ for every $0 < h \leq \delta_h$ and the limit

$$|c_h - c_*| + \|\Psi_h - \Psi_*\|_{H^1} + \left\|\partial_h^+ [\Psi_h - \Psi_*]\right\|_{H^1} + \left\|\partial_h^+ \partial_h^+ \partial_h^+ [\Psi_h - \Psi_*]\right\|_{L^2} \to 0 \tag{2.44}$$

holds as $h \downarrow 0$.

(v) Pick any $0 < h \leq \delta_h$ and consider a pair $(\widetilde{\Psi}, \tilde{c}) \in L^\infty \times \mathbb{R}$ that has $\widetilde{\Psi} - \Psi_* \in H^1$ with

$$|\tilde{c} - c_*| + \left\|\widetilde{\Psi} - \Psi_*\right\|_{H^1} + \left\|\partial_h^+ [\widetilde{\Psi} - \Psi_*]\right\|_{H^1} + \left\|\partial_h^+ \partial_h^+ \partial_h^+ [\widetilde{\Psi} - \Psi_*]\right\|_{L^2} < h^{3/4}. \tag{2.45}$$

Then the function $\widetilde{U} : \mathbb{R} \to \ell^\infty(h\mathbb{Z};\mathbb{R})$ *defined by*

$$\widetilde{U}_{jh}(t) = \widetilde{\Psi}_h\big(jh + \tilde{c}t\big) \tag{2.46}$$

satisfies the inclusion

$$t \mapsto \widetilde{U}(t) - H \in C^0\big(\mathbb{R}; \ell^2(h\mathbb{Z};\mathbb{R})\big), \tag{2.47}$$

together with the strict inequality $\left\|\partial^+ \widetilde{U}\right\|_\infty < 1$ *for all* $t \in \mathbb{R}$*. In addition, if* $\widetilde{U}$ *is a solution to the system (2.29) for all* $t \in \mathbb{R}$*, then we must have*

$$\big(\widetilde{\Psi}(\cdot), \tilde{c}\big) = \big(\Psi_h(\cdot + \vartheta), c_h\big) \tag{2.48}$$

for some $\vartheta \in \mathbb{R}$*.*

We emphasize that the location of the gridpoints for the waves (2.38) can be determined by using (2.30). In fact, our final result shows how these waves in the computational coordinates can be interpreted as wave-like objects in the original physical coordinates.

Corollary 2.7 (see §15). *Consider the setting of Theorem 2.6. Then there exists a constant* $0 < \widetilde{\delta}_h < \delta_h$ *so that for all* $0 < h \le \widetilde{\delta}_h$ *there exist pairs*

$$(\Psi_h^{(x)}, \Phi_h) \in C^1(\mathbb{R};\mathbb{R}) \times C^1(\mathbb{R};\mathbb{R}) \tag{2.49}$$

that satisfy the following properties.

(i) Upon writing

$$\begin{aligned} x_{jh}(t) &= jh + \Psi_h^{(x)}(jh + c_h t), \\ U_{jh}(t) &= \Psi_h(jh + c_h t), \end{aligned} \tag{2.50}$$

the adaptive grid equations (2.2) - (2.4) are satisfied for all $t \in \mathbb{R}$*.*

(ii) For every $t \in \mathbb{R}$ *and* $j \in \mathbb{Z}$*, the functions defined in (2.50) satisfy the relation*

$$U_{jh}(t) = \Phi_h\big(x_{jh}(t) + c_h t\big). \tag{2.51}$$

We remark that if (2.38) and (2.51) both hold, simple substitutions yield the identity

$$\begin{aligned} \Psi_h(jh + c_h t) &= U_{jh}(t) \\ &= \Phi_h\big(jh + \Psi_h^{(x)}(jh + c_h t) + c_h t\big) \\ &= \Phi_h\big(jh + c_h t + \Psi_h^{(x)}(jh + c_h t)\big). \end{aligned} \tag{2.52}$$

In particular, the main assertion in Corollary 2.7 is that the perturbed coordinate transformation

$$\xi_h(\tau) = \tau + \Psi_h^{(x)}(\tau) \tag{2.53}$$

is invertible for sufficiently small $h > 0$, allowing us to transfer the waves back to the original physical framework.

3 Stretched PDE waves

We recall the functions $\mathcal{A}(\xi)$ and ξ_* introduced in Lemma 2.5, which are related to the arclength parametrization of Φ_*. We also recall the stretched waveprofile

$$\Psi_*(\tau) = \Phi_*\big(\xi_*(\tau)\big) \tag{3.1}$$

and introduce the notation

$$\gamma_*(\tau) = \sqrt{1 - [\partial_\tau \Psi_*](\tau)^2} = \sqrt{1 - \Psi_*'(\tau)^2}. \tag{3.2}$$

Our first main result shows that γ_* is well-defined and that it can be used to translate the travelling wave equation for the continuum model (2.1) into the stretched computational coordinates.

Proposition 3.1. *Suppose that (Hg) and (HΦ_*) are satisfied. Then we have $\Psi_* \in C^5(\mathbb{R}, \mathbb{R})$ and there exists $\kappa > 0$ so that the bounds*

$$0 < \Psi_*'(\tau) < 1 - \kappa, \qquad \sqrt{\kappa} < \gamma_*(\tau) < 1 \tag{3.3}$$

hold for all $\tau \in \mathbb{R}$. In addition, there exists a constant $K > 0$ together with exponents $\eta_- > \max\{0, c_*\}$ and $\eta_+ > \max\{0, -c_*\}$ for which the bound

$$|\Psi_*(\tau)| + |\Psi_*'(\tau)| + |\Psi_*''(\tau)| + |\Psi_*'''(\tau)| + \left|\Psi_*^{(iv)}(\tau)\right| + \left|\Psi_*^{(v)}(\tau)\right| \;\; \leq \;\; K e^{-\eta_- |\tau|} \tag{3.4}$$

holds whenever $\tau < 0$, while the bound

$$|1 - \Psi_*(\tau)| + |\Psi_*'(\tau)| + |\Psi_*''(\tau)| + |\Psi_*'''(\tau)| + \left|\Psi_*^{(iv)}(\tau)\right| + \left|\Psi_*^{(v)}(\tau)\right| \;\; \leq \;\; K e^{-\eta_+ |\tau|} \tag{3.5}$$

holds for all $\tau \geq 0$. Finally, for every $\tau \in \mathbb{R}$ we have the identity

$$c_* \gamma_*^{-1}(\tau) \Psi_*'(\tau) \;\; = \;\; \gamma_*^{-4}(\tau) \Psi_*''(\tau) + g\big(\Psi_*(\tau)\big), \tag{3.6}$$

together with the differentiated version

$$c_* \gamma_*^{-3}(\tau) \Psi_*''(\tau) \;\; = \;\; \gamma_*^{-4}(\tau) \Psi_*'''(\tau) + 4\gamma_*^{-6}(\tau) \Psi_*''(\tau) \Psi_*'(\tau) \Psi_*''(\tau) + g'\big(\Psi_*(\tau)\big) \Psi_*'(\tau). \tag{3.7}$$

Inspired by (3.7), we introduce the linear operator $\mathcal{L}_{\mathrm{cmp}} : H^2 \to L^2$ given by

$$\mathcal{L}_{\mathrm{cmp}} v \;\; = \;\; -c_* \gamma_*^{-3} v' + \gamma_*^{-4} v'' + 4\gamma_*^{-6} \Psi_*' \Psi_*'' v' + g'(\Psi_*) v, \tag{3.8}$$

which corresponds with the linearization of (3.6) around Ψ_*. We also define the formal adjoint $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} : H^2 \to L^2$ that acts as

$$\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} w \;\; = \;\; c_* \partial_\tau [\gamma_*^{-3} w] + \partial_{\tau\tau} [\gamma_*^{-4} w] - \partial_\tau [4\gamma_*^{-6} \Psi_*' \Psi_*'' w] + g'(\Psi_*) w. \tag{3.9}$$

Indeed, one may easily verify that for any pair $(v, w) \in H^2 \times H^2$ we have

$$\langle \mathcal{L}_{\mathrm{cmp}} v, w \rangle_{L^2} = \langle v, \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} w \rangle_{L^2}. \tag{3.10}$$

Finally, we introduce the function

$$\Psi_*^{\mathrm{adj}}(\tau) = \Big[\int \gamma_*^{-1}(\tau') \Psi_*'(\tau') e^{-\int_0^{\tau'} c_* \gamma_*(s)\, ds} \Psi_*'(\tau')\, d\tau' \Big]^{-1} e^{-\int_0^{\tau} c_* \gamma_*(s)\, ds} \Psi_*'(\tau). \tag{3.11}$$

We note that the exponential bounds (3.4)-(3.5) together with (3.3) imply that Ψ_*^{adj} is a well-defined function that decays exponentially as $\tau \to \pm\infty$. The second main result in this section shows that we have now encountered all the kernel elements of $\mathcal{L}_{\mathrm{cmp}}$ and $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}$.

Proposition 3.2. *Suppose that (Hg) and (HΦ_*) both hold. Then the operators $\mathcal{L}_{\mathrm{cmp}} : H^2 \to L^2$ and $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} : H^2 \to L^2$ are both Fredholm with index zero. In addition, we have the identities*

$$\begin{aligned} \mathrm{Ker}\Big(\mathcal{L}_{\mathrm{cmp}}\Big) &= \mathrm{span}\{\Psi_*'\}, \\ \mathrm{Ker}\Big(\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}\Big) &= \mathrm{span}\{\Psi_*^{\mathrm{adj}}\}. \end{aligned} \tag{3.12}$$

In §14 we will see that the linearization of the adaptive grid problem leads naturally to a twisted version of $\mathcal{L}_{\mathrm{cmp}}$. To account for this, we introduce the notation

$$\Big[\int_- f\Big](\tau) = \int_{-\infty}^{\tau} f(\tau')\, d\tau', \qquad \Big[\int_+ f\Big](\tau) = \int_{\tau}^{\infty} f(\tau')\, d\tau' \tag{3.13}$$

for the bounded continuous functions that arise after integrating a function $f \in L^1$. This allows us to define the integral transforms

$$\begin{aligned} \mathcal{T}_* f &= \gamma_*^{-2}\big[f - \gamma_* \Psi_*' \textstyle\int_- \gamma_*^{-3} \Psi_*'' f\big], \\ \mathcal{T}_*^{\mathrm{adj}} f &= \gamma_*^{-2}\big[f - \gamma_*^{-1} \Psi_*'' \textstyle\int_+ \gamma_*^{-1} \Psi_*' f\big], \end{aligned} \tag{3.14}$$

now for any $f \in L^2$.

We give a detailed discussion of these transforms in §3.3 below. For now, we compute

$$\begin{aligned} \mathcal{T}_* \Psi_*' &= \gamma_*^{-2}\big[\Psi_*' - \gamma_* \Psi_*' \textstyle\int_- \gamma_*^{-3} \Psi_*'' \Psi_*'\big] \\ &= \gamma_*^{-2}\big[\Psi_*' - \gamma_* \Psi_*' [\gamma_*^{-1} - 1]\big] \\ &= \gamma_*^{-1} \Psi_*', \end{aligned} \tag{3.15}$$

which means that we have normalized Ψ_*^{adj} in such a way that

$$\langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* \Psi_*' \rangle = \langle \mathcal{T}_*^{\mathrm{adj}} \Psi_*^{\mathrm{adj}}, \Psi_*' \rangle = 1. \tag{3.16}$$

In particular, we see that $\lambda = 0$ is a simple eigenvalue for the twisted eigenvalue problem

$$\mathcal{L}_{\mathrm{cmp}} v = \lambda \mathcal{T}_* v. \tag{3.17}$$

This allows us to obtain the following essential estimate on the behaviour of $[\mathcal{L}_{\mathrm{cmp}} - \delta \mathcal{T}_*]^{-1}$ as $\delta \downarrow 0$, which will allow us to transfer the Fredholm properties of $\mathcal{L}_{\mathrm{cmp}}$ to its discrete twisted counterpart in §14.

Corollary 3.3. *Suppose that (Hg) and (HΦ_*) both hold. Then $\mathcal{L}_{\mathrm{cmp}} - \delta \mathcal{T}_*$ and $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} - \delta \mathcal{T}_*^{\mathrm{adj}}$ are both invertible as linear maps from H^2 into L^2 for all sufficiently small $\delta > 0$. In addition, there exists $K > 0$ so that the bounds*

$$\begin{aligned} \Big\| [\mathcal{L}_{\mathrm{cmp}} - \delta \mathcal{T}_*]^{-1} f + \delta^{-1} \Psi_*' \langle \Psi_*^{\mathrm{adj}}, f \rangle_{L^2} \Big\|_{H^2} &\le K \left\| f \right\|_{L^2}, \\ \Big\| [\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} - \delta \mathcal{T}_*^{\mathrm{adj}}]^{-1} f + \delta^{-1} \Psi_*^{\mathrm{adj}} \langle \Psi_*', f \rangle_{L^2} \Big\|_{H^2} &\le K \left\| f \right\|_{L^2} \end{aligned} \tag{3.18}$$

hold for all $f \in L^2$ and all sufficiently small $\delta > 0$.

3.1 Coordinate transformation

Consider two functions $f_{\mathrm{cmp}} : \mathbb{R} \to \mathbb{R}$ and $f_{\mathrm{phys}} : \mathbb{R} \to \mathbb{R}$. We introduce the stretching operator $\mathcal{S}_*$ and the compression operator $\mathcal{S}_*^{-1}$ that act as

$$[\mathcal{S}_* f_{\mathrm{phys}}](\tau) = f_{\mathrm{phys}}\big(\xi_*(\tau)\big), \qquad [\mathcal{S}_*^{-1} f_{\mathrm{cmp}}](\xi) = f_{\mathrm{cmp}}\big(\mathcal{A}(\xi)\big). \tag{3.19}$$

In particular, for any $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}$ we have the identities

$$[\mathcal{S}_*^{-1} f_{\mathrm{cmp}}]\big(\xi_*(\tau)\big) = f_{\mathrm{cmp}}(\tau), \qquad [\mathcal{S}_* f_{\mathrm{phys}}]\big(\mathcal{A}(\xi)\big) = f_{\mathrm{phys}}(\xi). \tag{3.20}$$

In order to understand the effect of these coordinate transformations on integrals and derivatives, we first need to understand ξ_*'.

Lemma 3.4. *Suppose that (Hg) and (HΦ_*) are satisfied. Then we have $\xi_* \in C^1(\mathbb{R};\mathbb{R})$. In addition, for any $\tau \in \mathbb{R}$ we have*

$$\begin{array}{rcl} \xi_*'(\tau) & = & \Big[1 + [\partial_\xi \Phi_*\big(\xi_*(\tau)\big)]^2\Big]^{-1/2} \\ & = & \gamma_*(\tau). \end{array} \tag{3.21}$$

Proof. The first identity in (3.21) follows by differentiating $\tau = \mathcal{A}(\xi_*(\tau))$ with respect to τ. Using the chain rule we compute

$$\begin{array}{rcl} \Psi_*'(\tau) & = & \partial_\tau [\Phi_*\big(\xi_*(\tau)\big)] \\ & = & [\partial_\xi \Phi_*]\big(\xi_*(\tau)\big)\xi_*'(\tau) \\ & = & [\partial_\xi \Phi_*]\big(\xi_*(\tau)\big)\big[1 + \partial_\xi \Phi_*\big(\xi_*(\tau)\big)^2\big]^{-1/2}. \end{array} \tag{3.22}$$

Squaring this identity yields

$$\Psi_*'(\tau)^2 \quad = \quad 1 - \big[1 + \partial_\xi \Phi_*\big(\xi_*(\tau)\big)^2\big]^{-1}, \tag{3.23}$$

which gives

$$\big[1 + \partial_\xi \Phi_*\big(\xi_*(\tau)\big)^2\big]^{-1} = 1 - \Psi_*'(\tau)^2 = \gamma_*(\tau)^2, \tag{3.24}$$

as desired. □

Corollary 3.5. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for any $f_{\mathrm{cmp}} \in C(\mathbb{R},\mathbb{R}) \cap L^2$ and $f_{\mathrm{phys}} \in C(\mathbb{R},\mathbb{R}) \cap L^2$ we have the identity*

$$\langle f_{\mathrm{phys}}, \mathcal{S}_*^{-1} f_{\mathrm{cmp}} \rangle_{L^2} = \langle \mathcal{S}_* f_{\mathrm{phys}}, \gamma_* f_{\mathrm{cmp}} \rangle_{L^2}, \tag{3.25}$$

together with

$$\langle \mathcal{S}_* f_{\mathrm{phys}}, f_{\mathrm{cmp}} \rangle_{L^2} = \langle f_{\mathrm{phys}}, \mathcal{S}_*^{-1}\big[\gamma_*^{-1} f_{\mathrm{cmp}}\big] \rangle_{L^2}. \tag{3.26}$$

In particular, $\mathcal{S}_*$ and $\mathcal{S}_*^{-1}$ can be interpreted as elements of $\mathcal{L}(L^2; L^2)$.

Proof. The substitution rule allows us to compute

$$\begin{array}{rcl} \langle f_{\mathrm{phys}}, \mathcal{S}_*^{-1} f_{\mathrm{cmp}} \rangle_{L^2} & = & \int f_{\mathrm{phys}}(\xi) f_{\mathrm{cmp}}\big(\mathcal{A}(\xi)\big)\, d\xi \\ & = & \int f_{\mathrm{phys}}\big(\xi_*(\tau)\big) f_{\mathrm{cmp}}\big(\mathcal{A}(\xi_*(\tau))\big)\xi_*'(\tau)\, d\tau \\ & = & \int f_{\mathrm{phys}}\big(\xi_*(\tau)\big) f_{\mathrm{cmp}}\big(\tau\big)\gamma_*(\tau)\, d\tau \\ & = & \langle \mathcal{S}_* f_{\mathrm{phys}}, \gamma_* f_{\mathrm{cmp}} \rangle_{L^2}. \end{array} \tag{3.27}$$

The second identity follows in a similar fashion. □

Corollary 3.6. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for any $f_{\mathrm{cmp}} \in H^1$, we have $\mathcal{S}_*^{-1} f_{\mathrm{cmp}} \in H^1$ with*

$$\partial_\xi [\mathcal{S}_*^{-1} f_{\mathrm{cmp}}] = \mathcal{S}_*^{-1}\big[\gamma_*^{-1} \partial_\tau f_{\mathrm{cmp}}\big]. \tag{3.28}$$

In addition, for any $f_{\mathrm{phys}} \in H^1$, we have $\mathcal{S}_* f_{\mathrm{phys}} \in H^1$ with

$$\partial_\tau [\mathcal{S}_* f_{\mathrm{phys}}] = \gamma_* \mathcal{S}_* [\partial_\xi f_{\mathrm{phys}}]. \tag{3.29}$$

Proof. For $f_{\mathrm{cmp}} \in C^1(\mathbb{R};\mathbb{R})$ we may use the chain rule to compute

$$\partial_\xi \big[f_{\mathrm{cmp}}\big(\mathcal{A}(\xi)\big)\big] = [\partial_\tau f_{\mathrm{cmp}}]\big(\mathcal{A}(\xi)\big)\partial_\xi \mathcal{A}(\xi) = [\partial_\tau f_{\mathrm{cmp}}]\big(\mathcal{A}(\xi)\big)\big[\xi_*'\big(\mathcal{A}(\xi)\big)\big]^{-1}. \tag{3.30}$$

In addition, for $f_{\mathrm{phys}} \in C^1(\mathbb{R};\mathbb{R})$ we compute

$$\partial_\tau [f_{\mathrm{phys}}\big(\xi_*(\tau)\big)] = [\partial_\xi f_{\mathrm{phys}}]\big(\xi_*(\tau)\big)\xi_*'(\tau). \tag{3.31}$$

The desired identities now follow from (3.19), (3.20) and (3.21). The final remark in Corollary 3.5 can be used to extend these results to $f_{\mathrm{cmp}} \in H^1$ and $f_{\mathrm{phys}} \in H^1$. □

The physical wave Φ_* satisfies the travelling wave ODE

$$c_* \partial_\xi \Phi_*(\xi) = \partial_{\xi\xi}\Phi_*(\xi) + g\big(\Phi_*(\xi)\big) \tag{3.32}$$

for all $\xi \in \mathbb{R}$. It is well known that the limiting behaviour of Φ_* as $\xi \to \pm\infty$ depends on the roots of the characteristic functions

$$\Delta_\pm(\eta) = -c_*\eta + \eta^2 + g'\big(\Phi_*(\pm\infty)\big). \tag{3.33}$$

In particular, upon writing

$$\eta_- = \frac{1}{2}c_* + \frac{1}{2}\sqrt{c_*^2 - 4g'(0)} > \frac{1}{2}c_* + \frac{1}{2}\left|c_*\right| \tag{3.34}$$

and

$$\eta_+ = -\Big[\frac{1}{2}c_* - \frac{1}{2}\sqrt{c_*^2 - 4g'(1)}\Big] > -\frac{1}{2}c_* + \frac{1}{2}\left|c_*\right|, \tag{3.35}$$

we have the bounds

$$|\partial_\xi \Phi_*(\xi)| \le K e^{-\eta_\pm |\xi|} \tag{3.36}$$

for $\xi \in \mathbb{R}_\pm$. In order to transfer this exponential bound to Ψ_*', we need to understand the differences $\xi_*(\tau) - \tau$.

Lemma 3.7. *Suppose that (Hg) and (HΦ_*) are satisfied. Then there exists $K > 0$ so that the inequality*

$$|\xi_*(\tau) - \tau| < K \tag{3.37}$$

holds for any $\tau \in \mathbb{R}$.

Proof. For any $x \in \mathbb{R}$ we have the standard inequality

$$\sqrt{1 + x^2} - 1 \le \frac{1}{2}x^2. \tag{3.38}$$

In particular, we see that

$$\begin{aligned} |\mathcal{A}(\xi) - \xi| &\le \tfrac{1}{2}\int_0^\xi \partial_{\xi'}\Phi_*(\xi')^2\, d\xi' \\ &\le \tfrac{1}{2}\left\|\partial_\xi \Phi_*\right\|_{L^2}^2, \end{aligned} \tag{3.39}$$

which gives

$$|\xi_*(\tau) - \tau| = \big|\xi_*(\tau) - \mathcal{A}\big(\xi_*(\tau)\big)\big| \le \frac{1}{2}\left\|\partial_\xi \Phi_*\right\|_{L^2}^2. \tag{3.40}$$

□

Proof of Proposition 3.1. Using $\Phi_* = \mathcal{S}_*^{-1}\Psi_*$ together with the commutation relation

$$g\big(\mathcal{S}_*^{-1}\Psi_*\big) = \mathcal{S}_*^{-1} g\big(\Psi_*\big), \tag{3.41}$$

we can apply Corollary 3.6 to the travelling wave ODE (3.32) to obtain

$$c_*\mathcal{S}_*^{-1}\big[\gamma_*^{-1}\Psi_*'\big] = \mathcal{S}_*^{-1}\big[\gamma_*^{-1}\partial_\tau[\gamma_*^{-1}\Psi_*']\big] + \mathcal{S}_*^{-1}[g(\Psi_*)]. \tag{3.42}$$

Using the identity

$$\gamma_*' = -\gamma_*^{-1}\Psi_*'\Psi_*'' \tag{3.43}$$

together with the definition $\gamma_*^2 = 1 - [\Psi_*']^2$, this gives

$$\begin{aligned} c_*\gamma_*^{-1}\Psi_*' &= \gamma_*^{-2}\Psi_*'' + \gamma_*^{-4}\big[\Psi_*'\big]^2\Psi_*'' + g(\Psi_*) \\ &= \gamma_*^{-4}\Psi_*'' + g(\Psi_*). \end{aligned} \tag{3.44}$$

A further differentiation yields

$$c_*\gamma_*^{-1}\Psi_*'' + c_*\gamma_*^{-3}\Psi_*'\Psi_*'\Psi_*'' \quad = \quad \gamma_*^{-4}\Psi_*''' + 4\gamma_*^{-6}\Psi_*''\Psi_*'\Psi_*'' + g'(\Psi_*)\Psi_*', \tag{3.45}$$

which can be simplified to (3.7).

The exponential bounds (3.4)-(3.5) now follow from Lemma 3.7 and (3.36), using (3.6) and its derivatives to understand the derivatives of order two and higher for $\Psi_*^{(i)}(\tau)$ for $2 \le i \le 5$. The inequality (3.3) for Ψ_*' follows directly from (3.23) and the fact that $\partial_\xi\Phi_*$ is uniformly bounded. Finally, the inequalities (3.3) for γ_* follow from

$$1 > \sqrt{1 - \Psi_*'(\tau)^2} > \sqrt{1 - (1-\kappa)^2} = \sqrt{2\kappa - \kappa^2} > \sqrt{\kappa}. \tag{3.46}$$

□

3.2 Linear operators

In principle, most of the statements in Proposition 3.2 can be obtained by an appeal to standard Sturm-Liouville theory. We pursue a more explicit approach here in the hope that it can play a role towards generalizing the theory developed in this paper to non-scalar systems.

Our first two results highlight the fact that our coordinate transformation does not simply map $\mathcal{L}_{\mathrm{cmp}}$ and $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}$ onto the standard linear operators

$$\begin{aligned} \mathcal{L}_{\mathrm{tw}} y &= -c_*\partial_\xi y + \partial_{\xi\xi} y + g'(\Phi_*)y, \\ \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} z &= +c_*\partial_\xi z + \partial_{\xi\xi} z + g'(\Phi_*)z \end{aligned} \tag{3.47}$$

obtained by linearizing the travelling wave ODE (3.32) around Φ_*. Indeed, the correct operators to consider are given by

$$\begin{aligned} \mathcal{L}_{\mathrm{phys}} y &= \mathcal{L}_{\mathrm{tw}} y + \big(\partial_\xi\Phi_*\big)^2 \tfrac{\partial_{\xi\xi}\Phi_*}{1+(\partial_\xi\Phi_*)^2}\partial_\xi\Big[\tfrac{y}{\partial_\xi\Phi_*}\Big], \\ \mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}} z &= \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}} z - \tfrac{1}{\partial_\xi\Phi_*}\partial_\xi\Big[\big(\partial_\xi\Phi_*\big)^2\tfrac{\partial_{\xi\xi}\Phi_*}{1+(\partial_\xi\Phi_*)^2} z\Big]. \end{aligned} \tag{3.48}$$

Lemma 3.8. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for any $v \in H^2$ we have the identity*

$$\mathcal{L}_{\mathrm{cmp}} v \quad = \quad \gamma_*^{-1}\mathcal{S}_*\mathcal{L}_{\mathrm{phys}}\mathcal{S}_*^{-1}[\gamma_*^{-1}v]. \tag{3.49}$$

Proof. We write $y = \mathcal{S}_*^{-1}[\gamma_*^{-1} v]$, so that $\gamma_*^{-1} v = \mathcal{S}_* y$. Using Corollary 3.6 we get

$$\begin{aligned} \mathcal{S}_* \partial_\xi y &= \gamma_*^{-1} \partial_\tau [\gamma_*^{-1} v] \\ &= \gamma_*^{-4} \Psi_*' \Psi_*'' v + \gamma_*^{-2} v'. \end{aligned} \tag{3.50}$$

In particular, (3.7) allows us to write

$$c_* \mathcal{S}_* \partial_\xi y = c_* \gamma_*^{-2} v' + \gamma_*^{-5} \Psi_*' \Psi_*''' v + 4 \gamma_*^{-7} (\Psi_*')^2 (\Psi_*'')^2 v + \gamma_*^{-1} g'(\Psi_*)(1 - \gamma_*^2) v. \tag{3.51}$$

In addition, we compute

$$\begin{aligned} \mathcal{S}_* \partial_{\xi\xi} y &= \gamma_*^{-1} \partial_\tau \big[\mathcal{S}_* \partial_\xi y \big] \\ &= 4 \gamma_*^{-7} (\Psi_*')^2 (\Psi_*'')^2 v + \gamma_*^{-5} (\Psi_*'' \Psi_*'' + \Psi_*' \Psi_*''') v + \gamma_*^{-5} \Psi_*' \Psi_*'' v' \\ &\quad + 2 \gamma_*^{-5} \Psi_*' \Psi_*'' v' + \gamma_*^{-3} v''. \end{aligned} \tag{3.52}$$

We hence see

$$\begin{aligned} \gamma_*^{-1} \mathcal{S}_* \mathcal{L}_{\mathrm{tw}} y &= -c_* \gamma_*^{-3} v' + \gamma_*^{-6} (\Psi_*'')^2 v + 3 \gamma_*^{-6} \Psi_*' \Psi_*'' v' + \gamma_*^{-4} v'' + g'(\Psi_*) v \\ &= \mathcal{L}_{\mathrm{cmp}} v + \gamma_*^{-6} (\Psi_*'')^2 v - \gamma_*^{-6} \Psi_*' \Psi_*'' v'. \end{aligned} \tag{3.53}$$

We now write

$$\mathcal{L}_{\mathrm{phys}} y = \mathcal{L}_{\mathrm{tw}} y + \partial_\xi \Phi_* \tfrac{\partial_{\xi\xi} \Phi_*}{1 + (\partial_\xi \Phi_*)^2} \partial_\xi y - \tfrac{(\partial_{\xi\xi} \Phi_*)^2}{1 + (\partial_\xi \Phi_*)^2} y. \tag{3.54}$$

Exploiting the identities

$$\begin{aligned} \mathcal{S}_* \big[\partial_\xi \Phi_* \big] &= \gamma_*^{-1} \Psi_*', \\ \mathcal{S}_* \big[1 + (\partial_\xi \Phi_*)^2 \big] &= \gamma_*^{-2}, \\ \mathcal{S}_* [\partial_{\xi\xi} \Phi_*] &= \gamma_*^{-4} \Psi_*'' \end{aligned} \tag{3.55}$$

together with (3.50), we may compute

$$\begin{aligned} \gamma_*^{-1} \mathcal{S}_* \mathcal{L}_{\mathrm{phys}} y &= \gamma_*^{-1} \mathcal{S}_* \mathcal{L}_{\mathrm{tw}} y + \gamma_*^{-4} \Psi_*' \Psi_*'' \big[\gamma_*^{-4} \Psi_*' \Psi_*'' v + \gamma_*^{-2} v' \big] \\ &\quad - \gamma_*^{-7} \Psi_*'' \big[\gamma_*^{-1} v \big] \\ &= \gamma_*^{-1} \mathcal{S}_* \mathcal{L}_{\mathrm{tw}} y - \gamma_*^{-6} (\Psi_*'')^2 v + \gamma_*^{-6} \Psi_*' \Psi_*'' v' \\ &= \mathcal{L}_{\mathrm{cmp}} v, \end{aligned} \tag{3.56}$$

as desired. □

Lemma 3.9. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for any $w \in H^2$ we have the identity*

$$\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}} w = \mathcal{S}_* \mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}} \mathcal{S}_*^{-1} [\gamma_*^{-2} w]. \tag{3.57}$$

Proof. Pick $v \in H^2$. Applying Corollary 3.5 twice, we compute

$$\begin{aligned} \langle \mathcal{L}_{\mathrm{cmp}} v, w \rangle_{L^2} &= \langle \gamma_*^{-1} \mathcal{S}_* \mathcal{L}_{\mathrm{phys}} \mathcal{S}_*^{-1} \gamma_*^{-1} v, w \rangle_{L^2} \\ &= \langle \mathcal{S}_* \mathcal{L}_{\mathrm{phys}} \mathcal{S}_*^{-1} \gamma_*^{-1} v, \gamma_*^{-1} w \rangle_{L^2} \\ &= \langle \mathcal{L}_{\mathrm{phys}} \mathcal{S}_*^{-1} \gamma_*^{-1} v, \mathcal{S}_*^{-1} [\gamma_*^{-2} w] \rangle_{L^2} \\ &= \langle \mathcal{S}_*^{-1} \gamma_*^{-1} v, \mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}} \mathcal{S}_*^{-1} [\gamma_*^{-2} w] \rangle_{L^2} \\ &= \langle v, \mathcal{S}_* \mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}} \mathcal{S}_*^{-1} [\gamma_*^{-2} w] \rangle_{L^2}. \end{aligned} \tag{3.58}$$

The result now follows from (3.10). □

The explicit form (3.48) allows one to immediately verify that

$$\mathcal{L}_{\mathrm{phys}}\partial_\xi\Phi_* = \mathcal{L}_{\mathrm{tw}}\partial_\xi\Phi_* = 0. \tag{3.59}$$

Upon defining

$$\Phi_*^{\mathrm{adj;tw}}(\xi) \quad = \quad e^{-c_*\xi}\partial_\xi\Phi_*(\xi), \tag{3.60}$$

it is a standard exercise to verify that $\mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}\Phi_*^{\mathrm{adj;tw}} = 0$. We now construct a kernel element for $\mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}}$ by writing

$$\Phi_*^{\mathrm{adj;phys}}(\xi) \quad = \quad \sqrt{1+\big(\partial_\xi\Phi_*(\xi)\big)^2}\,\Phi_*^{\mathrm{adj;tw}}(\xi). \tag{3.61}$$

Lemma 3.10. *Suppose that (Hg) and (HΦ_*) are satisfied. Then we have*

$$\mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}}\Phi_*^{\mathrm{adj;phys}} = 0. \tag{3.62}$$

Proof. We first compute

$$\begin{aligned} \mathcal{L}_{\mathrm{tw}}^{\mathrm{adj}}\Phi_*^{\mathrm{adj;phys}} \quad &= \quad c_*\tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}} + \partial_\xi\Big[\tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Big]\Phi_*^{\mathrm{adj;tw}} \\ &\qquad +2\tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\partial_\xi\Phi_*^{\mathrm{adj;tw}}. \end{aligned} \tag{3.63}$$

Upon writing

$$\mathcal{I} \quad = \quad \tfrac{1}{\partial_\xi\Phi_*}\partial_\xi\Big[\big(\partial_\xi\Phi_*\big)^2\tfrac{\partial_{\xi\xi}\Phi_*}{1+(\partial_\xi\Phi_*)^2}\Phi_*^{\mathrm{adj;phys}}\Big], \tag{3.64}$$

we also compute

$$\begin{aligned} \mathcal{I} \quad &= \quad \tfrac{1}{\partial_\xi\Phi_*}\partial_\xi\Big[\big(\partial_\xi\Phi_*\big)^2\tfrac{\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}}\Big] \\ &= \quad \partial_\xi\Big[\tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}}\Big] + \tfrac{(\partial_{\xi\xi}\Phi_*)^2}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}} + \tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\partial_\xi\Phi_*^{\mathrm{adj;tw}}. \end{aligned} \tag{3.65}$$

In particular, we find

$$\mathcal{L}_{\mathrm{phys}}^{\mathrm{adj}}\Phi_*^{\mathrm{adj;phys}} \quad = \quad c_*\tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}} + \tfrac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\partial_\xi\Phi_*^{\mathrm{adj;tw}} - \tfrac{(\partial_{\xi\xi}\Phi_*)^2}{\sqrt{1+(\partial_\xi\Phi_*)^2}}\Phi_*^{\mathrm{adj;tw}}. \tag{3.66}$$

The result now follows from the computation

$$\begin{aligned} \partial_\xi\Phi_*\partial_\xi\Phi^{\mathrm{adj;tw}} \quad &= \quad \partial_\xi\Phi_*\partial_\xi\big[e^{-c_*\cdot}\partial_\xi\Phi_*\big] \\ &= \quad -c_*\partial_\xi\Phi_*\Phi^{\mathrm{adj;tw}} + \Phi^{\mathrm{adj;tw}}\partial_{\xi\xi}\Phi_*. \end{aligned} \tag{3.67}$$

□

Lemma 3.11. *Suppose that (Hg) and (HΦ_*) are satisfied and recall the definition (3.11). Then the identity*

$$\Psi_*^{\mathrm{adj}} = \Big[\int \gamma_*^{-1}(\tau)\Psi_*'(\tau)e^{-\int_0^\tau c_*\gamma_*(s)\,ds}\Psi_*'(\tau)\,d\tau\Big]^{-1}\gamma_*^2\mathcal{S}_*[\Phi_*^{\mathrm{adj;phys}}] \tag{3.68}$$

holds. In particular, we have $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}\Psi_*^{\mathrm{adj}} = 0$.

Proof. This follows directly from

$$\begin{array}{rcl} \mathcal{S}_*[\partial_\xi \Phi_*] & = & \gamma_*^{-1}\Psi_*', \\ \mathcal{S}_*[\sqrt{1+(\partial_\xi \Phi_*)^2}] & = & \gamma_*^{-1}, \end{array} \tag{3.69}$$

together with the computation

$$\begin{array}{rcl} \mathcal{S}_*[\xi \mapsto e^{-c_*\xi}](\tau) & = & e^{-c_*\xi_*(\tau)} \\ & = & e^{-c_* \int_0^\tau \gamma_*(s)\, ds}. \end{array} \tag{3.70}$$

Here we used $\xi_*(0) = 0$ and $\xi_*'(s) = \gamma_*(s)$. □

Lemma 3.12. *Suppose that (Hg) and (HΦ_*) are satisfied. Then we have*

$$\operatorname{Ker} \mathcal{L}_{\mathrm{phys}} = \operatorname{span}\{\Phi_*'\}. \tag{3.71}$$

Proof. A potential second, linearly independent kernel element can be written as $\alpha\partial_\xi\Phi_*$ for some function α. We hence compute

$$\mathcal{L}_{\mathrm{phys}}[\alpha\partial_\xi\Phi_*] \quad = \quad -c_*\partial_\xi\alpha\partial_\xi\Phi_* + \partial_{\xi\xi}\alpha\partial_\xi\Phi_* + 2\partial_\xi\alpha\partial_{\xi\xi}\Phi_* + \big(\partial_\xi\Phi_*\big)^2 \tfrac{\partial_{\xi\xi}\Phi_*}{1+(\partial_\xi\Phi_*)^2}\partial_\xi\alpha. \tag{3.72}$$

Setting the right hand side to zero, we find

$$\begin{array}{rcl} \partial_{\xi\xi}\alpha & = & \Big[c_* - 2\frac{\partial_{\xi\xi}\Phi_*}{\partial_\xi\Phi_*} - \frac{\partial_\xi\Phi_*\partial_{\xi\xi}\Phi_*}{1+(\partial_\xi\Phi_*)^2}\Big]\partial_\xi\alpha \\ & = & \partial_\xi\Big[c_*\xi - 2\ln[\partial_\xi\Phi_*] - \frac{1}{2}\ln\big[1+(\partial_\xi\Phi_*)^2\big]\Big]\partial_\xi\alpha. \end{array} \tag{3.73}$$

Choosing an integration constant $\alpha_* \in \mathbb{R}$, this can be solved to yield

$$\partial_\xi\alpha = \alpha_*(\partial_\xi\Phi_*)^{-2}e^{c_*\xi}\frac{1}{\sqrt{1+(\partial_\xi\Phi_*)^2}}. \tag{3.74}$$

For $\alpha_* \neq 0$ it is clear that one can choose $\kappa > 0$ in such a way that

$$|\alpha(\xi)| \geq \kappa e^{2\eta_+\xi + c_*\xi} \tag{3.75}$$

holds for all sufficiently large $\xi \gg 1$. This prevents $\alpha\partial_\xi\Phi_*$ from being bounded. □

Proof of Proposition 3.2. Viewing $\mathcal{L}_{\mathrm{cmp}}$, $\mathcal{L}_{\mathrm{phys}}$ and $\mathcal{L}_{\mathrm{tw}}$ as operators in $\mathcal{L}(H^2; L^2)$, we observe that their essential spectral are equal. Indeed, the differential equations arising in the $\xi \to \pm\infty$ and $\tau \to \pm\infty$ limits agree with each other. In particular, all these operators are Fredholm with index zero. The description of $\operatorname{Ker} \mathcal{L}_{\mathrm{cmp}}$ follows directly from (3.71) and the correspondence (3.49). The description of $\operatorname{Ker} \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}$ follows directly from Lemma 3.11 and the fact that

$$0 = \operatorname{ind}(\mathcal{L}_{\mathrm{cmp}}) = \dim\Big(\operatorname{Ker} \mathcal{L}_{\mathrm{cmp}}\Big) - \dim\Big(\operatorname{Ker} \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}\Big). \tag{3.76}$$

□

3.3 Integral transforms

Our goals here are to discuss the integral transforms introduced in (3.14) and to prove Corollary 3.3. In particular, the integral transforms can be used to solve two integral equations that appear naturally when linearizing the adaptive grid equations around the stretched wave Ψ_*.

Lemma 3.13. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists $K > 0$ so that the bound*

$$\|\mathcal{T}_* f\|_{L^2} \le K \|f\|_{L^2} \tag{3.77}$$

holds for any $f \in L^2$, while the bound

$$\left\|\mathcal{T}_*^{\mathrm{adj}} f\right\|_{H^2} \le K \|f\|_{H^2} \tag{3.78}$$

holds for all $f \in H^2$.

Proof. The estimate (3.77) follows from the uniform bound (3.3) together with the inclusion $\Psi_* \in H^2$ and the inequality

$$\left\|\int_- \gamma_*^{-3}\Psi_*'' f\right\|_\infty \le \left\|\gamma_*^{-3}\right\|_\infty \|\Psi_*''\|_{L^2} \|f\|_{L^2}. \tag{3.79}$$

Writing $w = \mathcal{T}_*^{\mathrm{adj}} f$, we note that

$$\begin{array}{lcl} w' & = & [\gamma_*^{-2} f]' - [\gamma_*^{-3}\Psi_*'']' \int_+ \gamma_*^{-1}\Psi_*' f + \gamma_*^{-4}\Psi_*''\Psi_*' f, \\ w'' & = & [\gamma_*^{-2} f]'' - [\gamma_*^{-3}\Psi_*'']'' \int_+ \gamma_*^{-1}\Psi_*' f + [\gamma_*^{-3}\Psi_*'']'\gamma_*^{-1}\Psi_*' f \\ & & \quad +\left[\gamma_*^{-4}\Psi_*''\Psi_*'\right]' f + \gamma_*^{-4}\Psi_*''\Psi_*' f'. \end{array} \tag{3.80}$$

Exploiting the inclusion $\Psi_* \in H^4$ and the bound

$$\left\|\int_+ \gamma_*^{-1}\Psi_*' f\right\|_\infty \le \left\|\gamma_*^{-1}\right\|_\infty \|\Psi_*'\|_{L^2} \|f\|_{L^2}, \tag{3.81}$$

we see that indeed $w \in H^2$ and that the estimate (3.78) holds. □

Lemma 3.14. *Consider any pair $(w, f) \in L^2 \times L^2$. Then the identity*

$$\gamma_*^2 w + \Psi_*' \int_- \Psi_*'' w = f \tag{3.82}$$

holds if and only if

$$w = \mathcal{T}_* f = \gamma_*^{-2} f - \gamma_*^{-1}\Psi_*' \int_- \gamma_*^{-3}\Psi_*'' f. \tag{3.83}$$

Proof. Assuming (3.82) holds, we write

$$X = \int_- \Psi_*'' w \tag{3.84}$$

and compute

$$\begin{array}{lcl} X' & = & \Psi_*'' w \\ & = & \gamma_*^{-2}\Psi_*'' f - \gamma_*^{-2}\Psi_*'\Psi_*'' X. \end{array} \tag{3.85}$$

Recalling $\gamma_*' = -\gamma_*^{-1}\Psi_*'\Psi_*''$, we see that

$$[\gamma_*^{-1} X]' = \gamma_*^{-3}\Psi_*'' f. \tag{3.86}$$

Using the fact that $X(\tau) \to 0$ as $\tau \to -\infty$, this implies

$$X = \gamma_* \int_- \gamma_*^{-3}\Psi_*'' f \tag{3.87}$$

and hence

$$\gamma_*^2 w = f - \Psi_*' X = f - \gamma_* \Psi_*' \int_- \gamma_*^{-3} \Psi_*'' f. \tag{3.88}$$

On the other hand, assuming (3.83), we compute

$$\begin{aligned} \int_- \Psi_*'' w &= \int_- \gamma_*^{-2} \Psi_*'' f - \int_- \Big[\gamma_*^{-1} \Psi_*'' \Psi_*' \int_- \gamma_*^{-3} \Psi_*'' f \Big] \\ &= \int_- \gamma_*^{-2} \Psi_*'' f + \int_- \Big[\gamma_*' \int_- \gamma_*^{-3} \Psi_*'' f \Big] \\ &= \int_- \gamma_*^{-2} \Psi_*'' f + \gamma_* \int_- \gamma_*^{-3} \Psi_*'' f - \int_- \gamma_* \gamma_*^{-3} \Psi_*'' f \\ &= \gamma_* \int_- \gamma_*^{-3} \Psi_*'' f. \end{aligned} \tag{3.89}$$

Multiplying by Ψ_*', we hence see

$$\Psi_*' \int_- \Psi_*'' w = \gamma_* \Psi_*' \int_- \gamma_*^{-3} \Psi_*'' f = f - \gamma_*^2 w, \tag{3.90}$$

which yields (3.82). □

Lemma 3.15. *Consider any pair $(w, f) \in H^2 \times H^2$. Then the identity*

$$\gamma_*^2 w + \Psi_*'' \int_+ \Psi_*' w = f \tag{3.91}$$

holds if and only if

$$w = \mathcal{T}_*^{\mathrm{adj}} f = \gamma_*^{-2} \big[f - \gamma_*^{-1} \Psi_*'' \int_+ \gamma_*^{-1} \Psi_*' f \big]. \tag{3.92}$$

Proof. Assuming (3.91) holds, we write

$$Y = \int_+ \Psi_*' w \tag{3.93}$$

and compute

$$\begin{aligned} Y' &= -\Psi_*' w \\ &= -\Psi_*' \gamma_*^{-2} f + \gamma_*^{-2} \Psi_*' \Psi_*'' Y. \end{aligned} \tag{3.94}$$

In particular, we see that

$$[\gamma_* Y]' = -\gamma_*^{-1} \Psi_*' f. \tag{3.95}$$

We hence find

$$Y = \gamma_*^{-1} \int_+ \gamma_*^{-1} \Psi_*' f, \tag{3.96}$$

which yields

$$w = \gamma_*^{-2} \big[f - \Psi_*'' Y \big] = \gamma_*^{-2} \big[f - \gamma_*^{-1} \Psi_*'' \int_+ \gamma_*^{-1} \Psi_*' f \big]. \tag{3.97}$$

On the other hand, assuming (3.92) we compute

$$\begin{array}{rcl} \int_+ \Psi_*' w & = & \int_+ \gamma_*^{-2}\Psi_*' f - \int_+ \Big[\gamma_*^{-3}\Psi_*'\Psi_*'' \int_+ \gamma_*^{-1}\Psi_*' f\Big] \\ & = & \int_+ \gamma_*^{-2}\Psi_*' f - \int_+ \Big[[\gamma_*^{-1}]' \int_+ \gamma_*^{-1}\Psi_*' f\Big] \\ & = & \int_+ \gamma_*^{-2}\Psi_*' f + [\gamma_*^{-1}] \int_+ \gamma_*^{-1}\Psi_*' f - \int_+ \gamma_*^{-1}\gamma_*^{-1}\Psi_*' f \\ & = & \gamma_*^{-1} \int_+ \gamma_*^{-1}\Psi_*' f. \end{array} \tag{3.98}$$

Multiplying by Ψ_*'', we find

$$\Psi_*'' \int_+ \Psi_*' w = \gamma_*^{-1} \int_+ \gamma_*^{-1}\Psi_*' f = f - \gamma_*^2 w, \tag{3.99}$$

which yields (3.91). □

Proof of Corollary 3.3. We introduce the notation

$$\alpha_c[f] = \langle \Psi_*^{\mathrm{adj}}, f\rangle_{L^2} \tag{3.100}$$

and note that the normalization (3.16) implies that $\alpha_c[\mathcal{T}_*\Psi_*'] = 1$. In particular, the operator

$$\pi_c f = [\mathcal{T}_*\Psi_*']\alpha_c f \tag{3.101}$$

is a projection on L^2. Writing $\pi = I - \pi_c$, this yields the splitting $L^2 = R \oplus R_c$ with

$$R = \pi(L^2) = \mathcal{L}_{\mathrm{cmp}}(H^2), \qquad R_c = \pi_c(L^2). \tag{3.102}$$

Upon choosing a splitting

$$H^2 = \mathrm{span}\{\Psi_*'\} \oplus K_c, \tag{3.103}$$

we note that the linear map

$$\mathcal{L}_{\mathrm{cmp}} : K_c \to R \tag{3.104}$$

is invertible, which implies that the perturbed operators

$$\big[\mathcal{L}_{\mathrm{cmp}} - \delta\pi\mathcal{T}_*\big] : K_c \to R \tag{3.105}$$

are also invertible for small $\delta > 0$. For any $f \in R$, we introduce the function

$$L_\delta[f] = \big[\mathcal{L}_{\mathrm{cmp}} - \delta\pi\mathcal{T}_*\big]^{-1} f - \Psi_*'\alpha_c\Big[\mathcal{T}_*\big[\mathcal{L}_{\mathrm{cmp}} - \delta\pi\mathcal{T}_*\big]^{-1} f\Big] \tag{3.106}$$

and observe that

$$\begin{array}{rcl} [\mathcal{L}_{\mathrm{cmp}} - \delta\mathcal{T}_*]L_\delta f & = & f - \delta\mathcal{T}_*\Psi_*'\alpha_c[\mathcal{L}_{\mathrm{cmp}} - \delta\pi\mathcal{T}_*]^{-1} f + \delta\mathcal{T}_*\Psi_*'\alpha_c\Big[\mathcal{T}_*\big[\mathcal{L}_{\mathrm{cmp}} - \delta\pi\mathcal{T}_*\big]^{-1} f\Big] \\ & = & f. \end{array} \tag{3.107}$$

For any $f \in L^2$, this allows us to compute

$$\begin{array}{rcl} [\mathcal{L}_{\mathrm{cmp}} - \delta\mathcal{T}_*]\Big[-\delta^{-1}\Psi_*'\alpha_c[f] + L_\delta\pi[f]\Big] & = & \mathcal{T}_*\Psi_*'\alpha_c[f] + \pi[f] \\ & = & f, \end{array} \tag{3.108}$$

which provides an inverse for $\mathcal{L}_{\mathrm{cmp}} - \delta\mathcal{T}_*$. An analogous procedure can be used to obtain the result for $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}$. □

4 Preliminary identities

In this section we provide some basic identities concerning discrete differentiation and integration. In addition, we introduce notation for the gridpoint spacing functions $\sqrt{1-(\partial^{\pm}U)^2}$ and derive several useful identities for their discrete derivatives. This allows us to obtain expressions that are uniform in h for the derivative operator $\mathcal{F}^{\diamond\diamond_0}$ defined in (2.14) and the terms appearing in (2.16)-(2.17). This will turn out to be very convenient when performing estimates.

4.1 Discrete calculus

For any sequence $a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$ we introduce the notation $T^{\pm}a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$ to refer to the translated sequences

$$[T^{+}a]_{jh} = a_{(j+1)h}, \qquad [T^{-}a]_{jh} = a_{(j-1)h}. \tag{4.1}$$

In addition, we introduce the notation $S^{\pm}a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$ and $P^{\pm}a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$ to refer to the sum and product sequences

$$S^{\pm}a = a + T^{\pm}a, \qquad P^{\pm}a = aT^{\pm}a \tag{4.2}$$

Writing

$$[\partial^{0}\partial a]_{jh} = \frac{1}{2h}[\partial^{+}a - \partial^{-}a]_{jh} = \frac{1}{2h^{2}}\big[a_{(j+1)h} + a_{(j-1)h} - 2a_{jh}\big], \tag{4.3}$$

it is not hard to verify the basic identities

$$\begin{array}{rcl} \partial^{0}\partial a &=& \frac{1}{2}\partial^{+}\partial^{-}a, \\ \partial^{+}\partial^{0}a &=& S^{+}[\partial^{0}\partial a]. \end{array} \tag{4.4}$$

Consider two sequences $a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$ and $b \in \ell^{\infty}(h\mathbb{Z};\mathbb{R})$. One may easily compute

$$\begin{array}{rcl} \partial^{+}[ab] &=& \partial^{+}aT^{+}b + a\partial^{+}b \\ \partial^{0}[ab] &=& \partial^{0}aT^{+}b + T^{-}a\partial^{0}b, \\ \partial^{-}[ab] &=& [\partial^{-}a]b + \Big[T^{-}a\Big]\partial^{-}b, \end{array} \tag{4.5}$$

which yields

$$\begin{array}{rcl} \partial^{0}\partial[ab] &=& \frac{1}{2h}\Big[\partial^{+}[ab] - \partial^{-}[ab]\Big] \\ &=& (\partial^{0}\partial a)b + \frac{1}{2}\partial^{+}a\partial^{+}b + \frac{1}{2}\partial^{-}a\partial^{-}b + a\partial^{0}\partial b. \end{array} \tag{4.6}$$

In addition, if $b_{jh} \neq 0$ for all $j \in \mathbb{Z}$ then we have

$$\partial^{+}\big[\frac{a}{b}\big] = \frac{b\partial^{+}a - a\partial^{+}b}{P^{+}b}. \tag{4.7}$$

We often use the symmetrized versions

$$\begin{array}{rcl} \partial^{+}[ab] &=& \frac{1}{2}\partial^{+}aS^{+}b + \frac{1}{2}S^{+}a\partial^{+}b, \\ \partial^{+}\big[\frac{a}{b}\big] &=& \frac{S^{+}b\partial_{h}^{+}a}{2P^{+}b} - \frac{S^{+}a\partial_{h}^{+}b}{2P^{+}b}. \end{array} \tag{4.8}$$

For any sequence $a \in \ell^{1}(h\mathbb{Z};\mathbb{R})$, we define two new sequences

$$\sum_{-;h} a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R}), \qquad \sum_{+;h} a \in \ell^{\infty}(h\mathbb{Z};\mathbb{R}) \tag{4.9}$$

by writing

$$\begin{array}{rcl}\big[\sum_{-;h} a\big]_{jh} & = & \sum_{k>0} a_{(j-k)h}, \\ \big[\sum_{+;h} a\big]_{jh} & = & \sum_{k>0} a_{(j+k)h}.\end{array} \tag{4.10}$$

Using the fact that $\lim_{k\to\pm\infty} a_{kh} = 0$, one may readily verify the identities

$$\begin{array}{rcl}\partial^+\big[h\sum_{-;h} a\big]_{jh} & = & a_{jh}, \\ \partial^-\big[h\sum_{+;h} a\big]_{jh} & = & -a_{jh}.\end{array} \tag{4.11}$$

Finally, consider two sequences $a \in \ell^2(h\mathbb{Z};\mathbb{R})$ and $b \in \ell^2(h\mathbb{Z};\mathbb{R})$. Since $ab \in \ell^1(h\mathbb{Z};\mathbb{R})$, we may exploit (4.11) together with the identity

$$\partial^+\big[aT^-b\big] = [\partial^+a]b + a\partial^+\big[T^-b\big] = b\partial^+a + a\partial^-b \tag{4.12}$$

to obtain the discrete summation-by-parts formula

$$h\sum_{-;h} b\partial^+a = aT^-b - h\sum_{-;h} a\partial^-b. \tag{4.13}$$

In addition, we see that

$$h\partial^+\big[aT^-b\big] = b(T^+a - a) + a(b - T^-b) = bS^+a - aS^-b, \tag{4.14}$$

which gives a second summation-by-parts formula

$$h\sum_{-;h} bS^+a = haT^-b + h\sum_{-;h} aS^-b. \tag{4.15}$$

4.2 Gridpoint spacing

We first define

$$\begin{array}{rcl} r_U^+ & = & \sqrt{1-(\partial^+U)^2}, \\ r_U^- & = & \sqrt{1-(\partial^-U)^2}, \\ r_U^0 & = & \frac{1}{2}\sqrt{1-(\partial^+U)^2} + \frac{1}{2}\sqrt{1-(\partial^-U)^2} \\ & = & \frac{1}{2}[r_U^+ + r_U^-]. \end{array} \tag{4.16}$$

Notice that

$$T^+r_U^- = r_U^+, \qquad T^-r_U^+ = r_U^-. \tag{4.17}$$

In particular, we see that

$$\frac{r_U^+ - r_U^-}{h} = \partial^-r_U^+ = \partial^+r_U^-. \tag{4.18}$$

Lemma 4.1. *Consider any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ for which $\|\partial^+U\|_\infty < 1$. Then we have the identities*

$$\begin{array}{rcl} \partial^+r_U^- & = & -2[r_U^0]^{-1}\partial^0U\partial^0\partial U, \\ \partial^+r_U^0 & = & -S^+\Big[[r_U^0]^{-1}\partial^0U\partial^0\partial U\Big]. \end{array} \tag{4.19}$$

Proof. We compute

$$\begin{aligned} r_U^+ - r_U^- &= \sqrt{1-(\partial^+ U)^2} - \sqrt{1-(\partial^- U)^2} \\ &= \frac{(\partial^- U)^2 - (\partial^+ U)^2}{r^+ + r^-} \\ &= \frac{-(\partial^+ U - \partial^- U)(\partial^+ U + \partial^- U)}{2 r_U^0} \\ &= \frac{-2h\partial^0 \partial U (2\partial^0 U)}{2 r_U^0}, \end{aligned} \tag{4.20}$$

from which the first identity follows. In addition, we see that

$$\begin{aligned} h\partial^+[r_U^0] &= T^+ r_U^0 - r_U^0 \\ &= \tfrac{1}{2}\Big[T^+ r_U^+ + T^+ r_U^- - r_U^+ - r_U^-\Big] \\ &= \tfrac{1}{2}\Big[T^+ r_U^+ + r_U^+ - T^+ r_U^- - r_U^-\Big] \\ &= \tfrac{1}{2}T^+\big[r_U^+ - r_U^-\big] + \tfrac{1}{2}\big[r_U^+ - r_U^-\big] \\ &= \tfrac{1}{2}S^+\big[r_U^+ - r_U^-\big]. \end{aligned} \tag{4.21}$$

Using (4.18) we conclude $\partial^+[r_U^0] = \frac{1}{2}S^+[\partial^+ r_U^-]$, which yields the second identity. □

In order to break the directional biases appearing in the discrete derivatives in (4.16), it is convenient to define the sequence

$$\gamma_U = \sqrt{1-(\partial^0 U)^2}. \tag{4.22}$$

A short computation shows that

$$\begin{aligned} \gamma_{U^{(2)}} - \gamma_{U^{(1)}} &= \sqrt{1-(\partial^0 U^{(2)})^2} - \sqrt{1-(\partial^0 U^{(1)})^2} \\ &= -\frac{(\partial^0 U^{(2)})^2 - (\partial^0 U^{(1)})^2}{\sqrt{1-(\partial^0 U^{(1)})^2} + \sqrt{1-(\partial^0 U^{(2)})^2}} \\ &= -[\gamma_{U^{(1)}} + \gamma_{U^{(2)}}]^{-1}(\partial^0 U^{(1)} + \partial^0 U^{(2)})(\partial^0 U^{(2)} - \partial^0 U^{(1)}), \end{aligned} \tag{4.23}$$

which allows us to readily compute several useful discrete derivatives.

Lemma 4.2. *Consider any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ for which $\|\partial^+ U\|_\infty < 1$. Then we have the identities*

$$\begin{aligned} \partial^+[\gamma_U^{-4}] &= \frac{S^+[\partial^0 U]S^+[\partial^0\partial U]}{P^+[\gamma_U^2]}\frac{S^+[\gamma_U^2]}{P^+[\gamma_U^2]}, \\ \partial^+[\gamma_U^{-2}] &= \frac{S^+[\partial^0 U]S^+[\partial^0\partial U]}{P^+[\gamma_U^2]}, \\ \partial^+[\gamma_U^{-1}] &= \frac{S^+[\partial^0 U]S^+[\partial^0\partial U]}{S^+[\gamma_U]P^+[\gamma_U]}, \\ \partial^+[\gamma_U] &= -\frac{S^+[\partial^0 U]S^+[\partial^0\partial U]}{S^+\gamma_U}, \\ \partial^+[\gamma_U^2] &= -S^+[\partial^0 U]S^+[\partial^0\partial U]. \end{aligned} \tag{4.24}$$

Proof. Writing $U^{(2)} = T^+ U$ and $U^{(1)} = U$, we use (4.23) to compute

$$h\partial^+\gamma_U = -[S^+\gamma_U]^{-1}S^+[\partial^0 U]h\partial^+\partial^0 U \tag{4.25}$$

which yields the desired identity for $\partial^+\gamma_U$ upon remembering (4.4). We can now use the general identities

$$\begin{aligned} \partial^+[a^{-1}] &= -[P^+ a]^{-1}\partial^+ a, \\ \partial^+[a^2] &= \partial^+ a T^+ a + a\partial^+ a = \partial^+ a S^+ a, \end{aligned} \tag{4.26}$$

together with

$$S^+[a^{-1}] = \frac{S^+ a}{P^+ a} \tag{4.27}$$

to obtain the remaining expressions. □

4.3 Discrete derivatives

The definitions (4.16) allow us to rewrite the discrete first derivatives in (2.14) as

$$\begin{array}{rcl} \mathcal{F}^{\diamond_\pm}(U) & = & \frac{\partial^\pm U}{r_U^\pm}, \\ \mathcal{F}^{\diamond_0}(U) & = & \frac{\partial^0 U}{r_U^0}. \end{array} \tag{4.28}$$

This means that the identities (4.19) can be restated in the form

$$\begin{array}{rcl} \partial^+ r_U^- & = & -2\mathcal{F}^{\diamond_0}(U)\partial^0\partial U, \\ \partial^+ r_U^0 & = & -S^+\Big[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\Big]. \end{array} \tag{4.29}$$

We also introduce the second discrete derivatives

$$\begin{array}{rcl} \mathcal{F}^{\diamond_0;+}(U) & = & \partial^+\mathcal{F}^{\diamond_0}(U), \\ \mathcal{F}^{\diamond_-;+}(U) & = & \partial^+\mathcal{F}^{\diamond_-}(U), \\ \mathcal{F}^{\diamond_+;+}(U) & = & \partial^+\mathcal{F}^{\diamond_+}(U). \end{array} \tag{4.30}$$

Using the identities

$$T^+\mathcal{F}^{\diamond_-}(U) = \mathcal{F}^{\diamond_+}(U), \qquad T^-\mathcal{F}^{\diamond_+}(U) = \mathcal{F}^{\diamond_-}(U), \tag{4.31}$$

we readily see that

$$\frac{\mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_-}(U)}{h} = \mathcal{F}^{\diamond_-;+}(U), \tag{4.32}$$

which allows us to write

$$\mathcal{F}^{\diamond\diamond_0}(U) \quad = \quad \frac{1}{2r_U^0}\mathcal{F}^{\diamond_-;+}(U) \tag{4.33}$$

for the function $\mathcal{F}^{\diamond\diamond_0}$ appearing in (2.14). Finally, we introduce the third discrete derivative

$$\mathcal{F}^{\diamond\diamond_0;+}(U) \quad = \quad \partial^+[\mathcal{F}^{\diamond\diamond_0}(U)]. \tag{4.34}$$

Lemma 4.3. *Consider any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ for which $\|\partial^+ U\|_\infty < 1$. Then we have the identities*

$$\begin{array}{rcl} \mathcal{F}^{\diamond_-;+}(U) & = & \frac{2}{r_U^+}\big[1 + \mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big]\partial^0\partial U, \\ \mathcal{F}^{\diamond_0;+}(U) & = & \frac{1}{T^+ r_U^0}\Big[1 + \mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_0}(U)\Big][\partial^0\partial U] \\ & & \quad + \frac{1}{T^+ r_U^0}\Big[1 + \mathcal{F}^{\diamond_0}(U)T^+[\mathcal{F}^{\diamond_0}(U)]\Big]T^+[\partial^0\partial U]. \end{array} \tag{4.35}$$

Proof. Using (4.4), (4.7) and (4.29) we compute

$$\begin{array}{rcl} \mathcal{F}^{\diamond_-;+}(U) & = & [P^+ r_U^-]^{-1}\Big[r_U^-\partial^+\partial^- U - \partial^- U\partial^+ r_U^-\Big] \\ & = & [r_U^- r_U^+]^{-1}\Big[2r_U^-\partial^0\partial U + 2\partial^- U\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\Big] \\ & = & [r_U^+]^{-1}\Big[2\partial^0\partial U + 2\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\Big], \end{array} \tag{4.36}$$

together with

$$\begin{array}{lcl}\mathcal{F}^{\diamond_0;+}(U) & = & [P^+ r_U^0]^{-1}\big[r_U^0 \partial^+ \partial^0 U - \partial^0 U \partial^+ r_U^0\big] \\ & = & [r_U^0 T^+ r_U^0]^{-1}\Big[r_U^0 S^+ \partial^0 \partial U + \partial^0 U S^+\big[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\big]\Big] \\ & = & [T^+ r_U^0]^{-1}\Big[S^+\partial^0\partial U + \mathcal{F}^{\diamond_0}(U) S^+\big[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\big]\Big],\end{array} \tag{4.37}$$

from which the desired identities follow. □

In order to restate our result concerning $\mathcal{F}^{\diamond\diamond_0;+}$, we introduce the expressions

$$\begin{array}{lcl}\mathcal{I}_{0s}^{\diamond\diamond_0;+}(U) & = & \frac{1}{r_U^+ P^+ r_U^0}\mathcal{F}^{\diamond_0}(U)T^+\Big[1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\Big] \\ & & \quad +\frac{2}{r_U^0 r_U^+ r_U^+}T^+[\mathcal{F}^{\diamond_0}(U)]\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big) \\ & & \quad +\frac{1}{r_U^+ P^+ r_U^0}\mathcal{F}^{\diamond_-}(U)\big(1+\mathcal{F}^{\diamond_0}(U)^2\big), \\ \mathcal{I}_{ss}^{\diamond\diamond_0;+}(U) & = & \frac{2r_U^0 + T^+ r_U^+}{P^+ r_U^0 P^+ r_U^+}T^+\Big[\mathcal{F}^{\diamond_0}(U)\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big)\Big] \\ & & \quad +\frac{1}{r_U^+ P^+ r_U^0}\mathcal{F}^{\diamond_-}(U)\Big(1+\mathcal{F}^{\diamond_0}(U)T^+[\mathcal{F}^{\diamond_0}(U)]\Big), \\ \mathcal{I}_{+}^{\diamond\diamond_0;+}(U) & = & \frac{1}{r_U^0 r_U^+}\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big).\end{array} \tag{4.38}$$

These allow us to define the two components

$$\begin{array}{lcl}\mathcal{F}_a^{\diamond\diamond_0;+}(U) & = & \mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+[\partial^0\partial U], \\ \mathcal{F}_b^{\diamond\diamond_0;+}(U) & = & \mathcal{I}_{0s}^{\diamond\diamond_0;+}(U)\partial^0\partial U T^+[\partial^0\partial U] + \mathcal{I}_{ss}^{\diamond\diamond_0;+}(U)T^+[\partial^0\partial U]T^+[\partial^0\partial U].\end{array} \tag{4.39}$$

Lemma 4.4. *Consider any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ for which $\|\partial^+ U\|_\infty < 1$. Then we have the identity*

$$\mathcal{F}^{\diamond\diamond_0;+}(U) \quad = \quad \mathcal{F}_a^{\diamond\diamond_0;+}(U) + \mathcal{F}_b^{\diamond\diamond_0;+}(U). \tag{4.40}$$

Proof. Using (4.33) we may compute

$$\begin{array}{lcl}\partial^+\mathcal{F}^{\diamond\diamond_0}(U) & = & \partial^+\Big[\frac{1}{r_U^0 r_U^+}\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big)\partial^0\partial U\Big] \\ & = & \mathcal{I}_A + \mathcal{I}_B + \mathcal{I}_C,\end{array} \tag{4.41}$$

in which

$$\begin{array}{lcl}\mathcal{I}_A & = & \partial^+[\frac{1}{r_U^0 r_U^+}]T^+\Big[\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big)\partial^0\partial U\Big], \\ \mathcal{I}_B & = & \frac{1}{r_U^0 r_U^+}\big(\mathcal{F}^{\diamond_-;+}(U)T^+\mathcal{F}^{\diamond_0}(U) + \mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0;+}(U)\big)T^+\big[\partial^0\partial U\big], \\ \mathcal{I}_C & = & \frac{1}{r_U^0 r_U^+}\big(1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big)\partial^+[\partial^0\partial U].\end{array} \tag{4.42}$$

We immediately see that

$$\mathcal{I}_C = \mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+[\partial^0\partial U]. \tag{4.43}$$

In addition, we may use (4.7) and (4.29) to compute

$$\begin{array}{lcl}\partial^+\big[\frac{1}{r_U^0 r_U^+}\big] & = & -[P^+ r_U^0 P^+ r_U^+]^{-1}\Big[\partial^+ r_U^0 T^+ r^+ + r_U^0 \partial^+ r^+\Big] \\ & = & [P^+ r_U^0 P^+ r_U^+]^{-1}\Big[S^+[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U]T^+ r_U^+ + r_U^0 T^+[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U]\Big] \\ & = & \Big[\frac{1}{r_U^+ P^+ r_U^0}\mathcal{F}^{\diamond_0}(U)\Big]\partial^0\partial U + \Big[\frac{2r_U^0 + T^+ r_U^+}{P^+ r_U^0 P^+ r_U^+}\Big]T^+\big[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\big].\end{array} \tag{4.44}$$

Finally, Lemma 4.3 allows us to expand

$$\begin{aligned}\mathcal{I}_B &= 2[r_U^0 r_U^+ r_U^+]^{-1}\big[1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big][\partial^0\partial U]T^+\big[\mathcal{F}^{\diamond_0}(U)\partial^0\partial U\big] \\ &\quad +[r_U^+ P^+ r_U^0]^{-1}\big[1+\mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_0}(U)\big]\mathcal{F}^{\diamond_-}(U)[\partial^0\partial U]T^+[\partial^0\partial U] \\ &\quad +[r_U^+ P^+ r_U^0]^{-1}\big[1+\mathcal{F}^{\diamond_0}(U)T^+\mathcal{F}^{\diamond_0}(U)\big]\mathcal{F}^{\diamond_-}(U)T^+[\partial^0\partial U]T^+[\partial^0\partial U].\end{aligned} \tag{4.45}$$

The splitting (4.40) can now be read off directly. □

5 Sampling techniques

In order to link the continuum theory developed in §3 to the discrete setting of the adaptive grid, we often need to extract sequences from continuous functions and relate discrete derivatives to their continuous counterparts. In this section we collect several tools that will be useful for these procedures.

For any $h > 0$, we first introduce the Hilbert space ℓ_h^2 that is equal to $\ell^2(h\mathbb{Z};\mathbb{R})$ as a set, but is equipped with the rescaled inner product

$$\langle V, W\rangle_{\ell_h^2} = h\sum_{j\in\mathbb{Z}} V_{jh}W_{jh} \tag{5.1}$$

that compensates for the gridpoint density. In particular, for $V \in \ell_h^2$ we have

$$\|V\|_{\ell_h^2}^2 = h\sum_{j\in\mathbb{Z}} V_{jh}^2. \tag{5.2}$$

For convenience, we also introduce the alternative notation

$$\ell_h^\infty \quad = \quad \{V : h\mathbb{Z}\to\mathbb{R} \text{ for which } \|V\|_{\ell_h^\infty} := \sup_{j\in\mathbb{Z}}|V_{hj}| < \infty\} \tag{5.3}$$

for the usual set $\ell^\infty(h\mathbb{Z};\mathbb{R})$ with the supremum norm. For any $V \in \ell_h^2$, it is clear that also $V \in \ell_h^\infty$ and that we have the bound

$$\|V\|_{\ell_h^\infty} \le h^{-1/2}\|V\|_{\ell_h^2}. \tag{5.4}$$

In order to reduce the length of our expressions, we introduce the higher order norms

$$\begin{aligned}\|V\|_{\ell_h^{2;1}} &= \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2}, \\ \|V\|_{\ell_h^{2;2}} &= \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2}, \\ \|V\|_{\ell_h^{2;3}} &= \|V\|_{\ell_h^2} + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+\partial^+ V\|_{\ell_h^2},\end{aligned} \tag{5.5}$$

together with

$$\begin{aligned}\|V\|_{\ell_h^{\infty;1}} &= \|V\|_{\ell_h^\infty} + \|\partial^+ V\|_{\ell_h^\infty}, \\ \|V\|_{\ell_h^{\infty;2}} &= \|V\|_{\ell_h^\infty} + \|\partial^+ V\|_{\ell_h^\infty} + \|\partial^+\partial^+ V\|_{\ell_h^\infty}.\end{aligned} \tag{5.6}$$

We caution the reader that for fixed $h > 0$, these norms are equivalent to the norms on ℓ_h^2 respectively ℓ_h^∞. However, they do allow us to conveniently formulate estimates that are uniform in $h > 0$.

For any $f \in L^2$ and $h > 0$, we formally write

$$[\partial_h^+ f](\tau) = h^{-1}[f(\tau+h) - f(\tau)], \qquad [\partial_h^- f](\tau) = h^{-1}[f(\tau) - f(\tau-h)], \tag{5.7}$$

which obviously satisfy $\partial_h^{\pm} f \in L^2$. In a similar fashion, for any $f \in L^1$ and $h > 0$ we formally write

$$[\sum_{-;h} f](\tau) = \sum_{k>0} f(\tau - kh), \qquad [\sum_{+;h} f](\tau) = \sum_{k>0} f(\tau + kh), \tag{5.8}$$

noting that these functions are in L^1_{loc}.

In §5.1 we obtain several useful results that relate the ℓ_h^q-norms of sequences $v(h\mathbb{Z} + \vartheta)$ sampled from a function v back to L^q-norms of v and its derivatives. In §5.2 we introduce exponentially weighted norms on L^2 and discuss their impact on the summed functions (5.8). Finally, in §5.3 we discuss sequences of differences (5.7) and sums (5.8) for which $h \downarrow 0$. Upon taking weak limits, it is possible to recover the usual continuous derivatives and integrals.

5.1 Sampling estimates

For any bounded continuous function f, any $\vartheta \in \mathbb{R}$ and any $h > 0$, we write $\mathrm{ev}_\vartheta f \in \ell_h^\infty$ for the sequence

$$[\mathrm{ev}_\vartheta f]_{jh} = f(\vartheta + jh). \tag{5.9}$$

When the context is clear, we often simply write f to refer to the sampled sequence $\mathrm{ev}_0 f$. In this subsection we explore the relation between such sampled sequences and the original function.

Lemma 5.1. *Pick $q \in \{2, \infty\}$ and consider any $u \in W^{1,q}$. Then the estimates*

$$\left\| \partial_h^{\pm} u \right\|_{\ell_h^q} \leq \|u'\|_{L^q} \tag{5.10}$$

hold for any $h > 0$. If $q = 2$, then we also have

$$\left\| \partial_h^{\pm} u \right\|_{\ell_h^\infty} \leq h^{-1/2} \|u'\|_{L^2} \tag{5.11}$$

for all $h > 0$.

Proof. For $q = \infty$ the statement is immediate, so assume that $q = 2$. We may then compute

$$\begin{array}{lcl} \left\| \partial_h^{+} u \right\|_{\ell_h^2}^2 & = & h \sum_{j \in \mathbb{Z}} \frac{\big(u\big((j+1)h\big) - u(jh)\big)^2}{h^2} \\ & = & h \sum_{j \in \mathbb{Z}} h^{-2} \big[\int_0^h u'(jh + s)\, ds \big]^2 \\ & \leq & h \sum_{j \in \mathbb{Z}} h^{-2} h \int_0^h u'(jh + s)^2 \, ds \\ & = & \sum_{j \in \mathbb{Z}} \int_0^h u'(jh + s)^2 \, ds \\ & = & \|u'\|_{L^2}^2 . \end{array} \tag{5.12}$$

In addition, the identity (5.11) follows directly from (5.4). ☐

Lemma 5.2. *For any $u \in H^1$ and any $h > 0$ we have*

$$\|u\|_{\ell_h^2} \quad \leq \quad (2 + h) \|u\|_{H^1} . \tag{5.13}$$

Proof. We compute

$$\begin{array}{lcl} \|u\|_{\ell_h^2}^2 & = & h \sum_{j \in \mathbb{Z}} u(jh)^2 \\ & = & \sum_{j \in \mathbb{Z}} \int_0^h u(jh)^2 \, ds \\ & = & \sum_{j \in \mathbb{Z}} \int_0^h \Big[u(jh + s) - \int_0^s u'(jh + \sigma)\, d\sigma \Big]^2 \, ds. \end{array} \tag{5.14}$$

Using the standard bound $(a-b)^2 \le 2(a^2+b^2)$ we hence obtain

$$\begin{aligned}
\|u\|_{\ell_h^2}^2 &\le 2\sum_{j\in\mathbb{Z}} \int_0^h u(jh+s)^2\,ds \\
&\quad +2\sum_{j\in\mathbb{Z}} \int_0^h \Big[\int_0^s u'(jh+\sigma)^2\,d\sigma\Big]^2 ds \\
&\le 2\|u\|_{L^2}^2 + 2\sum_{j\in\mathbb{Z}} \int_0^h s\int_0^s u'(jh+\sigma)^2\,d\sigma\,ds \\
&= 2\|u\|_{L^2}^2 + 2\sum_{j\in\mathbb{Z}} \int_0^h u'(jh+\sigma)^2 \int_\sigma^h s\,ds\,d\sigma \\
&\le 2\|u\|_{L^2}^2 + h^2\sum_{j\in\mathbb{Z}} \int_0^h u'(jh+\sigma)^2\,d\sigma \\
&= 2\|u\|_{L^2}^2 + h^2\|u'\|_{L^2}^2 .
\end{aligned} \tag{5.15}$$

□

Corollary 5.3. *There exists $K>0$ so that for any $\vartheta\in\mathbb{R}$, any $v\in H^1$ and any $0<h<1$, we have the bounds*

$$\begin{aligned}
\|\mathrm{ev}_\vartheta v\|_{\ell_h^\infty} &\le K\|v\|_{H^1}, \\
\|\mathrm{ev}_\vartheta v\|_{\ell_h^{\infty;1}} &\le K\big[\|v\|_{H^1} + \|\partial_h^+ v\|_{H^1}\big], \\
\|\mathrm{ev}_\vartheta v\|_{\ell_h^{\infty;2}} &\le K\big[\|v\|_{H^1} + h^{-1/2}\|\partial_h^+ v\|_{H^1}\big],
\end{aligned} \tag{5.16}$$

together with

$$\begin{aligned}
\|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;1}} &\le K\|v\|_{H^1}, \\
\|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}} &\le K\big[\|v\|_{H^1} + \|\partial_h^+ v\|_{H^1}\big].
\end{aligned} \tag{5.17}$$

Proof. For convenience, pick $\vartheta=0$. Using Lemma 5.1 and the standard Sobolev bound $\|v\|_\infty \le C_1\|v\|_{H^1}$ for some $C_1>0$, we find

$$\begin{aligned}
\|v\|_{\ell_h^\infty} &\le C_1\|v\|_{H^1}, \\
\|\partial_h^+ v\|_{\ell_h^\infty} &\le C_1\|\partial_h^+ v\|_{H^1}, \\
\|\partial_h^+\partial_h^+ v\|_{\ell_h^\infty} &\le h^{-1/2}\|\partial_h^+ v'\|_{L^2} \\
&\le h^{-1/2}\|\partial_h^+ v\|_{H^1}.
\end{aligned} \tag{5.18}$$

In addition, using (5.13) we find

$$\begin{aligned}
\|v\|_{\ell_h^2} &\le 3\|v\|_{H^1}, \\
\|\partial_h^+ v\|_{\ell_h^2} &\le \|v'\|_{L^2} \\
&\le \|v\|_{H^1}, \\
\|\partial_h^+\partial_h^+ v\|_{\ell_h^2} &\le \|\partial_h^+ v'\|_{L^2} \\
&\le \|\partial_h^+ v\|_{H^1}.
\end{aligned} \tag{5.19}$$

□

We remark that the results above shows that we automatically have $\mathrm{ev}_\vartheta u \in \ell_h^2$ whenever $u\in H^1$. We exploit this in the next result, which shows how to recover L^2 norms from the individual grid evaluations. We note that a direct consequence of (5.10) and (i) below is that we have

$$\|\partial_h^\pm u\|_{L^q} \le \|u'\|_{L^q} \tag{5.20}$$

for any $u\in H^1$ and $q\in\{2,\infty\}$.

Lemma 5.4. *Consider any $f \in C(\mathbb{R};\mathbb{R})$ and any $g \in H^1$. Then the following properties hold for all $h > 0$.*

(i) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|g\|_\infty \tag{5.21}$$

holds for all $\vartheta \in [0, h]$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_\infty \,. \tag{5.22}$$

(ii) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|\mathrm{ev}_\vartheta g\|_{\ell_h^2} \tag{5.23}$$

holds for all $\vartheta \in [0, h]$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_{L^2} \,. \tag{5.24}$$

(iii) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|\mathrm{ev}_\vartheta g\|_{\ell_h^{2;2}} \tag{5.25}$$

holds for all $\vartheta \in (0, h)$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_{H^1} + \left\|\partial_h^+ \partial_h^+ g\right\|_{L^2} \,. \tag{5.26}$$

(iv) If the bound

$$\|\mathrm{ev}_\vartheta f\|_{\ell_h^2} \leq \|\mathrm{ev}_\vartheta g\|_{\ell_h^{2;3}} \tag{5.27}$$

holds for all $\vartheta \in [0, h]$, then $f \in L^2$ with

$$\|f\|_{L^2} \leq \|g\|_{H^1} + \left\|\partial_h^+ g\right\|_{H^1} + \left\|\partial_h^+ \partial_h^+ \partial_h^+ g\right\| \,. \tag{5.28}$$

Proof. We first note that

$$\begin{array}{rcl} \|f\|_{L^2}^2 & = & \int_{\mathbb{R}} f(x)^2 \, dx \\ & = & \sum_{k \in \mathbb{Z}} \int_0^h f(kh + \vartheta)^2 \, d\vartheta \\ & = & h^{-1} \int_0^h \|\mathrm{ev}_\vartheta f\|_{\ell_h^2}^2 \, d\vartheta. \end{array} \tag{5.29}$$

Item (i) and (ii) follow immediately from this.

For (iii), we note

$$\begin{array}{rcl} \|f\|_{L^2}^2 & \leq & h^{-1} \int_0^h \|\mathrm{ev}_\vartheta g\|_{\ell_h^{2;2}}^2 \, d\vartheta \\ & = & h^{-1} \int_0^h \left[\, \|\mathrm{ev}_\vartheta g\|_{\ell_h^2}^2 + \left\|\mathrm{ev}_\vartheta \partial_h^+ g\right\|_{\ell_h^2}^2 + \left\|\mathrm{ev}_\vartheta \partial_h^+ \partial_h^+ g\right\|_{\ell_h^2}^2 \, \right] d\vartheta \\ & = & \|g\|_{L^2}^2 + \left\|\partial_h^+ g\right\|_{L^2}^2 + \left\|\partial_h^+ \partial_h^+ g\right\|_{L^2}^2 \,. \end{array} \tag{5.30}$$

Exploiting (5.20), we obtain

$$\|f\|_{L^2}^2 \quad \leq \quad \|g\|_{H^1}^2 + \left\|\partial_h^+ \partial_h^+ g\right\|_{L^2}^2 \tag{5.31}$$

as desired.

To see (iv), we apply (5.20) to $\partial_h^+ g$ to obtain

$$\left\|\partial_h^+ \partial_h^+ g\right\|_{L^2} \leq \left\|\partial_h^+ g'\right\|_{L^2}. \tag{5.32}$$

This yields the desired bound

$$\begin{array}{rcl} \|f\|_{L^2}^2 & \leq & \|g\|_{L^2}^2 + \left\|\partial_h^+ g\right\|_{L^2}^2 + \left\|\partial_h^+ \partial_h^+ g\right\|_{L^2}^2 + \left\|\partial_h^+ \partial_h^+ \partial_h^+ g\right\|_{L^2}^2 \\ & \leq & \|g\|_{L^2}^2 + \left\|\partial_h^+ g\right\|_{L^2}^2 + \left\|\partial_h^+ g'\right\|_{L^2}^2 + \left\|\partial_h^+ \partial_h^+ \partial_h^+ g\right\|_{L^2}^2 \\ & \leq & \|g\|_{H^1}^2 + \left\|\partial_h^+ g\right\|_{H^1}^2 + \left\|\partial_h^+ \partial_h^+ \partial_h^+ g\right\|_{L^2}^2 . \end{array} \tag{5.33}$$

□

Lemma 5.5. *Pick $q \in \{2, \infty\}$ and consider any $u \in W^{2;q}$. Then the estimates*

$$\left\|\partial_h^{\pm} u - u'\right\|_{\ell_h^q} \leq h \left\|u''\right\|_{L^q} \tag{5.34}$$

hold for all $h > 0$.

Proof. Fix $h > 0$ and write $\mathcal{I}^{\pm} \in \ell_h^{\infty}$ for the sequences

$$\mathcal{I}_{jh}^{\pm} = [\partial_h^{\pm} u](jh) - u'(jh). \tag{5.35}$$

We may compute

$$\begin{array}{rcl} \mathcal{I}_{jh}^{+} & = & \frac{1}{h} \int_0^h [u'(jh+s) - u'(jh)]\, ds \\ & = & \int_0^1 [u'(jh+sh) - u'(jh)]\, ds \\ & = & \int_0^1 \int_0^{sh} u''(jh+s')\, ds'\, ds. \end{array} \tag{5.36}$$

For $q = \infty$ we hence see

$$\left|\mathcal{I}_{jh}^{+}\right| \leq \left\|u''\right\|_{L^\infty} \int_0^1 \int_0^{sh} ds'\, ds = \frac{1}{2} h \left\|u''\right\|_{L^\infty}. \tag{5.37}$$

For $q = 2$ we obtain the estimate

$$\begin{array}{rcl} \|\mathcal{I}^{+}\|_{\ell_h^2}^2 & = & h \sum_{j\in\mathbb{Z}} \left[\int_0^1 \int_0^{sh} u''(jh+s')\, ds'\, ds \right]^2 \\ & \leq & h \sum_{j\in\mathbb{Z}} \int_0^1 [\int_0^{sh} u''(jh+s')\, ds']^2\, ds \\ & \leq & h \sum_{j\in\mathbb{Z}} \int_0^1 sh \int_0^{sh} [u''(jh+s')]^2\, ds'\, ds \\ & \leq & h^2 \sum_{j\in\mathbb{Z}} \int_0^h [u''(jh+s')]^2\, ds' \\ & = & h^2 \left\|u''\right\|_{L^2}^2 . \end{array} \tag{5.38}$$

Similar computations can be used for $\mathcal{I}^{-}$. □

Corollary 5.6. *Pick $q \in \{2, \infty\}$ and consider any $u \in W^{3;q}$. Then the estimates*

$$\begin{array}{rcl} \left\|2[\partial^0 \partial]_h u - u''\right\|_{\ell_h^q} & \leq & 2h \left\|u'''\right\|_{L^q}, \\ \left\|2T^+[\partial^0 \partial]_h u - u''\right\|_{\ell_h^q} & \leq & 2h \left\|u'''\right\|_{L^q} \end{array} \tag{5.39}$$

hold for all $h > 0$.

Proof. We first compute

$$\begin{aligned} 2[\partial^0\partial]_h u - u'' &= \partial_h^+\partial_h^- u - u'' \\ &= \partial_h^+\partial_h^- u - \partial_h^- u' + \partial_h^- u' - u''. \end{aligned} \tag{5.40}$$

Applying Lemma 5.5 and (5.20) to $\partial_h^- u$ shows that

$$\left\|\partial_h^+\partial_h^- u - \partial_h^- u'\right\|_{\ell_h^q} \le h \left\|\partial_h^- u''\right\|_{L^q} \le h \left\|u'''\right\|_{L^q}. \tag{5.41}$$

Similarly, applying Lemma 5.5 to u' shows that

$$\left\|\partial_h^- u' - u''\right\|_{\ell_h^q} \le h \left\|u'''\right\|_{L^q}, \tag{5.42}$$

from which the first estimate follows. Upon writing

$$\begin{aligned} 2T^+[\partial^0\partial]_h u - u'' &= \partial_h^+\partial_h^+ u - u'' \\ &= \partial_h^+\partial_h^+ u - \partial_h^+ u' + \partial_h^+ u' - u'', \end{aligned} \tag{5.43}$$

the second estimate can be obtained in a similar fashion. □

Corollary 5.7. *Pick $q \in \{2, \infty\}$ and consider any $u \in W^{4;q}$. Then the estimate*

$$\left\|2\partial_h^+[\partial^0\partial]_h u - u'''\right\|_{\ell_h^2} \le 3h \left\|u^{(iv)}\right\|_{L^q} \tag{5.44}$$

holds for all $h > 0$.

Proof. Splitting up

$$\begin{aligned} 2\partial_h^+[\partial^0\partial]_h u - u''' &= \partial_h^+\partial_h^+\partial_h^- u - u''' \\ &= \partial_h^+\partial_h^+\partial_h^- u - \partial_h^+\partial_h^- u' \\ &\quad + \partial_h^+\partial_h^- u' - \partial_h^- u'' \\ &\quad + \partial_h^- u'' - u''', \end{aligned} \tag{5.45}$$

we can apply Lemma 5.5 to obtain

$$\left\|2\partial_h^+[\partial^0\partial]_h u - u'''\right\|_{\ell_h^q} \le h \left\|\partial_h^+\partial_h^- u''\right\|_{L^q} + h \left\|\partial_h^- u'''\right\|_{L^q} + h \left\|u''''\right\|_{L^2} \tag{5.46}$$

We can now repeatedly apply (5.20) to obtain the desired estimate. □

We recall the definitions (3.13). Our final result here is a standard approximation bound for discrete integration.

Lemma 5.8. *For any $f \in W^{1,1}$ and $h > 0$, we have the bounds*

$$\left\| h \sum_{\pm;h} f - \int_{\pm} f \right\|_{\ell_h^\infty} \le h \left\|f'\right\|_{L^1}. \tag{5.47}$$

Proof. Fixing $\tau \in \mathbb{R}$, we compute

$$\begin{aligned} \left[h \textstyle\sum_{+;h} f - \int_+ f\right](\tau) &= \textstyle\sum_{k>0} \int_0^h [f(\tau + kh) - f(\tau + (k-1)h + \sigma)]\, d\sigma \\ &= \textstyle\sum_{k\ge 0} \int_0^h [f(\tau + (k+1)h) - f(\tau + kh + \sigma)]\, d\sigma \\ &= \textstyle\sum_{k\ge 0} \int_0^h \int_0^\sigma f'(\tau + kh + \sigma')\, d\sigma'\, d\sigma \\ &= \textstyle\sum_{k\ge 0} \int_0^h \int_{\sigma'}^h f'(\tau + kh + \sigma')\, d\sigma\, d\sigma'. \end{aligned} \tag{5.48}$$

In particular, we obtain the estimate

$$\begin{array}{rcl}\Big|\big[h\sum_{+;h} f - \int_{+} f\big](\tau)\Big| & \leq & \sum_{k\geq 0} h \int_0^h |f'(\tau + kh + \sigma')| \, d\sigma' \\ & \leq & h\int_0^{\infty} |f'(\tau+\sigma')| \, d\sigma' \\ & \leq & h \, \|f'\|_{L^1} . \end{array} \tag{5.49}$$

□

5.2 Weighted norms

For any $\eta > 0$ we define the exponential weight function

$$e_\eta(\tau) = e^{-\eta|\tau|}. \tag{5.50}$$

This allows us to define an inner product

$$\langle a, b\rangle_{L^2_\eta} = \langle e_\eta a, e_\eta b\rangle_{L^2} = \langle e_{2\eta} a, b\rangle_{L^2}, \tag{5.51}$$

together with the associated Hilbert space

$$L^2_\eta = \{f \in L^1_{\mathrm{loc}} : \|f\|^2_{L^2_\eta} := \langle f, f\rangle_{L^2_\eta} < \infty\}. \tag{5.52}$$

Since $0 < e_\eta \leq 1$, we see that

$$\langle a, a\rangle_{L^2_\eta} \leq \langle a, a\rangle_{L^2} \tag{5.53}$$

for every $a \in L^2$. In particular, we have the continuous embedding

$$L^2 \subset L^2_\eta. \tag{5.54}$$

In addition, for any pair $(a, b) \in L^2_\eta \times L^2$, we have $e_\eta a \in L^2$ and hence also $e_{2\eta} a \in L^2$. This allows us to estimate

$$|\langle e_{2\eta} a, b\rangle_{L^2}| = \Big|\langle a, b\rangle_{L^2_\eta}\Big| \leq \|a\|_{L^2_\eta} \|b\|_{L^2_\eta} . \tag{5.55}$$

This weighted norm is very convenient when dealing with sampling sums.

Lemma 5.9. *Fix $\eta > 0$. There exists $K > 0$ so that for any $f \in L^2_\eta$ and any $0 < h < 1$, we have the estimate*

$$\Big\|h\sum_{-;h} e_{2\eta} f\Big\|_{L^2_\eta} \quad \leq \quad K \|f\|_{L^2_\eta} . \tag{5.56}$$

Proof. Using Cauchy-Schwartz, we compute

$$\begin{array}{rcl}\Big\|h\sum_{-;h} e_{2\eta} f\Big\|^2_{L^2_\eta} & = & \int e_{2\eta}(\tau)\big[h\sum_{-;h} e_{2\eta} f\big](\tau)^2 \, d\tau \\ & = & \int e_{2\eta}(\tau)\Big[h\sum_{k\geq 0} w^2_\eta(\tau - kh) f(\tau - kh)\Big]^2 \, d\tau \\ & \leq & \int e_{2\eta}(\tau)\Big[h\sum_{k\geq 0} e_{2\eta}(\tau - kh)\Big]\Big[h\sum_{k\geq 0} e_{2\eta}(\tau - kh) f(\tau - kh)^2\Big] \, d\tau. \end{array} \tag{5.57}$$

We note that there exists $C_1 > 0$ so that for all $0 < h \leq 1$ and all $\tau \in \mathbb{R}$ we have

$$\begin{array}{rcl} h\sum_{k\geq 0} e_{2\eta}(\tau - kh) & = & h\sum_{k\geq 0} e^{-2\eta|\tau - kh|} \\ & \leq & h\sum_{k\in\mathbb{Z}} e^{-2\eta|\tau - kh|} \\ & \leq & C_1. \end{array} \tag{5.58}$$

Using the substitution $\tau' = \tau - kh$, this allows us to compute

$$\begin{array}{rcl} \left\| h \sum_{-;h} e_{2\eta} f \right\|_{L^2_\eta}^2 & \leq & C_1 \int e_{2\eta}(\tau) \Big[h \sum_{k \geq 0} e_{2\eta}(\tau - kh) f(\tau - kh)^2 \Big] \, d\tau \\ & = & C_1 \int \Big[h \sum_{k \geq 0} e_{2\eta}(\tau' + kh) \Big] e_{2\eta}(\tau') f(\tau')^2 \, d\tau' \\ & \leq & C_1^2 \int e_{2\eta}(\tau') f(\tau')^2 \, d\tau' \\ & = & C_1^2 \, \|f\|_{L^2_\eta}^2 . \end{array} \tag{5.59}$$

□

5.3 Weak Limits

Our results here show how weak limits interact with discrete summation and differentiation. The first result concerns sequences that are bounded in H^1 and have bounded second differences, as described in the following assumption.

(hSeq) The sequence

$$\{(h_j, v_j)\}_{j>0} \subset (0,1) \times H^1 \tag{5.60}$$

satisfies $h_j \to 0$ as $j \to \infty$. In addition, there exists $K > 0$ so that the bound

$$\|v_j\|_{H^1} + \left\| \partial^+_{h_j} \partial^+_{h_j} v_j \right\|_{L^2} < K \tag{5.61}$$

holds for all $j > 0$.

The control on the second differences allows one to show that the weak limit is in fact in H^2. In addition, the first differences converge strongly on compact intervals.

Lemma 5.10. *Consider a sequence*

$$\{(h_j, v_j)\} \subset (0,1) \times H^1 \tag{5.62}$$

that satisfies (hSeq). Then there exists $V_* \in H^2$ so that, after passing to a subsequence, the following properties hold.

(i) We have the weak limit

$$v_j \rightharpoonup V_* \in H^1. \tag{5.63}$$

(ii) We have the weak limits

$$\partial^{\pm}_{h_j} v_j \rightharpoonup V_*' \in L^2. \tag{5.64}$$

(iii) We have the weak limit

$$2[\partial^0 \partial]_{h_j} v_j \rightharpoonup V_*'' \in L^2. \tag{5.65}$$

(iv) For any compact interval $\mathcal{I} \subset \mathbb{R}$, we have the strong convergences

$$v_j \to V_* \in L^2(\mathcal{I}), \qquad \partial^{\pm}_{h_j} v_j \to V_*' \in L^2(\mathcal{I}) \tag{5.66}$$

as $j \to \infty$.

Proof. Using (5.20) we obtain the uniform bound

$$\left\|\partial^{\pm}_{h_j} v_j\right\|_{L^2} < K \tag{5.67}$$

for all $j > 0$. In particular, after passing to a subsequence we can find a triplet

$$(V_*, V_*^{\pm}, V_*^{(2)}) \in H^1 \times L^2 \times L^2 \tag{5.68}$$

so that we have the weak convergences

$$v_j \rightharpoonup V_* \in H^1, \qquad \partial^{\pm}_{h_j} v_j \rightharpoonup V_*^{\pm} \in L^2, \qquad 2[\partial^0 \partial]_{h_j} v_j \rightharpoonup V_*^{(2)} \in L^2 \tag{5.69}$$

as $j \to \infty$.

Pick any test function $\zeta \in C_c^\infty$. We note that

$$\left\|\partial^-_{h_j} \zeta - \zeta'\right\|_{L^2} + \left\|2[\partial^0 \partial]_{h_j} \zeta - \zeta''\right\|_{L^2} \to 0 \tag{5.70}$$

as $j \to \infty$ by Lemma 5.5 and Corollary 5.6.

We now compute

$$\begin{array}{rcl} \langle \partial^+_{h_j} v_j, \zeta \rangle_{L^2} & = & -\langle v_j, \partial^-_{h_j} \zeta \rangle_{L^2} \\ & = & -\langle v_j, \zeta' \rangle_{L^2} + \langle v_j, \zeta' - \partial^-_{h_j} \zeta \rangle_{L^2} \\ & = & \langle v_j', \zeta \rangle_{L^2} + \langle v_j, \zeta' - \partial^-_{h_j} \zeta \rangle_{L^2}, \end{array} \tag{5.71}$$

together with

$$\begin{array}{rcl} \langle 2[\partial^0 \partial]_{h_j} v_j, \zeta \rangle_{L^2} & = & \langle v_j, 2[\partial^0 \partial]_{h_j} \zeta \rangle_{L^2} \\ & = & \langle v_j, \zeta'' \rangle_{L^2} + \langle v_j, 2[\partial^0 \partial \zeta]_{h_j} \zeta - \zeta'' \rangle_{L^2} \\ & = & -\langle v_j', \zeta' \rangle_{L^2} + \langle v_j, 2[\partial^0 \partial]_{h_j} \zeta - \zeta'' \rangle_{L^2}. \end{array} \tag{5.72}$$

The weak convergences $v_j' \rightharpoonup V_*' \in L^2$ and (5.69) imply that

$$\begin{array}{rclcrcl} \langle \partial^+_{h_j} v_j, \zeta \rangle_{L^2} & \to & \langle V_*', \zeta \rangle_{L^2}, & \quad & \langle \partial^+_{h_j} v_j, \zeta \rangle_{L^2} & \to & \langle V^+, \zeta \rangle_{L^2}, \\ \langle 2[\partial^0 \partial]_{h_j} v_j, \zeta \rangle_{L^2} & \to & -\langle V_*', \zeta' \rangle_{L^2}, & & \langle 2[\partial^0 \partial]_{h_j} v_j, \zeta \rangle_{L^2} & \to & \langle V_*^{(2)}, \zeta \rangle_{L^2} \end{array} \tag{5.73}$$

as $j \to \infty$. The density of C_c^∞ in L^2 now implies that $V^+ = V_*'$ and that $V_*' \in H^1$ with $V_*'' = V_*^{(2)}$. This yields (i), (ii) and (iii).

Turning to (iv), we pick a compact interval $\mathcal{I} \subset \mathbb{R}$. The compact embedding $H^1(\mathcal{I}) \subset L^2(\mathcal{I})$ allows us to pass to a subsequence for which

$$\left\|v_j - V_*\right\|_{L^2(\mathcal{I})} \to 0 \tag{5.74}$$

as $j \to \infty$. We compute

$$\begin{array}{rcl} \left\|\partial^+_{h_j} v_j - V_*'\right\|_{L^2(\mathcal{I})} & = & \langle \partial^+_{h_j} v_j - V_*', \partial^+_{h_j} v_j - V_*' \rangle_{L^2(\mathcal{I})} \\ & = & \langle \partial^+_{h_j} v_j - V_*', \partial^+_{h_j} v_j - \partial^+ V_* \rangle_{L^2(\mathcal{I})} + \langle \partial^+_{h_j} v_j - V_*', \partial^+_{h_j} V_* - V_*' \rangle_{L^2(\mathcal{I})} \\ & = & -\langle \partial^-_{h_j} \partial^+_{h_j} v_j - \partial^-_{h_j} V_*', v_j - V_* \rangle_{L^2(\mathcal{I})} \\ & & \quad + \langle \partial^+_{h_j} v_j - V_*', \partial^+_{h_j} V_* - V_*' \rangle_{L^2(\mathcal{I})}. \end{array} \tag{5.75}$$

Using (5.20) we see that

$$\left\|\partial_{h_j}^- V_*'\right\|_{L^2} \leq \|V_*''\|_{L^2}. \tag{5.76}$$

Together with (5.61), (5.67) and the identity

$$\partial_{h_j}^- \partial_{h_j}^+ v_j = 2[\partial^0 \partial]_{h_j} v_j, \tag{5.77}$$

this implies the uniform bound

$$\left\|\partial_{h_j}^- \partial_{h_j}^+ v_j\right\|_{L^2(\mathcal{I})} + \left\|\partial_{h_j}^+ v_j\right\|_{L^2(\mathcal{I})} + \left\|\partial_{h_j}^- V_*'\right\|_{L^2(\mathcal{I})} + \|V_*'\|_{L^2(\mathcal{I})} < C_1 \tag{5.78}$$

for some $C_1 > 0$. In particular, using Lemma 5.5 and (5.74), we see that

$$\begin{array}{rcl} \left\|\partial_{h_j}^+ v_j - V_*'\right\|_{L^2(\mathcal{I})} & \leq & 2C_1 \Big[\|v_j - V_*\|_{L^2(\mathcal{I})} + \left\|\partial_{h_j}^+ V_* - V_*'\right\|_{L^2(\mathcal{I})} \Big] \\ & \leq & 2C_1 \big[\|v_j - V_*\|_{L^2(\mathcal{I})} + h_j \|V_*''\|_{L^2} \big] \\ & \to & 0 \end{array} \tag{5.79}$$

as $j \to \infty$, as desired. A standard diagonalization argument now completes the proof. □

Lemma 5.11. *Consider a bounded sequence*

$$\{(h_j, f_j, \alpha_{1;j}, \alpha_{2;j}, \alpha_{3;j})\}_{j>0} \subset (0,1) \times L^2 \times H^1 \times H^1 \times H^1 \tag{5.80}$$

that satisfies the following properties.

(a) There exists $C > 0$ and $\eta > 0$ so that

$$|\alpha_{1;j}(\tau)| + |\alpha_{2;j}(\tau)| \leq C e_{2\eta}(\tau) \tag{5.81}$$

for all $\tau > 0$.

(b) There exists a triplet $(\alpha_{1;*}, \alpha_{2;*}, \alpha_{3;*}) \in H^1 \times H^1 \times H^1$ so that we have the strong convergence

$$(\alpha_{1;j}, \alpha_{2;j}, \alpha_{3;j}) \to (\alpha_{1;*}, \alpha_{2;*}, \alpha_{3;*}) \in H^1 \times H^1 \times H^1 \tag{5.82}$$

as $j \to \infty$.

(c) We have $h_j \to 0$ as $j \to \infty$.

Then, after passing to a subsequence, there exists $f_* \in L^2$ so that we have the weak convergences

$$f_j \rightharpoonup f_* \in L^2, \qquad \alpha_{3;j} f_j \rightharpoonup \alpha_{3;*} \in L^2, \tag{5.83}$$

together with

$$\alpha_{1;j} h_j \sum_{-;h_j} \alpha_{2;j} f_j \rightharpoonup \alpha_{1;*} \int_- \alpha_{2;*} f_* \in L^2 \tag{5.84}$$

as $j \to \infty$.

Proof. Writing

$$g_j = \alpha_{1;j} h_j \sum_{-;h_j} \alpha_{2;j} f_j, \tag{5.85}$$

we see that

$$\begin{array}{lcl} \|g_j\|_{L^2} & \leq & C^2 \left\| e_{2\eta} h_j \sum_{-;h_j} e_{2\eta} \, |f|_j \right\|_{L^2} \\ & \leq & C^2 \left\| e_{\eta} h_j \sum_{-;h_j} e_{2\eta} \, |f|_j \right\|_{L^2} \\ & = & C^2 \left\| h_j \sum_{-;h_j} e_{2\eta} \, |f|_j \right\|_{L^2_\eta} \\ & \leq & C_3' \, \|f_j\|_{L^2_\eta} \\ & \leq & C_3' \, \|f_j\|_{L^2} . \end{array} \tag{5.86}$$

In particular, after passing to a subsequence we have the weak convergences $f_j \rightharpoonup f_* \in L^2$ and $g_j \rightharpoonup g_* \in L^2$.

Pick any $\zeta \in C_c^\infty$ and write

$$\mathcal{I}_{\zeta;j} = \alpha_{2;j} h_j \sum_{+;h_j} \alpha_{1;j} \zeta - \alpha_{2;*} \int_+ \alpha_{1;*} \zeta, \tag{5.87}$$

which can be expanded as

$$\begin{array}{lcl} \mathcal{I}_{\zeta;j} & = & \big[\alpha_{2;j} - \alpha_{2;*}\big] h_j \sum_{+;h_j} \alpha_{1;j} \zeta \\ & & \quad + \alpha_{2;*} h_j \sum_{+;h_j} \big[\alpha_{1;j} - \alpha_{1;*}\big] \zeta \\ & & \quad + \alpha_{2;*} \Big[h_j \sum_{+;h_j} \alpha_{1;*} \zeta - \int_+ \alpha_{1;*} \zeta \Big]. \end{array} \tag{5.88}$$

Using the estimates (5.13) and (5.47) we see that

$$\begin{array}{lcl} \|\mathcal{I}_{\zeta,j}\|_{L^2} & \leq & \|\alpha_{2;j} - \alpha_{2;*}\|_{L^2} \, \|\alpha_{1;j}\|_{H^1} \, \|\zeta\|_{H^1} \\ & & \quad + \|\alpha_{2;*}\|_{L^2} \, \|\alpha_{1;j} - \alpha_{1;*}\|_{H^1} \, \|\zeta\|_{H^1} \\ & & \quad + \|\alpha_{2;*}\|_{L^2} \, |h_j| \, \|[\alpha_{1;*}\zeta]'\|_{L^1} . \end{array} \tag{5.89}$$

Observing that

$$\begin{array}{lcl} \|[\alpha_{1;*}\zeta]'\|_{L^1} & \leq & \left\|\alpha_{1;*}' \zeta\right\|_{L^1} + \|\alpha_{1;*}\zeta'\|_{L^1} \\ & \leq & \left\|\alpha_{1;*}'\right\|_{L^2} \|\zeta\|_{L^2} + \|\alpha_{1;*}\|_{L^2} \, \|\zeta'\|_{L^2} \\ & \leq & 2 \, \|\alpha_{1;*}\|_{H^1} \, \|\zeta\|_{H^1} , \end{array} \tag{5.90}$$

we see that $\|\mathcal{I}_{\zeta,j}\|_{L^2} \to 0$ as $j \to \infty$. In addition, we see that

$$\|(\alpha_{3;j} - \alpha_{3;*})\zeta\|_{L^2} \leq \|\alpha_{3;j} - \alpha_{3;*}\|_{L^\infty} \, \|\zeta\|_{L^2} \leq \|\alpha_{3;j} - \alpha_{3;*}\|_{H^1} \, \|\zeta\|_{L^2} \to 0 \tag{5.91}$$

as $j \to \infty$.

We now compute

$$\begin{array}{lcl} \langle g_j, \zeta \rangle_{L^2} & = & \langle \alpha_{1;j} h_j \sum_{-;h_j} \alpha_{2;j} f_j, \zeta \rangle_{L^2} \\ & = & \langle f_j, \alpha_{2;j} h_j \sum_{+;h_j} \alpha_{1;j} \zeta \rangle_{L^2} \\ & = & \langle f_j, \alpha_{2;*} \int_+ \alpha_{1;*} \zeta \rangle_{L^2} + \langle f_j, \mathcal{I}_{\zeta,j} \rangle_{L^2}, \end{array} \tag{5.92}$$

together with

$$\begin{array}{rcl} \langle \alpha_{3;j} f_j, \zeta \rangle_{L^2} & = & \langle f_j, \alpha_{3;j} \zeta \rangle_{L^2} \\ & = & \langle f_j, \alpha_* \zeta \rangle_{L^2} + \langle f_j, (\alpha_{3;j} - \alpha_{3;*}) \zeta \rangle_{L^2}. \end{array} \tag{5.93}$$

In particular, the weak convergence $f_j \rightharpoonup f_*$ implies that

$$\begin{array}{rcl} \langle g_j, \zeta \rangle_{L^2} & \to & \langle f_*, \alpha_{2;*} \int_+ \alpha_{1;*} \zeta \rangle_{L^2} \\ & = & \langle \alpha_{1;*} \int_- \alpha_{2;*} f_*, \zeta \rangle_{L^2} \end{array} \tag{5.94}$$

together with

$$\begin{array}{rcl} \langle \alpha_{3;j} f_j, \zeta \rangle_{L^2} & \to & \langle f_*, \alpha_{3;*} \rangle_{L^2} \\ & = & \langle \alpha_{3;*} f_*, \zeta \rangle_{L^2} \end{array} \tag{5.95}$$

as $j \to \infty$. The density of C_c^∞ in L^2 now implies the desired weak limits. □

6 Estimation techniques

In this section we introduce the basic framework that we use throughout the paper to estimate the terms featuring in our main equation (2.26). In particular, we introduce the sequence spaces on which the nonlinear terms can be conveniently estimated in a fashion that is uniform in $h > 0$. We also introduce several bookkeeping and approximation results that are essential to control convoluted expressions such as (2.17) in a feasible manner.

The Heaviside function H has $\|\partial^+ H\|_{\ell_h^\infty} = h^{-1}$, which makes it unsuitable for the bifurcation arguments used in ths paper. In order to smooth out the transition between the two stable equilibria of g, we pick a function $U_{\mathrm{ref};*} \in C^2(\mathbb{R}, [0,1])$ for which we have the identities

$$U_{\mathrm{ref};*}(\tau) = \begin{cases} 0 & \text{for all } \tau \le -2, \\ 1 & \text{for all } \tau \ge 2 \end{cases} \tag{6.1}$$

and for which the bounds

$$0 \le U'_{\mathrm{ref};*}(\tau) < 1, \qquad \left| U''_{\mathrm{ref};*}(\tau) \right| < 1 \tag{6.2}$$

hold for all $\tau \in \mathbb{R}$.

For any $\kappa > 0$ we subsequently write

$$U_{\mathrm{ref};\kappa}(\tau) = U_{\mathrm{ref};*}(\kappa\tau) \tag{6.3}$$

together with

$$\begin{array}{rcl} \mathcal{V}_{h;\kappa} & = & \{ V \in \ell_h^2 : \|V\|_{\ell_h^{2;2}} + \|V\|_{\ell_h^\infty} + \|\partial^+\partial^+ V\|_{\ell_h^\infty} < \frac{1}{2}\kappa^{-1} \\ & & \text{and } \|\partial^+ V\|_\infty < 1 - 2\kappa \}. \end{array} \tag{6.4}$$

As a consequence of the estimate (5.4), we see that $\mathcal{V}_{h;\kappa}$ is an open subset of ℓ_h^2.

Combining these two definitions allows us to introduce the sets

$$\Omega_{h;\kappa} = U_{\mathrm{ref};\kappa}(h\mathbb{Z}) + \mathcal{V}_{h;\kappa} \subset \ell_h^\infty. \tag{6.5}$$

Our first three results highlight the important role that these sets $\Omega_{h;\kappa}$ will play in the sequel. Indeed, the initial conditions referenced in the well-posedness result Proposition 2.4 can all be taken from such a set. In addition, we obtain a-priori bounds on almost all the terms that appear in the discrete derivatives defined in §4.3. The one exception is the third derivative $\partial^+\partial^0\partial U$, which will play a special role in our estimates.

Proposition 6.1. *Fix $h > 0$ and $0 < \kappa < \frac{1}{12}$. Then for any $U \in \Omega_{h;\kappa}$, we have the bound*

$$\|U\|_{\ell_h^\infty} + \left\|\partial^+ U\right\|_{\ell_h^2} + \left\|\partial^+\partial^+ U\right\|_{\ell_h^2} + \left\|\partial^+\partial^+ U\right\|_{\ell_h^\infty} < \kappa^{-1} \tag{6.6}$$

together with

$$\left\|\partial^+ U\right\|_{\ell_h^\infty} < 1 - \kappa. \tag{6.7}$$

In addition, we have

$$\|g(U)\|_{\ell_h^2} \quad \leq \quad 4\Big[\sup_{|u| \leq \kappa^{-1}} |g'(u)|\Big]\kappa^{-1}. \tag{6.8}$$

Proposition 6.2. *Fix $h > 0$ and consider any $U \in \ell_h^\infty$ for which $\|\partial^+ U\|_\infty < 1$ and $U - H \in \ell_h^2$. Then there exist $\epsilon_0 > 0$ and $\kappa_0 > 0$ so that for any $\widetilde{U} \in \ell_h^\infty$ that has $\left\|\widetilde{U} - U\right\|_{\ell_h^2} < \epsilon_0$, we have*

$$\widetilde{U} \in \Omega_{h;\kappa} \tag{6.9}$$

for all $0 < \kappa < \kappa_0$.

Proposition 6.3. *Consider any $u \in C(\mathbb{R};\mathbb{R})$ for which we have the inclusions*

$$u' \in H^2, \qquad\qquad u - U_{\mathrm{ref};*} \in L^2, \tag{6.10}$$

together with the bound $\|u'\|_{L^\infty} < 1$. Then there exist $\epsilon_0 > 0$ and $\kappa > 0$ so that for any $0 < h < 1$ and any $v \in H^1$ that has

$$\|v\|_{H^1} + h^{-1/2}\left\|\partial^+ v\right\|_{H^1} < \epsilon_0, \tag{6.11}$$

we have

$$\mathrm{ev}_\vartheta[u + v] \in \Omega_{h;\kappa} \tag{6.12}$$

for all $\vartheta \in [0, h]$.

We provide the proofs for these results in §6.3. In §6.1-§6.2 we introduce three convenient approximation results that allow us to considerably simplify the expressions that arise when linearizing (2.26) around the continuum wave Ψ_*.

Convention Throughout the remainder of this paper, we use the convention that primed constants (such as C_1', C_2' etc) that appear in proofs are positive and depend only on κ, the nonlinearity g and the wave Ψ_*, unless explicitly stated otherwise.

6.1 Approximate substitution

In this subsection, our goal is to consider composite functions $f \circ \phi$ in situations where it is convenient to approximate ϕ and $D\phi$ by ϕ_{apx} and ϕ_{lin}. These two approximants should be thought of as simplified versions of ϕ and $D\phi$ that are much easier to handle in computations, while still accurate to leading order in h. Our typical setup is described in the following assumption.

(hϕ) The set $K_f \subset \mathbb{R}^n$ is compact and we have the inclusion $\Omega_\phi \subset \mathcal{B}$, in which $\mathcal{B}$ is a Banach space. In addition, the function

$$\phi : \Omega_\phi \subset \mathcal{B} \to K_f \tag{6.13}$$

is Lipschitz continuous in the sense that there is $K_{\rm lip} > 1$ so that

$$|\phi(\omega_1) - \phi(\omega_2)| \le K_{\rm lip} \, \|\omega_1 - \omega_2\|_{\mathcal{B}} \tag{6.14}$$

holds for all $\omega_1, \omega_2 \in \Omega_\phi$. Finally, we have the inclusions

$$\phi_{\rm apx}(\omega) \in K_f, \qquad \phi_{{\rm lin};\omega} \in \mathcal{L}(\mathcal{B}; \mathbb{R}^n) \tag{6.15}$$

for every $\omega \in \Omega_\phi$, together with the uniform bound

$$\sup_{\omega \in \Omega_\phi} \|\phi_{{\rm lin};\omega}\|_{\mathcal{L}(\mathcal{B};\mathbb{R}^n)} < \infty. \tag{6.16}$$

Lemma 6.4. *Consider two triplets $(\phi, \phi_{\rm apx}, \phi_{\rm lin})$ and $(\Omega_\phi, \mathcal{B}, K_f)$ and suppose that $(h\phi)$ is satisfied. Suppose furthermore that there exists an open set $O_f \subset \mathbb{R}^n$ and a compact set $\overline{K}_f \subset \mathbb{R}^n$ for which*

$$K_f \subset O_f \subset \overline{K}_f. \tag{6.17}$$

Pick any $f \in C^2(\overline{K}_f; \mathbb{R})$ and consider the map

$$P : \Omega_\phi \subset \mathcal{B} \to \mathbb{R}, \qquad \omega \mapsto f\big(\phi(\omega)\big). \tag{6.18}$$

For any $\omega \in \Omega_\phi$ and $\beta \in \mathcal{B}$, write

$$\begin{array}{lcl} P_{\rm apx}(\omega) & = & f\big(\phi_{\rm apx}(\omega)\big), \\ P_{{\rm lin};\omega}[\beta] & = & Df\big(\phi_{\rm apx}(\omega)\big)\phi_{{\rm lin};\omega}[\beta]. \end{array} \tag{6.19}$$

In addition, for any $\omega \in \Omega_\phi$ and $\beta \in \mathcal{B}$ for which $\omega + \beta \in \Omega_\phi$, write

$$\begin{array}{lcl} \phi_{{\rm nl};\omega}(\beta) & = & \phi(\omega + \beta) - \phi(\omega) - \phi_{{\rm lin};\omega}[\beta], \\ P_{{\rm nl};\omega}(\beta) & = & P(\omega + \beta) - P(\omega) - P_{{\rm lin};\omega}[\beta]. \end{array} \tag{6.20}$$

Then there exists a constant $K > 0$ so that for any $\omega \in \Omega_\phi$ the bound

$$|P(\omega) - P_{\rm apx}(\omega)| \le K \, |\phi(\omega) - \phi_{\rm apx}(\omega)| \tag{6.21}$$

holds, while for any $\omega \in \Omega_\phi$ and $\beta \in \mathcal{B}$ for which $\omega + \beta \in \Omega_\phi$ we have the estimate

$$|P_{{\rm nl};\omega}(\beta)| \le K \Big[\, \|\beta\|_{\mathcal{B}}^2 + |\phi_{{\rm nl};\omega}(\beta)| + |\phi(\omega) - \phi_{\rm apx}(\omega)| \, \|\beta\|_{\mathcal{B}} \, \Big]. \tag{6.22}$$

Proof. The geometric condition (6.17) implies that f and Df are Lipschitz on K_f and that there is $C_1 > 0$ for which

$$\frac{|f(y) - f(x) - Df(x)(y - x)|}{|y - x|^2} \le C_1 \tag{6.23}$$

holds for all $(x, y) \in K_f \times K_f$ with $x \neq y$. Indeed, we can cover K_f completely with open balls in which the local versions of these properties follow from the C^2-smoothness of f on the larger set $\overline{K}_f$.

The inequality (6.21) follows directly from the fact that f is Lipschitz. Turning to (6.22), we decompose

$$P_{{\rm nl};\omega}(\beta) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \tag{6.24}$$

in which

$$\begin{array}{rcl}\mathcal{J}_1 &=& f\big(\phi(\omega+\beta)\big) - f\big(\phi(\omega)\big) - Df\big(\phi(\omega)\big)[\phi(\omega+\beta)-\phi(\omega)],\\ \mathcal{J}_2 &=& Df\big(\phi(\omega)\big)\phi_{\mathrm{nl};\omega}(\beta),\\ \mathcal{J}_3 &=& \big[Df\big(\phi(\omega)\big) - Df\big(\phi_{\mathrm{apx}}(\omega)\big)\big]\phi_{\mathrm{lin};\omega}[\beta].\end{array} \tag{6.25}$$

The bounds (6.14) and (6.23) imply

$$\begin{array}{rcl}|\mathcal{J}_1| &\leq& C_1\,|\phi(\omega+\beta)-\phi(\omega)|^2\\ &\leq& C_1 K_{\mathrm{lip}}^2\,\|\beta\|_{\mathcal{B}}^2\,,\end{array} \tag{6.26}$$

while the Lipschitz smoothness of Df yields

$$|\mathcal{J}_3| \leq C_2\,|\phi(\omega)-\phi_{\mathrm{apx}}(\omega)|\,\|\phi_{\mathrm{lin};\omega}\|_{\mathcal{L}(\mathcal{B};\mathbb{R}^n)}\,\|\beta\|_{\mathcal{B}} \tag{6.27}$$

for some $C_2 > 0$. The desired estimate (6.22) now follows from the uniform bound (6.16). □

Corollary 6.5. *Consider two triplets $(\phi, \phi_{\mathrm{apx}}, \phi_{\mathrm{lin}})$ and $(\Omega_\phi, \mathcal{B}, K_f)$ and suppose that $(h\phi)$ is satisfied. Suppose furthermore that there exists an open set $O_f \subset \mathbb{R}^n$ and a compact set $\overline{K}_f \subset \mathbb{R}^n$ for which*

$$K_f \subset O_f \subset \overline{K}_f. \tag{6.28}$$

Pick any $f \in C^2(\overline{K}_f;\mathbb{R})$, any Banach space $\mathcal{B}_L$, any $L \in \mathcal{L}(\mathcal{B}_L;\mathbb{R})$ and consider the map

$$P : \Omega_\phi \times \mathcal{B}_L \to \mathbb{R}, \qquad (\omega,\omega_L) \mapsto f\big(\phi(\omega)\big)L[\omega_L]. \tag{6.29}$$

For any $(\omega,\omega_L) \in \Omega_\phi \times \mathcal{B}_L$ and $(\beta,\beta_L) \in \mathcal{B}\times\mathcal{B}_L$, write

$$\begin{array}{rcl}P_{\mathrm{apx}}(\omega,\omega_L) &=& f\big(\phi_{\mathrm{apx}}(\omega)\big)L[\omega_L],\\ P_{\mathrm{lin};\omega,\omega_L}[\beta,\beta_L] &=& Df\big(\phi_{\mathrm{apx}}(\omega)\big)L[\omega_L]\phi_{\mathrm{lin};\omega}[\beta] + f\big(\phi_{\mathrm{apx}}(\omega)\big)L[\beta_L].\end{array} \tag{6.30}$$

In addition, for any $(\omega,\omega_L) \in \Omega_\phi \times \mathcal{B}_L$ and $(\beta,\beta_L) \in \mathcal{B}\times\mathcal{B}_L$ for which $\omega+\beta \in \Omega_\phi$, write

$$P_{\mathrm{nl};\omega,\omega_L}(\beta,\beta_L) \;=\; P(\omega+\beta,\omega_L+\beta_L) - P(\omega,\omega_L) - P_{\mathrm{lin};\omega,\omega_L}[\beta,\beta_L]. \tag{6.31}$$

Then there exists a constant $K > 0$ so that for any $(\omega,\omega_L) \in \Omega_\phi \times \mathcal{B}_L$ we have the bound

$$|P(\omega,\omega_L) - P_{\mathrm{apx}}(\omega,\omega_L)| \leq K\,|\phi(\omega)-\phi_{\mathrm{apx}}(\omega)|\,\|\omega_L\|_{\mathcal{B}_L}\,, \tag{6.32}$$

while for any for any $(\omega,\omega_L) \in \Omega_\phi \times \mathcal{B}_L$ and $(\beta,\beta_L) \in \mathcal{B}\times\mathcal{B}_L$ for which $\omega+\beta\in\Omega_\phi$ we have the bound

$$\begin{array}{rcl}|P_{\mathrm{nl};\omega,\omega_L}(\beta,\beta_L)| &\leq& K\Big[\,\|\beta\|_{\mathcal{B}}^2\,\|\omega_L\|_{\mathcal{B}_L} + \|\beta\|_{\mathcal{B}}\,\|\beta_L\|_{\mathcal{B}_L} + |\phi_{\mathrm{nl};\omega}(\beta)|\,\|\omega_L\|_{\mathcal{B}_L}\\ && \qquad + |\phi(\omega)-\phi_{\mathrm{apx}}(\omega)|\,\big[\,\|\beta\|_{\mathcal{B}}\,\|\omega_L\|_{\mathcal{B}_L} + \|\beta_L\|_{\mathcal{B}_L}\,\big]\Big].\end{array} \tag{6.33}$$

Proof. The bound (6.32) follows immediately from (6.21) together with the fact that $L \in \mathcal{L}(\mathcal{B}_L;\mathbb{R})$. Upon writing $P(\omega,\omega_L) = \widetilde{P}(\omega)L[\omega_L]$, we see that

$$\begin{array}{rcl}P_{\mathrm{nl};\omega,\omega_L}(\beta,\beta_L) &=& \widetilde{P}_{\mathrm{nl};\omega}(\beta)L[\omega_L] + \big[f\big(\phi(\omega+\beta)\big) - f\big(\phi_{\mathrm{apx}}(\omega)\big)\big]L[\beta_L]\\ &=& \widetilde{P}_{\mathrm{nl};\omega}(\beta)L[\omega_L] + \big[f\big(\phi(\omega+\beta)\big) - f\big(\phi(\omega)\big)\big]L[\beta_L]\\ && \quad +\big[f\big(\phi(\omega)\big) - f\big(\phi_{\mathrm{apx}}(\omega)\big)\big]L[\beta_L].\end{array} \tag{6.34}$$

In particular, exploiting the Lipschitz continuity of f and ϕ, we can find a constant $C_1 > 0$ for which

$$\begin{array}{rcl}|P_{\mathrm{nl};\omega,\omega_L}(\beta,\beta_L)| &\leq& C_1\,\Big|\widetilde{P}_{\mathrm{nl};\omega}(\beta)\Big|\,\|\omega_L\|_{\mathcal{B}_L} + C_1\,\|\beta\|_{\mathcal{B}}\,\|\beta_L\|_{\mathcal{B}_L}\\ && \quad +C_1\,|\phi(\omega)-\phi_{\mathrm{apx}}(\omega)|\,\|\beta_L\|_{\mathcal{B}_L}\,.\end{array} \tag{6.35}$$

Substituting the estimate (6.22) for $\widetilde{P}_{\mathrm{nl};\omega}(\beta)$ yields the desired bound (6.33). □

6.2 Approximate products

For any integer $k \geq 1$ and any sequence

$$\mathbf{q} = (q_1, q_2, \ldots, q_k) \in \{2, \infty\}^k, \tag{6.36}$$

we introduce the notation

$$\ell_h^{\mathbf{q}} = \ell_h^{q_1} \times \ell_h^{q_2} \times \ldots \times \ell_h^{q_k}. \tag{6.37}$$

Writing

$$\mathbf{q}_\pi = (q_{\pi;1}, q_{\pi;2}, \ldots, q_{\pi;k}) \in \{2, \infty\}^k, \tag{6.38}$$

we are interested in maps

$$\pi : \ell_h^{\mathbf{q}_\pi} \to \ell_h^2 \tag{6.39}$$

that are bounded and multi-linear in the following sense.

(hπ) Consider any set

$$\mathbf{v} = (v_1, \ldots, v_k) \in \ell_h^{\mathbf{q}_\pi}. \tag{6.40}$$

Then we have the estimate

$$\|\pi[\mathbf{v}]\|_{\ell_h^2} \leq K \|v_1\|_{\ell_h^{q_{\pi;1}}} \times \ldots \times \|v_k\|_{\ell_h^{q_{\pi;k}}} \tag{6.41}$$

for some constant $K > 0$ that does not depend on $\mathbf{v}$ and $h > 0$. In addition, if there is an integer $1 \leq i \leq k$ for which the decomposition

$$v_i = \lambda_A v_i^A + \lambda_B v_i^B \tag{6.42}$$

holds, with $v_i^{\#} \in \ell_h^{q_{\pi;i}}$ and $\lambda_{\#} \in \mathbb{R}$ for $\# \in \{A, B\}$, then we have

$$\pi[\mathbf{v}] = \lambda_A \pi[v_1, \ldots, v_i^A, \ldots, v_k] + \lambda_B \pi[v_1, \ldots, v_i^B, \ldots, v_k]. \tag{6.43}$$

We say that any sequence (6.36) is *admissable* for π if there is a constant $K > 1$ so that the bound

$$\|\pi[v]\|_{\ell_h^2} \leq K \|v_1\|_{\ell_h^{q_1}} \times \ldots \times \|v_k\|_{\ell_h^{q_k}} \tag{6.44}$$

holds for any

$$v \in \ell_h^{\mathbf{q}_\pi} \cap \ell_h^{\mathbf{q}} \tag{6.45}$$

and any $h > 0$.

As an example, we note that the two sequences

$$\mathbf{q}_{\pi_1} = (2, \infty), \qquad \mathbf{q}_{\pi_2} = (2, 2, \infty, 2) \tag{6.46}$$

with the accompanying maps

$$\pi_1[v_1, v_2] = v_1 v_2, \qquad \pi_2[v_1, v_2, v_3, v_4] = v_1 h \sum_{-;h} v_2 v_3 v_4 \tag{6.47}$$

both satisfy (hπ). In addition, $(\infty, 2)$ and $(2, \infty)$ are both admissable sequences for π_1, while

$$(2, \infty, 2, 2), \qquad (2, 2, \infty, 2), \qquad (2, 2, 2, \infty) \tag{6.48}$$

are all admissable for π_2.

Our goal is to study nonlinear functions of the form

$$\Omega_{h;\kappa} \ni U \mapsto \pi[f_1(U), \ldots, f_k(U)], \tag{6.49}$$

in which each nonlinearity f_i has zero-th and first order approximants f_{apx} and f_{lin} in the sense of (hϕ). In particular, we impose the following condition on each of the nonlinearities.

(hf) We have $Q_f \subset \{2, \infty\}$. For any $U \in \Omega_{h;\kappa}$ and $q \in Q_f$ we have the inclusions

$$f(U) \in \ell^q_h, \qquad f_{\mathrm{apx}}(U) \in \ell^q_h, \qquad f_{\mathrm{lin};U} \in \mathcal{L}(\ell^2_h; \ell^q_h). \tag{6.50}$$

In addition, for each $q \in Q_f$ there exists a constant $K_q > 0$ and a semi-norm $[\cdot]_{f;q,h}$ on ℓ^2_h so that the following properties are true.

(a) The inequality

$$\|f(U)\|_{\ell^q_h} + \|f_{\mathrm{apx}}(U)\|_{\ell^q_h} \quad \leq \quad K_q \tag{6.51}$$

holds for all $h > 0$ and $U \in \Omega_{h;\kappa}$.

(b) The inequality

$$[V]_{f;q,h} \quad \leq \quad K_q \tag{6.52}$$

holds for all $h > 0$ and $V \in \mathcal{V}_{h;\kappa}$.

(b) The Lipschitz estimate

$$\left\| f(U^{(1)}) - f(U^{(2)}) \right\|_{\ell^q_h} \leq K_q [U^{(1)} - U^{(2)}]_{f;q,h} \tag{6.53}$$

holds for all $h > 0$ and all pairs $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

(c) For every $h > 0$, the inequality

$$\|f_{\mathrm{lin};U}[V]\|_{\ell^q_h} \leq K_q [V]_{f;q,h} \tag{6.54}$$

holds for all $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

To obtain sharp estimates it is sometimes necessary to decompose the approximate linearization f_{lin} into two parts. Both parts can be evaluated in their own preferred norms, which do not necessarily have to be an element of the set Q_f discussed in (hf) above.

$(hf)_{\mathrm{lin}}$ We have $Q^A_{f;\mathrm{lin}} \subset \{2, \infty\}$ and $Q^B_{f;\mathrm{lin}} \subset \{2, \infty\}$. For all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we can make the decomposition

$$f_{\mathrm{lin};U}[V] = f^A_{\mathrm{lin};U}[V] + f^B_{\mathrm{lin};U}[V], \tag{6.55}$$

in which $f^A_{\mathrm{lin};U}[V] \in \ell^q_h$ for every $q \in Q^A_{f;\mathrm{lin}}$ and $f^B_{\mathrm{lin};U}[V] \in \ell^q_h$ for every $q \in Q^B_{f;\mathrm{lin}}$.

Our final condition concerns the residual term

$$f_{\mathrm{nl};U}(V) = f(U + V) - f(U) - f_{\mathrm{lin};U}[V], \tag{6.56}$$

which at times also needs to be decomposed into two parts that require separate norms.

$(hf)_{\mathrm{nl}}$ We have $Q^A_{f;\mathrm{nl}} \subset \{2, \infty\}$ and $Q^B_{f;\mathrm{nl}} \subset \{2, \infty\}$. For all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$, we can make the decomposition

$$f_{\mathrm{nl};U}(V) = f^A_{\mathrm{nl};U}(V) + f^B_{\mathrm{nl};U}(V), \tag{6.57}$$

in which $f^A_{\mathrm{nl};U}(V) \in \ell^q_h$ for every $q \in Q^A_{f;\mathrm{nl}}$ and $f^B_{\mathrm{nl};U}(V) \in \ell^q_h$ for every $q \in Q^B_{f;\mathrm{nl}}$.

Lemma 6.6. *Fix $k \geq 1$ and $0 < \kappa < \frac{1}{12}$, consider the pair $(\mathbf{q}_\pi, \pi)$ defined in (6.38)-(6.39) and assume that $(h\pi)$ holds. In addition, consider a set*

$$\{f_i, f_{i;\mathrm{apx}}, f_{i;\mathrm{lin}}, Q_{f_i}, Q^A_{f_i;\mathrm{lin}}, Q^A_{f_i;\mathrm{nl}}, Q^B_{f_i;\mathrm{lin}}, Q^B_{f_i;\mathrm{nl}}\}_{i=1}^k \tag{6.58}$$

of nonlinearities with their associated approximants and exponents that satisfy the following properties.

(a) For every $1 \leq i \leq k$, the set $\{f_i, f_{i;\mathrm{apx}}, f_{i;\mathrm{nl}}, Q_{f_i}\}$ satisfies (hf) and the set $\{f_{i;\mathrm{lin}}, Q^A_{f_i;\mathrm{lin}}, Q^B_{f_i;\mathrm{lin}}\}$ satisfies $(hf)_{\mathrm{lin}}$. In addition, recalling the definition (6.56), the set $\{f_{i;\mathrm{nl}}, Q^A_{f_i;\mathrm{nl}}, Q^B_{f_i;\mathrm{nl}}\}$ satisfies $(hf)_{\mathrm{nl}}$.

(b) We have $q_{\pi;i} \in Q_{f_i}$ for $1 \leq i \leq k$.

(c) For every $1 \leq i \leq k$, there are $\sigma^A_{i;\mathrm{nl}} \in Q^A_{f_i;\mathrm{nl}}$ and $\sigma^B_{i;\mathrm{nl}} \in Q^B_{f_i;\mathrm{nl}}$ together with sets

$$q^A_{i,\mathrm{nl}} = (q^A_{i,\mathrm{nl};1}, \dots, q^A_{i,\mathrm{nl};k}), \qquad q^B_{i,\mathrm{nl}} = (q^B_{i,\mathrm{nl};1}, \dots, q^B_{i,\mathrm{nl};k}) \tag{6.59}$$

that are admissible for π, which have

$$q^A_{i,\mathrm{nl};i} = \sigma^A_{i;\mathrm{nl}}, \qquad q^B_{i,\mathrm{nl};i} = \sigma^B_{i;\mathrm{nl}} \tag{6.60}$$

and

$$q^A_{i,\mathrm{nl};j} \in Q_{f_j}, \qquad q^B_{i,\mathrm{nl};j} \in Q_{f_j} \tag{6.61}$$

for all $j \neq i$.

(d) For every pair $(i,j) \in \{1, \dots, k\}^2$ with $i \neq j$, there are

$$\sigma^A_{ij;\mathrm{lin}} \in Q^A_{f_i;\mathrm{lin}}, \qquad \tau^A_{ij;\mathrm{lin}} \in Q_{f_j}, \qquad \sigma^B_{ij;\mathrm{lin}} \in Q^B_{f_i;\mathrm{lin}}, \qquad \tau^B_{ij;\mathrm{lin}} \in Q_{f_j}, \tag{6.62}$$

together with two sets

$$q^A_{ij,\mathrm{lin}} = (q^A_{ij,\mathrm{lin};1}, \dots, q^A_{ij,\mathrm{lin};k}), \qquad q^B_{ij,\mathrm{lin}} = (q^B_{ij,\mathrm{lin};1}, \dots, q^B_{ij,\mathrm{lin};k}) \tag{6.63}$$

that are admissible for π, which have

$$q^A_{ij,\mathrm{lin};i} = \sigma^A_{ij;\mathrm{lin}}, \qquad q^A_{ij,\mathrm{lin};j} = \tau^A_{ij;\mathrm{lin}} \qquad q^B_{ij,\mathrm{lin};i} = \sigma^B_{ij;\mathrm{lin}}, \qquad q^B_{ij,\mathrm{lin};j} = \tau^B_{ij;\mathrm{lin}} \tag{6.64}$$

and

$$q^A_{ij,\mathrm{lin};k'} \in Q_{f'_k}, \qquad q^B_{ij,\mathrm{lin};k'} \in Q_{f'_k} \tag{6.65}$$

for all $k' \notin \{i,j\}$.

Consider the map

$$P : \Omega_{h;\kappa} \to \ell^2_h, \qquad\qquad U \mapsto \pi[f_1(U), \dots, f_k(U)]. \tag{6.66}$$

For any $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, write

$$\begin{array}{rcl} P_{\mathrm{apx}}(U) & = & \pi[f_{1;\mathrm{apx}}(U), \dots, f_{k;\mathrm{apx}}(U)] \\ P_{\mathrm{lin};U}[V] & = & \pi[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx}}(U), \dots, f_{k;\mathrm{apx}}(U)] \\ & & \quad +\pi[f_{1;\mathrm{apx}}(U), f_{2;\mathrm{lin};U}[V], \dots, f_{k;\mathrm{apx}}(U)] \\ & & \quad +\pi\big[f_{1;\mathrm{apx}}(U), \dots, f_{k-1;\mathrm{apx}}(U), f_{k;\mathrm{lin};U}[V]\big]. \end{array} \tag{6.67}$$

In addition, for any $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$, write

$$P_{\mathrm{nl};U}(V) = P(U+V) - P(U) - P_{\mathrm{lin};U}[V]. \tag{6.68}$$

Then there exists a constant $K > 0$ so that for any $h > 0$ and $U \in \Omega_{h;\kappa}$ the bound

$$\|P(U) - P_{\mathrm{apx}}(U)\|_{\ell_h^2} \le K \sum_{i=1}^{k} \|f_i(U) - f_{i;\mathrm{apx}}(U)\|_{q_{\pi;i}} \tag{6.69}$$

holds, while for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$ we have the estimate

$$\|P_{\mathrm{nl};U}(V)\|_{\ell_h^2} \quad \le \quad K\mathcal{J}_{\mathrm{nl};U}(V) + K\mathcal{J}_{\mathrm{cross};U}(V) + K\mathcal{J}_{\mathrm{apx};U}(V). \tag{6.70}$$

Here we have introduced the expressions

$$\mathcal{J}_{\mathrm{nl};U}(V) \quad = \quad \textstyle\sum_{i=1}^{k} \big[\left\| f^A_{i;\mathrm{nl}}(V) \right\|_{\sigma^A_{i;\mathrm{nl}}} + \left\| f^B_{i;\mathrm{nl}}(V) \right\|_{\sigma^B_{i;\mathrm{nl}}} \big], \tag{6.71}$$

together with

$$\begin{aligned} \mathcal{J}_{\mathrm{cross};U}(V) \quad = \quad & \textstyle\sum_{i=1}^{k} \sum_{j \neq i} \left\| f^A_{i;\mathrm{lin}}[V] \right\|_{\sigma^A_{ij;\mathrm{lin}}} [V]_{f_j;\tau^A_{ij;\mathrm{lin}},h} \\ & + \textstyle\sum_{i=1}^{k} \sum_{j \neq i} \left\| f^B_{i;\mathrm{lin}}[V] \right\|_{\sigma^B_{ij;\mathrm{lin}}} [V]_{f_j;\tau^B_{ij;\mathrm{lin}},h} \end{aligned} \tag{6.72}$$

and finally

$$\begin{aligned} \mathcal{J}_{\mathrm{apx};U}(V) \quad = \quad & \textstyle\sum_{i=1}^{k} \sum_{j \neq i} \left\| f^A_{i;\mathrm{lin}}[V] \right\|_{\sigma^A_{ij;\mathrm{lin}}} \|f_j(U) - f_{j;\mathrm{apx}}(U)\|_{\tau^A_{ij;\mathrm{lin}}} \\ & + \textstyle\sum_{i=1}^{k} \sum_{j \neq i} \left\| f^B_{i;\mathrm{lin}}[V] \right\|_{\sigma^B_{ij;\mathrm{lin}}} \|f_j(U) - f_{j;\mathrm{apx}}(U)\|_{\tau^B_{ij;\mathrm{lin}}}. \end{aligned} \tag{6.73}$$

Proof. Pick $1 \le i \le k$, any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$. We remark that all the primed constants below are independent of these specific choices.

By definition, the condition on V means that

$$V = V^{(a)} - V^{(b)} \tag{6.74}$$

with $V^{(\#)} \in \mathcal{V}_{h;\kappa}$ for $\# \in \{a, b\}$. Exploiting (6.52), this shows that we have the uniform bound

$$[V]_{f_i;q,h} \le C_1' \tag{6.75}$$

for any $q \in Q_{f_i}$. In addition, we may use (6.53) and (6.54) to obtain the rough estimate

$$\begin{aligned} \|f_{i;\mathrm{nl}}(V)\|_{\ell_h^q} \quad & \le \quad \|f_i(U+V) - f_i(U)\|_q + \|f_{i;\mathrm{lin}}[V]\|_{\ell_h^q} \\ & \le \quad C_2'[V]_{i;q,h}, \end{aligned} \tag{6.76}$$

which gives

$$\|f_{i;\mathrm{lin};U}[V]\|_{\ell_h^q} + \|f_{i;\mathrm{nl};U}(V)\|_{\ell_h^q} \le C_3'[V]_{i;q,h}. \tag{6.77}$$

In addition, using (6.51) and (6.75), we obtain the uniform bound

$$\|f_i(U)\|_{\ell_h^q} + \|f_{i;\mathrm{lin};U}[V]\|_{\ell_h^q} + \|f_{i;\mathrm{nl};U}(V)\|_{\ell_h^q} \le C_4' + C_3'[V]_{i;q,h} \le C_5'. \tag{6.78}$$

Observe that

$$
\begin{aligned}
P(U+V)-P(U) &= \pi\big[f_1(U+V),\dots,f_k(U+V)\big]-\pi\big[f_1(U),\dots,f_k(U)\big] \\
&= \pi\big[f_1(U)+f_{1;\mathrm{lin};U}[V]+f_{1;\mathrm{nl};U}(V),\dots,f_k(U)+f_{k;\mathrm{lin};U}[V]+f_{k;\mathrm{nl};U}(V)\big] \\
&\qquad -\pi\big[f_1(U),\dots,f_k(U)\big].
\end{aligned}
\tag{6.79}
$$

In particular, writing

$$
\begin{aligned}
P_{\mathrm{lin};U;I}[V] &= \pi\big[f_{1;\mathrm{lin};U}[V],f_2(U),\dots,f_k(U)\big] \\
&\quad +\pi\big[f_1(U),f_{2;\mathrm{lin};U}[V],\dots,f_k(U)\big] \\
&\quad +\dots+\pi\big[f_1(U),\dots,f_{k-1}(U),f_{k;\mathrm{lin};U}[V]\big],
\end{aligned}
\tag{6.80}
$$

together with

$$P_{\mathrm{nl};U;I}(V)=P(U+V)-P(U)-P_{\mathrm{lin};U;I}[V], \tag{6.81}$$

the bounds (6.77) and (6.78) allow us to expand out $P_{\mathrm{nl};U;I}[V]$ and obtain

$$\|P_{\mathrm{nl};U;I}[V]\|_{\ell_h^2}\le C_6'\mathcal{J}_{\mathrm{nl};U}(V)+C_6'\mathcal{J}_{\mathrm{cross};U}(V). \tag{6.82}$$

Upon writing

$$\mathcal{J}_1 = \pi[f_{1;\mathrm{lin};U}[V],f_2(U),\dots,f_k(U)]-\pi[f_{1;\mathrm{lin};U}[V],f_{2;\mathrm{apx}}(U),\dots,f_{k;\mathrm{apx}}(U)], \tag{6.83}$$

we see by multi-linearity that

$$
\begin{aligned}
\mathcal{J}_1 &= \pi[f_{1;\mathrm{lin};U}[V],f_2(U)-f_{2;\mathrm{apx}}(U),\dots,f_k(U)] \\
&\quad +\pi[f_{1;\mathrm{lin};U}[V],f_{2;\mathrm{apx}}(U),f_3(U)-f_{3;\mathrm{apx}}(U),\dots,f_k(U)] \\
&\quad +\dots+\pi[f_{1;\mathrm{lin};U}[V],f_{2;\mathrm{apx}}(U),\dots,f_{k-1;\mathrm{apx}}(U),f_k(U)-f_{k;\mathrm{apx}}(U)].
\end{aligned}
\tag{6.84}
$$

In particular, exploiting (6.51), we obtain the bound

$$\|\mathcal{J}_1\|_{\ell_h^2}\le C_7'\mathcal{J}_{\mathrm{apx};U}(V). \tag{6.85}$$

Repeating this computation for the remaining indices shows that also

$$\|P_{\mathrm{lin};U}[V]-P_{\mathrm{lin};U;I}[V]\|_{\ell_h^2}\le C_8'\mathcal{J}_{\mathrm{apx};U}(V), \tag{6.86}$$

which establishes (6.70). The estimate (6.69) can be obtained in a similar, but much easier fashion. □

6.3 The reference function

We conclude this section by establishing Propositions 6.1-6.3 . The proofs are relatively direct, exploiting the following scaling results.

Lemma 6.7. *For any $0<\kappa<1$ and $h>0$, we have the bounds*

$$\|U_{\mathrm{ref};\kappa}\|_{\ell_h^\infty}\le 1,\qquad \big\|\partial^+U_{\mathrm{ref};\kappa}\big\|_{\ell_h^\infty}\le\kappa,\qquad \big\|\partial^+\partial^+U_{\mathrm{ref};\kappa}\big\|_{\ell_h^\infty}\le\kappa^2, \tag{6.87}$$

together with

$$\big\|\partial^+U_{\mathrm{ref};\kappa}\big\|_{\ell_h^2}\le 2\kappa^{1/2},\qquad \big\|\partial^+\partial^+U_{\mathrm{ref};\kappa}\big\|_{\ell_h^2}\le 2\kappa^{3/2} \tag{6.88}$$

and finally

$$\|U_{\mathrm{ref};\kappa}-H\|_{\ell_h^2}\le 2\sqrt{2}\kappa^{-1/2}. \tag{6.89}$$

Proof. The uniform bound on $U_{\mathrm{ref};\kappa}$ follows directly from the definition (6.1)-(6.2). Upon computing

$$U'_{\mathrm{ref};\kappa}(\xi) = \kappa U'_{\mathrm{ref};*}(\kappa\xi), \qquad U''_{\mathrm{ref};\kappa}(\xi) = \kappa^2 U''_{\mathrm{ref};*}(\kappa\xi), \tag{6.90}$$

the properties (6.2) immediately yield

$$\left\|U'_{\mathrm{ref};\kappa}\right\|_{L^\infty} \leq \kappa, \qquad \left\|U''_{\mathrm{ref};\kappa}\right\|_{L^\infty} \leq \kappa^2. \tag{6.91}$$

The bounds (6.87) now follow from (5.10) and (5.20).

It is easy to verify that

$$\left\|U'_{\mathrm{ref};*}\right\|^2_{L^2} \leq 4, \qquad \left\|U''_{\mathrm{ref};*}\right\|^2_{L^2} \leq 4. \tag{6.92}$$

This allows us to compute

$$\begin{aligned} \left\|U'_{\mathrm{ref};\kappa}\right\|^2_{L^2} &= \int \kappa^2 \left[U'_{\mathrm{ref};*}(\kappa\tau)\right]^2 d\tau \\ &= \kappa \int \left[U'_{\mathrm{ref};*}(\tau')\right]^2 d\tau' \\ &= \kappa \left\|U'_{\mathrm{ref};*}\right\|^2_{L^2} \\ &\leq 4\kappa. \end{aligned} \tag{6.93}$$

In a similar fashion, we obtain

$$\left\|U''_{\mathrm{ref};\kappa}\right\|^2_{L^2} = \kappa^3 \left\|U'_{\mathrm{ref};*}\right\|^2_{L^2} \leq 4\kappa^3. \tag{6.94}$$

We may now apply (5.10) and (5.20) once more to obtain (6.88).

Since $U_{\mathrm{ref};\kappa}$ is an increasing function, we see that

$$\begin{aligned} h\sum_{j<0} U_{\mathrm{ref};\kappa}(jh)^2 &\leq \int_{-\infty}^0 U_{\mathrm{ref};\kappa}(\tau+h)^2\, d\tau \\ &= \int_{-\infty}^0 U_{\mathrm{ref};*}(\kappa(\tau+h))^2\, d\tau \\ &= \kappa^{-1}\int_{-\infty}^{\kappa h} U_{\mathrm{ref};*}(\tau')^2\, d\tau' \\ &\leq \kappa^{-1}\int_{-2}^{2} U_{\mathrm{ref};*}(\tau')^2\, d\tau' \\ &\leq 4\kappa^{-1}. \end{aligned} \tag{6.95}$$

In a similar fashion, we find

$$h\sum_{j\geq 0}(U_{\mathrm{ref};\kappa}(jh)-1)^2 \quad \leq \quad 4\kappa^{-1} \tag{6.96}$$

and hence

$$\|U_{\mathrm{ref};\kappa} - H\|_{\ell^2_h} \leq 2\sqrt{2}\kappa^{-1/2}, \tag{6.97}$$

as desired. □

Proof of Proposition 6.1. Write $U = U_{\mathrm{ref};\kappa} + V$ with $V \in \mathcal{V}_{h;\kappa}$. Note that Lemma 6.7 implies that

$$\|U_{\mathrm{ref};\kappa}\|_{\ell^\infty_h} + \left\|\partial^+ U_{\mathrm{ref};\kappa}\right\|_{\ell^2_h} + \left\|\partial^+\partial^+ U_{\mathrm{ref};\kappa}\right\|_{\ell^2_h} + \left\|\partial^+\partial^+ U_{\mathrm{ref};\kappa}\right\|_{\ell^\infty_h} \leq 6. \tag{6.98}$$

In particular, we see that

$$\begin{aligned} \|U\|_{\ell^\infty_h} + \|\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+ U\|_{\ell^\infty_h} &< 6 + \tfrac{1}{2}\kappa^{-1} \\ &\leq \kappa^{-1} \end{aligned} \tag{6.99}$$

since $0 < \kappa \leq \frac{1}{12}$. In addition, we see that:

$$\left\|\partial^+ U\right\|_{\ell^\infty_h} \leq \left\|\partial^+ U_{\mathrm{ref};\kappa}\right\|_{\ell^\infty_h} + \left\|\partial^+ V\right\|_{\ell^\infty_h} < \kappa + 1 - 2\kappa = 1 - \kappa, \tag{6.100}$$

as desired.

Finally, we note that

$$g(U) = g(H) + g(U_{\mathrm{ref};\kappa}) - g(H) + g(U_{\mathrm{ref};\kappa} + V) - g(U_{\mathrm{ref};\kappa}). \tag{6.101}$$

Writing

$$M = \sup_{|u| \leq \kappa^{-1} |g'(u)|}, \tag{6.102}$$

we see that

$$\begin{array}{rcl} \|g(U_{\mathrm{ref};\kappa}) - g(H)\|_{\ell^2_h} & \leq & M \, \|U_{\mathrm{ref};\kappa} - H\|_{\ell^2_h} \\ & \leq & 2\sqrt{2} M \kappa^{-1/2}, \\ \|g(U_{\mathrm{ref};\kappa} + V) - g(U_{\mathrm{ref};\kappa})\|_{\ell^2_h} & \leq & M \, \|V\|_{\ell^2_h} \\ & \leq & \frac{1}{2} M \kappa^{-1}. \end{array} \tag{6.103}$$

The desired bound now follows from $g(H) = 0$. □

Proof of Proposition 6.2. Notice first that $\partial^+ H \in \ell^2_h$, which with $U - H \in \ell^2_h$ implies that $\partial^+ U \in \ell^2_h$. Pick $\kappa > 0$ to be so small that

$$\left\|\partial^+ U\right\|_{\ell^\infty_h} < 1 - 4\kappa \tag{6.104}$$

and also

$$\|U\|_{\ell^\infty_h} + \left\|\partial^+ U\right\|_{\ell^2_h} + \left\|\partial^+ \partial^+ U\right\|_{\ell^2_h} + \left\|\partial^+ \partial^+ U\right\|_{\ell^\infty_h} + 6 < \frac{1}{8}\kappa^{-1}. \tag{6.105}$$

In addition, pick $\epsilon_0 > 0$ to be so small that

$$\left\|\partial^+ \widetilde{U}\right\|_{\ell^\infty_h} < 1 - 3\kappa \tag{6.106}$$

and also

$$\left\|\widetilde{U}\right\|_{\ell^\infty_h} + \left\|\partial^+ \widetilde{U}\right\|_{\ell^2_h} + \left\|\partial^+ \partial^+ \widetilde{U}\right\|_{\ell^2_h} + \left\|\partial^+ \partial^+ \widetilde{U}\right\|_{\ell^\infty_h} + 6 < \frac{1}{4}\kappa^{-1} \tag{6.107}$$

whenever $\left\|\widetilde{U} - U\right\|_{\ell^2_h} < \epsilon_0$, which is possible because of the continuous embedding $\ell^2_h \subset \ell^\infty_h$.

For any such $\widetilde{U}$, we write

$$V_\kappa = \widetilde{U} - U_{\mathrm{ref};\kappa}. \tag{6.108}$$

We immediately see

$$\left\|\partial^+ V_\kappa\right\|_{\ell^\infty_h} \leq \left\|\partial^+ \widetilde{U}\right\|_{\ell^\infty_h} + \|U_{\mathrm{ref};\kappa}\|_{\ell^\infty_h} < 1 - 3\kappa + \kappa = 1 - 2\kappa. \tag{6.109}$$

In addition, we have:

$$\|V_\kappa\|_{\ell^\infty_h} + \|\partial^+ V_\kappa\|_{\ell^2_h} + \|\partial^+ \partial^+ V_\kappa\|_{\ell^2_h} + \|\partial^+ \partial^+ V_\kappa\|_{\ell^\infty_h} \quad < \quad \tfrac{1}{4}\kappa^{-1} - 6 + 6 = \tfrac{1}{4}\kappa^{-1}. \tag{6.110}$$

Finally, we note that

$$\begin{array}{lcl} \|V_\kappa\|_{\ell^2_h} & = & \left\|\widetilde{U} - U_{\mathrm{ref};\kappa}\right\|_{\ell^2_h} \\ & \leq & \left\|\widetilde{U} - U\right\|_{\ell^2_h} + \|U - H\|_{\ell^2_h} + \|U_{\mathrm{ref};\kappa} - H\|_{\ell^2_h} \\ & \leq & \epsilon_0 + \|U - H\|_{\ell^2_h} + 2\sqrt{2}[\kappa]^{-1/2}. \end{array} \tag{6.111}$$

By decreasing $\kappa > 0$ even further, which does not destroy the estimates above, we can hence obtain

$$\|V_\kappa\|_{\ell^2_h} < \frac{1}{4}\kappa^{-1}. \tag{6.112}$$

This shows that $V_\kappa \in \mathcal{V}_{h;\kappa}$, as desired. □

Proof of Proposition 6.3. Pick $\kappa > 0$ to be so small that

$$\|u'\|_{L^\infty} < 1 - 4\kappa \tag{6.113}$$

and also

$$\|u\|_{H^1} + \|u'\|_{L^2} + \|u''\|_{L^2} + \|u''\|_{H^1} + 6 \quad < \quad \tfrac{1}{8}\kappa^{-1}. \tag{6.114}$$

Using Lemma 5.1 and the inequality (5.20), we obtain

$$\left\|\mathrm{ev}_\vartheta \partial^+_h u\right\|_{\ell^\infty_h} \leq \|u'\|_{L^\infty} < 1 - 4\kappa \tag{6.115}$$

together with

$$\|\mathrm{ev}_\vartheta u\|_{\ell^\infty_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h u\right\|_{\ell^2_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h \partial^+_h u\right\|_{\ell^2_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h \partial^+_h u\right\|_{\ell^\infty_h} + 6 \quad < \quad \tfrac{1}{8}\kappa^{-1} \tag{6.116}$$

for any $\vartheta \in \mathbb{R}$.

Corollary 5.3 implies that we can pick a small constant $\epsilon_0 > 0$ in such a way that

$$\|\mathrm{ev}_\vartheta v\|_{\ell^{\infty;2}_h} + \|\mathrm{ev}_\vartheta v\|_{\ell^{2;2}_h} < \min\{\kappa, \frac{1}{8}\kappa^{-1}\} \tag{6.117}$$

holds for every $\vartheta \in \mathbb{R}$ and any $v \in H^1$ that satisfies (6.11). Upon writing $w = u + v$ for any such v, we see that

$$\|\mathrm{ev}_\vartheta w\|_{\ell^\infty_h} < 1 - 4\kappa + \kappa < 1 - 3\kappa \tag{6.118}$$

together with

$$\|\mathrm{ev}_\vartheta w\|_{\ell^\infty_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h w\right\|_{\ell^2_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h \partial^+_h w\right\|_{\ell^2_h} + \left\|\mathrm{ev}_\vartheta \partial^+_h \partial^+_h w\right\|_{\ell^\infty_h} + 6 < \frac{1}{4}\kappa^{-1} \tag{6.119}$$

for any $\vartheta \in \mathbb{R}$.

For any such w, we write

$$V_{\kappa;\vartheta} = \mathrm{ev}_\vartheta w - U_{\mathrm{ref};\kappa}. \tag{6.120}$$

We immediately see

$$\left\|\partial^+ V_{\kappa;\vartheta}\right\|_{\ell^\infty_h} \leq \left\|\mathrm{ev}_\vartheta \partial^+_h w\right\|_{\ell^\infty_h} + \left\|\partial^+_h U_{\mathrm{ref};\kappa}\right\|_{\ell^\infty_h} < 1 - 3\kappa + \kappa = 1 - 2\kappa. \tag{6.121}$$

In addition, we have

$$\|V_{\kappa;\vartheta}\|_{\ell^\infty_h} + \|\partial^+ V_{\kappa;\vartheta}\|_{\ell^2_h} + \|\partial^+\partial^+ V_{\kappa;\vartheta}\|_{\ell^2_h} + \|\partial^+\partial^+ V_{\kappa;\vartheta}\|_{\ell^\infty_h} \quad < \quad \tfrac{1}{4}\kappa^{-1} - 6 + 6 = \tfrac{1}{4}\kappa^{-1}. \tag{6.122}$$

Finally, we note that

$$\begin{array}{rcl} \|V_{\kappa;\vartheta}\|_{\ell^2_h} & = & \|\mathrm{ev}_\vartheta w - U_{\mathrm{ref};\kappa}\|_{\ell^2_h} \\ & \leq & \|\mathrm{ev}_\vartheta w - \mathrm{ev}_\vartheta u\|_{\ell^2_h} + \|\mathrm{ev}_\vartheta u - \mathrm{ev}_\vartheta U_{\mathrm{ref};*}\|_{\ell^2_h} \\ & & \qquad + \|\mathrm{ev}_\vartheta U_{\mathrm{ref};*} - U_{\mathrm{ref};*}\|_{\ell^2_h} + \|U_{\mathrm{ref};*} - H\|_{\ell^2_h} + \|U_{\mathrm{ref};\kappa} - H\|_{\ell^2_h} \\ & \leq & \epsilon_0 + 3\,\|u - U_{\mathrm{ref};*}\|_{H^1} + 3\,\|U_{\mathrm{ref};*}(\cdot + \vartheta) - U_{\mathrm{ref};*}(\cdot)\|_{H^1} + 2\sqrt{2} + 2\sqrt{2}[\kappa]^{-1/2}. \end{array} \tag{6.123}$$

By decreasing $\kappa > 0$ even further, which does not destroy the estimates above, we can hence obtain

$$\|V_{\kappa;\vartheta}\|_{\ell^2_h} < \frac{1}{4}\kappa^{-1} \tag{6.124}$$

for all $\vartheta \in [0, h]$. This shows that $V_{\kappa;\vartheta} \in \mathcal{V}_{h;\kappa}$ for all $\vartheta \in [0, h]$, as desired. □

7 Preliminary estimates

In this section we exploit the bounds in Proposition 6.1 to obtain a number of technical estimates on useful expressions that will help to streamline the arguments in the rest of the paper. In particular, in §7.1-§7.2 we derive preliminary estimates on the gridpoint spacing functions and discrete derivatives that were introduced in §4. In §7.3 we discuss two important error functions and in §7.4 we study several U-dependent linear operators on ℓ^2_h that are encountered when linearizing our main equation (2.26).

7.1 Gridpoint spacing estimates

Our first result here is crucial as it shows that the inverse functions $[r^{\pm}_U]^{-1}$ and γ_U^{-1} can be uniformly bounded on $\Omega_{h;\kappa}$ for all $h > 0$ simultaneously. We use it to simplify the expressions for γ_U^{-k} defined in Lemma 4.2 at the cost of an $O(h)$ error term.

Lemma 7.1. *Fix $h > 0$ and $0 < \kappa < \frac{1}{12}$. Then for any $U \in \Omega_{h;\kappa}$, we have the pointwise estimates*

$$\sqrt{\kappa} < r^{\pm}_U \leq 1, \qquad \sqrt{\kappa} < \gamma_U \leq 1. \tag{7.1}$$

Proof. We compute

$$1 \geq \sqrt{1 - (\partial^{\pm}U)^2} > \sqrt{1 - (1-\kappa)^2} = \sqrt{1 - 1 + 2\kappa - \kappa^2} \geq \sqrt{\kappa}. \tag{7.2}$$

□

Corollary 7.2. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$, we have the estimates*

$$\begin{array}{rcl} \|\gamma_{U^{(1)}} - \gamma_{U^{(2)}}\|_{\ell^\infty_h} & \leq & K\left\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\right\|_{\ell^\infty_h}, \\ \|\gamma_{U^{(1)}} - \gamma_{U^{(2)}}\|_{\ell^2_h} & \leq & K\left\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\right\|_{\ell^2_h}. \end{array} \tag{7.3}$$

Proof. These bounds are a direct consequence of the lower bounds in (7.1) and the representation (4.23). □

Corollary 7.3. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$ we have the bounds*

$$\begin{array}{rcl} \left|\partial^+ r^-_U\right| & \leq & K\left|\partial^0 \partial U\right|, \\ \left|\partial^+ r^0_U\right| & \leq & K\big[\left|\partial^0 \partial U\right| + T^+\left|\partial^0 \partial U\right|\big]. \end{array} \tag{7.4}$$

Proof. These estimates follow directly from Lemma 4.1. □

Lemma 7.4. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimates*

$$\begin{array}{lcl} \left|\partial^+[\gamma_U^2] + 2\partial^0 U S^+[\partial^0\partial U]\right| & \leq & Kh\Big[\left|\partial^0\partial U\right|^2 + T^+\left|\partial^0\partial U\right|^2\Big], \\ \left|\partial^+[\gamma_U] + \gamma_U^{-1}\partial^0 U S^+[\partial^0\partial U]\right| & \leq & Kh\Big[\left|\partial^0\partial U\right|^2 + T^+\left|\partial^0\partial U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-1}] - \gamma_U^{-3}\partial^0 U S^+[\partial^0\partial U]\right| & \leq & Kh\Big[\left|\partial^0\partial U\right|^2 + T^+\left|\partial^0\partial U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-2}] - 2\gamma_U^{-4}\partial^0 U S^+[\partial^0\partial U]\right| & \leq & Kh\Big[\left|\partial^0\partial U\right|^2 + T^+\left|\partial^0\partial U\right|^2\Big], \\ \left|\partial^+[\gamma_U^{-4}] - 4\gamma_U^{-6}\partial^0 U S^+[\partial^0\partial U]\right| & \leq & Kh\Big[\left|\partial^0\partial U\right|^2 + T^+\left|\partial^0\partial U\right|^2\Big]. \end{array} \tag{7.5}$$

Proof. Using the representation in Lemma 4.2, we see that

$$\left|\partial^+\gamma_U\right| \leq C_1'\Big[\left|\partial^0\partial U\right| + T^+\left|\partial^0\partial U\right|\Big], \tag{7.6}$$

together with

$$\left|\partial^+\partial^0 U\right| = \left|S^+[\partial^0\partial U]\right| \leq \left|\partial^0\partial U\right| + T^+\left|\partial^0\partial U\right|. \tag{7.7}$$

This implies that

$$\begin{array}{lcl} \left|S^+[\partial^0 U] - 2\partial^0 U\right| & \leq & C_2' h\big[\left|\partial^0\partial U\right| + T^+\left|\partial^0\partial U\right|\big], \\ \left|S^+[\gamma_U] - 2\gamma_U\right| + \left|S^+[\gamma_U^2] - 2\gamma_U^2\right| & \leq & C_2' h\big[\left|\partial^0\partial U\right| + T^+\left|\partial^0\partial U\right|\big], \\ \left|P^+[\gamma_U] - \gamma_U^2\right| + \left|P^+[\gamma_U^2] - \gamma_U^4\right| & \leq & C_2' h\big[\left|\partial^0\partial U\right| + T^+\left|\partial^0\partial U\right|\big]. \end{array} \tag{7.8}$$

Since the explicit expressions on the left hand side in (7.5) can be obtained from Lemma 4.2 by making the replacements

$$S^+[\partial^0 U] \mapsto 2\partial^0 U, \qquad S^+\gamma_U \mapsto 2\gamma_U, \qquad S^+\gamma_U^2 \mapsto 2\gamma_U^2, \qquad P^+\gamma_U \mapsto \gamma_U^2 \qquad P^+\gamma_U^2 \mapsto \gamma_U^4, \tag{7.9}$$

the desired estimates follow from the lower bounds for γ_U stated in (7.1). □

7.2 Discrete derivative estimates

In this subsection we obtain several preliminary estimates concerning the discrete derivatives introduced in §4.3 and the mixed expressions $\gamma_U^{-k}\partial^0 U$. We also consider approximations for three sums that can be seen as discrete versions of the integral identities

$$\begin{array}{lcl} \int_{-\infty}^{\tau} \frac{u'(\tau')u''(\tau')}{\sqrt{1-u'(\tau')^2}} d\tau' & = & 1 - \sqrt{1-u'(\tau)^2}, \\ \int_{-\infty}^{\tau} \frac{u'(\tau')u''(\tau')}{1-u'(\tau')^2}\, d\tau' & = & \frac{1}{2}\ln[1-u'(\tau)^2], \end{array} \tag{7.10}$$

together with

$$\int_{-\infty}^{\tau} \frac{u'(\tau')v''(\tau')}{\sqrt{1-u'(\tau')^2}} = \frac{u'(\tau)v'(\tau)}{\sqrt{1-u'(\tau)^2}} - \int_{-\infty}^{\tau} \frac{u''(\tau')v'(\tau')}{(1-u'(\tau')^2)^{3/2}}\, d\tau'. \tag{7.11}$$

Lemma 7.5. *Fix $h > 0$ and $0 < \kappa < \frac{1}{12}$. Then for any $U \in \Omega_{h;\kappa}$, we have the inclusions*

$$\big\{\mathcal{F}^{\diamond_\pm}(U), \mathcal{F}^{\diamond_0}(U), \mathcal{F}^{\diamond\diamond_0}(U), \mathcal{F}^{\diamond_0;+}(U), \mathcal{F}^{\diamond_-;+}(U), \mathcal{F}^{\diamond\diamond_0;+}(U)\big\} \subset \ell_h^2. \tag{7.12}$$

Proof. Proposition 6.1 implies that $\partial^{\pm} U \in \ell_h^2$, Together with Lemma 7.1 and the identity (4.28), this implies the inclusions

$$\mathcal{F}^{\diamond_\pm}(U) \in \ell_h^2, \qquad \mathcal{F}^{\diamond_0}(U) \in \ell_h^2. \tag{7.13}$$

Since $\partial^{\pm}(\ell_h^2) \subset \ell_h^2$, the remaining inclusions can be read off from the definitions (4.30), (4.33) and (4.34). □

Corollary 7.6. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise bounds*

$$\begin{array}{lcl} |\partial^+ \mathcal{F}^{\diamond_-}(U)| & \leq & K \left|\partial^0 \partial U\right|, \\ |\partial^+ \mathcal{F}^{\diamond_0}(U)| & \leq & K \big[\left|\partial^0 \partial U\right| + T^+ \left|\partial^0 \partial U\right| \big]. \end{array} \tag{7.14}$$

Proof. This follows directly from Lemma 4.3. □

Lemma 7.7. *Fix $h > 0$ and $0 < \kappa < \frac{1}{12}$. Then for any $U \in \Omega_{h;\kappa}$, we have the pointwise bounds*

$$\frac{1}{2}\kappa < \mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_\pm}(U) + 1 < \frac{3}{2}\kappa^{-1}. \tag{7.15}$$

Proof. We compute

$$\begin{array}{lcl} \mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_+}(U) + 1 & = & 1 + \frac{\partial^+ U + \partial^- U}{r_U^- + r_U^+} \frac{\partial^+ U}{r_U^+} \\ & = & \frac{r_U^+(r_U^- + r_U^+) + (\partial^+ U)^2 + \partial^- U \partial^+ U}{r_U^+(r_U^- + r_U^+)}. \end{array} \tag{7.16}$$

Since $(r_U^+)^2 + (\partial^+ U)^2 = 1$, we obtain

$$\mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_+}(U) + 1 = \frac{r_U^+ r_U^-}{r_U^+(r_U^- + r_U^+)} + \frac{1 + \partial^- U \partial^+ U}{r_U^+(r_U^- + r_U^+)}. \tag{7.17}$$

Observe that $|\partial^- U|\,|\partial^+ U| < 1$. In addition, Lemma 7.1 implies

$$2\kappa < r_U^+(r_U^- + r_U^+) \leq 2, \qquad \kappa < r_U^+ r_U^- \leq 1. \tag{7.18}$$

We hence find

$$\frac{\kappa}{2} < \mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_+}(U) + 1 < \frac{3}{2\kappa}, \tag{7.19}$$

as desired. The estimate involving $\mathcal{F}^{\diamond_-}(U)$ can be obtained in the same fashion. □

Lemma 7.8. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left|\partial^+ \big[\frac{\partial^0 U}{\gamma_U}\big] - \gamma_U^{-3} S^+[\partial^0 \partial U]\right| \leq K h \big[\left|\partial^0 \partial U\right| + T^+ \left|\partial^0 \partial U\right| \big]. \tag{7.20}$$

Proof. Using $\partial^+ \partial^0 U = S^+ \partial^0 \partial U$ and the definition (4.22) for γ_U, we compute

$$\begin{array}{lcl} \partial^+ \big[\frac{\partial^0 U}{\gamma_U}\big] & = & \partial^+[\gamma_U^{-1}] T^+ \partial^0 U + \gamma_U^{-1} \partial^+ \partial^0 U \\ & = & \partial^+[\gamma_U^{-1}] \partial^0 U + \mathcal{E}_1(U) + \gamma_U^{-1} \partial^+ \partial^0 U \\ & = & \gamma_U^{-3} \partial^0 U S^+[\partial^0 \partial U] \partial^0 U + \mathcal{E}_1(U) + \mathcal{E}_2(U) + \gamma_U^{-1} S^+[\partial^0 \partial U] \\ & = & \gamma_U^{-3} S^+[\partial^0 \partial U] + \mathcal{E}_1(U) + \mathcal{E}_2(U), \end{array} \tag{7.21}$$

in which

$$\begin{array}{rcl} \mathcal{E}_1(U) & = & h\partial^+[\gamma_U^{-1}]\partial^+\partial^0 U, \\ \mathcal{E}_2(U) & = & \Big[\partial^+[\gamma_U^{-1}] - \gamma_U^{-3}\partial^0 U S^+[\partial^0\partial U]\Big]\partial^0 U. \end{array} \tag{7.22}$$

The estimate (7.5) now yields the bounds

$$|\mathcal{E}_1(U)| + |\mathcal{E}_2(U)| \le C_1' h\big[\,\big|\partial^0\partial U\big| + T^+\big|\partial^0\partial U\big|\,\big], \tag{7.23}$$

which establishes (7.20). □

Lemma 7.9. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left|\partial^+\big[\frac{\partial^0 U}{\gamma_U^2}\big] - \gamma_U^{-4}(2-\gamma_U^2)S^+\partial^0\partial U\right| \le Kh\big[\,\big|\partial^0\partial U\big| + T^+\big|\partial^0\partial U\big|\,\big]. \tag{7.24}$$

Proof. Using $\partial^+\partial^0 U = S^+\partial^0\partial U$ and the definition (4.22) for γ_U, we compute

$$\begin{array}{rcl} \partial^+\big[\frac{\partial^0 U}{\gamma_U^2}\big] & = & \partial^+[\gamma_U^{-2}]T^+\partial^0 U + \gamma_U^{-2}\partial^+\partial^0 U \\ & = & \partial^+[\gamma_U^{-2}]\partial^0 U + \mathcal{E}_1(U) + \gamma_U^{-2}\partial^+\partial^0 U \\ & = & 2\gamma_U^{-4}\partial^0 U S^+[\partial^0\partial U]\partial^0 U + \mathcal{E}_1(U) + \mathcal{E}_2(U) + \gamma_U^{-2}S^+[\partial^0\partial U] \\ & = & \gamma_U^{-4}(2-\gamma_U^2)S^+[\partial^0\partial U] + \mathcal{E}_1(U) + \mathcal{E}_2(U), \end{array} \tag{7.25}$$

in which

$$\begin{array}{rcl} \mathcal{E}_1(U) & = & h\partial^+[\gamma_U^{-2}]\partial^+\partial^0 U, \\ \mathcal{E}_2(U) & = & \Big[\partial^+[\gamma_U^{-2}] - 2\gamma_U^{-4}\partial^0 U S^+[\partial^0\partial U]\Big]\partial^0 U. \end{array} \tag{7.26}$$

As in the proof of Lemma 7.8, the desired estimate now follows from the bounds (7.5). □

Lemma 7.10. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, the two linear expressions*

$$\begin{array}{rcl} S_{A;U}[V] & = & h\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^0\partial V, \\ S_{B:U}[V] & = & \frac{1}{2}\gamma_U^{-1}[\partial^0 U]\partial^0 V - h\sum_{-;h}\gamma_U^{-3}[\partial^0\partial U]\partial^0 V \end{array} \tag{7.27}$$

satisfy the pointwise estimate

$$|S_{B;U}[V] - S_{A;U}[V]| \quad \le \quad Kh\Big[T^-|\partial^- V| + |\partial^- V| + \big|\partial^0\partial V\big| + \|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2}\Big] \tag{7.28}$$

for all $V \in \ell_h^2$.

Proof. Using (4.11) we first observe that

$$\big|T^+S_{A;U}[V] - S_{A;U}[V]\big| = h\big|\partial^+ S_{A;U}[V]\big| \le C_1'\big|\partial^0\partial V\big|. \tag{7.29}$$

The summation-by-parts identity (4.13) allows us to compute

$$\begin{array}{rcl} T^+S_{A;U}[V] & = & T^+\Big[h\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^0\partial V\Big] \\ & = & T^+\Big[\frac{1}{2}h\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^+\partial^- V\Big] \\ & = & T^+\Big[\frac{1}{2}T^{-1}\Big[\gamma_U^{-1}\partial^0 U\Big]\partial^- V - \frac{1}{2}h\sum_{-;h}\partial^-[\gamma_U^{-1}\partial^0 U]\partial^- V\Big] \\ & = & \frac{1}{2}\gamma_U^{-1}[\partial^0 U]\partial^+ V - \frac{1}{2}h\sum_{-;h}\partial^+[\gamma_U^{-1}\partial^0 U]\partial^+ V. \end{array} \tag{7.30}$$

Upon writing

$$S_{A;U;I}[V] = \frac{1}{2}\gamma_U^{-1}[\partial^0 U]\partial^0 V - \frac{1}{2}h\sum_{-;h}\partial^+[\gamma_U^{-1}\partial^0 U]\partial^0 V, \tag{7.31}$$

we use the identity

$$\partial^+ V - \partial^0 V = h\partial^0\partial U \tag{7.32}$$

together with (7.20) to obtain

$$\left|S_{A;U;I}[V] - T^+ S_{A;U}[V]\right| \le C_2' h \left|\partial^0 \partial V\right| + C_2' h \left\|\partial^+\partial^+ V\right\|_{\ell_h^2}. \tag{7.33}$$

We now write

$$S_{A;U;II}[V] = \frac{1}{2}\gamma_U^{-1}[\partial^0 U]\partial^0 V - \frac{1}{2}h\sum_{-;h}\gamma_U^{-3}S^+[\partial^0\partial U]\partial^0 V, \tag{7.34}$$

which gives

$$S_{A;U;II}[V] - S_{A;U;I}[V] = -\frac{1}{2}h\sum_{-;h}\Big[\partial^+[\gamma_U^{-1}\partial^0 U] - \gamma_U^{-3}S^+[\partial^0\partial U]\Big]\partial^0 V. \tag{7.35}$$

In particular, (7.20) yields

$$|S_{A;U;II}[V] - S_{A;U;I}[V]| \le C_3' h \left\|\partial^+\partial^+ U\right\|_{\ell_h^2}\left\|\partial^0 V\right\|_{\ell_h^2} \le C_4' h\left\|\partial^+ V\right\|_{\ell_h^2}. \tag{7.36}$$

We now transfer the S^+ using the summation-by-parts identity (4.13) to obtain

$$S_{A;U;II}[V] = \frac{1}{2}\gamma_U^{-1}[\partial^0 U]\partial^0 V - \frac{1}{2}hT^-\left[\gamma_U^{-3}\partial^0 V\right]\partial^0\partial U - \frac{1}{2}h\sum_{-;h}S^-\left[\gamma_U^{-3}\partial^0 V\right]\partial^0\partial U. \tag{7.37}$$

We hence see that

$$S_{B;U}[V] - S_{A;U;II}[V] \quad = \quad \tfrac{1}{2}hT^-\left[\gamma_U^{-3}\partial^0 V\right] + \tfrac{1}{2}h\textstyle\sum_{-;h}h\partial^-\left[\gamma_U^{-3}\partial^0 V\right]\partial^0\partial U. \tag{7.38}$$

Using the fact that

$$\left\|\partial^-\left[\gamma_U^{-3}\partial^0 V\right]\right\|_{\ell_h^2} \le C_5'\left[\,\left\|\partial^+ V\right\|_{\ell_h^2} + \left\|\partial^+\partial^+ V\right\|_{\ell_h^2}\,\right] \tag{7.39}$$

the desired estimate follows. □

Lemma 7.11. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left|h\sum_{-;h}\gamma_U^{-1}[\partial^0 U]\partial^0\partial U - \frac{1}{2}(1-\gamma_U)\right| \le Kh. \tag{7.40}$$

Proof. Since $[\gamma_U]_{jh} \to 1$ as $j \to -\infty$, we have

$$\gamma_U - 1 \quad = \quad h\textstyle\sum_{-;h}\partial^+\gamma_U. \tag{7.41}$$

In particular, writing

$$\mathcal{S}_I \quad = \quad h\textstyle\sum_{-;h}\gamma_U^{-1}\partial^0 U S^+[\partial^0\partial U] \tag{7.42}$$

we may use the estimate (7.5) to obtain

$$\begin{array}{rcl} |\mathcal{S}_I - (1-\gamma_U)| & \leq & 2Kh \, \|\partial^+\partial^+ U\|^2_{\ell^2_h} \\ & \leq & C_1' h. \end{array} \tag{7.43}$$

Using the summation-by-parts identity (4.15), we can transfer the S^+ to obtain

$$\mathcal{S}_I \quad = \quad h \sum_{-;h} S^- \big[\gamma_U^{-1} \partial^0 U\big] \partial^0 \partial U + h \partial^0 \partial U T^- \big[\gamma_U^{-1} \partial^0 U\big]. \tag{7.44}$$

In particular, writing

$$\mathcal{I} = \mathcal{S}_I - 2h \sum_{-;h} \gamma_U^{-1} [\partial^0 U] \partial^0 \partial U, \tag{7.45}$$

we see that

$$\mathcal{I} \quad = \quad -h \sum_{-;h} h \partial^- \big[\gamma_U^{-1} \partial^0 U\big] \partial^0 \partial U + h \partial^0 \partial U T^- \big[\gamma_U^{-1} \partial^0 U\big]. \tag{7.46}$$

Using Lemma 7.8 we see that

$$|\mathcal{I}| \leq C_1' h \left\|\partial^+ \partial^+ U\right\|^2_{\ell^2_h} + C_2' h, \tag{7.47}$$

from which the desired estimate follows. □

Lemma 7.12. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for any $h > 0$ and any $U \in \Omega_{h;\kappa}$, we have the pointwise estimate*

$$\left| h \sum_{-;h} \gamma_U^{-2} \partial^0 U S^+ [\partial^0 \partial U] + \ln[\gamma_U] \right| \quad \leq \quad Kh. \tag{7.48}$$

Proof. We first compute

$$\begin{array}{rcl} \partial^+ [\ln \gamma_U] & = & \frac{1}{h} \ln T^+ \gamma_U - \frac{1}{h} \ln \gamma_U \\ & = & \frac{1}{h} \ln \frac{T^+ \gamma_U}{\gamma_U}. \end{array} \tag{7.49}$$

The bounds in Lemma 7.1 imply that

$$\frac{T^+ \gamma_U}{\gamma_U} \geq \sqrt{\kappa}. \tag{7.50}$$

We recall that

$$|\ln(1+x) - x| \leq C_1' \, |x|^2 \tag{7.51}$$

holds for all $x \in \mathbb{R}$ that have $1 + x \geq \sqrt{\kappa} > 0$. Applying this estimate with

$$x = \frac{T^+ \gamma_U}{\gamma_U} - 1 = h \gamma_U^{-1} \partial^+ [\gamma_U], \tag{7.52}$$

we conclude that the sequence

$$\mathcal{I}_1 = \partial^+ [\ln \gamma_U] - \gamma_U^{-1} \partial^+ [\gamma_U] \tag{7.53}$$

satisfies the pointwise bound

$$|\mathcal{I}_1| \quad \leq \quad C_1' h^{-1} \Big[h \gamma_U^{-1} \left| \partial^+ [\gamma_U] \right| \Big]^2. \tag{7.54}$$

Using the explicit expression for $\partial^+[\gamma_U]$ in Lemma 4.2, we conclude

$$|\mathcal{I}_1| \leq C_2' h\Big[\big|\partial^0 \partial U\big|^2 + T^+ \big|\partial^0 \partial U\big|^2 \Big]. \quad (7.55)$$

Writing

$$\mathcal{I}_2 = \partial^+[\ln \gamma_U] + \gamma_U^{-2} \partial^0 U S^+[\partial^0 \partial U], \quad (7.56)$$

the estimate (7.5) implies that also

$$|\mathcal{I}_2| \leq C_3' h\Big[\big|\partial^0 \partial U\big|^2 + T^+ \big|\partial^0 \partial U\big|^2 \Big]. \quad (7.57)$$

In particular, we see that

$$\begin{aligned} \Big|h \textstyle\sum_{-;h} \mathcal{I}_2\Big| &\leq 2C_3' h \, \|\partial^+ \partial^+ U\|_{\ell_h^2}^2 \\ &\leq C_4' h. \end{aligned} \quad (7.58)$$

Since $[\gamma_U]_{jh} \to 1$ as $j \to -\infty$, we conclude that

$$\ln[\gamma_U] = h \sum_{-;h} \partial^+ \big[\ln \gamma_U \big] \quad (7.59)$$

must hold pointwise. The desired estimate follows directly from this identity and the bound (7.58).
□

7.3 Error functions

In the sequel we will encounter two error functions that are small when applied to Ψ_*, but that need to be controlled for arbitrary $U \in \Omega_{h;\kappa}$. In particular, we define the function

$$\mathcal{E}_{\rm sm}(U) = h \partial^- \Big[\gamma_U^{-4} (2 - \gamma_U^2) S^+[\partial^0 \partial U] \Big], \quad (7.60)$$

which measures the smoothness of U in some sense. In addition, we define

$$\mathcal{E}_{\rm tw}(U) = 2\gamma_U^{-4} \partial^0 \partial U + g(U; a) - c_* \tfrac{\partial^0 U}{\gamma_U}, \quad (7.61)$$

which measures the error when U is substituted into a discretization of the travelling wave equation (3.6). Finally, we introduce the function

$$\begin{aligned} \mathcal{E}^+_{\rm tw;apx}(U) = \; & 8\gamma_U^{-6} \partial^0 U S^+[\partial^0 \partial U] T^+[\partial^0 \partial U] + 2\gamma_U^{-4} \partial^+ \partial^0 \partial U \\ & + g'(U) \partial^0 U - c_* \gamma_U^{-3} S^+[\partial^0 \partial U], \end{aligned} \quad (7.62)$$

which can be used to approximate the discrete derivative of (7.61).

Proposition 7.13. *Assume that (Hg) and (HΦ_*) are satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists $K > 0$ so that for any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the a-priori bounds*

$$\begin{aligned} \|\mathcal{E}_{\rm sm}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\rm sm}(U)\|_{\ell_h^2} &\leq K, \\ \|\mathcal{E}_{\rm tw}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\rm tw}(U)\|_{\ell_h^2} &\leq K \end{aligned} \quad (7.63)$$

together with the estimate

$$\big\|\partial^+[\mathcal{E}_{\rm tw}(U)] - \mathcal{E}^+_{\rm tw;apx}(U)\big\|_{\ell_h^\infty} + \big\|\partial^+[\mathcal{E}_{\rm tw}(U)] - \mathcal{E}^+_{\rm tw;apx}(U)\big\|_{\ell_h^2} \leq Kh, \quad (7.64)$$

while for any $U^{(1)} \in \Omega_{h;\kappa}$ *and* $U^{(2)} \in \Omega_{h;\kappa}$ *we have the Lipschitz bounds*

$$\begin{array}{lcl} \left\|\mathcal{E}_{\rm sm}(U^{(1)}) - \mathcal{E}_{\rm sm}(U^{(2)})\right\|_{\ell^2_h} & \leq & K\Big[\left\|\partial^+ U^{(2)} - \partial^+ U^{(1)}\right\|_{\ell^2_h} + \left\|\partial^+\partial^+ U^{(2)} - \partial^+\partial^+ U^{(1)}\right\|_{\ell^2_h} \Big], \\ \left\|\mathcal{E}_{\rm tw}(U^{(1)}) - \mathcal{E}_{\rm tw}(U^{(2)})\right\|_{\ell^2_h} & \leq & K \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{2;2}_h}. \end{array} \tag{7.65}$$

Proof. The bounds in (7.63) and (7.65) follow directly from $\|h\partial^-\|_{\mathcal{L}(\ell^2_h,\ell^2_h)} \leq 2$, the Lipschitz bounds in Corollary 7.2, the estimate (6.8) on $g(U)$ and the pointwise inequality

$$\left|g(U^{(2)}) - g(U^{(1)})\right| \leq \Big[\sup_{|u|\leq\kappa^{-1}} |g'(u)|\Big] \left|U^{(2)} - U^{(1)}\right|. \tag{7.66}$$

In order to establish (7.64), we compute

$$\partial^+[\mathcal{E}_{\rm tw}(U)] \quad = \quad 2\partial^+[\gamma_U^{-4}]T^+[\partial^0\partial U] + 2\gamma_U^{-4}\partial^+\partial^0\partial U + \partial^+[g(U)] - c_*\partial^+[\gamma_U^{-1}\partial^0 U] \tag{7.67}$$

and notice that

$$\partial^+[g(U)] - g'(U)\partial^0 U = \partial^+[g(U)] - g'(U)\partial^+ U + hg'(U)\partial^0\partial U. \tag{7.68}$$

Upon estimating

$$\begin{array}{lcl} |\partial^+[g(U)] - g'(U)\partial^+ U| & = & h^{-1}\,|g(U + h\partial^+ U) - g(U) - g'(U)h\partial^+ U| \\ & \leq & \frac{1}{2}\Big[\sup_{|u|\leq\kappa^{-1}} |g''(u)|\Big] h^{-1}\,|h\partial^+ U|^2 \\ & = & \frac{1}{2}h\Big[\sup_{|u|\leq\kappa^{-1}} |g''(u)|\Big]\,|\partial^+ U|^2, \end{array} \tag{7.69}$$

we can use (7.5) together with (7.20) to obtain the desired bound. □

Proposition 7.14. *Assume that (Hg) and (HΦ_*) are satisfied. Then there exists $K > 0$ so that for any $h > 0$ we have the estimates*

$$\begin{array}{lcl} \|\mathcal{E}_{\rm sm}(\Psi_*)\|_{\ell^\infty_h} + \|\mathcal{E}_{\rm sm}(\Psi_*)\|_{\ell^2_h} & \leq & Kh, \\ \|\mathcal{E}_{\rm tw}(\Psi_*)\|_{\ell^\infty_h} + \|\mathcal{E}_{\rm tw}(\Psi_*)\|_{\ell^2_h} & \leq & Kh, \\ \|\partial^+[\mathcal{E}_{\rm tw}(\Psi_*)]\|_{\ell^\infty_h} + \|\partial^+[\mathcal{E}_{\rm tw}(\Psi_*)]\|_{\ell^2_h} & \leq & Kh. \end{array} \tag{7.70}$$

Proof. We have $\Psi_* \in W^{3;q}$ for $q \in \{2,\infty\}$, which allows us to apply Lemma 5.1 and (5.20) to obtain

$$\begin{array}{lcl} \|\mathcal{E}_{\rm sm}(\Psi_*)\|_{\ell^2_h} & \leq & C_1' h\,\|\partial^-\partial^+\partial^-\Psi_*\|_{\ell^q_h} \\ & \leq & C_1' h\,\|\Psi_*'''\|_{L^q}. \end{array} \tag{7.71}$$

This yields the first bound.

Since γ_* and γ_{Ψ_*} are both uniformly bounded away from zero, we can estimate

$$\left|\gamma_*^{-1} - \gamma_{\Psi_*}^{-1}\right| + \left|\gamma_*^{-3} - \gamma_{\Psi_*}^{-3}\right| + \left|\gamma_*^{-4} - \gamma_{\Psi_*}^{-4}\right| + \left|\gamma_*^{-6} - \gamma_{\Psi_*}^{-6}\right| \leq C_1' \left|\partial^0\Psi_* - \Psi_*'\right|. \tag{7.72}$$

Exploiting the fact that Ψ_*', Ψ_*'', , γ_*^{-1} $\gamma_{\Psi_*}^{-1}$, $\partial^0\Psi_*$ and $\partial^0\partial\Psi_*$ are all uniformly bounded, we now see that

$$\begin{array}{lcl} \left\|\gamma_*^{-1}\Psi_*' - \gamma_{\Psi_*}^{-1}\partial^0\Psi_*\right\|_{\ell^q_h} & \leq & C_2' \left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} \\ \left\|\gamma_*^{-4}\Psi_*'' - 2\gamma_{\Psi_*}^{-1}\partial^0\partial\Psi_*\right\|_{\ell^q_h} & \leq & C_2'\big[\left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} + \left\|2\partial^0\partial\Psi_* - \Psi_*''\right\|_{\ell^q_h} \big] \\ \left\|2\gamma_{\Psi_*}^{-4}\partial^+\partial^0\partial\Psi_* - \gamma_*^{-4}\Psi_*'''\right\|_{\ell^q_h} & \leq & C_2'\big[\left\|\partial^0\Psi_* - \Psi_*'\right\|_{\ell^q_h} + \left\|2\partial^+\partial^0\partial\Psi_* - \Psi_*'''\right\|_{\ell^q_h} \big] \end{array} \tag{7.73}$$

for $q \in \{2, \infty\}$.

Since $\Psi_* \in W^{3,2} \cap W^{3,\infty}$, we may apply Lemma 5.5, and Corollary 5.6 to obtain

$$\big\|\gamma_*^{-1}\Psi_*' - \gamma_{\Psi_*}^{-1}\partial^0\Psi_*\big\|_{\ell^q_h} + \big\|\gamma_*^{-4}\Psi_*'' - 2\gamma_{\Psi_*}^{-4}\partial^0\partial\Psi_*\big\|_{\ell^q_h} + \big\|\gamma_*^{-4}\Psi_*''' - 2\gamma_{\Psi_*}^{-4}\partial^0\partial\Psi_*\big\|_{\ell^q_h} \quad \leq \quad C_3'h \tag{7.74}$$

for $q \in \{2, \infty\}$. The travelling wave equation (3.6) allows us to write

$$\mathcal{E}_{\mathrm{tw}}(\Psi_*) \quad = \quad 2\gamma_{\Psi_*}^{-4}\partial^0\partial\Psi_* - \gamma_*^{-4}\Psi_*'' - c_*\gamma_{\Psi_*}^{-1}\partial^0\Psi_* + c_*\gamma_*^{-1}\Psi_*', \tag{7.75}$$

which using (7.74) yields the second bound.

Using the fact that $\Psi_* \in W^{4,2} \cap W^{4,\infty}$, which allows us to apply Corollary 5.7, we may argue in a similar fashion as above to conclude

$$\begin{array}{lcl} \big\|\gamma_*^{-6}\Psi_*''\Psi_*'\Psi_*'' - 2\gamma_{\Psi_*}^{-6}\partial^0\Psi_* S^+[\partial^0\partial\Psi_*]T^+[\partial^0\partial\Psi_*]\big\|_{\ell^q_h} & \leq & C_4'h, \\ \big\|\gamma_*^{-3}\Psi_*'' - \gamma_{\Psi_*}^{-3}S^+[\partial^0\partial\Psi_*]\big\|_{\ell^q_h} & \leq & C_4'h, \\ \big\|\gamma_*^{-4}\Psi_*''' - 2\gamma_{\Psi_*}^{-4}\partial^+\partial^0\partial\Psi_*\big\|_{\ell^q_h} & \leq & C_4'h, \end{array} \tag{7.76}$$

for $q \in \{2, \infty\}$. The differentiated travelling wave equation (3.7) allows us to write

$$\begin{array}{lcl} \mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*) & = & 8\gamma_{\Psi_*}^{-6}\partial^0\Psi_* S^+[\partial^0\partial\Psi_*]T^+[\partial^0\partial\Psi_*] - 4\gamma_*^{-6}\Psi_*''\Psi_*'\Psi_*'' \\ & & +2\gamma_{\Psi_*}^{-4}\partial^+\partial^0\partial\Psi_* - \gamma_*^{-4}\Psi_*''' \\ & & +g'(\Psi_*)\partial^0\Psi_* - g'(\Psi_*)\Psi_*' \\ & & -c_*\gamma_{\Psi_*}^{-3}S^+[\partial^0\partial\Psi_*] + c_*\gamma_*^{-3}\Psi_*''. \end{array} \tag{7.77}$$

Using (7.76) together with (7.64) we may hence conclude

$$\begin{array}{lcl} \|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)]\|_{\ell^q_h} & \leq & \big\|\mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*)\big\|_{\ell^q_h} + \big\|\partial^+[\mathcal{E}_{\mathrm{tw}}(\Psi_*)] - \mathcal{E}^+_{\mathrm{tw;apx}}(\Psi_*)\big\|_{\ell^q_h} \\ & \leq & C_3'h, \end{array} \tag{7.78}$$

which yields the third bound. □

7.4 The M terms

In the sequel we will often encounter the quantities

$$\begin{array}{lclclcl} M_{U;A}[V] & = & 8\gamma_U^{-4}\partial^0 U[\partial^0\partial U]\partial^0 V, & & M_{U;C}[V] & = & \gamma_U^2 g'(U)V, \\ M_{U;B}[V] & = & 2\gamma_U^{-2}\partial^0\partial V, & & M_{U;D}[V] & = & -c_*\gamma_U^{-1}\partial^0 V. \end{array} \tag{7.79}$$

The discrete derivatives of these terms can be approximated by

$$\begin{array}{lcl} M^+_{U;A;\mathrm{apx}}[V] & = & 16(4\gamma_U^{-6} - 3\gamma_U^{-4})[\partial^0\partial U]^2\partial^0 V + 8\gamma_U^{-4}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V \\ & & +16\gamma_U^{-4}\partial^0 U[\partial^0\partial U]\partial^0\partial V, \\ M^+_{U;B;\mathrm{apx}}[V] & = & 8\gamma_U^{-4}\partial^0 U[\partial^0\partial U]\partial^0\partial V + 2\gamma_U^{-2}\partial^+\partial^0\partial V, \\ M^+_{U;C;\mathrm{apx}}[V] & = & -4\partial^0 U[\partial^0\partial U]g'(U)V + \gamma_U^2 g''(U)[\partial^0 U]V + \gamma_U^2 g'(U)\partial^0 V, \\ M^+_{U;D;\mathrm{apx}}[V] & = & -2c_*\gamma_U^{-3}\partial^0 U[\partial^0\partial U]\partial^0 V - 2c_*\gamma_U^{-1}\partial^0\partial V. \end{array} \tag{7.80}$$

We are specifically interested in the combinations

$$\begin{array}{lcl} M_U[V] & = & M_{U;A}[V] + M_{U;B}[V] + M_{U;C}[V] + M_{U;D}[V], \\ M^+_{U;\mathrm{apx}}[V] & = & M^+_{U;A;\mathrm{apx}}[V] + M^+_{U;B;\mathrm{apx}}[V] + M^+_{U;C;\mathrm{apx}}[V] + M^+_{U;D;\mathrm{apx}}[V], \end{array} \tag{7.81}$$

for which we obtain the following bounds.

Proposition 7.15. *Assume that* (Hg) *is satisfied and fix* $\kappa > 0$. *There exists* $K > 0$ *so that for any* $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$ *we have the a-priori bounds*

$$\begin{array}{lcl} \|M_U[V]\|_{\ell^2_h} & \leq & K\|V\|_{\ell^{2;2}_h}, \\ \|\partial^+ M_U[V]\|_{\ell^2_h} & \leq & K\|V\|_{\ell^{2;3}_h} + K\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|\partial^+ V\|_{\ell^2_h}, \\ \|\partial^+ M_U[V] - M_U[\partial^+ V]\|_{\ell^2_h} & \leq & K\|V\|_{\ell^{2;2}_h} + K\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|\partial^+ V\|_{\ell^2_h}, \end{array} \tag{7.82}$$

together with the estimate

$$\left\|\partial^+ M_U[V] - M^+_{U;\mathrm{apx}}[V]\right\|_{\ell^2_h} \leq Kh\|V\|_{\ell^{2;3}_h} + Kh\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|V\|_{\ell^{2;2}_h}. \tag{7.83}$$

In addition, for any $h > 0$, *any pair* $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ *and any* $V \in \ell^2_h$, *we have the Lipschitz bound*

$$\|M_{U^{(2)}}[V] - M_{U^{(1)}}[V]\|_{\ell^2_h} \leq \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{2;2}_h}\|V\|_{\ell^{\infty;1}_h} + \left\|U^{(2)} - U^{(1)}\right\|_{\ell^{\infty;1}_h}\|V\|_{\ell^{2;2}_h}. \tag{7.84}$$

We will also encounter the expressions

$$\widetilde{M}_{U;\#}[V] = \gamma_U^{-2} M^+_{U;\#;\mathrm{apx}}[V] + 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U] M_{U;\#}[V] \tag{7.85}$$

for $\# \in \{A, B, C, D\}$, together with

$$\widetilde{M}_{U;E}[V] = 8\gamma_U^{-6}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V + 2\gamma_U^{-4}\partial^+\partial^0\partial V. \tag{7.86}$$

The relevant combinations are evaluated explicitly in the second main result of this subsection.

Proposition 7.16. *For any* $\kappa > 0$, $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$, *we have the identities*

$$\begin{array}{lcl} \widetilde{M}_{U;A}[V] + \widetilde{M}_{U;B}[V] + \widetilde{M}_{U;C}[V] & = & 16[6\gamma_U^{-8} - 5\gamma_U^{-6}][\partial^0\partial U]^2\partial^0 V \\ & & +32\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V \\ & & +g''(U)[\partial^0 U]V + g'(U)\partial^0 V, \\ & & +\widetilde{M}_{U;E}[V] \\ \widetilde{M}_{U;D}[V] & = & -6c_*\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V - 2c_*\gamma_U^{-3}\partial^0\partial V. \end{array} \tag{7.87}$$

In the remainder of this subsection we set out to establish these results. We will treat each of the components separately, using the estimates (7.5) to approximate the $\partial^+[\gamma_U^{-k}]$ terms.

Lemma 7.17. *Fix* $\kappa > 0$. *There exist* $K > 0$ *so that for any* $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$ *we have the bound*

$$\|\partial^+ M_{U;A}[V] - M_{U;A}[\partial^+ V]\|_{\ell^2_h} \leq K\|\partial^+ V\|_{\ell^2_h} + K\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|\partial^+ V\|_{\ell^2_h}, \tag{7.88}$$

together with the estimate

$$\begin{array}{lcl} \left\|\partial^+ M_{U;A}[V] - M^+_{U;A;\mathrm{apx}}[V]\right\|_{\ell^2_h} & \leq & Kh\Big[\|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ V\|_{\ell^2_h}\Big] \\ & & +Kh\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|\partial^+ V\|_{\ell^2_h}. \end{array} \tag{7.89}$$

Proof. We compute

$$\begin{array}{lcl} \partial^+ M_{U;A}[V] & = & 8\partial^+[\gamma_U^{-4}]T^+\Big[\partial^0 U[\partial^0\partial U]\partial^0 V\Big] \\ & & +8\gamma_U^{-4}S^+[\partial^0\partial U]T^+[\partial^0\partial U]T^+\partial^0 V \\ & & +8\gamma_U^{-4}\partial^0 U\partial^+[\partial^0\partial U]T^+\partial^0 V \\ & & +8\gamma_U^{-4}\partial^0 U[\partial^0\partial U]S^+[\partial^0\partial V], \end{array} \tag{7.90}$$

together with

$$M_{U;A}[\partial^+ V] \quad = \quad 8\gamma_U^{-4}\partial^0 U[\partial^0 \partial U]S^+[\partial^0 \partial V]. \tag{7.91}$$

The estimate (7.88) now follows directly from inspection.

Upon making the replacements

$$\partial^+[\gamma_U^{-4}] \mapsto 8\gamma_U^{-6}\partial^0 U[\partial^0 \partial U], \qquad T^+ \mapsto I, \qquad S^+ \mapsto 2I, \tag{7.92}$$

we readily see that $\partial^+\big[M_{U;A}[V]\big]$ agrees with $M^+_{U;A;\mathrm{apx}}[V]$. In particular, applying these replacements to each of the four terms in (7.90) separately, we may write

$$\partial^+\big[M_{U;A}[V]\big] - M^+_{U;A;\mathrm{apx}}[V] = \mathcal{J}_a + \mathcal{J}_b + \mathcal{J}_c + \mathcal{J}_d, \tag{7.93}$$

in which

$$\begin{aligned} \mathcal{J}_a \quad = \quad & 8\Big[\partial^+[\gamma_U^{-4}] - 4\gamma_U^{-6}\partial^0 U S^+[\partial^0\partial U]\Big]T^+\Big[\partial^0 U[\partial^0\partial U]\partial^0 V\Big] \\ & +32h\gamma_U^{-6}\partial^0 U\partial^+[\partial^0\partial U]T^+\Big[\partial^0 U[\partial^0\partial U]\partial^0 V\Big] \\ & +64h\gamma_U^{-6}\partial^0 U[\partial^0\partial U]S^+[\partial^0\partial U]T^+[\partial^0\partial U]T^+[\partial^0 V] \\ & +64h\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0 U\partial^+[\partial^0\partial U]T^+[\partial^0 V] \\ & +64h\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0 U[\partial^0\partial U]S^+[\partial^0\partial V], \end{aligned} \tag{7.94}$$

together with

$$\begin{aligned} \mathcal{J}_b \quad = \quad & 8h\gamma_U^{-4}\partial^+[\partial^0\partial U]T^+[\partial^0\partial U]T^+[\partial^0 V] \\ & +16h\gamma_U^{-4}[\partial^0\partial U]\partial^+[\partial^0\partial U]T^+[\partial^0 V] \\ & +16h\gamma_U^{-4}[\partial^0\partial U][\partial^0\partial U]S^+[\partial^0\partial V] \end{aligned} \tag{7.95}$$

and finally

$$\begin{aligned} \mathcal{J}_c \quad &= \quad 8h\gamma_U^{-4}\partial^0 U\partial^+[\partial^0\partial U]S^+[\partial^0\partial V], \\ \mathcal{J}_d \quad &= \quad 8h\gamma_U^{-4}\partial^0 U[\partial^0\partial U]\partial^+[\partial^0\partial V]. \end{aligned} \tag{7.96}$$

The desired estimate (7.89) follows from (7.5) and inspection of the above identities. □

Lemma 7.18. *Fix $\kappa > 0$. There exist $K > 0$ so that for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bound*

$$\|\partial^+ M_{U;B}[V] - M_{U;B}[\partial^+ V]\|_{\ell^2_h} \quad \leq \quad K\,\|\partial^+\partial^+ V\|_{\ell^2_h}\,, \tag{7.97}$$

together with the estimate

$$\begin{aligned} \Big\|\partial^+ M_{U;B}[V] - M^+_{U;B;\mathrm{apx}}[V]\Big\|_{\ell^2_h} \quad \leq \quad & Kh\Big[\,\|\partial^+\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ V\|_{\ell^2_h}\,\Big] \\ & +Kh\,\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\,\|\partial^+\partial^+ V\|_{\ell^2_h}\,. \end{aligned} \tag{7.98}$$

Proof. We compute

$$\partial^+ M_{U;B}[V] \quad = \quad 2\partial^+[\gamma_U^{-2}]T^+[\partial^0\partial V] + 2\gamma_U^{-2}\partial^+\partial^0\partial V, \tag{7.99}$$

together with

$$M_{U;B}[\partial^+ V] \quad = \quad 2\gamma_U^{-2}\partial^+\partial^0\partial V. \tag{7.100}$$

The estimate (7.97) now follows directly from inspection.

Upon making the replacements

$$\partial^+[\gamma_U^{-2}] \mapsto 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U], \qquad T^+ \mapsto I, \tag{7.101}$$

we readily see that $\partial^+\big[M_{U;B}[V]\big]$ agrees with $M^+_{U;B;\mathrm{apx}}[V]$. In particular, we may write

$$\partial^+\big[M_{U;B}[V]\big] - M^+_{U;B;\mathrm{apx}}[V] = \mathcal{J}_a, \tag{7.102}$$

in which

$$\begin{array}{lcl} \mathcal{J}_a & = & 2\Big[\partial^+[\gamma_U^{-2}] - 2\gamma_U^{-4}\partial^0 U S^+[\partial^0\partial U]\Big]T^+[\partial^0\partial V] \\ & & +4h\gamma_U^{-4}\partial^0 U\partial^+[\partial^0\partial U]T^+[\partial^0\partial V] \\ & & +8h\gamma_U^{-4}\partial^0 U[\partial^0\partial U]\partial^+[\partial^0\partial V]. \end{array} \tag{7.103}$$

The desired estimate (7.98) follows from (7.5) and inspection of the above identity. □

Lemma 7.19. *Assume that (Hg) is satisfied and fix $\kappa > 0$. There exist $K > 0$ so that for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bound*

$$\|\partial^+ M_{U;C}[V] - M_{U;C}[\partial^+ V]\|_{\ell^2_h} \quad \le \quad K\|V\|_{\ell^2_h}, \tag{7.104}$$

together with the estimate

$$\begin{array}{lcl} \Big\|\partial^+ M_{U;C}[V] - M^+_{U;C;\mathrm{apx}}[V]\Big\|_{\ell^2_h} & \le & Kh\Big[\|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h}\Big] \\ & & +Kh\|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\|V\|_{\ell^2_h}. \end{array} \tag{7.105}$$

Proof. We compute

$$\begin{array}{lcl} \partial^+ M_{U;C}[V] & = & \partial^+[\gamma_U^2]T^+[g'(U)V] \\ & & +\gamma_U^2\partial^+[g'(U)]T^+V \\ & & +\gamma_U^2 g'(U)\partial^+ V, \end{array} \tag{7.106}$$

together with

$$M_{U;C}[\partial^+ V] \quad = \quad \gamma_U^2 g'(U)\partial^+ V. \tag{7.107}$$

The estimate (7.104) now follows directly from inspection.

Upon making the replacements

$$\partial^+[\gamma_U^{-2}] \mapsto 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U], \qquad T^+ \mapsto I, \qquad \partial^+[g'(U)] \mapsto g''(U)\partial^0 U \tag{7.108}$$

we readily see that $\partial^+\big[M_{U;C}[V]\big]$ agrees with $M^+_{U;C;\mathrm{apx}}[V]$. In particular, applying these replacements to each of the three terms in (7.106) separately, we may write

$$\partial^+\big[M_{U;C}[V]\big] - M^+_{U;C;\mathrm{apx}}[V] = \mathcal{J}_a + \mathcal{J}_b + \mathcal{J}_c, \tag{7.109}$$

in which

$$\begin{array}{lcl} \mathcal{J}_a & = & \Big[\partial^+[\gamma_U^2] + 2\partial^0 U S^+[\partial^0\partial U]\Big]T^+\Big[g'(U)V\Big] \\ & & -2h\partial^0 U\partial^+[\partial^0\partial U]T^+\Big[g'(U)V\Big] \\ & & -4h\partial^0 U[\partial^0\partial U]\partial^+[g'(U)]T^+V \\ & & -4h\partial^0 U[\partial^0\partial U]g'(U)\partial^+ V, \end{array} \tag{7.110}$$

together with

$$\begin{array}{rcl}\mathcal{J}_b & = & \gamma_U^2\big[\partial^+ g'(U) - g''(U)\partial^0 U]T^+ V \\ & & +\gamma_U^2 g''(U)[\partial^+ U - \partial^0 U]T^+ V \\ & & +h\gamma_U^2 g''(U)\partial^0 U \partial^+ V\end{array} \tag{7.111}$$

and finally

$$\begin{array}{rcl}\mathcal{J}_c & = & \gamma_U^2 g'(U)[\partial^+ V - \partial^0 V] \\ & = & h\gamma_U^2 g'(U)\partial^0\partial V.\end{array} \tag{7.112}$$

In order to estimate $\|\mathcal{J}_b\|_{\ell_h^2}$, we recall that $\partial^+ U - \partial^0 U = h\partial^0\partial U$ and compute

$$\begin{array}{rcl}|\partial^+ g'(U) - g''(U)\partial^+ U| & = & h^{-1}\,|g'(U + h\partial^+ U) - g'(U) - g''(U)h\partial^+ U| \\ & \le & \frac{1}{2}\Big[\sup_{|u|\le\kappa^{-1}}|g'''(u)|\Big]h^{-1}\,|h\partial^+ U|^2 \\ & = & \frac{1}{2}h\Big[\sup_{|u|\le\kappa^{-1}}|g'''(u)|\Big]\,|\partial^+ U|^2\,.\end{array} \tag{7.113}$$

The desired estimate (7.105) now follows from (7.5) and inspection of the above identities. □

Lemma 7.20. *Fix $\kappa > 0$. There exist $K > 0$ so that for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ we have the bound*

$$\|\partial^+ M_{U;D}[V] - M_{U;D}[\partial^+ V]\|_{\ell_h^2} \quad \le \quad K\,\|\partial^+ V\|_{\ell_h^2}\,, \tag{7.114}$$

together with the estimate

$$\begin{array}{rcl}\Big\|\partial^+ M_{U;D}[V] - M^+_{U;D;\mathrm{apx}}[V]\Big\|_{\ell_h^2} & \le & Kh\Big[\,\|\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+ V\|_{\ell_h^2} + \|\partial^+\partial^+\partial^+ V\|_{\ell_h^2}\,\Big] \\ & & +Kh\,\|\partial^+\partial^+\partial^+ U\|_{\ell_h^\infty}\,\|\partial^+ V\|_{\ell_h^2}\,.\end{array} \tag{7.115}$$

Proof. We compute

$$\partial^+ M_{U;D}[V] \quad = \quad -c_*\partial^+[\gamma_U^{-1}]T^+\partial^0 V - c_*\gamma_U^{-1}S^+[\partial^0\partial V], \tag{7.116}$$

together with

$$M_{U;D}[\partial^+ V] \quad = \quad -c_*\gamma_U^{-1}S^+[\partial^0\partial V]. \tag{7.117}$$

The estimate (7.114) now follows directly from inspection.

Upon making the replacements

$$\partial^+[\gamma_U^{-1}] \mapsto 2\gamma_U^{-3}\partial^0 U[\partial^0\partial U], \qquad T^+ \mapsto I, \qquad S^+ \mapsto 2I \tag{7.118}$$

we readily see that $\partial^+\big[M_{U;D}[V]\big]$ agrees with $M^+_{U;D;\mathrm{apx}}[V]$. In particular, applying these replacements to each of the two terms in (7.106) separately, we see that

$$\partial^+\big[M_{U;D}[V]\big] - M^+_{U;D;\mathrm{apx}}[V] = \mathcal{J}_a + \mathcal{J}_b, \tag{7.119}$$

in which

$$\begin{array}{rcl}\mathcal{J}_a & = & -c_*\Big[\partial^+[\gamma_U^{-1}] - \gamma_U^{-3}\partial^0 U S^+[\partial^0\partial U]\Big]T^+[\partial^0 V] \\ & & -c_* h\gamma_U^{-3}\partial^0 U\partial^+[\partial^0\partial U]T^+[\partial^0 V] \\ & & -2c_* h\gamma_U^{-3}\partial^0 U[\partial^0\partial U]S^+[\partial^0\partial V],\end{array} \tag{7.120}$$

together with

$$\mathcal{J}_b \quad = \quad -c_* h\partial^+[\partial^0\partial V]. \tag{7.121}$$

The desired estimate (7.115) now follows from (7.5) and inspection of the above identities. □

Proof of Proposition 7.15. The bound for $\|M_U[V]\|_{\ell_h^2}$ and the Lipschitz bound (7.84) follow directly by inspecting the definitions (7.79). The remaining bounds follow from Lemma's 7.17 - 7.20. $\square$

Proof of Proposition 7.16. Direct computations yield

$$\begin{aligned} \widetilde{M}_{U;A}[V] &= 16(4\gamma_U^{-8} - 3\gamma_U^{-6})[\partial^0 \partial U]^2 \partial^0 V \\ &\quad +8\gamma_U^{-6}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V \\ &\quad +16\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V \\ &\quad +32\gamma_U^{-8}\partial^0 U[\partial^0\partial U]\partial^0 U[\partial^0\partial U]\partial^0 V \\ &= 16(6\gamma_U^{-8} - 5\gamma_U^{-6})[\partial^0\partial U]^2\partial^0 V \\ &\quad +8\gamma_U^{-6}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V \\ &\quad +16\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V, \end{aligned} \tag{7.122}$$

together with

$$\begin{aligned} \widetilde{M}_{U;B}[V] &= 8\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V + 2\gamma_U^{-4}\partial^+\partial^0\partial V + 8\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V \\ &= 16\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V + 2\gamma_U^{-4}\partial^+\partial^0\partial V \end{aligned} \tag{7.123}$$

and finally

$$\begin{aligned} \widetilde{M}_{U;C}[V] &= -4\gamma_U^{-2}\partial^0 U[\partial^0\partial U]g'(U)V + g''(U)[\partial^0 U]V + g'(U)\partial^0 V \\ &\quad +4\gamma_U^{-2}\partial^0 U[\partial^0\partial U]g'(U)V \\ &= g''(U)[\partial^0 U]V + g'(U)\partial^0 V. \end{aligned} \tag{7.124}$$

The first identity follows directly from these expressions. To obtain the second identity we compute

$$\begin{aligned} \widetilde{M}_{U;D}[V] &= -2c_*\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V - 2c_*\gamma_U^{-3}\partial^0\partial V \\ &\quad -4c_*\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V \\ &= -6c_*\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V - 2c_*\gamma_U^{-3}\partial^0\partial V. \end{aligned} \tag{7.125}$$

$\square$

8 Gridpoint behaviour

In this section we derive the reduced equation (2.26) by analyzing the function $\mathcal{Y}(U)$ defined in (2.17) and showing that the speed of the gridpoints satisfies

$$\dot{x} = \mathcal{Y}(U). \tag{8.1}$$

In particular, we establish Lemma 2.1 together with Propositions 2.2 and 2.3.

In order to clean up the expressions (2.16)-(2.17), we introduce the functions

$$\begin{aligned} \widetilde{p}(U) &= \tfrac{1}{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)}, \\ p(U) &= \mathcal{F}^{\diamond_+}(U)\widetilde{p}(U), \end{aligned} \tag{8.2}$$

together with

$$q(U) = h^{-1}\ln\big[1 + hp(U)\mathcal{F}^{\diamond_0;+}(U)\big]. \tag{8.3}$$

In addition, we introduce the functions

$$\begin{array}{lcl} \mathcal{Z}^+(U) & = & \exp[\mathcal{Q}(U)], \\ \mathcal{Z}^-(U) & = & \exp[-\mathcal{Q}(U)]. \end{array} \tag{8.4}$$

Our first main result states that these expressions are well-defined, allowing us to obtain reasonably compact expressions for $\mathcal{Y}(U)$ and its discrete derivative.

Proposition 8.1. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. For any $U \in \Omega_{h;\kappa}$, we have the inclusions*

$$\widetilde{p}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad p(U) \in \ell^2(h\mathbb{Z};\mathbb{R}), \qquad q(U) \in \ell^1(h\mathbb{Z};\mathbb{R}), \tag{8.5}$$

together with

$$\mathcal{Q}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad \mathcal{Z}^\pm(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad \mathcal{Y}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}). \tag{8.6}$$

In addition, we have the identity

$$\mathcal{Q}(U) \quad = \quad h\textstyle\sum_{-;h} q(U), \tag{8.7}$$

together with

$$\begin{array}{lcl} \mathcal{Y}(U) & = & -\mathcal{Z}^-(U)h\sum_{-;h} p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big], \\ \partial^+\mathcal{Y}(U) & = & -\frac{1}{2}T^+\big[\mathcal{Z}^-(U)\big]p(U)\mathcal{F}^{\diamond_0;+}(U)S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)] \\ & & \quad -\frac{1}{2}S^+[\mathcal{Z}^-(U)]p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \end{array} \tag{8.8}$$

Finally, for every $U \in \Omega_{h;\kappa}$ we have the limit

$$\lim_{j \to -\infty} \mathcal{Y}_{jh}(U) = 0. \tag{8.9}$$

Our second main result shows that we indeed have $\dot{x} = \mathcal{Y}(U)$, irrespective of whether the full equation (2.2) or the reduced system (2.26) is satisfied.

Proposition 8.2. *Suppose that (Hg) is satisfied. Consider a function $U : [0,T] \to \ell^\infty_h$ for which $U - H \in C^1([0,T];\ell^2_h)$ and*

$$\big\|\partial^+U(t)\big\|_{\ell^\infty_h} < 1 \tag{8.10}$$

for all $0 \le t \le T$. Write

$$\begin{array}{lcl} x(t) & = & x_{\mathrm{eq};h} + h\sum_{-;h}(r^+_{U(t)} - 1) \\ & = & x_{\mathrm{eq};h} - h\sum_{-;h} \frac{\left(\partial^+U(t)\right)^2}{r^+_{U(t)}+1}. \end{array} \tag{8.11}$$

Suppose furthermore that at least one of the following two conditions holds.

(a) The function U satisfies (2.26) on $[0,T]$.

(b) The pair (U,x) satisfies (2.2) on $[0,T]$.

Then there exists $0 < \kappa < \frac{1}{12}$ so that for every $0 \le t \le T$ we have the inclusion

$$U(t) \in \Omega_{h;\kappa}, \tag{8.12}$$

together with the identity

$$\dot{x}(t) = \mathcal{Y}\big(U(t)\big). \tag{8.13}$$

8.1 Basic properties for $\mathcal{Z}$ and $\mathcal{Y}$

In this subsection we show that the definitions above are well-posed. In addition, we establish some basic identities for the discrete derivatives of $\mathcal{Z}$ and $\mathcal{Y}$ that allow us to establish Proposition 8.1.

Lemma 8.3. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. For any $U \in \Omega_{h;\kappa}$ we have the identity*

$$hq(U) \quad = \quad \ln\Big[1 + \mathcal{F}^{\diamond_+}(U) T^+ \mathcal{F}^{\diamond_0}(U)\Big] - \ln\Big[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)\Big], \tag{8.14}$$

together with the inequality

$$\begin{array}{rcl} \exp[hq(U)] & = & 1 + hp(U)\mathcal{F}^{\diamond_0;+}(U) \\ & \geq & \frac{1}{3}\kappa^2. \end{array} \tag{8.15}$$

Proof. We compute

$$\begin{array}{rcl} \exp[hq(U)] & = & 1 + hp(U)\partial^+ \mathcal{F}^{\diamond_0}(U) \\ & = & 1 + \frac{\mathcal{F}^{\diamond_+}(U) h \partial^+ \mathcal{F}^{\diamond_0}(U)}{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)} \\ & = & \frac{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)+\mathcal{F}^{\diamond_+}(U)h\partial^+\mathcal{F}^{\diamond_0}(U)}{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)} \\ & = & \frac{1+\mathcal{F}^{\diamond_+}(U)T^+\mathcal{F}^{\diamond_0}(U)}{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)}, \end{array} \tag{8.16}$$

which directly implies (8.14). In addition, we may use Lemma 7.7 to conclude

$$\begin{array}{rcl} \exp\big[hq(U)\big] & = & \frac{T^+\Big[1+\mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\Big]}{1+\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)} \\ & \geq & \frac{\frac{1}{2}\kappa}{\frac{3}{2}\kappa^{-1}} \\ & = & \frac{1}{3}\kappa^2. \end{array} \tag{8.17}$$

□

Lemma 8.4. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. For any $U \in \Omega_{h;\kappa}$, we have the inclusions*

$$\widetilde{p}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad p(U) \in \ell^2(h\mathbb{Z};\mathbb{R}), \qquad q(U) \in \ell^1(h\mathbb{Z};\mathbb{R}), \tag{8.18}$$

together with the identity

$$\mathcal{Q}(U) \quad = \quad h\textstyle\sum_{-;h} q(U). \tag{8.19}$$

Proof. Note that Lemma 7.7 yields

$$\widetilde{p}(U) = [1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)]^{-1} \in \ell^\infty(h\mathbb{Z};\mathbb{R}). \tag{8.20}$$

Together with Lemma 7.5 this shows that $p(U) \in \ell^2(h\mathbb{Z};\mathbb{R})$.

Since $\frac{d}{dx}[\ln(x)]$ can be uniformly bounded on sets of the form $x \geq \frac{1}{3}\kappa^2 > 0$, the bound (8.15) implies that there exists $C_1' > 1$ for which

$$|hq(U)| \leq C_1' h\, |p(U)|\, \big|\mathcal{F}^{\diamond_0;+}(U)\big|\,. \tag{8.21}$$

Lemma 7.5 implies that $\mathcal{F}^{\diamond_0;+}(U) \in \ell^2(h\mathbb{Z};\mathbb{R})$, allowing us to apply Cauchy-Schwartz to conclude that $q(U) \in \ell^1(h\mathbb{Z};\mathbb{R})$. Finally, the identity (8.19) follows directly from (8.14). □

Lemma 8.5. *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$ *and* $h > 0$*. For any* $U \in \Omega_{h;\kappa}$*, we have the inclusions*

$$\mathcal{Q}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad \mathcal{Z}^\pm(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad \mathcal{Y}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}). \tag{8.22}$$

together with the identity

$$\mathcal{Y}(U) \quad = \quad -\mathcal{Z}^-(U) h \textstyle\sum_{-;h} p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \tag{8.23}$$

In addition, the limit (8.9) holds.

Proof. The inclusions (8.22) for $\mathcal{Q}$ and $\mathcal{Z}^\pm$ follow immediately from Lemma 8.4 and the definitions (8.4). The expression (8.23) follows immediately from the definition (2.17).

We note that we have the inclusions $p(U) \in \ell^2_h$, $\mathcal{F}^{\diamond\diamond_0}(U) \in \ell^2_h$ and $g(U) \in \ell^2_h$ by Lemma 8.4, Lemma 7.5 and Proposition 6.1 respectively. In particular, writing

$$\mathcal{H}(U) = p(U)\mathcal{Z}^+(U)\partial^+[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)] \tag{8.24}$$

we may use the fact that ∂^+ is a bounded operator on $\ell^2(h\mathbb{Z};\mathbb{R})$ to conclude by Cauchy-Schwarz that $\mathcal{H}(U) \in \ell^1(h\mathbb{Z};\mathbb{R})$. The inclusion $\mathcal{Y}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ and the limit (8.9) follow directly from this. □

We remark that we explicitly constructed $\mathcal{Z}^+(U)$ with the aim of satisfying the first identity in (8.26). Indeed, writing $Z = \mathcal{Z}^+$ and attempting to solve this equation, we compute

$$h\partial^+ \ln(Z) = \ln(T^+Z) - \ln(Z) = \ln(1 + hZ^{-1}\partial^+ Z) = \ln[1 + p(U)h\mathcal{F}^{\diamond_0;+}(U)], \tag{8.25}$$

which leads naturally to (8.4). This choice will allow us to use $\mathcal{Z}$ as a discrete version of an integrating factor; see Lemma 8.10 below.

Lemma 8.6. *Fix* $0 < \kappa < \frac{1}{12}$ *and* $h > 0$*. For any* $U \in \Omega_{h;\kappa}$*, we have the identities*

$$\begin{array}{lcl} \partial^+[\mathcal{Z}^+(U)] & = & \mathcal{Z}^+(U)p(U)\mathcal{F}^{\diamond_0;+}(U), \\ \partial^+[\mathcal{Z}^-(U)] & = & -T^+\big[\mathcal{Z}^-(U)\big]p(U)\mathcal{F}^{\diamond_0;+}(U). \end{array} \tag{8.26}$$

Proof. For any $U \in \ell^\infty(h\mathbb{Z};\mathbb{R})$ we observe that

$$\begin{array}{lcl} \partial^+\big[\exp[U]\big] & = & h^{-1}\exp[U]\Big[\exp\big[T^+[U] - U\big] - 1\Big] \\ & = & h^{-1}\exp[U]\Big[\exp\big[h\partial^+[U]\big] - 1\Big]. \end{array} \tag{8.27}$$

This allows us to compute

$$\begin{array}{lcl} \partial^+\mathcal{Z}^+(U) & = & h^{-1}\mathcal{Z}^+(U)\Big[\exp\big[h\partial^+[\mathcal{Q}(U)]\big] - 1\Big] \\ & = & h^{-1}\mathcal{Z}^+(U)\Big[\exp\big[\ln[1 + hp(U)\mathcal{F}^{\diamond_0;+}(U)]\big] - 1\Big] \\ & = & h^{-1}\mathcal{Z}^+(U)\Big[hp(U)\mathcal{F}^{\diamond_0;+}(U)\Big], \end{array} \tag{8.28}$$

which yields the first identity. Using (4.7) we compute

$$\begin{array}{lcl} \partial^+\big[\mathcal{Z}^-(U)\big] & = & \partial^+\big[\frac{1}{\mathcal{Z}^+(U)}\big] \\ & = & -\frac{\mathcal{Z}^+(U)p(U)\mathcal{F}^{\diamond_0;+}(U)}{P^+\mathcal{Z}^+(U)}, \end{array} \tag{8.29}$$

which yields the second identity. □

Lemma 8.7. *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$ *and* $h > 0$. *For any* $U \in \Omega_{h;\kappa}$, *we have the identities*

$$\begin{array}{rcl} \partial^+[\mathcal{Z}^+(U)\mathcal{Y}(U)] & = & -p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big], \\ \partial^+[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)] & = & \widetilde{p}(U)T^+\big[\mathcal{Z}^-(U)\big]\mathcal{F}^{\diamond_0;+}(U). \end{array} \tag{8.30}$$

Proof. The first identity follows from $\mathcal{Z}^+(U)\mathcal{Z}^-(U) = 1$ together with the expression (8.23) for $\mathcal{Y}(U)$. In addition, using (8.26) and (4.5) we compute

$$\begin{array}{rcl} \partial^+[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)] & = & \mathcal{F}^{\diamond_0;+}(U)T^+[\mathcal{Z}^-(U)] - \mathcal{F}^{\diamond_0}(U)T^+\big[\mathcal{Z}^-(U)\big]p(U)\mathcal{F}^{\diamond_0;+}(U) \\ & = & \big(1 - \mathcal{F}^{\diamond_0}(U)p(U)\big)T^+\big[\mathcal{Z}^-(U)\big]\mathcal{F}^{\diamond_0;+}(U). \end{array} \tag{8.31}$$

The second identity follows from

$$\mathcal{F}^{\diamond_0}(U)p(U) = 1 - \widetilde{p}(U). \tag{8.32}$$

□

Proof of Proposition 8.1. In view of Lemma's 8.4 and 8.5, it suffices to establish the identity for $\partial^+\mathcal{Y}$. Exploiting (4.8), (8.26) and (8.30), we compute

$$\begin{array}{rcl} \partial^+\mathcal{Y}(U) & = & \partial^+[\mathcal{Z}^-(U)\mathcal{Z}^+(U)\mathcal{Y}(U)] \\ & = & \frac{1}{2}\partial^+[\mathcal{Z}^-(U)]S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)] \\ & & \quad -\frac{1}{2}S^+[\mathcal{Z}^-(U)]p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big] \\ & = & -\frac{1}{2}\Big[T^+[\mathcal{Z}^-(U)]p(U)\mathcal{F}^{\diamond_0;+}(U)\Big]S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)] \\ & & \quad -\frac{1}{2}S^+[\mathcal{Z}^-(U)]p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big], \end{array} \tag{8.33}$$

as desired. □

8.2 Gridpoint speed

In this subsection we use the discrete derivatives (8.26) and (8.30) to analyze the discrete differential equations that govern the behaviour of the gridpoints. This allows us to establish Proposition 8.2 and the first three main results from §2.

Lemma 8.8. *Consider the setting of Proposition 8.2, but without requiring (a) or (b) to hold. Then there exists* $0 < \kappa < \frac{1}{12}$ *for which the inclusion*

$$U(t) \in \Omega_{h;\kappa} \tag{8.34}$$

holds for all $0 \leq t \leq T$. *In addition, we have*

$$x - x_{\mathrm{eq};h} \in C^1\Big([0,T]; \ell_h^\infty\Big), \tag{8.35}$$

with

$$\dot{x} \quad = \quad -h\textstyle\sum_{-;h}\mathcal{F}^{\diamond_+}(U)\partial^+\dot{U}. \tag{8.36}$$

Finally, we have the limit

$$\lim_{j\to-\infty}\dot{x}_{jh}(t) = 0 \tag{8.37}$$

for every $0 \leq t \leq T$.

Proof. Since $U(t) - H$ is continuous in ℓ_h^2, and the interval $[0, T]$ is compact, the existence of the constant $\kappa > 0$ can be deduced from Proposition 6.2.

Taking a pointwise derivative, we may compute

$$\dot{r}_U^+ = -\mathcal{F}^{\diamond +}(U)\partial^+\dot{U}. \tag{8.38}$$

Writing

$$p_1(U) = \frac{\partial^+ U}{\sqrt{1-(\partial^+U)^2}+1} = \frac{\partial^+ U}{r_U^+ + 1} \tag{8.39}$$

and taking a pointwise derivative, we obtain the identity

$$\begin{array}{rcl} \dot{p}_1(U) & = & \frac{\partial^+\dot{U}}{r_U^+ + 1} + \frac{\partial^+ U}{(r_U^+ + 1)^2}\mathcal{F}^{\diamond +}(U)\partial^+\dot{U} \\ & = & \frac{r_U^+ + 1 + \partial^+ U \mathcal{F}^{\diamond +}(U)}{(r_U^+ + 1)^2}\partial^+\dot{U} \\ & = & \frac{r_U^+ + 1 + [r_U^+]^{-1}(1-(r_U^+)^2)}{(r_U^+ + 1)^2}\partial^+\dot{U} \\ & = & \frac{1}{r_U^+(r_U^+ + 1)}\partial^+\dot{U}. \end{array} \tag{8.40}$$

The embedding $\ell_h^2 \subset \ell_h^\infty$ together with the smoothness assumption on U implies that

$$\begin{array}{rcccl} t & \mapsto & p_1\big(U(t)\big) & \in & C^1([0,T];\ell_h^2), \\ t & \mapsto & \partial^+ U(t) & \in & C^1([0,T];\ell_h^2). \end{array} \tag{8.41}$$

In particular, since the map

$$\pi : \ell_h^2 \times \ell_h^2 \to \ell_h^\infty, \qquad (V^{(1)}, V^{(2)}) \mapsto h\sum_{-;h} V^{(1)}V^{(2)} \tag{8.42}$$

is a bounded bilinear map, we see that

$$t \mapsto \pi\Big[p_1\big(U(t)\big), \partial^+ U(t)\Big] \in C^1([0,T];\ell_h^\infty). \tag{8.43}$$

Since we have

$$x(t) = x_{\mathrm{eq};h} - \pi\big[p_1\big(U(t)\big), \partial^+U(t)\big], \tag{8.44}$$

we may compute

$$\begin{array}{rcl} \dot{x}(t) & = & -\frac{d}{dt}\pi\Big[p_1\big(U(t)\big), \partial^+ U(t)\Big] \\ & = & -\pi\Big[\dot{p}_1\big(U(t)\big), \partial^+ U(t)\Big] - \pi\Big[p_1\big(U(t)\big), \partial^+ \dot{U}(t)\Big] \\ & = & -\pi\Big[\frac{1}{r_{U(t)}^+(r_{U(t)}^+ + 1)}\partial^+\dot{U}(t), \partial^+ U(t)\Big] - \pi\Big[\frac{\partial^+ U(t)}{r_{U(t)}^+ + 1}, \partial^+\dot{U}(t)\Big] \\ & = & -\pi\Big[\big[\frac{1}{r_{U(t)}^+(r_{U(t)}^+ + 1)} + \frac{1}{r_{U(t)}^+ + 1}\big]\partial^+ U(t), \partial^+\dot{U}(t)\Big] \\ & = & -\pi\Big[\mathcal{F}^{\diamond +}\big(U(t)\big), \partial^+\dot{U}(t)\Big], \end{array} \tag{8.45}$$

which gives the desired expression. Finally, the limit (8.37) follows directly from the fact that $\mathcal{F}^{\diamond +}\big(U(t)\big) \in \ell_h^2$ and $\partial^+\dot{U}(t) \in \ell_h^2$, which means that the product is in $\ell^1(h\mathbb{Z};\mathbb{R})$. □

Lemma 8.9. *Consider the setting of Proposition 8.2 and suppose that (a) holds. Then we have* $\dot{x}(t) = \mathcal{Y}\big(U(t)\big)$ *for all* $0 \le t \le T$.

Proof. Exploiting the identity (8.36), we compute

$$\begin{aligned} \dot{x} &= -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+\dot{U} \\ &= -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+\big[\mathcal{F}^{\diamond_0}(U)\mathcal{Y}(U) + 2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big] \\ &= -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+\big[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)\mathcal{Z}^+(U)\mathcal{Y}(U) + 2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big] \\ &= -\tfrac{1}{2}h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+\big[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)\big]S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)\big] \\ &\quad -\tfrac{1}{2}h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)S^+\big[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)\big]\partial^+[\mathcal{Z}^+(U)\mathcal{Y}(U)] \\ &\quad -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \end{aligned} \tag{8.46}$$

Using the definition (8.2) and the identities (8.30), we find

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}h\sum_{-;h} p(U)T^+\big[\mathcal{Z}^-(U)\big]\mathcal{F}^{\diamond_0;+}(U)S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)\big] \\ &\quad +\tfrac{1}{2}h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)S^+\big[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)\big]p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big] \\ &\quad -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \end{aligned} \tag{8.47}$$

Writing

$$\mathcal{H}(U) = \tfrac{1}{2}S^+[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)]p(U)\mathcal{Z}^+(U) - 1, \tag{8.48}$$

we see

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}h\sum_{-;h} p(U)T^+\big[\mathcal{Z}^-(U)\big]\mathcal{F}^{\diamond_0;+}(U)S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)\big] \\ &\quad +h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\mathcal{H}(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \end{aligned} \tag{8.49}$$

Using (8.30) we now compute

$$\begin{aligned} 2\mathcal{H}(U) &= \Big[S^+[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)]p(U) - 2\mathcal{Z}^-(U)\Big]\mathcal{Z}^+(U) \\ &= \Big[2\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)p(U) + h\partial^+[\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)]p(U) - 2\mathcal{Z}^-(U)\Big]\mathcal{Z}^+(U) \\ &= \Big[2\mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U)p(U) + h\widetilde{p}(U)T^+[\mathcal{Z}^-(U)]\mathcal{F}^{\diamond_0;+}(U)p(U) - 2\mathcal{Z}^-(U)\Big]\mathcal{Z}^+(U). \end{aligned} \tag{8.50}$$

Exploiting (8.32) we obtain

$$2\mathcal{H}(U) = \Big[-2\widetilde{p}(U)\mathcal{Z}^-(U) + h\widetilde{p}(U)T^+[\mathcal{Z}^-(U)]\mathcal{F}^{\diamond_0;+}(U)p(U)\Big]\mathcal{Z}^+(U), \tag{8.51}$$

which using (8.26) yields

$$\begin{aligned} 2\mathcal{H}(U) &= \Big[-2\mathcal{Z}^-(U) - h\partial^+[\mathcal{Z}^-(U)]\Big]\widetilde{p}(U)\mathcal{Z}^+(U) \\ &= -S^+[\mathcal{Z}^-(U)]\widetilde{p}(U)\mathcal{Z}^+(U). \end{aligned} \tag{8.52}$$

In particular, recalling (8.8) we see

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}h\sum_{-;h} p(U)T^+\big[\mathcal{Z}^-(U)\big]\mathcal{F}^{\diamond_0;+}(U)S^+[\mathcal{Z}^+(U)\mathcal{Y}(U)\big] \\ &\quad -\tfrac{1}{2}h\sum_{-;h} S^+[\mathcal{Z}^-(U)]p(U)\mathcal{Z}^+(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big] \\ &= h\sum_{-;h} \partial^+[\mathcal{Y}(U)]. \end{aligned} \tag{8.53}$$

The desired conclusion $\dot{x} = \mathcal{Y}(U)$ now follows from the limit (8.9). $\square$

Lemma 8.10. *Consider the setting of Proposition 8.2 and suppose that (b) holds. Then we have* $\dot{x}(t) = \mathcal{Y}\big(U(t)\big)$ *for all* $0 \le t \le T$.

Proof. Exploiting the identity (8.36), we compute

$$\begin{array}{lcl} \dot{x} & = & -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+ \dot{U} \\ & = & -h\sum_{-;h} \mathcal{F}^{\diamond_+}(U)\partial^+ \big[\mathcal{F}^{\diamond_0}(U)\dot{x} + 2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \end{array} \tag{8.54}$$

Taking a difference, we obtain

$$\partial^+ \dot{x} + \mathcal{F}^{\diamond_+}(U)\partial^+[\mathcal{F}^{\diamond_0}(U)\dot{x}] \quad = \quad -\mathcal{F}^{\diamond_+}(U)\partial^+\big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\big]. \tag{8.55}$$

Using (8.26), we now observe that

$$\begin{array}{lcl} \partial^+[\mathcal{Z}^+(U)\dot{x}] & = & \partial^+[\mathcal{Z}^+(U)]T^+\dot{x} + \mathcal{Z}^+(U)\partial^+[\dot{x}] \\ & = & \mathcal{Z}^+(U)p(U)\mathcal{F}^{\diamond_0;+}(U)T^+\dot{x} + \mathcal{Z}^+(U)\partial^+[\dot{x}] \\ & = & \mathcal{Z}^+(U)p(U)\Big[\partial^+[\mathcal{F}^{\diamond_0}(U)\dot{x}] - \mathcal{F}^{\diamond_0}(U)\partial^+\dot{x}\Big] \\ & & \quad +\mathcal{Z}^+(U)\partial^+[\dot{x}] \\ & = & Z^+(U)\Big[\big(1 - p(U)\mathcal{F}^{\diamond_0}(U)\big)\partial^+[\dot{x}] + p(U)\partial^+[\mathcal{F}^{\diamond_0}(U)\dot{x}]\Big]. \end{array} \tag{8.56}$$

In particular, recalling (8.32) we see that

$$\begin{array}{lcl} \partial^+[\mathcal{Z}^+(U)\dot{x}] & = & Z^+(U)\Big[\widetilde{p}(U)\partial^+[\dot{x}] + p(U)\partial^+[\mathcal{F}^{\diamond_0}(U)\dot{x}]\Big] \\ & = & Z^+(U)\widetilde{p}(U)\Big[\partial^+[\dot{x}] + \mathcal{F}^{\diamond_+}(U)\partial^+[\mathcal{F}^{\diamond_0}(U)\dot{x}]\Big]. \end{array} \tag{8.57}$$

Substituting (8.55), we find

$$\partial^+[\mathcal{Z}^+(U)\dot{x}] \quad = \quad -\mathcal{Z}^+(U)p(U)\partial^+\Big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\Big]. \tag{8.58}$$

The limit (8.37) together with the inclusion $\mathcal{Z}^+(U) \in \ell^\infty$ implies that

$$\lim_{j \to -\infty} \mathcal{Z}^+_{jh}(U)\dot{x}_{jh} = 0. \tag{8.59}$$

In particular, we obtain

$$\mathcal{Z}^+(U)\dot{x} \quad = \quad -h\sum_{-;h} \mathcal{Z}^+(U)p(U)\partial^+\Big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U)\Big], \tag{8.60}$$

as desired. □

Proof of Propostion 8.2. The result follows immediately from Lemma's 8.8-8.10. □

Proof of Lemma 2.1. The statements follow directly from Propositions 6.1 and 8.1. □

Proof of Proposition 2.2. Suppose that (b) and (c) are satisfied. Writing

$$y_{jh}(t) = x_{jh}(t) - jh \tag{8.61}$$

we see that

$$\lim_{j \to -\infty} y_{jh}(t) = 0 \tag{8.62}$$

and

$$\partial^+ y(t) = \sqrt{1 - [\partial^+ U(t)]^2} - 1. \tag{8.63}$$

This means that

$$y(t) = h \sum_{-;h} \partial^+ y(t) = h \sum_{-;h} \big[\sqrt{1 - [\partial^+ U(t)]^2} - 1 \big] \tag{8.64}$$

and hence x must satisfy (8.11). Together with (a) and (d), this allows us to apply Proposition 8.2 and conclude that $\dot{x} = \mathcal{Y}(U)$. Item (e) now directly implies that (2.26) holds. □

Proof of Proposition 2.3. Items (a') and (b') together with (8.11) allow us to apply Proposition 8.2 and conclude that $\dot{x} = \mathcal{Y}(U)$. Together with (c') this implies that (2.25) holds. Item (a) follows from Lemma 8.8, (b) and (d) are immediate and finally (c) follows from the fact that $r_U^+ - 1 \in \ell^1(h\mathbb{Z};\mathbb{R})$. □

9 The full nonlinearity

In this section we study the function

$$\mathcal{G}(U) \quad = \quad \mathcal{F}^{\diamond_0}(U)\mathcal{Y}(U) + 2\mathcal{F}^{\diamond\diamond_0}(U) + g(U), \tag{9.1}$$

which contains all the terms on the right-hand side of our main reduced equation (2.26). In addition, we study the discrete derivative

$$\mathcal{G}^+(U) = \partial^+[\mathcal{G}(U)]. \tag{9.2}$$

In principle the results in §4 and §8 provide explicit expressions for all these terms, but the main issue here is that the expression (8.8) features a third order derivative that cannot be controlled uniformly for $U \in \Omega_{h;\kappa}$ and $h > 0$. This is particularly dangerous since we can only except our linear operator to generate two derivatives, in line with the continuous theory developed in §3.

This can be repaired by a discrete summation-by-parts procedure that we carry out in this section. Naturally, the term $\mathcal{G}^+(U)$ will feature third derivatives, but as a consequence of the discrete differentiation the linear operator also generates an extra derivative.

In order to state our results, we need to introduce the three auxiliary functions

$$\begin{aligned} p_A^{\diamond +}(U) &= \tfrac{S^+[1+\mathcal{F}^{\diamond +}(U)\mathcal{F}^{\diamond_0}(U)]}{2P^+[1+\mathcal{F}^{\diamond +}(U)\mathcal{F}^{\diamond_0}(U)]}, \\ p_B^{\diamond +}(U) &= -\tfrac{S^+\mathcal{F}^{\diamond +}(U)S^+\mathcal{F}^{\diamond_0}(U)}{4P^+[1+\mathcal{F}^{\diamond +}(U)\mathcal{F}^{\diamond_0}(U)]}, \\ p^{\diamond_0}(U) &= -\tfrac{S^+\mathcal{F}^{\diamond +}(U)S^+\mathcal{F}^{\diamond +}(U)}{4P^+[1+\mathcal{F}^{\diamond +}(U)\mathcal{F}^{\diamond_0}(U)]} \end{aligned} \tag{9.3}$$

together with the convenient shorthand

$$p^{\diamond +}(U) = p_A^{\diamond +}(U) + p_B^{\diamond +}(U). \tag{9.4}$$

Our first main result here shows how these functions can be used to describe $\partial^+ p(U)$.

Proposition 9.1. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. Then for any $U \in \Omega_{h;\kappa}$ we have the inclusions*

$$p_A^{\diamond +}(U) \in \ell^\infty(h\mathbb{Z};\mathbb{R}), \qquad p_B^{\diamond +}(U) \in \ell^2(h\mathbb{Z};\mathbb{R}), \qquad p^{\diamond_0}(U) \in \ell^2(h\mathbb{Z};\mathbb{R}). \tag{9.5}$$

In addition, we have the identity

$$\partial^+ p(U) = p^{\diamond +}(U)\mathcal{F}^{\diamond +;+}(U) + p^{\diamond_0}(U)\mathcal{F}^{\diamond_0;+}(U). \tag{9.6}$$

We now have all the necessary ingredients to define the functions

$$\begin{aligned} \mathcal{Y}_1(U) &= \mathcal{F}^{\diamond_0}(U)\mathcal{Z}^-(U), \\ \mathcal{Y}_2(U) &= 2\mathcal{F}^{\diamond\diamond_0}(U) + g(U), \end{aligned} \tag{9.7}$$

together with

$$\begin{aligned} \mathcal{X}_A(U) &= p(U)\mathcal{Z}^+(U), \\ \mathcal{X}_B(U) &= S^+[\mathcal{Z}^+(U)]p^{\diamond_+}(U), \\ \mathcal{X}_C(U) &= S^+[\mathcal{Z}^+(U)]p^{\diamond_0}(U), \\ \mathcal{X}_D(U) &= S^+[p(U)]\mathcal{Z}^+(U)p(U). \end{aligned} \tag{9.8}$$

Our second main result shows that these functions can be used to split $\mathcal{G}(U)$ into the four components

$$\begin{aligned} \mathcal{G}_A(U) &= \Big[1 - \mathcal{Y}_1(U)T^-\big[\mathcal{X}_A\big]\Big]\mathcal{Y}_2(U), \\ \mathcal{G}_B(U) &= \tfrac{1}{2}\mathcal{Y}_1(U)h\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_B(U)\mathcal{F}^{\diamond_+;+}(U)\Big], \\ \mathcal{G}_C(U) &= \tfrac{1}{2}\mathcal{Y}_1(U)h\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_C(U)\mathcal{F}^{\diamond_0;+}(U)\Big], \\ \mathcal{G}_D(U) &= \tfrac{1}{2}\mathcal{Y}_1(U)h\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_D(U)\mathcal{F}^{\diamond_0;+}(U)\Big]. \end{aligned} \tag{9.9}$$

Proposition 9.2. *Suppose that* (Hg) *is satisfied and fix* $0 < \kappa < \frac{1}{12}$ *and* $h > 0$*. Then for any* $U \in \Omega_{h;\kappa}$ *we have the identity*

$$\mathcal{G}(U) = \mathcal{G}_A(U) + \mathcal{G}_B(U) + \mathcal{G}_C(U) + \mathcal{G}_D(U). \tag{9.10}$$

Turning to $\mathcal{G}^+(U)$, we introduce the functions

$$\mathcal{Y}_1^+(U) = \partial^+[\mathcal{Y}_1(U)], \qquad \mathcal{Y}_2^+(U) = \partial^+[\mathcal{Y}_2(U)]. \tag{9.11}$$

Using (8.30), one readily obtains the identities

$$\begin{aligned} \mathcal{Y}_1^+(U) &= \widetilde{p}(U)\mathcal{F}^{\diamond_0;+}(U)T^+[\mathcal{Z}^-(U)], \\ \mathcal{Y}_2^+(U) &= 2\mathcal{F}^{\diamond\diamond_0;+}(U) + \partial^+[g(U)]. \end{aligned} \tag{9.12}$$

In order to isolate the third derivative in $\mathcal{Y}_2^+$, we write

$$\begin{aligned} \mathcal{Y}_{2a}^+(U) &= 2\mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+\partial^0\partial U, \\ \mathcal{Y}_{2b}^+(U) &= 2\Big[\mathcal{F}^{\diamond\diamond_0;+}(U) - \mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+\partial^0\partial U\Big] + \partial^+[g(U)]. \end{aligned} \tag{9.13}$$

Our third main result shows that $\mathcal{G}^+(U)$ can be decomposed into the components

$$\begin{aligned} \mathcal{G}_{A'a}^+(U) &= \Big[1 - \mathcal{Y}_1(U)\mathcal{X}_A(U)\Big]\mathcal{Y}_{2a}^+(U), \\ \mathcal{G}_{A'b}^+(U) &= \Big[1 - \mathcal{Y}_1(U)\mathcal{X}_A(U)\Big]\mathcal{Y}_{2b}^+(U), \\ \mathcal{G}_{A'c}^+(U) &= -\mathcal{Y}_1^+(U)\mathcal{X}_A(U)T^+\big[\mathcal{Y}_2(U)\big], \end{aligned} \tag{9.14}$$

together with

$$\begin{aligned} \mathcal{G}_{B'}^+(U) &= \tfrac{1}{2}\mathcal{Y}_1^+(U)hT^+\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_B(U)\mathcal{F}^{\diamond_+;+}(U)\Big], \\ \mathcal{G}_{C'}^+(U) &= \tfrac{1}{2}\mathcal{Y}_1^+(U)hT^+\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_C(U)\mathcal{F}^{\diamond_0;+}(U)\Big], \\ \mathcal{G}_{D'}^+(U) &= \tfrac{1}{2}\mathcal{Y}_1^+(U)hT^+\textstyle\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_C(U)\mathcal{F}^{\diamond_0;+}(U)\Big]. \end{aligned} \tag{9.15}$$

Proposition 9.3. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. Then for any $U \in \Omega_{h;\kappa}$ we have the identity*

$$\mathcal{G}^+(U) \quad = \quad \mathcal{G}^+_{A'a}(U) + \mathcal{G}^+_{A'b}(U) + \mathcal{G}^+_{A'c}(U) + \mathcal{G}^+_{B'}(U) + \mathcal{G}^+_{C'}(U) + \mathcal{G}^+_{D'}(U). \tag{9.16}$$

We provide the proof for these three results in §9.1. We conclude in §9.2 by analyzing the structure of our decompositions. In particular, each term can be written as a sum of products that can be described in the terminology of §6.2.

9.1 Summation by parts

Proof of Proposition 9.1. The inclusions follow directly from Lemma's 7.5 and 7.7. In addition, we may use (4.8) to compute

$$\begin{array}{rcl} \partial^+ p(U) & = & \partial^+ \big[\frac{\mathcal{F}^{\diamond_+}(U)}{1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)} \big] \\ & = & \frac{S^+[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)]\partial^+[\mathcal{F}^{\diamond_+}(U)]}{2P^+[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)]} - \frac{S^+[\mathcal{F}^{\diamond_+}(U)]\partial^+[\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)]}{2P^+[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U)]} \\ & = & \frac{1}{2}[P^+\widetilde{p}(U)]^{-1} S^+ \big[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U) \big] \mathcal{F}^{\diamond_+;+}(U) \\ & & \quad - \frac{1}{2}[P^+\widetilde{p}(U)]^{-1} S^+ \big[\mathcal{F}^{\diamond_+}(U) \big] \partial^+ \big[\mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U) \big]. \end{array} \tag{9.17}$$

Applying (4.8) once more we obtain the desired decomposition

$$\begin{array}{rcl} \partial^+ p(U) & = & \frac{1}{2}[P^+\widetilde{p}(U)]^{-1} S^+ \big[1 + \mathcal{F}^{\diamond_+}(U)\mathcal{F}^{\diamond_0}(U) \big] \mathcal{F}^{\diamond_+;+}(U) \\ & & \quad - \frac{1}{4}[P^+\widetilde{p}(U)]^{-1} S^+ \big[\mathcal{F}^{\diamond_+}(U) \big] S^+[\mathcal{F}^{\diamond_0}(U)] \mathcal{F}^{\diamond_+;+}(U) \\ & & \quad - \frac{1}{4}[P^+\widetilde{p}(U)]^{-1} S^+ \big[\mathcal{F}^{\diamond_+}(U) \big] S^+ \big[\mathcal{F}^{\diamond_+}(U) \big] \mathcal{F}^{\diamond_0;+}(U). \end{array} \tag{9.18}$$

□

Lemma 9.4. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. Then for any $U \in \Omega_{h;\kappa}$ we have the identity*

$$\partial^+[\mathcal{X}_A(U)] \quad = \quad \tfrac{1}{2}\mathcal{X}_B(U)\mathcal{F}^{\diamond_+;+}(U) + \tfrac{1}{2}\mathcal{X}_C(U)\mathcal{F}^{\diamond_0;+}(U) + \tfrac{1}{2}\mathcal{X}_D(U)\mathcal{F}^{\diamond_0;+}(U). \tag{9.19}$$

Proof. Appying (4.8) and (8.26), we compute

$$\begin{array}{rcl} \partial^+[\mathcal{X}_A(U)] & = & \partial^+[p(U)\mathcal{Z}^+(U)] \\ & = & \frac{1}{2}\partial^+[p(U)]S^+[\mathcal{Z}^+(U)] + \frac{1}{2}S^+[p(U)]\partial^+[\mathcal{Z}^+(U)] \\ & = & \frac{1}{2}[p_A^{\diamond_+}(U) + p_B^{\diamond_+}(U)]\mathcal{F}^{\diamond_+;+}(U)S^+[\mathcal{Z}^+(U)] \\ & & \quad + \frac{1}{2}p^{\diamond_0}(U)\mathcal{F}^{\diamond_0;+}(U)S^+[\mathcal{Z}^+(U)] \\ & & \quad + \frac{1}{2}S^+[p(U)]p(U)\mathcal{Z}^+(U)\mathcal{F}^{\diamond_0;+}(U), \end{array} \tag{9.20}$$

which yields the desired result. □

Proof of Proposition 9.2. Applying the discrete summation-by-parts formula (4.13) to the expression (8.8) for $\mathcal{Y}$, we obtain

$$\begin{array}{rcl} \mathcal{Y}(U) & = & -\mathcal{Z}^-(U)T^- \big[p(U)\mathcal{Z}^+(U) \big] \big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U) \big] \\ & & \quad + \mathcal{Z}^-(U)h \sum_{-;h} \big[2\mathcal{F}^{\diamond\diamond_0}(U) + g(U) \big] \partial^- \big[p(U)\mathcal{Z}^+(U) \big]. \end{array} \tag{9.21}$$

Exploiting the definitions (9.7)-(9.8), this allows us to write

$$\begin{array}{lcl}\mathcal{G}(U) & = & -\mathcal{Y}_1(U)\mathcal{Y}_2(U)T^-\big[\mathcal{X}_A(U)\big] \\ & & \quad +\mathcal{Y}_1(U)h\sum_{-;h}\mathcal{Y}_2(U)\partial^-\mathcal{X}_A \\ & & \quad +\mathcal{Y}_2(U).\end{array} \tag{9.22}$$

Applying (9.19), we find

$$\begin{array}{lcl}\mathcal{G}(U) & = & \Big[1-\mathcal{Y}_1(U)T^-\big[\mathcal{X}_A(U)\big]\Big]\mathcal{Y}_2(U) \\ & & \quad +\frac{1}{2}\mathcal{Y}_1(U)h\sum_{-;h}\mathcal{Y}_2(U)T^-\Big[\mathcal{X}_B(U)\mathcal{F}^{\diamond_+;+}(U)\Big] \\ & & \quad +\frac{1}{2}\mathcal{Y}_1(U)h\sum_{-;h}\mathcal{Y}_2(U)\Big[\mathcal{X}_C(U)\mathcal{F}^{\diamond_0;+}(U)+\mathcal{X}_D(U)\mathcal{F}^{\diamond_0;+}(U)\Big],\end{array} \tag{9.23}$$

as desired. □

Proof of Proposition 9.3. We use the preliminary expression (9.22) together with (4.5) to compute

$$\begin{array}{lcl}\partial^+[\mathcal{G}(U)] & = & -\mathcal{Y}_1^+(U)T^+[\mathcal{Y}_2(U)]\mathcal{X}_A(U)-\mathcal{Y}_1(U)\mathcal{Y}_2^+(U)\mathcal{X}_A(U)-\mathcal{Y}_1(U)\mathcal{Y}_2(U)\partial^-[\mathcal{X}_A(U)] \\ & & \quad +\mathcal{Y}_1^+(U)hT^+\sum_{-;h}\mathcal{Y}_2(U)\partial^-\mathcal{X}_A+\mathcal{Y}_1(U)\mathcal{Y}_2(U)\partial^-\mathcal{X}_A \\ & & \quad +\mathcal{Y}_2^+(U) \\ & = & -\mathcal{Y}_1^+(U)\mathcal{X}_A(U)T^+\mathcal{Y}_2(U)+\big(1-\mathcal{Y}_1(U)\mathcal{X}_A(U)\big)\mathcal{Y}_2^+(U) \\ & & \quad +\mathcal{Y}_1^+(U)hT^+\sum_{-;h}\mathcal{Y}_2(U)\partial^-\mathcal{X}_A.\end{array} \tag{9.24}$$

Applying (9.19) now yields the desired decomposition. □

9.2 Product structure

The first two results below describe the two types of products that appear in our decompositions of $\mathcal{G}(U)$ and $\mathcal{G}^+(U)$. Both types can be covered by the theory developed in §6.2.

Lemma 9.5. *Pick $k \geq 1$. Assume that*

$$\mathbf{q}_\pi = (q_{\pi;1}, \ldots, q_{\pi;k}) \in \{2, \infty\}^k \tag{9.25}$$

is a sequence containing precisely one 2 and suppose that the map

$$\pi : \ell^{\mathbf{q}_\pi} \to \ell_h^2 \tag{9.26}$$

is given by

$$\pi[v_1, \ldots, v_k] = v_1 v_2 \cdots v_k. \tag{9.27}$$

Then the pair $(\mathbf{q}_\pi, \pi)$ satisfies $(h\pi)$.

Proof. This follows directly from the bound

$$\|v_1 \cdots v_k\|_{\ell_h^2} \leq \|v_1\|_{\ell_h^2} \|v_2\|_{\ell_h^\infty} \cdots \|v_k\|_{\ell_h^\infty} \tag{9.28}$$

and rearrangements thereof. □

Lemma 9.6. *Pick $k_1 \geq 1$ and $k_2 \geq 2$ and write $k = k_1 + k_2$. Assume that*

$$\mathbf{q}_\pi = (q_{\pi;1}, \ldots, q_{\pi;k}) \in \{2, \infty\}^k \tag{9.29}$$

is a sequence containing precisely one 2 in the first k_1 positions and precisely two 2's in the last k_2 positions. Assume also that the map

$$\pi : \ell_h^{\mathbf{q}_\pi} \to \ell_h^2 \tag{9.30}$$

is given by

$$\pi[v_1, \ldots, v_k] = v_1 \cdots v_{k_1} h \sum_{-;h} v_{k_1+1} \cdots v_k. \tag{9.31}$$

Then the pair $(\mathbf{q}_\pi, \pi)$ satisfies $(h\pi)$.

Proof. This follows directly from the bound

$$\|\pi[v_1, \ldots, v_k]\|_{\ell_h^2} \leq \|v_1\|_{\ell_h^2} \|v_2\|_{\ell_h^\infty} \cdots \|v_{k_1}\|_{\ell_h^\infty} \|v_{k_1+1}\|_{\ell_h^2} \|v_{k_1+2}\|_{\ell_h^2} \|v_{k_1+3}\|_{\ell_h^\infty} \cdots \|v_k\|_{\ell_h^\infty} \tag{9.32}$$

and rearrangements thereof. □

We now define the set of nonlinearities

$$\mathcal{S}_{\mathrm{nl}} = \{\mathcal{F}^{\diamond_0}, p, p^{\diamond_0}, p^{\diamond_+}, \mathcal{F}^{\diamond\diamond_0}, \mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}, \mathcal{Z}^+, \mathcal{Z}^-, g\}. \tag{9.33}$$

In addition, for each $f \in \mathcal{S}_{\mathrm{nl}}$ we define a set of preferred exponents $Q_{f;\mathrm{pref}} \subset \{2, \infty\}$ via

$$Q_{f;\mathrm{pref}} = \begin{cases} \{2\} & \text{for } f \in \{\mathcal{F}^{\diamond\diamond_0}, \mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}, g\}, \\ \{\infty\} & \text{for } f \in \{p, p^{\diamond_0}, p^{\diamond_+}, \mathcal{Z}^+, \mathcal{Z}^-\}, \\ \{2, \infty\} & \text{for } f \in \{\mathcal{F}^{\diamond_0}\}. \end{cases} \tag{9.34}$$

Introducing the notation $g^+(U) = \partial^+ g(U)$, we also define

$$\overline{\mathcal{S}}_{\mathrm{nl}} = \mathcal{S}_{\mathrm{nl}} \cup \{\widetilde{p}, \mathcal{I}_{0s}^{\diamond\diamond_0;+}, \mathcal{I}_{ss}^{\diamond\diamond_0;+}, g^+, \partial^0\partial\}, \tag{9.35}$$

together with the preferred exponent sets

$$\overline{Q}_{f;\mathrm{pref}} = \begin{cases} \{2\} & \text{for } f \in \{\mathcal{F}^{\diamond\diamond_0}, \mathcal{F}^{\diamond_-;+}, g, g^+\}, \\ \{\infty\} & \text{for } f \in \{\widetilde{p}, p, p^{\diamond_0}, p^{\diamond_+}, \mathcal{I}_{0s}^{\diamond\diamond_0;+}, \mathcal{I}_{0s}^{\diamond\diamond_0;+}, \mathcal{Z}^+, \mathcal{Z}^-\}, \\ \{2, \infty\} & \text{for } f \in \{\mathcal{F}^{\diamond_0}, \partial^0\partial, \mathcal{F}^{\diamond_0;+}\}. \end{cases} \tag{9.36}$$

Comparing with (9.34), we remark that ∞ was added to $Q_{\mathcal{F}^{\diamond_0;+};\mathrm{pref}}$. This is motivated by the fact that the $\mathcal{G}^+_{A'c}(U)$ term contains a product of this nonlinearity with $\mathcal{F}^{\diamond\diamond_0}$. In any case, we note that for any $f \in \mathcal{S}_{\mathrm{nl}}$ we have

$$Q_{f;\mathrm{pref}} \subset \overline{Q}_{f;\mathrm{pref}}. \tag{9.37}$$

Notice that we are excluding the third derivative from the set $\overline{S}_{\mathrm{nl}}$. Recalling the identity

$$\mathcal{Z}^-(U)\mathcal{Z}^+(U) = 1 \tag{9.38}$$

and using (8.32), we obtain the simplification

$$\begin{aligned} \mathcal{G}^+_{A'a}(U) &= 2[1 - \mathcal{F}^{\diamond_0}(U)p(U)]\mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+\partial^0\partial U \\ &= 2\widetilde{p}(U)\mathcal{I}_+^{\diamond\diamond_0;+}(U)\partial^+\partial^0\partial U. \end{aligned} \tag{9.39}$$

The third derivative requires special attention, but appears here in a relatively straightforward fashion. For this reason, we exclude it from our general statements here and analyze it directly in the sequel.

The following two results state that $\mathcal{G}(U)$ and $\mathcal{G}^+(U)-\mathcal{G}^+_{A'a}(U)$ can be decomposed into products of the two types discussed above. In addition, every product can be estimated in ℓ^2_h by only using norms $\|f(U)\|_{\ell^q_h}$ for which $q \in Q_{f;\mathrm{pref}}$.

Lemma 9.7. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. There exists an integer $N > 1$ together with integers $k_i \geq 1$, sequences*

$$\begin{array}{rclcl} \mathbf{q}_{\pi_i} & = & (q_{\pi_i;1}, \dots, q_{\pi_i;k_i}) & \in & \{2,\infty\}^k, \\ \mathbf{f}_i & = & (f_{i;1}, \dots, f_{i;k_i}) & \in & \mathcal{S}^k_{\mathrm{nl}} \end{array} \tag{9.40}$$

and maps

$$\pi_i : \ell^{\mathbf{q}_{\pi_i}}_h \to \ell^2_h, \tag{9.41}$$

each defined for $1 \leq i \leq N$, so that the following properties hold true.

(i) For each $U \in \Omega_{h;\kappa}$ we have the decomposition

$$\mathcal{G}(U) = \sum_{i=1}^{N} \pi_i\big[f_{i;1}(U), \dots, f_{i;k_i}(U)\big]. \tag{9.42}$$

(ii) For each $1 \leq i \leq N$ the pair $(\mathbf{q}_{\pi_i}, \pi_i)$ satisfies the conditions of either Lemma 9.5 or Lemma 9.6.

(iii) For each $1 \leq i \leq N$ and $1 \leq j \leq k_i$ we have the inclusion

$$q_{\pi_i;j} \in Q_{f_{i;j};\mathrm{pref}}. \tag{9.43}$$

Proof. The desired decomposition can be read off directly from the structure of the terms defined in (9.9). □

Lemma 9.8. *Fix $0 < \kappa < \frac{1}{12}$ and $h > 0$. There exists an integer $N > 1$ together with integers $k_i \geq 1$, sequences*

$$\begin{array}{rclcl} \mathbf{q}_{\pi_i} & = & (q_{\pi_i;1}, \dots, q_{\pi_i;k_i}) & \in & \{2,\infty\}^k, \\ \mathbf{f}_i & = & (f_{i;1}, \dots, f_{i;k_i}) & \in & \overline{\mathcal{S}}^k_{\mathrm{nl}} \end{array} \tag{9.44}$$

and maps

$$\pi_i : \ell^{\mathbf{q}_{\pi_i}}_h \to \ell^2_h, \tag{9.45}$$

each defined for $1 \leq i \leq N$, so that the following properties hold true.

(i) For each $U \in \Omega_{h;\kappa}$ we have the decomposition

$$\mathcal{G}^+(U) - G^+_{A'a}(U) = \sum_{i=1}^{N} \pi_i\big[f_{i;1}(U), \dots, f_{i;k_i}(U)\big]. \tag{9.46}$$

(ii) For each $1 \leq i \leq N$ the pair $(\mathbf{q}_{\pi_i}, \pi_i)$ satisfies the conditions of either Lemma 9.5 or Lemma 9.6.

(iii) For each $1 \le i \le N$ and $1 \le j \le k_i$ we have the inclusion

$$q_{\pi_i;j} \in \overline{Q}_{f_{i;j};\mathrm{pref}}. \tag{9.47}$$

Proof. The desired composition can be read off directly from the structure of the terms defined in (9.14) and (9.15). □

Our final result allows us to construct admissable sequences for our multi-linear maps by simply swapping suitable exponents. This will allow us to deviate from the exponents $Q_{f;\mathrm{pref}}$ defined above in a strategic fashion, which is crucial to obtain useful error bounds.

Lemma 9.9. *Consider the setting of either Lemma 9.5 or Lemma 9.6. Pick any integer $1 \le i_* \le k$ for which $q_{\pi;i_*} = \infty$. Then there is an integer*

$$1 \le j_*[i_*] \le k \tag{9.48}$$

that has

$$q_{\pi;j_*[i_*]} = 2 \tag{9.49}$$

and for which the swapped sequence

$$\mathbf{q}_{i_*} = (q_{i_*;1}, \dots, q_{i_*;k}) \tag{9.50}$$

defined by

$$q_{i_*;j} = \begin{cases} q_{\pi;j} & \text{if } j \notin \{i_*, j_*\}, \\ 2 & \text{if } j = i_*, \\ \infty & \text{if } j = j_*, \end{cases} \tag{9.51}$$

is admissable for π.

Proof. This follows directly by inspecting (9.28) and (9.32). □

10 Component estimates

Our goal in this section is to analyze the nonlinearities $f \in \mathcal{S}_{\mathrm{nl}} \cup \overline{S}_{\mathrm{nl}}$ and introduce the terminology that allows the conditions (hf), $(hf)_{\mathrm{lin}}$ and $(hf)_{\mathrm{nl}}$ to be verified. In particular, we construct suitable approximants f_{apx} and f_{lin} that are accurate to leading order in h, but also tractable to use in our subsequent computations.

In order to apply Lemma 6.6 in a streamlined fashion, we state our estimates that are relevant for (6.72) in terms of the quantities

$$\begin{array}{rclrcl} S_{\mathrm{full}}(V) & = & \|V\|_{\ell_h^{2;2}} + \|\partial^+ V\|_{\ell_h^\infty}, & \overline{S}_{\mathrm{full}}(V) & = & S_{\mathrm{full}}(V) + \|\partial^+\partial^+ V\|_{\ell_h^\infty}, \\ S_{2;\mathrm{fix}}(V) & = & \|V\|_{\ell_h^{2;2}}, & \overline{S}_{2;\mathrm{fix}}(V) & = & S_{2;\mathrm{fix}}(V) \end{array} \tag{10.1}$$

related to the seminorms in (hf), together with the expressions

$$\begin{array}{rclrcl} T_{\mathrm{safe}}(V) & = & \|V\|_{\ell_h^{2;2}}, & \overline{T}_{\mathrm{safe}}(V) & = & T_{\mathrm{safe}}(V), \\ T_{\infty;\mathrm{opt}}(V) & = & \|\partial^+ V\|_{\ell_h^\infty}, & \overline{T}_{\infty;\mathrm{opt}}(V) & = & T_{\infty;\mathrm{opt}}(V) + \|\partial^+\partial^+ V\|_{\ell_h^\infty} \end{array} \tag{10.2}$$

associated to the linear terms in $(hf)_{\mathrm{lin}}$. Finally, we use the functions

$$\begin{array}{rcl} \mathcal{E}_{\mathrm{nl}}(V) & = & (\|V\|_{\ell_h^{2;2}} + \|V\|_{\ell_h^{\infty;1}} + h)\,\|V\|_{\ell_h^{2;2}}, \\ \overline{\mathcal{E}}_{\mathrm{nl}}(V) & = & \mathcal{E}_{\mathrm{nl}}(V) \end{array} \quad (10.3)$$

to control the nonlinear terms (6.71).

We divide our nonlinearities into five distinct groups that are fully described by Propositions 10.1-10.5 in §10.1. In Corollaries 10.6-10.9 we subsequently discuss a number of bookkeeping issues that in §12 will allow us to control the cross-terms (6.72) for $\mathcal{G}(U)$ by

$$\mathcal{J}_{\mathrm{cross};U}(V) = T_{\mathrm{safe}}(V)S_{\mathrm{full}}(V) + T_{\infty;\mathrm{opt}}(V)S_{2;\mathrm{fix}}(V). \quad (10.4)$$

Naturally, the related estimate for $\mathcal{G}^+(U) - \mathcal{G}^+_{A'a}(U)$ will also hold.

The proofs for our estimates can be found in §10.2-§10.6. The main idea is to apply the substitution techniques from §6.1 to the explicit identities derived in §4.

10.1 Estimate summary

The first set of nonlinearities is given by

$$\mathcal{S}_{\mathrm{nl};I} = \{\mathcal{F}^{\diamond_0}\}. \quad (10.5)$$

We define

$$\mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U) \;=\; \gamma_U^{-1}\partial^0 U, \qquad \mathcal{F}^{\diamond_0}_{\mathrm{lin};U}[V] \;=\; \gamma_U^{-3}\partial^0 V. \quad (10.6)$$

For any $f \in \mathcal{S}_{\mathrm{nl};I}$, we write

$$Q_f = \{2,\infty\}, \qquad Q^A_{f;\mathrm{lin}} = Q^B_{f;\mathrm{lin}} = \{2,\infty\}, \qquad Q^A_{f;\mathrm{nl}} = Q^B_{f;\mathrm{nl}} = \{2\} \quad (10.7)$$

and recall that $Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = \{2,\infty\}$.

Proposition 10.1. *Assume that (Hg) is satisfied, fix $0 < \kappa < \frac{1}{12}$ and pick any nonlinearity $f \in \mathcal{S}_{\mathrm{nl};I}$. Then there exists a constant $K > 0$ so that the following properties are true.*

(i) Upon introducing the seminorms

$$\begin{array}{rcccl} [V]_{f;2,h} & = & \|\partial^+ V\|_{\ell_h^2} & \leq & S_{\mathrm{full}}(V), \\ [V]_{f;\infty,h} & = & \|\partial^+ V\|_{\ell_h^\infty} & \leq & S_{\mathrm{full}}(V), \end{array} \quad (10.8)$$

the conditions in (hf) are all satisfied.

(ii) For every $q \in Q_f$, $U \in \Omega_{h;\kappa}$ and $h > 0$ we have the estimate

$$\|f(U) - f_{\mathrm{apx}}(U)\|_{\ell_h^q} \leq Kh. \quad (10.9)$$

(iii) Upon writing $f^B_{\mathrm{lin};U} = 0$, the conditions in $(hf)_{\mathrm{lin}}$ are satisfied. In addition, the bounds

$$\begin{array}{rcccl} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell_h^2} & \leq & K\,\|\partial^+ V\|_{\ell_h^2} & \leq & KT_{\mathrm{safe}}(V), \\ \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell_h^\infty} & \leq & K\,\|\partial^+ V\|_{\ell_h^\infty} & \leq & KT_{\infty;\mathrm{opt}}(V) \end{array} \quad (10.10)$$

hold for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell_h^2$.

(iv) Upon writing $f^B_{\mathrm{nl};U} = 0$, the conditions in $(hf)_{\mathrm{nl}}$ are satisfied. In addition, we have the bound

$$\begin{array}{lcl} \left\| f^A_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} & \leq & K \left\|\partial^+ V\right\|_\infty \left\|\partial^+ V\right\|_{\ell^2_h} + Kh\big[\left\|\partial^+ V\right\|_{\ell^2_h} + \left\|\partial^+\partial^+ V\right\|_{\ell^2_h} \big] \\ & \leq & K\mathcal{E}_{\mathrm{nl}}(V) \end{array} \tag{10.11}$$

for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

The second set of nonlinearities is given by

$$\mathcal{S}_{\mathrm{nl};II} = \{\widetilde{p}, p, p^{\diamond_0}, p^{\diamond_+}, \mathcal{I}^{\diamond\diamond_0;+}_{0s}, \mathcal{I}^{\diamond\diamond_0;+}_{ss}, \mathcal{I}^{\diamond\diamond_0;+}_{+}\}. \tag{10.12}$$

We remark that $\mathcal{I}^{\diamond\diamond_0;+}_{+} \notin \overline{\mathcal{S}}_{\mathrm{nl}}$, but we do need the bounds stated below in order to estimate $\mathcal{G}^+_{A'a}$. We write

$$\begin{array}{lcllcl} \widetilde{p}_{\mathrm{apx}}(U) & = & \gamma_U^2, & \widetilde{p}_{\mathrm{lin};U}[V] & = & -2\partial^0 U \partial^0 V, \\ p_{\mathrm{apx}}(U) & = & \gamma_U \partial^0 U, & p_{\mathrm{lin};U}[V] & = & \gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V, \\ p^{\diamond_0}_{\mathrm{apx}}(U) & = & \gamma_U^2(\gamma_U^2 - 1), & p^{\diamond_0}_{\mathrm{lin};U}[V] & = & (2 - 4\gamma_U^2)\partial^0 U \partial^0 V, \\ p^{\diamond_+}_{\mathrm{apx}}(U) & = & \gamma_U^4, & p^{\diamond_+}_{\mathrm{lin};U}[V] & = & -4\gamma_U^2 \partial^0 U \partial^0 V, \\ I^{\diamond\diamond_0;+}_{0s;\mathrm{apx}}(U) & = & 4\gamma_U^{-6}\partial^0 U, & I^{\diamond\diamond_0;+}_{0s;\mathrm{lin};U}[V] & = & 4[6\gamma_U^{-8} - 5\gamma_U^{-6}]\partial^0 V, \\ I^{\diamond\diamond_0;+}_{ss;\mathrm{apx}}(U) & = & 4\gamma_U^{-6}\partial^0 U, & I^{\diamond\diamond_0;+}_{ss;\mathrm{lin};U}[V] & = & 4[6\gamma_U^{-8} - 5\gamma_U^{-6}]\partial^0 V, \\ I^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U) & = & \gamma_U^{-4}, & I^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V] & = & 4\gamma_U^{-6}\partial^0 U \partial^0 V. \end{array} \tag{10.13}$$

In addition, we write

$$Q_f = \{\infty\}, \qquad Q^A_{f;\mathrm{lin}} = Q^B_{f;\mathrm{lin}} = \{2, \infty\}, \qquad Q^A_{f;\mathrm{nl}} = Q^B_{f;\mathrm{nl}} = \{2\} \tag{10.14}$$

for each $f \in \mathcal{S}_{\mathrm{nl};II}$. We recall that $Q_{f;\mathrm{pref}} = \{\infty\}$ for $f \in \mathcal{S}_{\mathrm{nl};II} \cap \mathcal{S}_{\mathrm{nl}}$ and $\overline{Q}_{f;\mathrm{pref}} = \{\infty\}$ for $f \in \mathcal{S}_{\mathrm{nl};II} \cap \overline{\mathcal{S}}_{\mathrm{nl}}$.

For later use, we recall the definitions (4.39) and remark that we can formally write

$$\begin{array}{lcl} \mathcal{F}^{\diamond\diamond_0;+}_{a;\mathrm{apx}}(U) & = & \gamma_U^{-4}\partial^+\partial^0\partial U, \\ \mathcal{F}^{\diamond\diamond_0;+}_{b;\mathrm{apx}}(U) & = & 4\gamma_U^{-6}\partial^0 U S^+[\partial^0\partial U]T^+[\partial^0\partial U], \end{array} \tag{10.15}$$

together with

$$\begin{array}{lcl} \mathcal{F}^{\diamond\diamond_0;+}_{a;\mathrm{lin};U}[V] & = & 4\gamma_U^{-6}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V + \gamma_U^{-4}\partial^+\partial^0\partial V, \\ \mathcal{F}^{\diamond\diamond_0;+}_{b;\mathrm{lin};U}[V] & = & 4[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^0\partial U]T^+[\partial^0\partial U]\partial^0 V \\ & & \quad +4\gamma_U^{-6}\partial^0 U\Big[T^+[\partial^0\partial U]S^+[\partial^0\partial V] + S^+[\partial^0\partial U]T^+[\partial^0\partial V]\Big]. \end{array} \tag{10.16}$$

Proposition 10.2. *Assume that (Hg) is satisfied, fix $0 < \kappa < \frac{1}{12}$ and pick any nonlinearity $f \in \mathcal{S}_{\mathrm{nl};II}$. Then there exists a constant $K > 0$ so that the following properties are true.*

(i) Upon introducing the seminorm

$$[V]_{f;\infty,h} \quad = \quad \left\|\partial^+ V\right\|_{\ell^\infty_h} \quad \leq \quad S_{\mathrm{full}}(V), \tag{10.17}$$

the conditions in (hf) are all satisfied.

(ii) For every $q \in Q_f$, $U \in \Omega_{h;\kappa}$ and $h > 0$ we have the estimate

$$\|f(U) - f_{\mathrm{apx}}(U)\|_{\ell^q_h} \le Kh. \tag{10.18}$$

(iii) Upon writing $f^B_{\mathrm{lin};U} = 0$, the conditions in $(hf)_{\mathrm{lin}}$ are satisfied. In addition, the bounds

$$\begin{array}{lllll} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \le & K \|\partial^+ V\|_{\ell^2_h} & \le & K T_{\mathrm{safe}}(V), \\ \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \le & K \|\partial^+ V\|_{\ell^\infty_h} & \le & K T_{\infty;\mathrm{opt}}(V) \end{array} \tag{10.19}$$

hold for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

(iv) Upon writing $f^B_{\mathrm{nl};U} = 0$, the conditions in $(hf)_{\mathrm{nl}}$ are satisfied. In addition, we have the bound

$$\begin{array}{lll} \left\| f^A_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} & \le & K \|\partial^+ V\|_{\ell^\infty_h} \|\partial^+ V\|_{\ell^2_h} + Kh\big[\|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \big] \\ & \le & K \mathcal{E}_{\mathrm{nl}}(V) \end{array} \tag{10.20}$$

for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

The third set of nonlinearities is given by

$$\mathcal{S}_{\mathrm{nl};III} = \{\mathcal{F}^{\diamond\diamond_0}, \mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}, \partial^0\partial\}. \tag{10.21}$$

We write

$$\begin{array}{llllll} \mathcal{F}^{\diamond\diamond_0}_{\mathrm{apx}}(U) & = & \gamma_U^{-4}\partial^0\partial U, & \mathcal{F}^{\diamond\diamond_0}_{\mathrm{lin};U}[V] & = & 4\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0 V + \gamma_U^{-4}\partial^0\partial V, \\ \mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U) & = & \gamma_U^{-3}S^+[\partial^0\partial U], & \mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V] & = & 3\gamma_U^{-5}\partial^0 U S^+[\partial^0\partial U]\partial^0 V + \gamma_U^{-3}S^+[\partial^0\partial V], \\ \mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U) & = & 2\gamma_U^{-3}\partial^0\partial U, & \mathcal{F}^{\diamond_-;+}_{\mathrm{lin};U}[V] & = & 6\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V + 2\gamma_U^{-3}\partial^0\partial V, \\ [\partial^0\partial]_{\mathrm{apx}}(U) & = & \partial^0\partial U, & [\partial^0\partial]_{\mathrm{lin};U}[V] & = & \partial^0\partial V. \end{array} \tag{10.22}$$

In addition, for each $f \in \mathcal{S}_{\mathrm{nl};III}$ we write

$$Q_f = \{2, \infty\}, \qquad Q^A_{f;\mathrm{lin}} = Q^B_{f;\mathrm{lin}} = \{2, \infty\}, \qquad Q^A_{f;\mathrm{nl}} = Q^B_{f;\mathrm{nl}} = \{2\}. \tag{10.23}$$

We recall that $Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = \{2\}$ for $f \in \{\mathcal{F}^{\diamond\diamond_0}, \mathcal{F}^{\diamond_-;+}\}$. For $f = \mathcal{F}^{\diamond_0;+}$ we have $Q_{f;\mathrm{pref}} = \{2\}$ and for $f \in \{\mathcal{F}^{\diamond_0;+}, \partial^0\partial\}$ we have $\overline{Q}_{f;\mathrm{pref}} = \{2, \infty\}$.

Proposition 10.3. *Assume that (Hg) is satisfied, fix $0 < \kappa < \frac{1}{12}$ and pick any nonlinearity $f \in \mathcal{S}_{\mathrm{nl};III}$. Then there exists a constant $K > 0$ so that the following properties are true.*

(i) Upon introducing the seminorms

$$\begin{array}{lllll} [V]_{f;2,h} & = & \|\partial^+ V\|_{\ell^2_h} + \left\|\partial^0\partial V\right\|_{\ell^2_h} & \le & \min\{S_{\mathrm{full}}(V), S_{2;\mathrm{fix}}(V)\}, \\ [V]_{f;\infty,h} & = & \|\partial^+ V\|_{\ell^\infty_h} + \|\partial^+\partial^+ V\|_{\ell^\infty_h} & \le & \overline{S}_{\mathrm{full}}(V), \end{array} \tag{10.24}$$

the conditions in (hf) are all satisfied.

(ii) For every $q \in Q_f$, $U \in \Omega_{h;\kappa}$ and $h > 0$ we have the estimate

$$\|f(U) - f_{\mathrm{apx}}(U)\|_{\ell^q_h} \le Kh. \tag{10.25}$$

(iii) Upon writing $f^B_{\mathrm{lin};U} = 0$, the conditions in $(hf)_{\mathrm{lin}}$ are satisfied. In addition, the bounds

$$\begin{array}{lclcl} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \leq & K\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \,\big] & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq & K\big[\, \|\partial^+ V\|_{\ell^\infty_h} + \|\partial^+\partial^+ V\|_{\ell^\infty_h} \,\big] & \leq & K \overline{T}_{\infty;\mathrm{opt}}(V) \end{array} \tag{10.26}$$

hold for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

(iv) Upon writing $f^B_{\mathrm{nl};U} = 0$, the conditions in $(hf)_{\mathrm{nl}}$ are satisfied. In addition, we have the bound

$$\begin{array}{lcl} \left\| f^A_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} & \leq & K \, \|\partial^+ V\|_\infty \big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \,\big] \\ & & \quad + K h \big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \,\big] \\ & \leq & K \mathcal{E}_{\mathrm{nl}}(V) \end{array} \tag{10.27}$$

for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

The fourth set of nonlinearities is given by

$$\mathcal{S}_{\mathrm{nl};IV} = \{ \mathcal{Z}^+, \mathcal{Z}^- \}. \tag{10.28}$$

We write

$$\begin{array}{lclclcl} \mathcal{Z}^+_{\mathrm{apx}}(U) & = & \gamma_U^{-1}, & & \mathcal{Z}^+_{\mathrm{lin};U}[V] & = & \gamma_U^{-3} \partial^0 U \partial^0 V + \gamma_U^{-1} h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V, \\ \mathcal{Z}^-_{\mathrm{apx}}(U) & = & \gamma_U, & & \mathcal{Z}^-_{\mathrm{lin};U}[V] & = & -\gamma_U^{-1} \partial^0 U \partial^0 V - \gamma_U h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V. \end{array} \tag{10.29}$$

In addition, for every $f \in \mathcal{S}_{\mathrm{nl};IV}$ we write

$$Q_f = \{\infty\}, \qquad Q^A_{f;\mathrm{lin}} = \{\infty\}, \qquad Q^B_{f;\mathrm{lin}} = \{2, \infty\}, \qquad Q^A_{f;\mathrm{nl}} = \{\infty\}, \qquad Q^B_{f;\mathrm{nl}} = \{2\} \tag{10.30}$$

and recall that $Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = \{\infty\}$.

Proposition 10.4. *Assume that (Hg) is satisfied, fix $0 < \kappa < \frac{1}{12}$ and pick any nonlinearity $f \in \mathcal{S}_{\mathrm{nl};IV}$. Then there exists a constant $K > 0$ so that the following properties are true.*

(i) Upon introducing the seminorm

$$[V]_{f;\infty,h} \quad = \quad \|\partial^+ V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^\infty_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \quad \leq \quad S_{\mathrm{full}}(V), \tag{10.31}$$

the conditions in (hf) are all satisfied.

(ii) For every $q \in Q_f$, $U \in \Omega_{h;\kappa}$ and $h > 0$ we have the estimate

$$\|f(U) - f_{\mathrm{apx}}(U)\|_{\ell^q_h} \leq K h. \tag{10.32}$$

(iii) The conditions in $(hf)_{\mathrm{lin}}$ are satisfied. In addition, the bounds

$$\begin{array}{lclcl} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq & K \, \|\partial^+ V\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \leq & K \, \|\partial^+ V\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq & K \, \|\partial^+ V\|_{\ell^\infty_h} & \leq & K T_{\infty;\mathrm{opt}}(V) \end{array} \tag{10.33}$$

hold for all $U \in \Omega_{h;\kappa}$, $h > 0$ and $V \in \ell^2_h$.

(iv) The conditions in $(hf)_{\rm nl}$ *are satisfied. In addition, we have the bounds*

$$\begin{array}{lcl} \left\| f^A_{{\rm nl};U}(V) \right\|_{\ell^\infty_h} & \leq & K\big[\, \|\partial^+ V\|^2_{\ell^2_h} + \|\partial^+\partial^+ V\|^2_{\ell^2_h} \, \big] \\ & & \quad +Kh\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \, \big] \\ & \leq & K\mathcal{E}_{\rm nl}(V), \\ \left\| f^B_{{\rm nl};U}(V) \right\|_{\ell^2_h} & \leq & Kh \, \|\partial^+ V\|_{\ell^2_h} \\ & \leq & K\mathcal{E}_{\rm nl}(V) \end{array} \tag{10.34}$$

for all $U \in \Omega_{h;\kappa}$*,* $h > 0$ *and* $V \in \ell^2_h$*.*

Recalling the notation $g^+(U) = \partial^+ g(U)$, the final set of nonlinearities is given by

$$\mathcal{S}_{{\rm nl};V} = \{g, g^+\}. \tag{10.35}$$

We write

$$\begin{array}{lclclcl} g_{\rm apx}(U) & = & g(U), & \quad & g_{{\rm lin};U}[V] & = & g'(U)V, \\ g^+_{\rm apx}(U) & = & g'(U)\partial^0 U, & & g^+_{{\rm lin};U}[V] & = & g''(U)[\partial^0 U]V + g'(U)\partial^0 V. \end{array} \tag{10.36}$$

In addition, for every $f \in \mathcal{S}_{{\rm nl};V}$ we write

$$Q_f = \{2, \infty\}, \qquad Q^A_{f;{\rm lin}} = Q^B_{f;{\rm lin}} = \{2\}, \qquad Q^A_{f;{\rm nl}} = Q^B_{f;{\rm nl}} = \{2\}. \tag{10.37}$$

We recall that $Q_{g;{\rm pref}} = \overline{Q}_{g;{\rm pref}} = \{2\}$ and $\overline{Q}_{g^+;{\rm pref}} = \{2\}$.

Proposition 10.5. *Assume that (Hg) is satisfied, fix* $0 < \kappa < \frac{1}{12}$ *and pick any nonlinearity* $f \in \mathcal{S}_{{\rm nl};V}$*. Then there exists a constant* $K > 0$ *so that the following properties are true.*

(i) Upon introducing the seminorms

$$\begin{array}{lclcl} [V]_{f;2,h} & = & \|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} & \leq & \min\{S_{\rm full}(V), S_{2;{\rm fix}}(V)\}, \\ [V]_{f;\infty,h} & = & \|V\|_{\ell^\infty_h} + \|\partial^+ V\|_{\ell^\infty_h} \, , & & \end{array} \tag{10.38}$$

the conditions in (hf) *are all satisfied.*

(ii) For every $q \in Q_f$*,* $U \in \Omega_{h;\kappa}$ *and* $h > 0$ *we have the estimate*

$$\|f(U) - f_{\rm apx}(U)\|_{\ell^q_h} \leq Kh. \tag{10.39}$$

(iii) Upon writing $f^B_{{\rm lin};U} = 0$*, the conditions in* $(hf)_{\rm lin}$ *are satisfied. In addition, the bound*

$$\left\| f^A_{{\rm lin};U}[V] \right\|_{\ell^2_h} \quad \leq \quad K\big[\, \|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} \, \big] \quad \leq \quad KT_{\rm safe}(V) \tag{10.40}$$

holds for all $U \in \Omega_{h;\kappa}$*,* $h > 0$ *and* $V \in \ell^2_h$*.*

(iv) Upon writing $f^B_{{\rm nl};U} = 0$*, the conditions in* $(hf)_{\rm nl}$ *are satisfied. In addition, we have the bound*

$$\begin{array}{lcl} \left\| f^A_{{\rm nl};U}(V) \right\|_{\ell^2_h} & \leq & K\Big[\, \|V\|_{\ell^\infty_h} + \|\partial^+ V\|_{\ell^\infty_h} \Big] \big[\, \|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} \, \big] \\ & & \quad Kh\big[\, \|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \, \big] \\ & \leq & K\mathcal{E}_{\rm nl}(V) \end{array} \tag{10.41}$$

for all $U \in \Omega_{h;\kappa}$*,* $h > 0$ *and* $V \in \ell^2_h$*.*

Corollary 10.6. *For every $f \in \mathcal{S}_{\mathrm{nl}}$ we have $\infty \in Q_f$ together with $Q_{f;\mathrm{pref}} \subset Q^A_{f;\mathrm{lin}}$ and $Q_{f;\mathrm{pref}} \subset Q^B_{f;\mathrm{lin}}$. The same properties hold upon replacing $(\mathcal{S}_{\mathrm{nl}}, Q_{f;\mathrm{pref}})$ by $(\overline{\mathcal{S}}_{\mathrm{nl}}, \overline{Q}_{f;\mathrm{pref}})$.*

Proof. The result can be readily verified by inspecting the identities above. ☐

Corollary 10.7. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. For every $f \in \mathcal{S}_{\mathrm{nl}}$ and $q \in Q_{f;\mathrm{pref}}$ we have*

$$[V]_{f;q,h} \leq S_{\mathrm{full}}(V) \tag{10.42}$$

for any $h > 0$ and $V \in \ell^2_h$. In addition, if $2 \in Q_{f;\mathrm{pref}}$ then at least one of the following two properties hold true.

(a) We have

$$[V]_{f;2,h} \leq S_{2;\mathrm{fix}}(V) \tag{10.43}$$

for every $h > 0$ and $V \in \ell^2_h$.

(b) We have

$$[V]_{f;\infty,h} \leq S_{\mathrm{full}}(V) \tag{10.44}$$

for every $h > 0$ and $V \in \ell^2_h$.

The same properties hold upon replacing $(\mathcal{S}_{\mathrm{nl}}, Q_{f;\mathrm{pref}}, S_{\mathrm{full}}, S_{2;\mathrm{fix}})$ by $(\overline{\mathcal{S}}_{\mathrm{nl}}, \overline{Q}_{f;\mathrm{pref}}, \overline{S}_{\mathrm{full}}, \overline{S}_{2;\mathrm{fix}})$.

Proof. The result can be readily verified by inspecting the identities and estimates above. ☐

Corollary 10.8. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. For any $f \in \mathcal{S}_{\mathrm{nl}}$, any $\# \in \{A, B\}$ and any $q \in Q_{f;\mathrm{pref}}$, at least one of the following two properties hold true.*

(a) There exists $K > 0$ so that

$$\left\| f^{\#}_{\mathrm{lin};U}[V] \right\|_{\ell^q_h} \leq K T_{\mathrm{safe}}(V) \tag{10.45}$$

holds for every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

(b) We have $q = \infty$ and there exists $K > 0$ so that the bounds

$$\begin{aligned} \left\| f^{\#}_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} &\leq K T_{\mathrm{safe}}(V), \\ \left\| f^{\#}_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} &\leq K T_{\infty;\mathrm{opt}}(V) \end{aligned} \tag{10.46}$$

hold for every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

The same properties hold upon replacing $(\mathcal{S}_{\mathrm{nl}}, Q_{f;\mathrm{pref}}, T_{\mathrm{safe}}, T_{\infty;\mathrm{opt}})$ by $(\overline{\mathcal{S}}_{\mathrm{nl}}, \overline{Q}_{f;\mathrm{pref}}, \overline{T}_{\mathrm{safe}}, \overline{T}_{\infty;\mathrm{opt}})$.

Proof. The result can be readily verified by inspecting the identities and estimates above. ☐

Corollary 10.9. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Consider any $f \in \mathcal{S}_{\mathrm{nl}}$ and any $\# \in \{A, B\}$. Then if $2 \in Q_{f;\mathrm{pref}}$, there exists a constant $K > 0$ so that*

$$\left\| f^{\#}_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} \leq K \mathcal{E}_{\mathrm{nl}}(V) \tag{10.47}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Otherwise, there exists $q \in \{2, \infty\}$ together with a constant $K > 0$ so that

$$\left\| f^{\#}_{\mathrm{nl};U}(V) \right\|_{\ell^q_h} \leq K \mathcal{E}_{\mathrm{nl}}(V) \tag{10.48}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$. The same properties hold upon replacing $(\mathcal{S}_{\mathrm{nl}}, Q_{f;\mathrm{pref}}, \mathcal{E}_{\mathrm{nl}})$ by $(\overline{\mathcal{S}}_{\mathrm{nl}}, \overline{Q}_{f;\mathrm{pref}}, \overline{\mathcal{E}}_{\mathrm{nl}})$.

Proof. The result can be readily verified by inspecting the identities and estimates above. □

10.2 Gridpoint spacing

We define the approximate derivative

$$r_{\mathrm{lin};U}[V] = -\gamma_U^{-1} \partial^0 U \partial^0 V \tag{10.49}$$

together with the nonlinear residuals

$$\begin{array}{rcl} r^{\pm}_{\mathrm{nl};U}(V) & = & r^{\pm}_{U+V} - r^{\pm}_U - r_{\mathrm{lin};U}[V], \\ r^{0}_{\mathrm{nl};U}(V) & = & r^{0}_{U+V} - r^{0}_U - r_{\mathrm{lin};U}[V]. \end{array} \tag{10.50}$$

Lemma 10.10. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise bounds*

$$\begin{array}{rcl} \left|r^0_U - \gamma_U\right| + \left|r^+_U - \gamma_U\right| + \left|r^-_U - \gamma_U\right| & \leq & Kh \left|\partial^0 \partial U\right|, \\ \left|r^0_{\mathrm{nl};U}(V)\right| + \left|r^+_{\mathrm{nl};U}(V)\right| + \left|r^+_{\mathrm{nl};U}(V)\right| & \leq & K \Big[\left|\partial^+ V\right|^2 + \left|\partial^- V\right|^2 \Big] \\ & & + Kh \Big[\left|\partial^+ V\right| + \left|\partial^- V\right| + \left|\partial^0 \partial V\right| \Big] \end{array} \tag{10.51}$$

hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. We consider only the statements concerning r^+_U as the functions r^-_U and r^0_U can be treated in a similar fashion. Writing $f(x) = \sqrt{1 - x^2}$ and $\phi(\partial^- U, \partial^+ U) = \partial^+ U$, we see that

$$r^+_U = f\big(\phi(\partial^- U, \partial^+ U)\big). \tag{10.52}$$

We include the redundant variable $\partial^- U$ here because it will be used for our approximate function

$$\begin{array}{rcl} \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & \frac{1}{2}\partial^+ U + \frac{1}{2}\partial^-(U) \\ & = & \partial^0 U \end{array} \tag{10.53}$$

and our approximate derivative

$$\phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] \quad = \quad \partial^0 V. \tag{10.54}$$

An easy computation shows that

$$\begin{array}{rcl} \phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & \partial^+ U - \partial^0 U \\ & = & h \partial^0 \partial U, \end{array} \tag{10.55}$$

together with

$$\begin{array}{rcl} \phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V) & = & \partial^+(U + V) - \partial^+ U - \partial^0 V \\ & = & \partial^+ V - \partial^0 V \\ & = & h \partial^0 \partial V. \end{array} \tag{10.56}$$

The a-priori estimate (6.7) ensures that the geometric condition (6.17) can be satisfied. In particular, the bounds now follow directly from Lemma 6.4 and the observations

$$\begin{array}{rcl} f\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big) & = & \gamma_U, \\ Df\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big) & = & -\gamma_U^{-1}\partial^0 U. \end{array} \tag{10.57}$$

□

10.3 Discrete derivatives

We write

$$\begin{array}{rcccccl} \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U) & = & \mathcal{F}^{\diamond_-}_{\mathrm{apx}}(U) & = & \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U) & = & \gamma_U^{-1}\partial^0 U, \\ \mathcal{F}^{\diamond_+}_{\mathrm{lin};U}[V] & = & \mathcal{F}^{\diamond_-}_{\mathrm{lin};U}[V] & = & \mathcal{F}^{\diamond_0}_{\mathrm{lin};U}[V] & = & \gamma_U^{-3}\partial^0 V \end{array} \tag{10.58}$$

and introduce the nonlinear residuals

$$\begin{array}{rcl} \mathcal{F}^{\diamond_\pm}_{\mathrm{nl};U}(V) & = & \mathcal{F}^{\diamond_\pm}(U+V) - \mathcal{F}^{\diamond_\pm}(U) - \mathcal{F}^{\diamond_\pm}_{\mathrm{lin};U}[V], \\ \mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V) & = & \mathcal{F}^{\diamond_0}(U+V) - \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{lin};U}[V]. \end{array} \tag{10.59}$$

Lemma 10.11. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise bounds*

$$\big|\mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U)\big| + \big|\mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U)\big| + \big|\mathcal{F}^{\diamond_-}(U) - \mathcal{F}^{\diamond_-}_{\mathrm{apx}}(U)\big| \quad \le \quad Kh\,\big|\partial^0\partial U\big| \tag{10.60}$$

and

$$\begin{array}{rcl} \Big|\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V)\Big| + \Big|\mathcal{F}^{\diamond_+}_{\mathrm{nl};U}(V)\Big| + \Big|\mathcal{F}^{\diamond_-}_{\mathrm{nl};U}(V)\Big| & \le & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2\,\Big] \\ & & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + \big|\partial^0\partial V\big|\,\Big] \end{array} \tag{10.61}$$

hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. We consider only the statements concerning $\mathcal{F}^{\diamond_+}$ as the functions $\mathcal{F}^{\diamond_-}$ and $\mathcal{F}^{\diamond_0}$ can be treated in a similar fashion. Recalling the fact that r_U^+ depends only on $\partial^+ U$, we abuse notation slightly to write

$$\phi(\partial^- U, \partial^+ U) = (\partial^+ U, r_U^+). \tag{10.62}$$

Upon introducing $f(x,y) = x/y$, we see that

$$\mathcal{F}^{\diamond_+}(U) = f\big(\phi(\partial^- U, \partial^+ U)\big). \tag{10.63}$$

We now define the approximants

$$\begin{array}{rcl} \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & (\partial^0 U, \gamma_U), \\ \phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] & = & (\partial^0 V, -\gamma_U^{-1}\partial^0 U \partial^0 V) \end{array} \tag{10.64}$$

and compute

$$\begin{array}{rcl} \phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & \big(\partial^+ U - \partial^0 U, r_U^+ - \gamma_U\big) \\ & = & \big(h\partial^0\partial U, r_U^+ - \gamma_U\big) \end{array} \tag{10.65}$$

together with

$$\begin{array}{rcl}\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V) & = & \big(\partial^+ V - \partial^0 V, r^+_{\mathrm{nl};U}(V)\big) \\ & = & \big(h\partial^0 \partial V, r^+_{\mathrm{nl};U}(V)\big).\end{array} \tag{10.66}$$

In particular, Lemma 10.10 provides the bound

$$|\phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)| \quad \leq \quad C_1' h \left|\partial^0 \partial U\right| \tag{10.67}$$

together with

$$|\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V)| \quad \leq \quad C_1' \Big[|\partial^- V|^2 + |\partial^+ V|^2 \Big] + C_1' h \Big[|\partial^- V| + |\partial^+ V| + \left|\partial^0 \partial V\right| \Big]. \tag{10.68}$$

Upon computing

$$\begin{array}{lcl} f\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big) & = & \gamma_U^{-1} \partial^0 U, \\ Df\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big)\phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] & = & \gamma_U^{-1} \partial^0 V - \partial^0 U \gamma_U^{-2}(-\gamma_U^{-1} \partial^0 U \partial^0 V) \\ & = & \big[\gamma_U^{-1} + (\partial^0 U)^2 \gamma_U^{-3}\big] \partial^0 V \\ & = & \gamma_U^{-3} \partial^0 V, \end{array} \tag{10.69}$$

the desired bounds follow directly from Lemma 6.4. □

Proof of Proposition 10.1. The results follow directly from Lemma 10.11. □

Turning to second derivatives, we recall the definitions

$$\begin{array}{rcl}\mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U) & = & 2\gamma_U^{-3} \partial^0 \partial U, \\ \mathcal{F}^{\diamond_-;+}_{\mathrm{lin};U}[V] & = & 6\gamma_U^{-5} \partial^0 U [\partial^0 \partial U] \partial^0 V + 2\gamma_U^{-3} \partial^0 \partial V\end{array} \tag{10.70}$$

and write

$$\mathcal{F}^{\diamond_-;+}_{\mathrm{nl};U}(V) \quad = \quad \mathcal{F}^{\diamond_-;+}(U+V) - \mathcal{F}^{\diamond_-;+}(U) - \mathcal{F}^{\diamond_-;+}_{\mathrm{lin};U}[V]. \tag{10.71}$$

Lemma 10.12. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\left|\mathcal{F}^{\diamond_-;+}(U) - \mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U)\right| \quad \leq \quad Kh \left|\partial^0 \partial U\right| \tag{10.72}$$

and the residual bound

$$\begin{array}{rcl}\left|\mathcal{F}^{\diamond_-;+}_{\mathrm{nl};U}(V)\right| & \leq & K\Big[|\partial^- V|^2 + |\partial^+ V|^2 + |\partial^- V| \left|\partial^0 \partial V\right| + |\partial^+ V| \left|\partial^0 \partial V\right| \Big] \\ & & +Kh\Big[|\partial^- V| + |\partial^+ V| + \left|\partial^0 V\right| \Big]\end{array} \tag{10.73}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Motivated by the identity

$$\mathcal{F}^{\diamond_-;+}(U) = \frac{2}{r_U^+}\big[1 + \mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big] \partial^0 \partial U \tag{10.74}$$

derived in Lemma 4.3, we write

$$f(y, z_-, z_0) = \frac{2}{y}\big[1 + z_- z_0\big] \tag{10.75}$$

together with

$$\phi(\partial^- U, \partial^+ U) = \big(r_U^+, \mathcal{F}^{\diamond_-}(U), \mathcal{F}^{\diamond_0}(U)\big) \tag{10.76}$$

and finally

$$P(\partial^- U, \partial^+ U, \partial^0 \partial U) = f\big(\phi(\partial^- U, \partial^+ U)\big)\partial^0 \partial U. \tag{10.77}$$

One readily verifies that

$$\mathcal{F}^{\diamond_-;+}(U) = P(\partial^- U, \partial^+ U, \partial^0 \partial U). \tag{10.78}$$

We now define the approximants

$$\begin{array}{rcl} \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & \big(\gamma_U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] & = & \big(-\gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big) \end{array} \tag{10.79}$$

and compute

$$\phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) \quad = \quad \big(r_U^+ - \gamma_U^{-1}, \mathcal{F}^{\diamond_-}(U) - \mathcal{F}^{\diamond}_{\mathrm{apx}}, \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond}_{\mathrm{apx}}\big) \tag{10.80}$$

together with

$$\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V) \quad = \quad \big(r^+_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_-}_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V)\big). \tag{10.81}$$

In particular, Lemma's 10.10 and 10.11 provide the bound

$$|\phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)| \quad \leq \quad C_1' h \left|\partial^0 \partial U\right|, \tag{10.82}$$

together with

$$\begin{array}{rcl} |\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V)| & \leq & C_1' \Big[\left|\partial^- V\right|^2 + \left|\partial^+ V\right|^2 \Big] \\ & & + C_1' h \Big[|\partial^- V| + |\partial^+ V| + \left|\partial^0 \partial V\right| \Big]. \end{array} \tag{10.83}$$

Introducing the compressed nonlinearity

$$\overline{f}(y, z) = f(y, z, z) = \frac{2}{y}(1 + z^2) \tag{10.84}$$

together with the compressed approximants

$$\begin{array}{rcl} \overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U) & = & (\gamma_U, \gamma_U^{-1}\partial^0 U), \\ \overline{\phi}_{\mathrm{lin};U}[\partial^- V, \partial^+ V] & = & \big(-\gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V\big), \end{array} \tag{10.85}$$

we see that

$$\begin{array}{rcl} f\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big) & = & \overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big), \\ Df\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big)\phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] & = & D\overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big)\overline{\phi}_{\mathrm{lin};U}[\partial^- V, \partial^+ V]. \end{array} \tag{10.86}$$

Upon computing

$$D\overline{f}(y,z) = \big(-\frac{2}{y^2}(1+z^2), 4\frac{z}{y}\big), \tag{10.87}$$

we hence see that the functions defined in (6.30) satisfy

$$\begin{aligned} P_{\mathrm{apx}}(U) &= 2\gamma_U^{-1}\big(1+(\partial^0 U)^2\gamma_U^{-2}\big)\partial^0\partial U \\ &= 2\gamma_U^{-3}\partial^0\partial U, \end{aligned} \tag{10.88}$$

together with

$$\begin{aligned} P_{\mathrm{lin};U}[V] &= -2\gamma_U^{-2}\big(1+(\partial^0 U)^2\gamma_U^{-2}\big)(-\gamma_U^{-1}\partial^0 U)\partial^0 V(\partial^0\partial U) \\ &\quad +4\gamma_U^{-2}\partial^0 U(\gamma_U^{-3}\partial^0 V)(\partial^0\partial U) \\ &\quad +2\gamma_U^{-1}\big(1+(\partial^0 U)^2\gamma_U^{-2}\big)\partial^0\partial V \\ &= 6\gamma_U^{-5}\partial^0 U\partial^0\partial U\partial^0 V + 2\gamma_U^{-3}\partial^0\partial V. \end{aligned} \tag{10.89}$$

The desired estimates now follow directly from Corollary 6.5. □

We also recall the definitions

$$\begin{aligned} \mathcal{F}^{\diamond\diamond_0}_{\mathrm{apx}}(U) &= \gamma_U^{-4}\partial^0\partial U, \\ \mathcal{F}^{\diamond\diamond_0}_{\mathrm{lin};U}[V] &= 4\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0 V + \gamma_U^{-4}\partial^0\partial V \end{aligned} \tag{10.90}$$

and write

$$\mathcal{F}^{\diamond\diamond_0}_{\mathrm{nl};U}(V) = \mathcal{F}^{\diamond\diamond_0}(U+V) - \mathcal{F}^{\diamond\diamond_0}(U) - \mathcal{F}^{\diamond\diamond_0}_{\mathrm{lin};U}[V]. \tag{10.91}$$

Lemma 10.13. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\big|\mathcal{F}^{\diamond\diamond_0}(U) - \mathcal{F}^{\diamond\diamond_0}_{\mathrm{apx}}(U)\big| \le Kh\,\big|\partial^0\partial U\big| \tag{10.92}$$

and the residual bound

$$\begin{aligned} \Big|\mathcal{F}^{\diamond\diamond_0}_{\mathrm{nl};U}(V)\Big| \le\; & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + |\partial^- V|\,\big|\partial^0\partial V\big| + |\partial^+ V|\,\big|\partial^0\partial V\big|\,\Big] \\ & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + \big|\partial^0\partial V\big|\,\Big] \end{aligned} \tag{10.93}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Motivated by the identity

$$\mathcal{F}^{\diamond\diamond_0}(U) = \frac{1}{r_U^+ r_U^0}\big[1 + \mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big]\partial^0\partial U \tag{10.94}$$

derived in Lemma 4.3 and (4.33), we write

$$f(y_+, y_0, z_-, z_0) = \frac{1}{y_+ y_0}\big[1 + z_- z_0\big] \tag{10.95}$$

together with

$$\phi(\partial^- U, \partial^+ U) = \big(r_U^+, r_U^0, \mathcal{F}^{\diamond_-}(U), \mathcal{F}^{\diamond_0}(U)\big) \tag{10.96}$$

and finally

$$P(\partial^- U, \partial^+ U, \partial^0 \partial U) = f\big(\phi(\partial^- U, \partial^+ U)\big)\partial^0 \partial U. \tag{10.97}$$

One readily verifies that

$$\mathcal{F}^{\diamond\diamond_0}(U) = P(\partial^- U, \partial^+ U, \partial^0 \partial U). \tag{10.98}$$

We now define the approximants

$$\begin{array}{lcl} \phi_{\rm apx}(\partial^- U, \partial^+ U) & = & \big(\gamma_U, \gamma_U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{{\rm lin};U}[\partial^- V, \partial^+ V] & = & \big(- \gamma_U^{-1}\partial^0 U \partial^0 V, -\gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big). \end{array} \tag{10.99}$$

This allows us to compute

$$\phi(\partial^- U, \partial^+ U) - \phi_{\rm apx}(\partial^- U, \partial^+ U) \;\; = \;\; \big(r_U^+ - \gamma_U^{-1}, r_U^0 - \gamma_U^{-1}, \mathcal{F}^{\diamond_-}(U) - \mathcal{F}^{\diamond}_{\rm apx}, \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond}_{\rm apx}\big), \tag{10.100}$$

together with

$$\phi_{{\rm nl};U}(\partial^- V, \partial^+ V) \;\; = \;\; \big(r^+_{{\rm nl};U}(V), r^0_{{\rm nl};U}(V)\mathcal{F}^{\diamond_-}_{{\rm nl};U}(V), \mathcal{F}^{\diamond_0}_{{\rm nl};U}(V)\big). \tag{10.101}$$

In particular, the bounds (10.82)-(10.83) remain valid.

This allows us to repeat the procedure in the proof of Lemma 10.12 with the compressed approximants (10.85) and the compressed nonlinearity

$$\overline{f}(y, z) = f(y, y, z, z) = \frac{1}{y^2}(1 + z^2), \tag{10.102}$$

for which we have

$$D\overline{f}(y, z) = \big(- \frac{2}{y^3}(1 + z^2), 2\frac{z}{y^2}\big). \tag{10.103}$$

The functions defined in (6.30) hence satisfy

$$\begin{array}{lcl} P_{\rm apx}(U) & = & \gamma_U^{-2}\big(1 + (\partial^0 U)^2 \gamma_U^{-2}\big)\partial^0 \partial U \\ & = & \gamma_U^{-4}\partial^0 \partial U, \end{array} \tag{10.104}$$

together with

$$\begin{array}{lcl} P_{{\rm lin};U}[V] & = & -2\gamma_U^{-3}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big)(-\gamma_U^{-1}\partial^0 U)\partial^0 V(\partial^0 \partial U) \\ & & +2\gamma_U^{-3}\partial^0 U(\gamma_U^{-3}\partial^0 V)(\partial^0 \partial U) \\ & & \quad +\gamma_U^{-2}\big(1 + (\partial^0 U)^2 \gamma_U^{-2}\big)\partial^0 \partial V \\ & = & 4\gamma_U^{-6}\partial^0 U[\partial^0 \partial U]\partial^0 V + \gamma_U^{-4}\partial^0 \partial V. \end{array} \tag{10.105}$$

The desired estimates again follow directly from Corollary 6.5. □

We now recall the definitions

$$\begin{array}{lcl} \mathcal{F}^{\diamond_0;+}_{\rm apx}(U) & = & \gamma_U^{-3}S^+[\partial^0 \partial U], \\ \mathcal{F}^{\diamond_0;+}_{{\rm lin};U}[V] & = & 3\gamma_U^{-5}\partial^0 U[S^+\partial^0 \partial U]\partial^0 V + \gamma_U^{-3}S^+[\partial^0 \partial V] \end{array} \tag{10.106}$$

and write

$$\mathcal{F}^{\diamond_0;+}_{{\rm nl};U}(V) \;\; = \;\; \mathcal{F}^{\diamond_0;+}(U + V) - \mathcal{F}^{\diamond_0;+}(U) - \mathcal{F}^{\diamond_0;+}_{{\rm lin};U}[V]. \tag{10.107}$$

Lemma 10.14. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\big|\mathcal{F}^{\diamond_0;+}(U) - \mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\big| \quad \le \quad Kh\big[\,\big|\partial^0 \partial U\big| + T^+\,\big|\partial^0\partial U\big|\,\big] \tag{10.108}$$

and the residual bound

$$\begin{array}{lcl} \Big|\mathcal{F}^{\diamond_0;+}_{\mathrm{nl};U}(V)\Big| & \le & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + T^+\,|\partial^+ V|^2\,\Big] \\ & & +K\Big[\,|\partial^- V| + |\partial^+ V| + T^+\,|\partial^+ V|\,\Big]\Big[\,\big|\partial^0\partial V\big| + T^+\,\big|\partial^0 \partial V\big|\,\Big] \\ & & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + T^+\,|\partial^+ V| + \big|\partial^0 V\big| + T^+\,\big|\partial^0\partial V\big|\,\Big] \end{array} \tag{10.109}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Motivated by the identity

$$\begin{array}{lcl} \mathcal{F}^{\diamond_0;+}(U) & = & \frac{1}{T^+ r^0_U}\Big[1 + \mathcal{F}^{\diamond_0}(U)\mathcal{F}^{\diamond_0}(U)\Big][\partial^0\partial U] \\ & & +\frac{1}{T^+ r^0_U}\Big[1 + \mathcal{F}^{\diamond_0}(U) T^+[\mathcal{F}^{\diamond_0}(U)]\Big]T^+[\partial^0\partial U] \end{array} \tag{10.110}$$

derived in Lemma 4.3, we write

$$\begin{array}{lcl} f_1(y_s, z, z_s) & = & \frac{1}{y_s}\big[1 + z^2\big], \\ f_2(y_s, z, z_s) & = & \frac{1}{y_s}\big[1 + z z_s\big] \end{array} \tag{10.111}$$

together with

$$\phi(\partial^- U, \partial^+ U, T^+\partial^+ U) = \big(T^+ r^0_U, \mathcal{F}^{\diamond_0}(U), T^+\mathcal{F}^{\diamond_0}(U)\big) \tag{10.112}$$

and finally

$$\begin{array}{lcl} P_1(\partial^- U, \partial^+ U, T^+\partial^+ U, \partial^0\partial U) & = & f_1\big(\phi(\partial^- U, \partial^+ U, T^+\partial^+ U)\big)\partial^0\partial U, \\ P_2(\partial^- U, \partial^+ U, T^+\partial^+ U, T^+\partial^0\partial U) & = & f_2\big(\phi(\partial^- U, \partial^+ U, T^+\partial^+ U)\big)T^+\partial^0\partial U. \end{array} \tag{10.113}$$

For convenience, we introduce the shorthand

$$\omega_U = (\partial^- U, \partial^+ U, T^+\partial^+ U). \tag{10.114}$$

One readily verifies that

$$\mathcal{F}^{\diamond_0;+}(U) = P_1(\omega_U, \partial^0\partial U) + P_2(\omega_U, T^+\partial^0\partial U). \tag{10.115}$$

We now define the approximants

$$\begin{array}{lcl} \phi_{\mathrm{apx}}(\omega_U) & = & \big(\gamma_U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{\mathrm{lin};U}[\omega_V] & = & \big(-\gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big). \end{array} \tag{10.116}$$

This allows us to compute

$$\begin{array}{lcl} \phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U) & = & \big(T^+ r^0_U - \gamma_U, \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond}_{\mathrm{apx}}(U), T^+\mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond}_{\mathrm{apx}}(U)\big) \\ & = & \big(r^0_U - \gamma_U, \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U)\big) \\ & & \quad +\big(h\partial^+[r^0_U], 0, h\mathcal{F}^{\diamond_0;+}(U)\big), \end{array} \tag{10.117}$$

together with

$$\begin{array}{rcl}\phi_{\mathrm{nl};U}(\omega_V) & = & \Big(T^+ r^0_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), T^+\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V)\Big) \\ & & +h\Big(- \partial^+[\gamma_U^{-1}\partial^0 U \partial^0 V], 0, \partial^+[\gamma_U^{-3}\partial^0 V]\Big).\end{array} \tag{10.118}$$

In particular, Lemma's 10.10 and 10.11 together with Corollaries 7.3 and 7.6 provide the bound

$$|\phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U)| \quad \leq \quad C_1' h\big[\,\big|\partial^0\partial U\big| + T^+\,\big|\partial^0\partial U\big|\,\big], \tag{10.119}$$

together with

$$\begin{array}{rcl}|\phi_{\mathrm{nl};U}(\omega_V)| & \leq & C_1'\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + T^+\,|\partial^+ V|^2\,\Big] \\ & & +C_1' h\Big[\,|\partial^- V| + |\partial^+ V| + T^+\,|\partial^+ V| + \big|\partial^0\partial V\big| + T^+\,\big|\partial^0\partial V\big|\,\Big].\end{array} \tag{10.120}$$

Introducing the compressed nonlinearity

$$\overline{f}(y,z) = f_1(y,z,z) = f_2(y,z,z) = \frac{1}{y}(1+z^2), \tag{10.121}$$

together with the compressed approximants

$$\begin{array}{rcl}\overline{\phi}_{\mathrm{apx}}(\omega_U) & = & (\gamma_U, \gamma_U^{-1}\partial^0 U), \\ \overline{\phi}_{\mathrm{lin};U}[\omega_V] & = & \big(- \gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V\big),\end{array} \tag{10.122}$$

we see that the identities

$$\begin{array}{rcl}f_i\big(\phi_{\mathrm{apx}}(\omega_U)\big) & = & \overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big), \\ Df_i\big(\phi_{\mathrm{apx}}(\omega_U)\big)\phi_{\mathrm{lin};U}[\omega_V] & = & D\overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big)\overline{\phi}_{\mathrm{lin};U}[\omega_V]\end{array} \tag{10.123}$$

hold for $i = 1, 2$. Upon computing

$$D\overline{f}(y,z) = \big(- \frac{1}{y^2}(1+z^2), 2\frac{z}{y}\big), \tag{10.124}$$

we hence see that the functions defined in (6.30) satisfy

$$\begin{array}{rcl}P_{1;\mathrm{apx}}(\omega_U, \partial^0\partial U) & = & \gamma_U^{-1}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big)\partial^0\partial U \\ & = & \gamma_U^{-3}\partial^0\partial U, \\ P_{2;\mathrm{apx}}(\omega_U, T^+\partial^0\partial U) & = & \gamma_U^{-3} T^+\partial^0\partial U,\end{array} \tag{10.125}$$

together with

$$\begin{array}{rcl}P_{1;\mathrm{lin};U}[\omega_V, \partial^0\partial V] & = & -\gamma_U^{-2}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big)(-\gamma_U^{-1}\partial^0 U)\partial^0 V(\partial^0\partial U) \\ & & +2\gamma_U^{-2}\partial^0 U(\gamma_U^{-3}\partial^0 V)(\partial^0\partial U) \\ & & +\gamma_U^{-1}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big)\partial^0\partial V \\ & = & 3\gamma_U^{-5}\partial^0 U\partial^0\partial U\partial^0 V + \gamma_U^{-3}\partial^0\partial V, \\ P_{2;\mathrm{lin};U}[\omega_V, T^+\partial^0\partial V] & = & 3\gamma_U^{-5}\partial^0 U T^+[\partial^0\partial U]\partial^0 V + \gamma_U^{-3} T^+[\partial^0\partial V].\end{array} \tag{10.126}$$

The desired estimates again follow directly from Corollary 6.5. □

Proof of Proposition 10.3. The results follow directly from Lemma's 10.12, 10.13 and 10.14. ☐

We recall the definitions

$$\begin{array}{rcl} \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U) & = & \gamma_U^{-4}, \\ \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V] & = & 4\gamma_U^{-6}\partial^0 U \partial^0 V \end{array} \tag{10.127}$$

and write

$$\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V) \;=\; \mathcal{I}^{\diamond\diamond_0;+}_{+}(U+V) - \mathcal{I}^{\diamond\diamond_0;+}_{+}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V]. \tag{10.128}$$

Lemma 10.15. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\left|\mathcal{I}^{\diamond\diamond_0;+}_{+}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\right| \;\leq\; Kh\left|\partial^0\partial U\right| \tag{10.129}$$

and the residual bound

$$\begin{array}{rcl} \left|\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V)\right| & \leq & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2\,\Big] \\ & & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + \left|\partial^0\partial V\right|\,\Big] \end{array} \tag{10.130}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Motivated by the identity

$$\mathcal{I}^{\diamond\diamond_0;+}_{+}(U) = \frac{1}{r_U^+ r_U^0}\big[1 + \mathcal{F}^{\diamond_-}(U)\mathcal{F}^{\diamond_0}(U)\big] \tag{10.131}$$

derived in Lemma 4.4, we may reuse the functions f, ϕ, ϕ_{apx} and ϕ_{lin} defined in the proof of Lemma 10.13. Writing

$$P(\partial^- U, \partial^+ U) = f\big(\phi(\partial^- U, \partial^+ U)\big), \tag{10.132}$$

one readily verifies that

$$\mathcal{I}^{\diamond\diamond_0;+}_{+}(U) = P(\partial^- U, \partial^+ U). \tag{10.133}$$

Reusing the computations in the proof of Lemma 10.13, we see that the functions defined in (6.19) satisfy

$$\begin{array}{rcl} P_{\mathrm{apx}}(U) & = & \gamma_U^{-2}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big) \\ & = & \gamma_U^{-4}, \end{array} \tag{10.134}$$

together with

$$\begin{array}{rcl} P_{\mathrm{lin};U}[V] & = & -2\gamma_U^{-3}\big(1 + (\partial^0 U)^2\gamma_U^{-2}\big)(-\gamma_U^{-1}\partial^0 U)\partial^0 V \\ & & +2\gamma_U^{-3}\partial^0 U(\gamma_U^{-3}\partial^0 V) \\ & = & 4\gamma_U^{-6}\partial^0 U\partial^0 V. \end{array} \tag{10.135}$$

The desired estimates now follow from Lemma 6.4 and the bounds (10.82)-(10.83). ☐

We recall the definitions

$$\begin{array}{rcccl} \mathcal{I}^{\diamond\diamond_0;+}_{0s;\mathrm{apx}}(U) & = & \mathcal{I}^{\diamond\diamond_0;+}_{ss;\mathrm{apx}}(U) & = & 4\gamma_U^{-6}\partial^0 U, \\ \mathcal{F}^{\diamond\diamond_0;+}_{0s;\mathrm{lin};U}[V] & = & \mathcal{F}^{\diamond\diamond_0;+}_{ss;\mathrm{lin};U}[V] & = & 4\big[6\gamma_U^{-8} - 5\gamma_U^{-6}\big]\partial^0 V \end{array} \tag{10.136}$$

and write

$$\begin{array}{rcl} \mathcal{I}^{\diamond\diamond_0;+}_{0s;\mathrm{nl};U}(V) & = & \mathcal{I}^{\diamond\diamond_0;+}_{0s}(U+V) - \mathcal{I}^{\diamond\diamond_0;+}_{0s}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{0s;\mathrm{lin};U}[V], \\ \mathcal{I}^{\diamond\diamond_0;+}_{ss;\mathrm{nl};U}(V) & = & \mathcal{I}^{\diamond\diamond_0;+}_{ss}(U+V) - \mathcal{I}^{\diamond\diamond_0;+}_{ss}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{ss;\mathrm{lin};U}[V]. \end{array} \tag{10.137}$$

Lemma 10.16. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\big|\mathcal{I}^{\diamond\diamond_0;+}_{0s}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{s;\mathrm{apx}}(U)\big| + \big|\mathcal{I}^{\diamond\diamond_0;+}_{ss}(U) - \mathcal{I}^{\diamond\diamond_0;+}_{s;\mathrm{apx}}(U)\big| \quad \leq \quad Kh\big[\,\big|\partial^0\partial U\big| + T^+\big|\partial^0\partial U\big|\,\big] \tag{10.138}$$

and the residual bound

$$\begin{array}{rcl} \Big|\mathcal{I}^{\diamond\diamond_0;+}_{0s;\mathrm{nl};U}(V)\Big| + \Big|\mathcal{I}^{\diamond\diamond_0;+}_{ss;\mathrm{nl};U}(V)\Big| & \leq & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + T^+|\partial^+ V|^2\,\Big] \\ & & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + T^+|\partial^+ V| + \big|\partial^0\partial V\big| + T^+\big|\partial^0\partial V\big|\,\Big] \end{array} \tag{10.139}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. For convenience, we introduce the shorthand

$$\omega_U = (\partial^- U, \partial^+ U, T^+\partial^+ U). \tag{10.140}$$

Motivated by the identities derived in Lemma 4.4, we write

$$\begin{array}{rcl} f_{0s}(y_0, y_{0s}, y_+, y_{+s}, z_0, z_{0s}, z_-, z_+) & = & \frac{1}{y_+ y_0 y_{0s}} z_0(1 + z_+ z_{0s}) \\ & & +\frac{2}{y_0 y_+ y_+} z_{0s}(1 + z_- z_0) \\ & & +\frac{1}{y_+ y_0 y_{0s}} z_-(1 + z_0^2), \\ f_{ss}(y_0, y_{0s}, y_+, y_{+s}, z_0, z_{0s}, z_-, z_+) & = & \frac{2y_0 + y_{+s}}{y_0 y_{0s} y_+ y_{+s}} z_{0s}(1 + z_+ z_{0s}) \\ & & +\frac{1}{y_+ y_0 y_{0s}} z_-(1 + z_0 z_{0s}), \end{array} \tag{10.141}$$

together with

$$\phi(\omega_U) = \big(r_U^0, T^+ r_U^0, r_U^+, T^+ r_U^+, \mathcal{F}^{\diamond_0}(U), T^+\mathcal{F}^{\diamond_0}(U), \mathcal{F}^{\diamond_-}(U), \mathcal{F}^{\diamond_+}(U)\big) \tag{10.142}$$

and finally

$$\begin{array}{rcl} P_{0s}(\omega_U) & = & f_{0s}\big(\phi(\omega_U)\big), \\ P_{ss}(\omega_U) & = & f_{ss}\big(\phi(\omega_U)\big). \end{array} \tag{10.143}$$

One readily verifies that

$$\begin{array}{l} \mathcal{I}^{\diamond\diamond_0;+}_{0s}(U) = P_{0s}(\omega_U), \\ \mathcal{I}^{\diamond\diamond_0;+}_{ss}(U) = P_{ss}(\omega_U). \end{array} \tag{10.144}$$

We now define the approximants

$$\begin{array}{lcl}\phi_{\mathrm{apx}}(\omega_U) & = & \big(\gamma_U, \gamma_U, \gamma_U, \gamma_U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{\mathrm{lin};U}[\omega_V] & = & \big(- \gamma_U^{-1}\partial^0 U \partial^0 V, -\gamma_U^{-1}\partial^0 U \partial^0 V, -\gamma_U^{-1}\partial^0 U \partial^0 V, -\gamma_U^{-1}\partial^0 U \partial^0 V, \\ & & \qquad \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big).\end{array} \tag{10.145}$$

This allows us to compute

$$\begin{array}{lcl}\phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U) & = & \Big(r_U^0 - \gamma_U, r_U^0 - \gamma_U, r_U^+ - \gamma_U, r_U^+ - \gamma_U, \\ & & \quad \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \\ & & \quad \mathcal{F}^{\diamond_-}(U) - \mathcal{F}^{\diamond_-}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U)\Big) \\ & & + h\Big(0, \partial^+[r_U^0], 0, \partial^+[r_U^+], 0, \mathcal{F}^{\diamond_0;+}(U), 0, 0\Big),\end{array} \tag{10.146}$$

together with

$$\begin{array}{lcl}\phi_{\mathrm{nl};U}(\omega_V) & = & \Big(r^0_{\mathrm{nl};U}(V), T^+ r^0_{\mathrm{nl};U}(V), r^+_{\mathrm{nl};U}(V), T^+ r^+_{\mathrm{nl};U}(V), \\ & & \quad \mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), T^+\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_-}_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_+}_{\mathrm{nl};U}(V)\Big) \\ & & + h\Big(0, -\partial^+[\gamma_U^{-1}\partial^0 U \partial^0 V], 0, -\partial^+[\gamma_U^{-1}\partial^0 U \partial^0 V], \\ & & \quad 0, \partial^+[\gamma_U^{-3}\partial^0 V], 0, 0\Big).\end{array} \tag{10.147}$$

In particular, Lemma's 10.10 and 10.11 together with Corollaries 7.3 and 7.6 provide the bound

$$|\phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U)| \quad \le \quad C_1' h\Big[\big|\partial^0 \partial U\big| + T^+ \big|\partial^0\partial U\big| \Big] \tag{10.148}$$

together with

$$\begin{array}{lcl}|\phi_{\mathrm{nl};U}(\omega_V)| & \le & C_1'\Big[|\partial^- V|^2 + |\partial^+ V|^2 + T^+ |\partial^+ V|^2 \Big] \\ & & + C_1' h\Big[|\partial^- V| + |\partial^+ V| + T^+ |\partial^+ V| + \big|\partial^0 \partial V\big| + T^+ \big|\partial^0 \partial V\big| \Big].\end{array} \tag{10.149}$$

Introducing the compressed nonlinearity

$$\overline{f}(y,z) = f_{0s}(y,y,y,y,z,z,z,z) = f_{ss}(y,y,y,y,z,z,z,z) = \frac{4z}{y^3} + \frac{4z^3}{y^3}, \tag{10.150}$$

together with the compressed approximants

$$\begin{array}{lcl}\overline{\phi}_{\mathrm{apx}}(\omega_U) & = & (\gamma_U, \gamma_U^{-1}\partial^0 U), \\ \overline{\phi}_{\mathrm{lin};U}[\omega_V] & = & \big(- \gamma_U^{-1}\partial^0 U \partial^0 V, \gamma_U^{-3}\partial^0 V\big),\end{array} \tag{10.151}$$

we see that the identities

$$\begin{array}{lcl}f_\#\big(\phi_{\mathrm{apx}}(\omega_U)\big) & = & \overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big), \\ Df_\#\big(\phi_{\mathrm{apx}}(\omega_U)\big)\phi_{\mathrm{lin};U}[\omega_V] & = & D\overline{f}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big)\overline{\phi}_{\mathrm{lin};U}[\omega_V]\end{array} \tag{10.152}$$

hold for $\# \in \{0s, ss\}$. Upon computing

$$D\overline{f}(y,z) = \big(- 12\frac{z + z^3}{y^4}, \frac{4 + 12z^2}{y^3}\big), \tag{10.153}$$

we hence see that the functions defined in (6.19) satisfy

$$\begin{array}{rcl} P_{0s;\mathrm{apx}}(\omega_U) & = & P_{ss;\mathrm{apx}}(\omega_U) \\ & = & 4\gamma_U^{-4}\partial^0 U + 4\gamma_U^{-6}(\partial^0 U)^3 \\ & = & 4\gamma_U^{-6}\partial^0 U, \end{array} \tag{10.154}$$

together with

$$\begin{array}{rcl} P_{0s;\mathrm{lin};U}[\omega_V] & = & P_{ss;\mathrm{lin};U}[\omega_V] \\ & = & -12\gamma_U^{-5}\partial^0 U\big(1+\gamma_U^{-2}(\partial^0 U)^2\big)(-\gamma_U^{-1}\partial^0 U\partial^0 V) \\ & & \quad +\Big(4\gamma_U^{-3}+12\gamma_U^{-5}(\partial^0 U)^2\Big)\gamma_U^{-3}\partial^0 V \\ & = & \Big[12\gamma_U^{-8}(1-\gamma_U^2)+4\gamma_U^{-6}+12\gamma_U^{-8}(1-\gamma_U^2)\Big]\partial^0 V \\ & = & \Big[-20\gamma_U^{-6}+24\gamma_U^{-8}\Big]\partial^0 V. \end{array} \tag{10.155}$$

The desired estimates now follow from Lemma 6.4. □

10.4 Auxiliary functions

We recall the definitions

$$\begin{array}{rclcrcl} \widetilde{p}_{\mathrm{apx}}(U) & = & \gamma_U^2, & \qquad & \widetilde{p}_{\mathrm{lin};U}[V] & = & -2\partial^0\partial V, \\ p_{\mathrm{apx}}(U) & = & \gamma_U\partial^0 U, & & p_{\mathrm{lin};U}[V] & = & \gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V \end{array} \tag{10.156}$$

and write

$$\begin{array}{rcl} \widetilde{p}_{\mathrm{nl};U}(V) & = & \widetilde{p}(U+V)-\widetilde{p}(U)-\widetilde{p}_{\mathrm{lin};U}[V], \\ p_{\mathrm{nl};U}(V) & = & p(U+V)-p(U)-p_{\mathrm{lin};U}[V]. \end{array} \tag{10.157}$$

Lemma 10.17. *Fix $0<\kappa<\frac{1}{12}$. Then there exists $K>0$ so that the pointwise approximation estimate*

$$|\widetilde{p}(U)-\widetilde{p}_{\mathrm{apx}}(U)|+|p(U)-p_{\mathrm{apx}}(U)| \quad \le \quad Kh\left|\partial^0\partial U\right| \tag{10.158}$$

and the residual bound

$$\begin{array}{rcl} |\widetilde{p}_{\mathrm{nl};U}(V)|+|p_{\mathrm{nl};U}(V)| & \le & K\big[\,|\partial^- V|^2+|\partial^+ V|^2\,\big] \\ & & +Kh\big[\,|\partial^- V|+|\partial^+ V|+\left|\partial^0\partial V\right|\,\big] \end{array} \tag{10.159}$$

both hold for any $h>0$, any $U\in\Omega_{h;\kappa}$ and any $V\in\ell_h^2$ for which $U+V\in\Omega_{h;\kappa}$.

Proof. Motivated by the definitions (8.2), we write

$$\begin{array}{rcl} f_{\widetilde{p}}(z_0,z_+) & = & \frac{1}{1+z_+z_0}, \\ f_p(z_0,z_+) & = & \frac{z_+}{1+z_+z_0}, \end{array} \tag{10.160}$$

together with

$$\phi(\partial^- U,\partial^+ U)=\big(\mathcal{F}^{\diamond_0}(U),\mathcal{F}^{\diamond_+}(U)\big) \tag{10.161}$$

and finally

$$\begin{aligned} \widetilde{P}(\partial^- U, \partial^+ U) &= f_{\widetilde{p}}\big(\phi(\partial^- U, \partial^+ U)\big), \\ P(\partial^- U, \partial^+ U) &= f_p\big(\phi(\partial^- U, \partial^+ U)\big). \end{aligned} \tag{10.162}$$

One readily verifies that

$$\begin{aligned} \widetilde{p}(U) &= \widetilde{P}(\partial^- U, \partial^+ U), \\ p(U) &= P(\partial^- U, \partial^+ U). \end{aligned} \tag{10.163}$$

We now define the approximants

$$\begin{aligned} \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) &= \big(\gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] &= \big(\gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big). \end{aligned} \tag{10.164}$$

This allows us to compute

$$\phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U) = \Big(\mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U)\Big), \tag{10.165}$$

together with

$$\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V) = \Big(\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_+}_{\mathrm{nl};U}(V)\Big). \tag{10.166}$$

In particular, Lemma 10.11 provides the bound

$$|\phi(\partial^- U, \partial^+ U) - \phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)| \leq Kh\left|\partial^0 \partial U\right| \tag{10.167}$$

together with

$$\begin{aligned} |\phi_{\mathrm{nl};U}(\partial^- V, \partial^+ V)| \leq{} & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2\,\Big] \\ & + Kh\Big[\,|\partial^- V| + |\partial^+ V| + \big|\partial^0 \partial V\big|\,\Big]. \end{aligned} \tag{10.168}$$

Introducing the compressed nonlinearities

$$\begin{aligned} \overline{f}_{\widetilde{p}}(z) &= f_{\widetilde{p}}(z, z) = \tfrac{1}{1+z^2}, \\ \overline{f}_p(z) &= f_p(z, z) = \tfrac{z}{1+z^2}, \end{aligned} \tag{10.169}$$

together with the compressed approximants

$$\begin{aligned} \overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U) &= \gamma_U^{-1}\partial^0 U, \\ \overline{\phi}_{\mathrm{lin};U}[\partial^- V, \partial^+ V] &= \gamma_U^{-3}\partial^0 V, \end{aligned} \tag{10.170}$$

we see that the identities

$$\begin{aligned} f_{\#}\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big) &= \overline{f}_{\#}\big(\overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big), \\ Df_{\#}\big(\phi_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big)\phi_{\mathrm{lin};U}[\partial^- V, \partial^+ V] &= D\overline{f}_{\#}\big(\overline{\phi}_{\mathrm{apx}}(\partial^- U, \partial^+ U)\big)\overline{\phi}_{\mathrm{lin};U}[\partial^- V, \partial^+ V] \end{aligned} \tag{10.171}$$

hold for $\# \in \{\widetilde{p}, p\}$. Upon computing

$$\begin{aligned} \overline{f}'_{\widetilde{p}}(z) &= -\tfrac{2z}{(1+z^2)^2}, \\ \overline{f}'_p(z) &= \tfrac{1-z^2}{(1+z^2)^2}, \end{aligned} \tag{10.172}$$

we hence see that the functions defined in (6.19) satisfy

$$\begin{array}{rcl} \widetilde{P}_{\mathrm{apx}}(U) & = & [1+\gamma_U^{-2}(\partial^0 U)^2]^{-1} \\ & = & \gamma_U^2, \\ P_{\mathrm{apx}}(U) & = & \gamma_U^{-1}\partial^0 U[1+\gamma_U^{-2}(\partial^0 U)^2]^{-1} \\ & = & \gamma_U \partial^0 U, \end{array} \tag{10.173}$$

together with

$$\begin{array}{rcl} \widetilde{P}_{\mathrm{lin};U}[V] & = & -2\gamma_U^{-1}\partial^0 U[1+\gamma_U^{-2}(\partial^0 U)^2]^{-2}\gamma_U^{-3}\partial^0 V \\ & = & -2\partial^0 U\partial^0 V, \\ P_{\mathrm{lin};U}[V] & = & (1-\gamma_U^{-2}(\partial^0 U)^2)[1+\gamma_U^{-2}(\partial^0 U)^2]^{-2}\gamma_U^{-3}\partial^0 V \\ & = & (2-\gamma_U^{-2})\gamma_U\partial^0 V \\ & = & \gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V. \end{array} \tag{10.174}$$

The desired estimates now follow from Lemma 6.4. □

We recall the definitions

$$\begin{array}{rclcrcl} p^{\diamond_0}_{\mathrm{apx}}(U) & = & \gamma_U^2(\gamma_U^2-1), & & p^{\diamond_0}_{\mathrm{lin};U}[V] & = & (2-4\gamma_U^2)\partial^0 U\partial^0 V, \\ p^{\diamond_+}_{\mathrm{apx}}(U) & = & \gamma_U^4, & & p^{\diamond_+}_{\mathrm{lin};U}[V] & = & -4\gamma_U^2\partial^0 U\partial^0 V \end{array} \tag{10.175}$$

and write

$$\begin{array}{rcl} p^{\diamond_0}_{\mathrm{nl};U}(V) & = & p^{\diamond_0}(U+V)-p^{\diamond_0}(U)-p^{\diamond_0}_{\mathrm{lin};U}[V], \\ p^{\diamond_+}_{\mathrm{nl};U}(V) & = & p^{\diamond_+}(U+V)-p^{\diamond_+}(U)-p^{\diamond_+}_{\mathrm{lin};U}[V]. \end{array} \tag{10.176}$$

Lemma 10.18. *Fix $0<\kappa<\frac{1}{12}$. Then there exists $K>0$ so that the pointwise approximation estimate*

$$\left|p^{\diamond_+}(U)-p^{\diamond_+}_{\mathrm{apx}}(U)\right|+\left|p^{\diamond_0}(U)-p^{\diamond_0}_{\mathrm{apx}}(U)\right| \quad \le \quad Kh\big[\left|\partial^0\partial U\right|+T^+\left|\partial^0\partial U\right|\big] \tag{10.177}$$

and the residual bound

$$\begin{array}{rcl} \left|p^{\diamond_+}_{\mathrm{nl};U}(V)\right|+\left|p^{\diamond_0}_{\mathrm{nl};U}(V)\right| & \le & K\big[\left|\partial^- V\right|^2+\left|\partial^+ V\right|^2+T^+\left|\partial^+ V\right|^2\big] \\ & & +Kh\big[\left|\partial^- V\right|+\left|\partial^+ V\right|+T^+\left|\partial^+ V\right|+\left|\partial^0\partial V\right|+T^+\left|\partial^0\partial V\right|\big] \end{array} \tag{10.178}$$

both hold for any $h>0$, any $U\in\Omega_{h;\kappa}$ and any $V\in\ell^2_h$ for which $U+V\in\Omega_{h;\kappa}$.

Proof. For convenience, we introduce the shorthand

$$\omega_U=(\partial^- U,\partial^+ U,T^+\partial^+ U). \tag{10.179}$$

Motivated by the definitions (9.3), we write

$$\begin{array}{rcl} f^{\diamond_+}_A(z_0,z_{0s},z_+,z_{+s}) & = & \frac{2+z_+z_0+z_{+s}z_{0s}}{2(1+z_+z_0)(1+z_{+s}z_{0s})}, \\ f^{\diamond_+}_B(z_0,z_{0s},z_+,z_{+s}) & = & -\frac{(z_++z_{+s})(z_0+z_{0s})}{4(1+z_+z_0)(1+z_{+s}z_{0s})}, \\ f^{\diamond_0}(z_0,z_{0s},z_+,z_{+s}) & = & -\frac{(z_++z_{+s})^2}{4(1+z_+z_0)(1+z_{+s}z_{0s})}, \end{array} \tag{10.180}$$

together with

$$\phi(\omega_U) = \big(\mathcal{F}^{\diamond_0}(U), T^+\mathcal{F}^{\diamond_0}(U), \mathcal{F}^{\diamond_+}(U), T^+\mathcal{F}^{\diamond_+}(U)\big) \tag{10.181}$$

and finally

$$\begin{array}{lcl} P_A^{\diamond_+}(\omega_U) & = & f_A^{\diamond_+}\big(\phi(\omega_U)\big), \\ P_B^{\diamond_+}(\omega_U) & = & f_B^{\diamond_+}\big(\phi(\omega_U)\big), \\ P^{\diamond_0}(\omega_U) & = & f^{\diamond_0}\big(\phi(\omega_U)\big). \end{array} \tag{10.182}$$

One readily verifies that

$$\begin{array}{lcl} p_A^{\diamond_+}(U) & = & P_A^{\diamond_+}(\omega_U), \\ p_B^{\diamond_+}(U) & = & P_B^{\diamond_+}(\omega_U), \\ p^{\diamond_0}(U) & = & P^{\diamond_0}(\omega_U). \end{array} \tag{10.183}$$

We now define the approximants

$$\begin{array}{lcl} \phi_{\mathrm{apx}}(\omega_U) & = & \big(\gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U, \gamma_U^{-1}\partial^0 U\big), \\ \phi_{\mathrm{lin};U}[\omega_V] & = & \big(\gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V, \gamma_U^{-3}\partial^0 V\big). \end{array} \tag{10.184}$$

This allows us to compute

$$\begin{array}{lcl} \phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U) & = & \Big(\mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_0}(U) - \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U), \\ & & \quad \mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U), \mathcal{F}^{\diamond_+}(U) - \mathcal{F}^{\diamond_+}_{\mathrm{apx}}(U)\Big) \\ & & +h\Big(0, \mathcal{F}^{\diamond_0;+}(U), 0, \mathcal{F}^{\diamond_+;+}(U)\Big), \end{array} \tag{10.185}$$

together with

$$\begin{array}{lcl} \phi_{\mathrm{nl};U}(\omega_V) & = & \Big(\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), T^+\mathcal{F}^{\diamond_0}_{\mathrm{nl};U}(V), \mathcal{F}^{\diamond_+}_{\mathrm{nl};U}(V), T^+\mathcal{F}^{\diamond_+}_{\mathrm{nl};U}(V)\Big) \\ & & +h\Big(0, \partial^+[\gamma_U^{-3}\partial^0 V], 0, \partial^+[\gamma_U^{-3}\partial^0 V]\Big). \end{array} \tag{10.186}$$

In particular, Lemma 10.11 provides the bound

$$|\phi(\omega_U) - \phi_{\mathrm{apx}}(\omega_U)| \quad \leq \quad Kh\big[\,\big|\partial^0\partial U\big| + T^+\big|\partial^0\partial U\big|\,\big], \tag{10.187}$$

together with

$$\begin{array}{lcl} |\phi_{\mathrm{nl};U}(\omega_V)| & \leq & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + T^+|\partial^+ V|^2\,\Big] \\ & & +Kh\Big[\,|\partial^- V| + |\partial^+ V| + T^+|\partial^+ V| + \big|\partial^0\partial V\big| + T^+\big|\partial^0\partial V\big|\,\Big]. \end{array} \tag{10.188}$$

Introducing the compressed nonlinearities

$$\begin{array}{lcl} \overline{f}_A^{\diamond_+}(z) & = & f_A^{\diamond_+}(z,z,z,z) \\ & = & \frac{1}{1+z^2}, \\ \overline{f}_B^{\diamond_+}(z) & = & f_B^{\diamond_+}(z,z,z,z) \\ & = & -\frac{z^2}{(1+z^2)^2} \\ & = & -\frac{1}{1+z^2} + \frac{1}{(1+z^2)^2}, \\ \overline{f}^{\diamond_0}(z) & = & f^{\diamond_0}(z,z,z,z) \\ & = & \overline{f}_B^{\diamond_+}(z), \end{array} \tag{10.189}$$

together with the compressed approximants

$$\begin{array}{lll} \overline{\phi}_{\mathrm{apx}}(\omega_U) & = & \gamma_U^{-1}\partial^0 U, \\ \overline{\phi}_{\mathrm{lin};U}[\omega_V] & = & \gamma_U^{-3}\partial^0 V, \end{array} \tag{10.190}$$

we see that the identities

$$\begin{array}{lll} f_{\#}^{\diamond +}\big(\phi_{\mathrm{apx}}(\omega_U)\big) & = & \overline{f}_{\#}^{\diamond +}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big), \\ Df_{\#}^{\diamond +}\big(\phi_{\mathrm{apx}}(\omega_U)\big)\phi_{\mathrm{lin};U}[\omega_V] & = & D\overline{f}_{\#}^{\diamond +}\big(\overline{\phi}_{\mathrm{apx}}(\omega_U)\big)\overline{\phi}_{\mathrm{lin};U}[\omega_V] \end{array} \tag{10.191}$$

hold for $\# \in \{A, B\}$, together with similar identities for $f^{\diamond_0}$. Upon computing

$$\begin{array}{lll} D\overline{f}_A^{\diamond +}(z) & = & \frac{-2z}{(1+z^2)^2}, \\ D\overline{f}_B^{\diamond +}(z) & = & \frac{2z}{(1+z^2)^2} - \frac{4z}{(1+z^2)^3}, \end{array} \tag{10.192}$$

we hence see that the functions defined in (6.19) satisfy

$$\begin{array}{lll} P_{A;\mathrm{apx}}^{\diamond +}(U) & = & [1+\gamma_U^{-2}(\partial^0 U)^2]^{-1} \\ & = & \gamma_U^2, \\ P_{B;\mathrm{apx}}^{\diamond +}(U) & = & P_{\mathrm{apx}}^{\diamond_0}(U) \\ & = & -\gamma_U^{-2}(\partial^0 U)^2[1+\gamma_U^{-2}(\partial^0 U)^2]^{-2} \\ & = & -\gamma_U^2(1-\gamma_U^2) \\ & = & \gamma_U^4 - \gamma_U^2, \end{array} \tag{10.193}$$

together with

$$\begin{array}{lll} P_{A;\mathrm{lin};U}^{\diamond +}[V] & = & -2\gamma_U^{-1}\partial^0 U[1+\gamma_U^{-2}(\partial^0 U)^2]^{-2}\gamma_U^{-3}\partial^0 V \\ & = & -2\partial^0 U\partial^0 V, \\ P_{B;\mathrm{lin};U}^{\diamond +}[V] & = & P_{\mathrm{lin};U}^{\diamond_0}[V] \\ & = & \gamma_U^{-1}\partial^0 U\Big[2[1+\gamma_U^{-2}(\partial^0 U)^2]^{-2} - 4[1+\gamma_U^{-2}(\partial^0 U)^2]^{-3}\Big]\gamma_U^{-3}\partial^0 V \\ & = & \big[2-4\gamma_U^2\big]\partial^0 U\partial^0 V. \end{array} \tag{10.194}$$

The desired estimates now follow from Lemma 6.4. □

Proof of Proposition 10.2. The results follow directly from Lemma's 10.15-10.18. □

10.5 Estimates for $\mathcal{Z}$

We introduce the function

$$\widetilde{q}(U) = p(U)\mathcal{F}^{\diamond_0;+}(U) \tag{10.195}$$

and write

$$\begin{array}{lll} \widetilde{q}_{\mathrm{apx}}(U) & = & \gamma_U^{-2}\partial^0 U S^+[\partial^0\partial U], \\ \widetilde{q}_{\mathrm{lin};U}[V] & = & \gamma_U^{-4}(2-\gamma_U^2)S^+[\partial^0\partial U]\partial^0 V + \gamma_U^{-2}\partial^0 U S^+[\partial^0\partial V], \end{array} \tag{10.196}$$

together with

$$\widetilde{q}_{\mathrm{nl};U}(V) = \widetilde{q}(U+V) - \widetilde{q}(U) - \widetilde{q}_{\mathrm{lin};U}[V]. \tag{10.197}$$

Lemma 10.19. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$|\widetilde{q}(U) - \widetilde{q}_{\mathrm{apx}}(U)| \quad \leq \quad Kh\big[\, |\partial^+ U| + \big|\partial^0 \partial U\big| \,\big]\big[\, \big|\partial^0 \partial U\big| + T^+ \left|\partial^0 \partial U\right| \,\big] \tag{10.198}$$

and the residual bound

$$\begin{aligned} |\widetilde{q}_{\mathrm{nl};U}(V)| \quad \leq \quad & K\Big[\, |\partial^- V|^2 + |\partial^+ V|^2 + T^+ \left|\partial^+ V\right|^2 \,\Big] \\ & +K\Big[\, |\partial^- V| + |\partial^+ V| + T^+ \left|\partial^+ V\right| \,\Big]\Big[\, \big|\partial^0 \partial V\big| + T^+ \left|\partial^0 \partial V\right| \,\Big] \\ & +Kh\Big[\, |\partial^- U| + |\partial^+ U| + T^+ \left|\partial^+ U\right| + \big|\partial^0 \partial U\big| + T^+ \left|\partial^0 \partial U\right| \,\Big] \\ & \qquad \times\Big[\, |\partial^- V| + |\partial^+ V| + T^+ \left|\partial^+ V\right| + \big|\partial^0 \partial V\big| + T^+ \left|\partial^0 \partial V\right| \,\Big] \end{aligned} \tag{10.199}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. We first note that

$$\begin{aligned} p_{\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U) \quad &= \quad \gamma_U \partial^0 U \gamma_U^{-3} S^+[\partial^0 \partial U] \\ &= \quad \gamma_U^{-2} \partial^0 U S^+[\partial^0 \partial U] \\ &= \quad \widetilde{q}_{\mathrm{apx}}(U), \end{aligned} \tag{10.200}$$

while also

$$\begin{aligned} p_{\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V] + p_{\mathrm{lin};U}[V]\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U) \quad &= \quad \gamma_U \partial^0 U\Big[3\gamma_U^{-5} \partial^0 U S^+[\partial^0 \partial U]\partial^0 V + \gamma_U^{-3} S^+[\partial^0 \partial V]\Big] \\ & \qquad +\gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V \gamma_U^{-3} S^+[\partial^0 \partial U] \\ &= \quad \big(\tfrac{2\gamma_U^2 - 1}{\gamma_U^4} + 3\gamma_U^{-4}(\partial^0 U)^2\big) S^+[\partial^0 \partial U]\partial^0 V \\ & \qquad +\gamma_U^{-2} \partial^0 U S^+[\partial^0 \partial V] \\ &= \quad \widetilde{q}_{\mathrm{lin};U}[V]. \end{aligned} \tag{10.201}$$

Lemma 4.3 and the definition (8.2) yield the bound

$$|p(U)| + \big|\mathcal{F}^{\diamond_0;+}(U)\big| \leq C_1' \big[\, \big|\partial^+ U\big| + \big|\partial^0 \partial U\big| + T^+ \left|\partial^0 \partial U\right| \,\big]. \tag{10.202}$$

Observing that

$$\begin{aligned} |\widetilde{q}(U) - \widetilde{q}_{\mathrm{apx}}(U)| \quad \leq \quad & |p(U) - p_{\mathrm{apx}}(U)| \, |\mathcal{F}^{\diamond_0;+}(U)| \\ & + |p_{\mathrm{apx}}(U)| \, \big|\mathcal{F}^{\diamond_0;+}(U) - \mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\big| \,, \end{aligned} \tag{10.203}$$

we may hence exploit Lemma's 10.14 and 10.17 to obtain the first desired estimate.

In addition, the computation

$$\begin{aligned} \widetilde{q}_{\mathrm{nl};U}(V) \quad &= \quad p(U+V)\mathcal{F}^{\diamond_0;+}(U+V) - p(U)\mathcal{F}^{\diamond_0;+}(U) - \widetilde{q}_{\mathrm{lin};U}[V] \\ &= \quad [p(U) + p_{\mathrm{lin};U}[V] + p_{\mathrm{nl};U}(V)][\mathcal{F}^{\diamond_0;+}(U) + \mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V] + \mathcal{F}^{\diamond_0;+}_{\mathrm{nl};U}(V)] \\ & \qquad -p(U)\mathcal{F}^{\diamond_0;+}(U) - \widetilde{q}_{\mathrm{lin};U}[V] \\ &= \quad p_{\mathrm{lin};U}[V]\Big(\mathcal{F}^{\diamond_0;+}(U) - \mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big) + \Big(p(U) - p_{\mathrm{apx}}(U)\Big)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V] \\ & \qquad +p_{\mathrm{nl};U}(V)\mathcal{F}^{\diamond_0;+}(U) + p(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{nl};U}(V) \\ & \qquad \Big(p_{\mathrm{lin};U}[V] + p_{\mathrm{nl};U}(V)\Big)\Big(\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V] + \mathcal{F}^{\diamond_0;+}_{\mathrm{nl};U}(V)\Big) \end{aligned} \tag{10.204}$$

together with the bounds in Lemma's 10.14 and 10.17 yields the second desired estimate. □

We now write

$$\begin{array}{rcl} q_{\rm apx}(U) & = & \widetilde{q}_{\rm apx}(U), \\ q_{{\rm lin};U}[V] & = & \widetilde{q}_{{\rm lin};U}[V], \end{array} \tag{10.205}$$

together with

$$q_{{\rm nl};U}(V) \quad = \quad q(U+V) - q(U) - q_{{\rm lin};U}[V]. \tag{10.206}$$

Lemma 10.20. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$|q(U) - q_{\rm apx}(U)| \quad \leq \quad Kh\big[\,|\partial^+ U| + \big|\partial^0 \partial U\big|\,\big]\big[\,\big|\partial^0 \partial U\big| + T^+\big|\partial^0 \partial U\big|\,\big] \tag{10.207}$$

and the residual bound

$$\begin{array}{rcl} |q_{{\rm nl};U}(V)| & \leq & K\Big[\,|\partial^- V|^2 + |\partial^+ V|^2 + T^+ |\partial^+ V|^2\,\Big] \\ & & +K\Big[\,|\partial^- V| + |\partial^+ V| + T^+ |\partial^+ V|\,\Big]\Big[\,\big|\partial^0 \partial V\big| + T^+\big|\partial^0 \partial V\big|\,\Big] \\ & & +Kh\Big[\,|\partial^- U| + |\partial^+ U| + T^+|\partial^+ U| + \big|\partial^0 \partial U\big| + T^+\big|\partial^0 \partial U\big|\,\Big] \\ & & \quad \times\Big[\,|\partial^- V| + |\partial^+ V| + T^+|\partial^+ V| + \big|\partial^0 \partial V\big| + T^+\big|\partial^0 \partial V\big|\,\Big] \end{array} \tag{10.208}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. We recall that for every $\tau > 0$ there exists $C_\tau > 0$ so that the inequalities

$$\begin{array}{rcl} \Big|\ln(1+x) - \ln(1+y) - \frac{1}{1+y}(x-y)\Big| & \leq & C_\tau\,|x-y|^2, \\ \big|[1+x]^{-1} - 1\big| & \leq & C_\tau\,|x| \end{array} \tag{10.209}$$

hold whenever $x + 1 \geq \tau$ and $y + 1 \geq \tau$.

We now write

$$\mathcal{I}_0 = q(U+V) - q(U) - \widetilde{q}(U+V) + \widetilde{q}(U). \tag{10.210}$$

Recalling the definition

$$\begin{array}{rcl} q(U) & = & h^{-1}\ln\big[1 + hp(U)\mathcal{F}^{\diamond_0;+}(U)\big] \\ & = & h^{-1}\ln\big[1 + h\widetilde{q}(U)\big], \end{array} \tag{10.211}$$

we may compute

$$\begin{array}{rcl} \mathcal{I}_0 & = & h^{-1}\Big[\ln[1+h\widetilde{q}(U+V)] - \ln[1+h\widetilde{q}(U)]\Big] - h^{-1}[1+h\widetilde{q}(U)]^{-1}[h\widetilde{q}(U+V) - h\widetilde{q}(U)] \\ & & +\Big[[1+h\widetilde{q}(U)]^{-1} - 1\Big][\widetilde{q}(U+V) - \widetilde{q}(U)]. \end{array} \tag{10.212}$$

The uniform estimate (8.15) allows us to apply (10.209) with $\tau = \frac{1}{3}\kappa^2$ to obtain

$$\begin{array}{rcl} |\mathcal{I}_0| & \leq & h^{-1}C_2' h^2\,|\widetilde{q}(U+V) - \widetilde{q}(U)|^2 \\ & & \quad +C_2' h\,|\widetilde{q}(U)|\,|\widetilde{q}(U+V) - \widetilde{q}(U)|\,. \end{array} \tag{10.213}$$

Exploiting Lemma 10.19 and inspecting (10.195), we see that

$$\begin{array}{lll}|\mathcal{I}_0| & \leq & C_3' h\Big[|\partial^- V|^2 + |\partial^+ V|^2 + T^+ |\partial^+ V|^2 + \big|\partial^0 \partial V\big|^2 + T^+ \big|\partial^0 \partial V\big|^2 \Big] \\ & & +C_3' h\big[\big|\partial^0 \partial U\big| + T^+ \big|\partial^0 \partial U\big| \big] \big[|\partial^- V| + |\partial^+ V| + T^+ |\partial^+ V| + \big|\partial^0 \partial V\big| + T^+ \big|\partial^0 \partial V\big| \big].\end{array} \tag{10.214}$$

The bound (10.208) now follows from the observation

$$\begin{array}{lll} q_{\mathrm{nl};U}(V) & = & \mathcal{I}_0 + \widetilde{q}(U+V) - \widetilde{q}(U) - q_{\mathrm{lin};U}[V] \\ & = & \mathcal{I}_0 + \widetilde{q}(U+V) - \widetilde{q}(U) - \widetilde{q}_{\mathrm{lin};U}[V] \\ & = & \mathcal{I}_0 + \widetilde{q}_{\mathrm{nl};U}(V). \end{array} \tag{10.215}$$

Applying (10.209) with $y = 0$ and using

$$|\widetilde{q}(U)| \leq C_4' \min\{\big|\partial^+ U\big|, \big|\partial^0 \partial U\big| + T^+ \big|\partial^0 \partial U\big|\}, \tag{10.216}$$

we find

$$\begin{array}{lll} |q(U) - \widetilde{q}(U)| & \leq & h^{-1} C_5' h^2 |\widetilde{q}(U)|^2 \\ & \leq & h C_6' |\partial^+ U| \big[\big|\partial^0 \partial U\big| + T^+ \big|\partial^0 \partial U\big| \big]. \end{array} \tag{10.217}$$

The desired bound (10.207) now follows from the identity

$$q(U) - q_{\mathrm{apx}}(U) = q(U) - \widetilde{q}_{\mathrm{apx}}(U) = q(U) - \widetilde{q}(U) + \widetilde{q}(U) - \widetilde{q}_{\mathrm{apx}}(U). \tag{10.218}$$

□

We now turn our attention to the function

$$\mathcal{Q}(U) = h \sum_{-;h} q(U). \tag{10.219}$$

Recalling the definition (7.60), we write

$$\begin{array}{lll} \mathcal{Q}_{\mathrm{apx}}(U) & = & -\ln \gamma_U, \\ \mathcal{Q}_{\mathrm{lin};U}[V] & = & \gamma_U^{-2} \partial^0 U \partial^0 V + h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V, \end{array} \tag{10.220}$$

together with

$$\mathcal{Q}_{\mathrm{nl};U}(V) = \mathcal{Q}(U+V) - \mathcal{Q}(U) - \mathcal{Q}_{\mathrm{lin};U}[V]. \tag{10.221}$$

Lemma 10.21. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$|\mathcal{Q}(U) - \mathcal{Q}_{\mathrm{apx}}(U)| \quad \leq \quad Kh \tag{10.222}$$

holds for all $h > 0$ and all $U \in \Omega_{h;\kappa}$.

Proof. Writing

$$\begin{array}{lll} \mathcal{Q}_{\mathrm{apx};I}(U) & = & h \sum_{-;h} q_{\mathrm{apx}}(U) \\ & = & h \sum_{-;h} \gamma_U^{-2} \partial^0 U S^+ [\partial^0 \partial U], \end{array} \tag{10.223}$$

Lemma 10.20 implies that

$$\begin{array}{rcl} |\mathcal{Q}(U) - \mathcal{Q}_{\mathrm{apx};I}(U)| & \leq & h \sum_{-;h} |q(U) - q_{\mathrm{apx}}(U)| \\ & \leq & C_1' h \Big[\|\partial^+ U\|_{\ell_h^2} + \|\partial^+ \partial^+ U\|_{\ell_h^2} \Big] \|\partial^+ \partial^+ U\|_{\ell_h^2} \\ & \leq & C_2' h. \end{array} \tag{10.224}$$

On the other hand, Lemma 7.12 yields the bound

$$\begin{array}{rcl} |\mathcal{Q}_{\mathrm{apx};I}(U) + \ln[\gamma_U]| & = & |\mathcal{Q}_{\mathrm{apx};I}(U) - \mathcal{Q}_{\mathrm{apx}}(U)| \\ & \leq & C_3' h, \end{array} \tag{10.225}$$

which completes the proof. □

Lemma 10.22. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the the residual bound*

$$|\mathcal{Q}_{\mathrm{nl};U}(V)| \quad \leq \quad K \Big[\|\partial^+ V\|_{\ell_h^2} + h \Big] \Big[\|\partial^+ V\|_{\ell_h^2} + \|\partial^+ \partial^+ V\|_{\ell_h^2} \Big] + K h \left| \partial^0 V \right| \tag{10.226}$$

holds for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Writing

$$\begin{array}{rcl} \mathcal{Q}_{\mathrm{lin};U;I}[V] & = & h \sum_{-;h} q_{\mathrm{lin};U}[V] \\ & = & h \sum_{-;h} \Big[\gamma_U^{-4} (2 - \gamma_U^2) S^+ [\partial^0 \partial U] \partial^0 V + \gamma_U^{-2} \partial^0 U S^+ [\partial^0 \partial V] \Big], \end{array} \tag{10.227}$$

we compute

$$\begin{array}{rcl} |\mathcal{Q}(U+V) - \mathcal{Q}(U) - \mathcal{Q}_{\mathrm{lin};U;I}[V]| & \leq & h \sum_{-;h} |q_{\mathrm{nl};U}(V)| \\ & \leq & K \Big[\|\partial^+ V\|_{\ell_h^2} + h \Big] \Big[\|\partial^+ V\|_{\ell_h^2} + \|\partial^+ \partial^+ V\|_{\ell_h^2} \Big]. \end{array} \tag{10.228}$$

Recalling the definition (7.60), we see that

$$\begin{array}{rcl} \mathcal{Q}_{\mathrm{lin};U;I}[V] & = & h \sum_{-;h} \Big[T^- \big[\gamma_U^{-4} (2 - \gamma_U^2) S^+ [\partial^0 \partial U] \big] \partial^0 V + \gamma_U^{-2} \partial^0 U S^+ [\partial^0 \partial V] \Big] \\ & & + h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V. \end{array} \tag{10.229}$$

The summation-by-parts identity (4.13) implies that

$$\begin{array}{rcl} h \sum_{-;h} \gamma_U^{-2} \partial^0 U S^+ [\partial^0 \partial V] & = & h \sum_{-;h} \gamma_U^{-2} [\partial^0 U] \partial^+ \partial^0 V \\ & = & T^- \big[\gamma_U^{-2} \partial^0 U \big] \partial^0 V \\ & & \quad - h \sum_{-;h} \partial^- \big[\gamma_U^{-2} \partial^0 U \big] \partial^0 V \\ & = & \gamma_U^{-2} \partial^0 U \partial^0 V - h \partial^- \big[\gamma_U^{-2} \partial^0 U \big] \partial^0 V \\ & & \quad - h \sum_{-;h} T^- \Big[\partial^+ \big[\gamma_U^{-2} \partial^0 U \big] \Big] \partial^0 V. \end{array} \tag{10.230}$$

In particular, upon writing

$$\begin{array}{rcl} \mathcal{Q}_{\mathrm{lin};U;II}[V] & = & \sum_{-;h} T^- \Big[\gamma_U^{-4} (2 - \gamma_U^2) S^+ [\partial^0 \partial U] - \partial^+ \big[\gamma_U^{-2} \partial^0 U \big] \Big] \partial^0 V \\ & & + \gamma_U^{-2} \partial^0 U \partial^0 V + h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V, \end{array} \tag{10.231}$$

we see that

$$\begin{array}{rcl} |Q_{\mathrm{lin};U;II}[V] - \mathcal{Q}_{\mathrm{lin};U;I}[V]| & = & h \left| \partial^- \left[\gamma_U^{-2} \partial^0 U \right] \partial^0 V \right| \\ & \leq & h C_1' \left| \partial^0 V \right| . \end{array} \tag{10.232}$$

Observing that

$$Q_{\mathrm{lin};U;II}[V] - \mathcal{Q}_{\mathrm{lin};U}[V] = \sum_{-;h} T^- \Big[\gamma_U^{-4} (2 - \gamma_U^2) S^+ [\partial^0 \partial U] - \partial^+ \left[\gamma_U^{-2} \partial^0 U \right] \Big] \partial^0 V, \tag{10.233}$$

we may apply Lemma 7.9 to conclude

$$|Q_{\mathrm{lin};U;II}[V] - \mathcal{Q}_{\mathrm{lin};U}[V]| \quad \leq \quad h C_2' \left\| \partial^+ V \right\|_{\ell_h^2}, \tag{10.234}$$

as desired. □

We now recall the definitions

$$\begin{array}{rclrcl} \mathcal{Z}^+_{\mathrm{apx}}(U) & = & \gamma_U^{-1}, & \mathcal{Z}^+_{\mathrm{lin};U}[V] & = & \gamma_U^{-3} \partial^0 U \partial^0 V + \gamma_U^{-1} h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V, \\ \mathcal{Z}^-_{\mathrm{apx}}(U) & = & \gamma_U, & \mathcal{Z}^-_{\mathrm{lin};U}[V] & = & -\gamma_U^{-1} \partial^0 U \partial^0 V - \gamma_U h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \end{array} \tag{10.235}$$

and write

$$\mathcal{Z}^{\pm}_{\mathrm{nl};U}(V) \quad = \quad \mathcal{Z}^{\pm}(U+V) - \mathcal{Z}^{\pm}(U) - \mathcal{Z}^{\pm}_{\mathrm{lin};U}[V]. \tag{10.236}$$

Lemma 10.23. *Fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the pointwise approximation estimate*

$$\left| \mathcal{Z}^+(U) - \mathcal{Z}^+_{\mathrm{apx}}(U) \right| + \left| \mathcal{Z}^-(U) - \mathcal{Z}^-_{\mathrm{apx}}(U) \right| \quad \leq \quad Kh \tag{10.237}$$

and the residual bound

$$\begin{array}{rcl} \left| \mathcal{Z}^+_{\mathrm{nl}}(V) \right| + \left| \mathcal{Z}^-_{\mathrm{nl}}(V) \right| & \leq & K \Big[\left\| \partial^+ V \right\|^2_{\ell_h^2} + \left\| \partial^+ \partial^+ V \right\|^2_{\ell_h^2} \Big] \\ & & + Kh \Big[\left\| \partial^+ V \right\|_{\ell_h^2} + \left\| \partial^+ \partial^+ V \right\|_{\ell_h^2} \Big] + Kh \left| \partial^0 V \right| \end{array} \tag{10.238}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Motivated by the definitions (8.4), we write

$$f^+(x) = \exp[x], \qquad f^-(x) = \exp[-x], \tag{10.239}$$

together with

$$\phi(\partial^+ U, \partial^0 \partial U) \quad = \quad \mathcal{Q}(U) \tag{10.240}$$

and finally

$$P^{\pm}(\partial^+ U, \partial^0 \partial U) = f^{\pm}\big(\phi(\partial^+ U, \partial^0 \partial U) \big). \tag{10.241}$$

One readily verifies that

$$\mathcal{Z}^{\pm}(U) = P^{\pm}(\partial^+ U, \partial^0 \partial U). \tag{10.242}$$

We now define the approximants

$$\begin{aligned} \phi_{\rm apx}(\partial^+ U, \partial^0 \partial U) &= -\ln \gamma_U, \\ \phi_{{\rm lin};U}[\partial^+ V, \partial^0 \partial V] &= \gamma_U^{-2} \partial^0 U \partial^0 V + h \textstyle\sum_{-;h} \mathcal{E}_{\rm sm}(U) \partial^0 V. \end{aligned} \tag{10.243}$$

Lemma 10.22 provides the pointwise bound

$$\begin{aligned} \big|\phi(\partial^+ U, \partial^0 \partial U) - \phi_{\rm apx}(\partial^+ U, \partial^0 \partial U)\big| &= |\mathcal{Q}(U) - \mathcal{Q}_{\rm apx}(U)| \\ &\leq C_1' h, \end{aligned} \tag{10.244}$$

together with

$$\begin{aligned} |\phi_{{\rm nl};U}[V]| &= |\mathcal{Q}_{{\rm nl};U}(V)| \\ &\leq C_1' \Big[\, \|\partial^+ V\|_{\ell^2_h} + h \Big] \Big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+ \partial^+ V\|_{\ell^2_h} \, \Big] + C_1' h \, \big|\partial^0 V\big| \, . \end{aligned} \tag{10.245}$$

Noting that $Df^{\pm}(x) = \pm f^{\pm}(x)$, we see that the functions defined in (6.19) satisfy

$$\begin{aligned} P^{\pm}_{\rm apx}(U) &= f^{\pm}\big(\phi_{\rm apx}(\partial^+ U, \partial^0 \partial U)\big) \\ &= \exp\big[\mp \ln[\gamma_U]\big] \\ &= \gamma_U^{\mp 1}, \end{aligned} \tag{10.246}$$

together with

$$\begin{aligned} P^{\pm}_{{\rm lin};U}[V] &= \pm f^{\pm}(-\ln[\gamma_U]) \Big[\phi_{{\rm lin};U}[V]\Big] \\ &= \pm \gamma_U^{\mp 1} \Big[\gamma_U^{-2} \partial^0 U \partial^0 V + h \textstyle\sum_{-;h} \mathcal{E}_{\rm sm}(U) \partial^0 V\Big] \\ &= \pm \gamma_U^{-(2 \pm 1)} \partial^0 U \partial^0 V \pm \gamma_U^{\mp 1} h \textstyle\sum_{-;h} \mathcal{E}_{\rm sm}(U) \partial^0 V. \end{aligned} \tag{10.247}$$

The desired estimates now follow from Lemma 6.4. □

Proof of Proposition 10.4. The results follow directly from Lemma 10.23. □

10.6 Estimates for g

We recall the notation $g^+(U) = \partial^+ g(U)$ together with the definitions

$$\begin{aligned} g_{\rm apx}(U) &= g(U), & g_{{\rm lin};U}[V] &= g'(U) V, \\ g^+_{\rm apx}(U) &= g'(U) \partial^0 U, & g^+_{{\rm lin};U}[V] &= g''(U)[\partial^0 U] V + g'(U) \partial^0 V \end{aligned} \tag{10.248}$$

and write

$$\begin{aligned} g_{{\rm nl};U}(V) &= g(U+V) - g(U) - g_{{\rm lin};U}[V], \\ g^+_{{\rm nl};U}(V) &= g^+(U+V) - g^+(U) - g^+_{{\rm lin};U}[V]. \end{aligned} \tag{10.249}$$

Lemma 10.24. *Suppose that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$*. Then there exists* $K > 0$ *so that the pointwise approximation estimate*

$$\big|g^+(U) - g^+_{\rm apx}(U)\big| \quad \leq \quad K h \big[\, \big|\partial^0 \partial U\big| + |\partial^+ U| \, \big] \tag{10.250}$$

and the residual bounds

$$
\begin{array}{lcl}
|g_{\mathrm{nl};U}(V)| & \leq & K\,|V|^2\,, \\
\left|g^{+}_{\mathrm{nl};U}(V)\right| & \leq & K\big[\,|V|^2 + T^{+}\,|V|^2 + |\partial^{+}V|^2\,\big] \\
 & & \qquad +Kh\big[\,|V| + T^{+}\,|V| + |\partial^{+}V| + \left|\partial^{0}\partial V\right|\,\big]
\end{array}
\qquad (10.251)
$$

all hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. We first note that

$$
\begin{array}{lcl}
g_{\mathrm{nl};U}(V) & = & g(U+V) - g(U) - g'(U)V \\
 & = & \int_0^1 \int_0^{\sigma} g''(U+\sigma' V)V^2\, d\sigma' d\sigma,
\end{array}
\qquad (10.252)
$$

which yields the desired estimate for g_{nl}. In addition, for any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$, the C^3-smoothness of g implies the pointwise bound

$$
\left|g_{\mathrm{nl};U^{(1)}}(V) - g_{\mathrm{nl};U^{(2)}}(V)\right| \leq C_1' \left|U^{(1)} - U^{(2)}\right| V^2. \qquad (10.253)
$$

Finally, upon writing

$$
g^{(1)}_{\mathrm{nl};U}(V) = g'(U+V) - g'(U) - g''(U)V, \qquad (10.254)
$$

the C^3-smoothness of g implies the bound

$$
\left|g^{(1)}_{\mathrm{nl};U}(V)\right| \leq C_2' \,|V|^2\,. \qquad (10.255)
$$

We now compute

$$
\begin{array}{lcl}
g^{+}(U) & = & h^{-1}\big[g(T^{+}U) - g(U)\big] \\
 & = & g'(U)\partial^{+}U + h^{-1}g_{\mathrm{nl};U}(h\partial^{+}U) \\
 & = & g'(U)\partial^{0}U + hg'(U)\partial^{0}\partial U + h^{-1}g_{\mathrm{nl};U}(h\partial^{+}U),
\end{array}
\qquad (10.256)
$$

which yields (10.250). In addition, we compute

$$
\begin{array}{lcl}
g^{+}(U+V) & = & h^{-1}\big[g(T^{+}U + T^{+}V) - g(U+V)\big] \\
 & = & h^{-1}\big[g(T^{+}U + T^{+}V) - g(U+T^{+}V)\big] \\
 & & \qquad +h^{-1}\big[g(U+T^{+}V) - g(U+V)\big] \\
 & = & \mathcal{I}_A + \mathcal{I}_B,
\end{array}
\qquad (10.257)
$$

in which we have

$$
\begin{array}{lcl}
\mathcal{I}_A & = & h^{-1}\big[g'(U+T^{+}V)h\partial^{+}U + g_{\mathrm{nl};U+T^{+}V}(h\partial^{+}U)\big], \\
\mathcal{I}_B & = & h^{-1}\big[g'(U+V)h\partial^{+}V + g_{\mathrm{nl};U+V}(h\partial^{+}V)\big].
\end{array}
\qquad (10.258)
$$

We compute

$$
\begin{array}{lcl}
\mathcal{I}_A & = & g'(U)\partial^{+}U + [g'(U+T^{+}V) - g'(U)]\partial^{+}U + h^{-1}g_{\mathrm{nl};U}(h\partial^{+}U) \\
 & & \qquad +h^{-1}\big[g_{\mathrm{nl};U+T^{+}V}(h\partial^{+}U) - g_{\mathrm{nl};U}(h\partial^{+}U)\big] \\
 & = & g^{+}(U) + g^{(1)}_{\mathrm{nl};U}(T^{+}V)\partial^{+}U + g''(U)\partial^{+}UT^{+}V \\
 & & \qquad +h^{-1}\big[g_{\mathrm{nl};U+T^{+}V}(h\partial^{+}U) - g_{\mathrm{nl};U}(h\partial^{+}U)\big],
\end{array}
\qquad (10.259)
$$

together with

$$\mathcal{I}_B \quad = \quad g'(U)\partial^+ V + [g'(U+V) - g'(U)]\partial^+ V + h^{-1} g_{\mathrm{nl};U+V}(h\partial^+ V). \tag{10.260}$$

In particular, we see that

$$\begin{aligned} g^+_{\mathrm{nl};U}(V) \quad = \quad & g''(U)\Big[\partial^+ U T^+ V - [\partial^0 U]V\Big] + g'(U)\big[\partial^+ V - \partial^0 V\big] \\ & + g^{(1)}_{\mathrm{nl};U}(T^+ V)\partial^+ U + h^{-1}\big[g_{\mathrm{nl};U+T^+V}(h\partial^+ U) - g_{\mathrm{nl};U}(h\partial^+ U)\big] \\ & + [g'(U+V) - g'(U)]\partial^+ V + h^{-1} g_{\mathrm{nl};U+V}(h\partial^+ V). \end{aligned} \tag{10.261}$$

Using (10.253) and (10.255), the desired estimate can now be read off from this identity. □

Proof of Proposition 10.5. The results follow directly from Lemma 10.24. □

11 Component estimates - II

In this section, we are interested in the set of nonlinearities

$$\mathcal{S}_{\mathrm{nl;short}} \quad = \quad \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}, \mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}, \tag{11.1}$$

which contains all the components featuring in the decomposition (9.9) for $\mathcal{G}$. In addition, we consider the set

$$\overline{\mathcal{S}}_{\mathrm{nl;short}} \quad = \quad \mathcal{S}_{\mathrm{nl;short}} \cup \{\mathcal{Y}^+_1, \mathcal{Y}^+_{2b}\}, \tag{11.2}$$

which contains all the components that feature in the decompositions (9.14) and (9.15) for $\mathcal{G}^+(U)$, with the exception of $\mathcal{Y}^+_{2a}$.

Exploiting the definitions (9.7) and (9.11), we define the standard approximants

$$\begin{aligned} \mathcal{Y}_{1;\mathrm{apx}}(U) \quad &= \quad \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U)\mathcal{Z}^-_{\mathrm{apx}}(U), \\ \mathcal{Y}_{2;\mathrm{apx}}(U) \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0}_{\mathrm{apx}}(U) + g(U), \\ \mathcal{Y}^+_{1;\mathrm{apx}}(U) \quad &= \quad \widetilde{p}_{\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)T^+[\mathcal{Z}^-_{\mathrm{apx}}(U)], \\ \mathcal{Y}^+_{2a;\mathrm{apx}}(U) \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0;+}_{a;\mathrm{apx}}(U), \\ \mathcal{Y}^+_{2b;\mathrm{apx}}(U) \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0;+}_{b;\mathrm{apx}}(U) + g^+_{\mathrm{apx}}(U), \end{aligned} \tag{11.3}$$

together with

$$\begin{aligned} \mathcal{Y}_{1;\mathrm{lin};U}[V] \quad &= \quad \mathcal{F}^{\diamond_0}_{\mathrm{lin};U}[V]\mathcal{Z}^-_{\mathrm{apx}}(U) + \mathcal{F}^{\diamond_0}_{\mathrm{apx}}(U)\mathcal{Z}^-_{\mathrm{lin};U}[V], \\ \mathcal{Y}_{2;\mathrm{lin};U}[V] \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0}_{\mathrm{lin};U}[V] + g'(U)V, \\ \mathcal{Y}^+_{1;\mathrm{lin};U}[V] \quad &= \quad \widetilde{p}_{\mathrm{lin};U}[V]\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)T^+[\mathcal{Z}^-_{\mathrm{apx}}(U)] + \widetilde{p}_{\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V]T^+[\mathcal{Z}^-_{\mathrm{apx}}(U)] \\ & \qquad + \widetilde{p}_{\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)T^+\big[\mathcal{Z}^-_{\mathrm{lin};U}[V]\big], \\ \mathcal{Y}^+_{2a;\mathrm{lin};U}[V] \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0;+}_{a;\mathrm{lin};U}[V], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] \quad &= \quad 2\mathcal{F}^{\diamond\diamond_0;+}_{b;\mathrm{lin};U}[V] + g^+_{\mathrm{lin};U}[V]. \end{aligned} \tag{11.4}$$

In addition, exploiting the definitions (9.8), we write

$$\begin{aligned} \mathcal{X}_{A;\mathrm{apx}}(U) \quad &= \quad p_{\mathrm{apx}}(U)\mathcal{Z}^+_{\mathrm{apx}}(U), \\ \mathcal{X}_{B;\mathrm{apx}}(U) \quad &= \quad S^+[\mathcal{Z}^+_{\mathrm{apx}}(U)]p^{\diamond+}_{\mathrm{apx}}(U), \\ \mathcal{X}_{C;\mathrm{apx}}(U) \quad &= \quad S^+[\mathcal{Z}^+_{\mathrm{apx}}(U)]p^{\diamond_0}_{\mathrm{apx}}(U), \\ \mathcal{X}_{D;\mathrm{apx}}(U) \quad &= \quad S^+[p_{\mathrm{apx}}(U)]\mathcal{Z}^+_{\mathrm{apx}}(U)p_{\mathrm{apx}}(U), \end{aligned} \tag{11.5}$$

together with

$$
\begin{aligned}
\mathcal{X}_{A;\mathrm{lin};U}[V] &= p_{\mathrm{lin};U}[V]\mathcal{Z}^{+}_{\mathrm{apx}}(U) + p_{\mathrm{apx}}(U)\mathcal{Z}^{+}_{\mathrm{lin};U}[V],\\
\mathcal{X}_{B;\mathrm{lin};U}[V] &= S^{+}\big[\mathcal{Z}^{+}_{\mathrm{lin};U}[V]\big]p^{\diamond_{+}}_{\mathrm{apx}}(U) + S^{+}[\mathcal{Z}^{+}_{\mathrm{apx}}(U)]p^{\diamond_{+}}_{\mathrm{lin};U}[V],\\
\mathcal{X}_{C;\mathrm{lin};U}[V] &= S^{+}\big[\mathcal{Z}^{+}_{\mathrm{lin};U}[V]\big]p^{\diamond_{0}}_{\mathrm{apx}}(U) + S^{+}[\mathcal{Z}^{+}_{\mathrm{apx}}(U)]p^{\diamond_{0}}_{\mathrm{lin};U}[V],\\
\mathcal{X}_{D;\mathrm{lin};U}[V] &= S^{+}\big[p_{\mathrm{lin};U}[V]\big]\mathcal{Z}^{+}_{\mathrm{apx}}(U)p_{\mathrm{apx}}(U) + S^{+}[p_{\mathrm{apx}}(U)]\mathcal{Z}^{+}_{\mathrm{lin};U}[V]p_{\mathrm{apx}}(U)\\
&\quad + S^{+}\big[p_{\mathrm{apx}}(U)\big]\mathcal{Z}^{+}_{\mathrm{apx}}(U)p_{\mathrm{lin};U}[V].
\end{aligned}
\tag{11.6}
$$

These approximants can be used as building blocks for the expressions P_{apx} and $P_{\mathrm{lin};U}$ defined in (6.67) that arise when applying Lemma 6.6 to $\mathcal{G}$ and $\mathcal{G}(U)$.

Using the expressions introduced in §10 all these approximants can be explicitly evaluated. However, as we shall see in the sequel, the resulting identities are not always easy to handle. Our goal in this section is to introduce a framework that allows us to keep track of the errors that arise when simplifying these expressions. In particular, for any $f \in \mathcal{S}_{\mathrm{nl;short}} \cup \overline{\mathcal{S}}_{\mathrm{nl;short}}$ we introduce decompositions

$$
\begin{aligned}
f_{\mathrm{apx}}(U) &= f_{\mathrm{apx;expl}}(U) + f_{\mathrm{apx;sh}}(U) + f_{\mathrm{apx;rem}}(U),\\
f_{\mathrm{lin};U}[V] &= f_{\mathrm{lin};U;\mathrm{expl}}[V] + f_{\mathrm{lin};U;\mathrm{sh}}[V] + f_{\mathrm{lin};U;\mathrm{rem}}[V].
\end{aligned}
\tag{11.7}
$$

The expressions with the label 'expl' are the actual simplifications. The label 'sh' is used for terms which are always small, while the label 'rem' is used for terms which are small when using $U = \Psi_*$.

In addition, we define sets

$$
Q_{f;\mathrm{pref}} \subset \{2, \infty\}
\tag{11.8}
$$

for $f \in \mathcal{S}_{\mathrm{nl;short}}$, together with sets

$$
\overline{Q}_{f;\mathrm{pref}} \subset \{2, \infty\}
\tag{11.9}
$$

for $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$, so that the decomposition Lemma's 9.7 and 9.8 remain valid upon replacing $(\mathcal{S}_{\mathrm{nl}}, \overline{\mathcal{S}}_{\mathrm{nl}})$ by $(\mathcal{S}_{\mathrm{nl;short}}, \overline{\mathcal{S}}_{\mathrm{nl;short}})$. Finally, for each $f \in \mathcal{S}_{\mathrm{nl;short}} \cup \overline{\mathcal{S}}_{\mathrm{nl;short}}$, we define sets

$$
Q_f \subset \{2, \infty\}, \qquad Q^{A}_{f;\mathrm{lin}} \subset \{2, \infty\}, \qquad Q^{B}_{f;\mathrm{lin}} \subset \{2, \infty\}
\tag{11.10}
$$

so that the main spirit of the framework developed in §10 transfers directly to the estimates considered here. Indeed, for $f \in \{\mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}\}$ these sets are identical to those defined earlier.

11.1 Summary of estimates

In order to state our results, we introduce the expressions

$$
\begin{aligned}
S_{\mathrm{sh;full}}(U) &= h, & S_{\mathrm{sh;2;fix}}(U) &= 0,\\
\overline{S}_{\mathrm{sh;full}}(U) &= h\big[1 + \|\partial^{+}\partial^{+}\partial^{+}U\|_{\ell^2_h} + \|\partial^{+}\partial^{+}\partial^{+}U\|_{\ell^\infty_h}\big], & \overline{S}_{\mathrm{sh;2;fix}}(U) &= 0,
\end{aligned}
\tag{11.11}
$$

together with

$$
\begin{aligned}
S_{\mathrm{rem;full}}(U) &= \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h}, & S_{\mathrm{rem;2;fix}}(U) &= 0,\\
\overline{S}_{\mathrm{rem;full}}(U) &= S_{\mathrm{rem;full}}(U) + \|\partial^{+}[\mathcal{E}_{\mathrm{tw}}(U)]\|_{\ell^2_h}, & \overline{S}_{\mathrm{rem;2;fix}}(U) &= 0
\end{aligned}
\tag{11.12}
$$

and finally

$$\begin{aligned} S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) &= \big\|U^{(2)} - U^{(1)}\big\|_{\ell_h^{2;2}} + \big\|U^{(2)} - U^{(1)}\big\|_{\ell_h^{\infty;1}} , \\ S_{\mathrm{diff;2;fix}}(U^{(1)}, U^{(2)}) &= \big\|U^{(2)} - U^{(1)}\big\|_{\ell_h^{2;2}} . \end{aligned} \tag{11.13}$$

These expressions are all related to the f_{apx} functions and play a very similar role as the quantities S_{full} and $S_{2;\mathrm{fix}}$ that were defined in §10.

We state our estimates for the approximants f_{lin} in terms of the quantities

$$\begin{aligned} \mathcal{E}_{\mathrm{sh};U}(V) &= h \|V\|_{\ell_h^{2;2}} , \\ \overline{\mathcal{E}}_{\mathrm{sh};U}(V) &= h \|V\|_{\ell_h^{3;2}} + \big[\|\partial^+\partial^+\partial^+ U\|_{\ell_h^2} + \|\partial^+\partial^+\partial^+ U\|_{\ell_h^\infty} \big] \|V\|_{\ell_h^{2;2}} , \end{aligned} \tag{11.14}$$

together with

$$\begin{aligned} \mathcal{E}_{\mathrm{rem};U}(V) &= \|V\|_{\ell_h^{2;2}} \Big[\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2} \Big], \\ \overline{\mathcal{E}}_{\mathrm{rem};U}(V) &= \|V\|_{\ell_h^{2;2}} \Big[\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2} \Big] \\ &\qquad + \|V\|_{\ell_h^{2;1}} \|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\|_{\ell_h^\infty} \end{aligned} \tag{11.15}$$

and finally

$$\begin{aligned} \mathcal{E}_{\mathrm{prod}}(W^{(1)}, W^{(2)}) &= \big\|W^{(1)}\big\|_{\ell_h^{2;2}} \big\|W^{(2)}\big\|_{\ell_h^{2;2}} \\ &\qquad + \big\|W^{(1)}\big\|_{\ell_h^{2;2}} \big\|W^{(2)}\big\|_{\ell_h^{\infty;1}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;1}} \big\|W^{(2)}\big\|_{\ell_h^{2;2}} . \end{aligned} \tag{11.16}$$

These expressions should be compared to $\mathcal{E}_{\mathrm{nl}}$ in §10.

Our main results summarize the structure that the decompositions described in the sequel will adhere to. Propositions 11.1 and 11.2 state that the approximants $f_{\mathrm{apx};\#}$ are all uniformly bounded and that the full linear approximants $f_{\mathrm{lin};U}$ share the structure and estimates of the nonlinearities in $\mathcal{S}_{\mathrm{nl}} \cup \overline{\mathcal{S}}_{\mathrm{nl}}$. Propositions 11.3-11.4 should be seen as the equivalents of of Corollary 10.7, while Propositions 11.5-11.6 are the equivalents of Corollary 10.9.

Proposition 11.1. *The statements in Corollary 10.6 also hold upon replacing* $(\mathcal{S}_{\mathrm{nl}}, \overline{\mathcal{S}}_{\mathrm{nl}})$ *by their counterparts* $(\mathcal{S}_{\mathrm{nl;short}}, \overline{\mathcal{S}}_{\mathrm{nl;short}})$*. In addition, for every* $f \in \mathcal{S}_{\mathrm{nl}} \cup \overline{\mathcal{S}}_{\mathrm{nl}}$ *there exists* $K > 0$ *so that for each* $q \in Q_f$*, the bound*

$$\|f_{\mathrm{apx;expl}}(U)\|_{\ell_h^q} + \|f_{\mathrm{apx;sh}}(U)\|_{\ell_h^q} + \|f_{\mathrm{apx;rem}}(U)\|_{\ell_h^q} \le K \tag{11.17}$$

holds for all $h > 0$ *and* $U \in \Omega_{h;\kappa}$*.*

Proposition 11.2. *The statements in Corollary 10.8 also hold upon replacing* $(\mathcal{S}_{\mathrm{nl}}, \overline{\mathcal{S}}_{\mathrm{nl}})$ *by their counterparts* $(\mathcal{S}_{\mathrm{nl;short}}, \overline{\mathcal{S}}_{\mathrm{nl;short}})$ *and picking* $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}}$*.*

Proposition 11.3. *Assume that (Hg) is satisfied and fix* $0 < \kappa < \frac{1}{12}$*. Then there exists* $K > 0$ *so that for every* $f \in \mathcal{S}_{\mathrm{nl}}$*,* $q \in Q_{f;\mathrm{pref}}$ *and* $\# \in \{\mathrm{sh}, \mathrm{rem}\}$*, we have*

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell_h^q} \quad \le \quad K S_{\#;\mathrm{full}}(U) \tag{11.18}$$

for any $h > 0$ *and* $U \in \Omega_{h;\kappa}$*.*

In addition, if $2 \in Q_{f;\mathrm{pref}}$ *then for every* $\# \in \{\mathrm{sh}, \mathrm{rem}\}$ *at least one of the following two properties hold true.*

(a) There exists $K > 0$ so that

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell^2_h} \quad \leq \quad K S_{\#;2;\mathrm{fix}}(U) \tag{11.19}$$

holds for every $h > 0$ and $U \in \Omega_{h;\kappa}$.

(b) There exists $K > 0$ so that

$$\|f_{\mathrm{apx};\#}(U)\|_{\ell^\infty_h} \quad \leq \quad K S_{\#;\mathrm{full}}(U) \tag{11.20}$$

holds for every $h > 0$ and $U \in \Omega_{h;\kappa}$.

The same properties hold upon making the replacement

$$(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}}, S_{\#;full}, S_{\#;2;\mathrm{fix}}) \mapsto (\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}}, \overline{S}_{\#;\mathrm{full}}, \overline{S}_{\#;2;\mathrm{fix}}). \tag{11.21}$$

Proposition 11.4. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that for every $f \in \mathcal{S}_{\mathrm{nl}}$, $q \in Q_{f;\mathrm{pref}}$ and $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$, we have*

$$\left\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\right\|_{\ell^q_h} \quad \leq \quad K S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) \tag{11.22}$$

for any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

In addition, if $2 \in Q_{f;\mathrm{pref}}$, then for every $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$ at least one of the following two properties hold true.

(a) There exists $K > 0$ so that

$$\left\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\right\|_{\ell^2_h} \quad \leq \quad K S_{\mathrm{diff};2;\mathrm{fix}}(U^{(1)}, U^{(2)}) \tag{11.23}$$

holds for every $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

(b) There exists $K > 0$ so that

$$\left\|f_{\mathrm{apx};\#}(U^{(2)}) - f_{\mathrm{apx};\#}(U^{(1)})\right\|_{\ell^\infty_h} \quad \leq \quad K S_{\mathrm{diff;full}}(U^{(1)}, U^{(2)}) \tag{11.24}$$

holds for every $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

Proposition 11.5. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Consider any $f \in \mathcal{S}_{\mathrm{nl}}$ and any $\# \in \{\mathrm{sh}, \mathrm{rem}\}$. Then if $2 \in Q_{f;\mathrm{pref}}$, there exists a constant $K > 0$ so that*

$$\|f_{\mathrm{lin};U;\#}(V)\|_{\ell^2_h} \leq K\mathcal{E}_{\#;U}(V) \tag{11.25}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Otherwise, there exists $q \in \{2, \infty\}$ together with a constant $K > 0$ so that

$$\|f_{\mathrm{lin};U;\#}(V)\|_{\ell^q_h} \leq K\mathcal{E}_{\#;U}(V) \tag{11.26}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$. The same properties hold upon making the replacement

$$(\mathcal{S}_{\mathrm{nl;short}}, Q_{f;\mathrm{pref}}, \mathcal{E}_{\#}) \mapsto (\overline{\mathcal{S}}_{\mathrm{nl;short}}, \overline{Q}_{f;\mathrm{pref}}, \overline{\mathcal{E}}_{\#}). \tag{11.27}$$

Proposition 11.6. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Consider any $f \in \mathcal{S}_{\mathrm{nl}}$ and any $\# \in \{\mathrm{expl}, \mathrm{sh}, \mathrm{rem}\}$. Then if $2 \in Q_{f;\mathrm{pref}}$, there exists a constant $K > 0$ so that*

$$\left\|f_{\mathrm{lin};U^{(2)};\#}(V) - f_{\mathrm{lin};U^{(1)};\#}(V)\right\|_{\ell^2_h} \leq K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V) \tag{11.28}$$

holds for all $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$.

Otherwise, there exists $q \in \{2, \infty\}$ together with a constant $K > 0$ so that

$$\left\|f_{\mathrm{lin};U^{(2)};\#}(V) - f_{\mathrm{lin};U^{(1)};\#}(V)\right\|_{\ell^q_h} \leq K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V) \tag{11.29}$$

holds for all $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$.

11.2 Decomposition for $\mathcal{Y}_1$ and $\mathcal{X}_A$

Identities for $\mathcal{Y}_1$ Substituting the relevant expressions from §10 into (11.3)-(11.4), we compute

$$\begin{array}{rcl} \mathcal{Y}_{1;\mathrm{apx}}(U) & = & \gamma_U^{-1}\partial^0 U \gamma_U \\ & = & \partial^0 U, \\ \mathcal{Y}_{1;\mathrm{lin};U}[V] & = & \gamma_U^{-3}\partial^0 V \gamma_U - \gamma_U^{-2}(\partial^0 U)^2 \partial^0 V \\ & & \quad -\gamma_U^{-1}\partial^0 U \gamma_U h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U)\partial^0 V \\ & = & \partial^0 V - \partial^0 U \big[h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \big]. \end{array} \tag{11.30}$$

We realize the splittings (11.7) by writing

$$\begin{array}{rclrcl} \mathcal{Y}_{1;\mathrm{apx;expl}}(U) & = & \partial^0 U, & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V] & = & \partial^0 V, \\ \mathcal{Y}_{1;\mathrm{apx;sh}}(U) & = & 0, & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{Y}_{1;\mathrm{apx;rem}}(U) & = & 0, & \mathcal{Y}_{1;\mathrm{lin};U;\mathrm{rem}}[V] & = & -\partial^0 U \big[h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \big]. \end{array} \tag{11.31}$$

In addition, we introduce the sets

$$Q_{\mathcal{Y}_1;\mathrm{pref}} = \overline{Q}_{\mathcal{Y}_1;\mathrm{pref}} = \{2, \infty\}, \tag{11.32}$$

together with

$$Q_{\mathcal{Y}_1} = \{2, \infty\}, \qquad Q^A_{\mathcal{Y}_1;\mathrm{lin}} = Q^B_{\mathcal{Y}_1;\mathrm{lin}} = \{2, \infty\}. \tag{11.33}$$

Identities for $\mathcal{X}_A$ Substituting the relevant expressions from §10 into (11.5)-(11.6), we compute

$$\begin{array}{rcl} \mathcal{X}_{A;\mathrm{apx}}(U) & = & \gamma \partial^0 U \gamma_U^{-1} \\ & = & \partial^0 U, \\ \mathcal{X}_{A;\mathrm{lin};U}[V] & = & \gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V \gamma_U^{-1} + \gamma_U \partial^0 U \gamma_U^{-3} \partial^0 U \partial^0 V \\ & & \quad +\gamma_U \partial^0 U \gamma_U^{-1} \big[h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \big] \\ & = & \partial^0 V + \partial^0 U \big[h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \big]. \end{array} \tag{11.34}$$

We realize the splittings (11.7) by writing

$$\begin{array}{rclrcl} \mathcal{X}_{A;\mathrm{apx;expl}}(U) & = & \partial^0 U, & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V] & = & \partial^0 V, \\ \mathcal{X}_{A;\mathrm{apx;sh}}(U) & = & 0, & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{X}_{A;\mathrm{apx;rem}}(U) & = & 0, & \mathcal{X}_{A;\mathrm{lin};U;\mathrm{rem}}[V] & = & +\partial^0 U \big[h \sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U) \partial^0 V \big]. \end{array} \tag{11.35}$$

In addition, we introduce the sets

$$Q_{\mathcal{X}_A;\mathrm{pref}} = \overline{Q}_{\mathcal{X}_A;\mathrm{pref}} = \{\infty\}, \tag{11.36}$$

together with

$$Q_{\mathcal{X}_A} = \{\infty\}, \qquad Q^A_{\mathcal{X}_A;\mathrm{lin}} = \{\infty\}, \qquad Q^B_{\mathcal{X}_A;\mathrm{lin}} = \{2, \infty\}. \tag{11.37}$$

Estimates

Lemma 11.7. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. Then the conditions in $(hf)_{\mathrm{lin}}$ are satisfied with $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}}$ and there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{lclcl} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \leq & K \|\partial^+ V\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq & K \|\partial^+ V\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \leq & K \|\partial^+ V\|_{\ell^2_h} & \leq & K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq & K \|\partial^+ V\|_{\ell^\infty_h} & \leq & K T_{\infty;\mathrm{opt}}(V) \end{array} \tag{11.38}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. Writing $f^B_{\mathrm{lin};U}[V] = f_{\mathrm{lin};U;\mathrm{expl}}[V]$, the bounds follow from inspection. □

Lemma 11.8. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|f_{\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^2_h} + \|f_{\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^\infty_h} \quad \leq \quad K. \tag{11.39}$$

(ii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{lcl} \left\| f_{\mathrm{apx};\mathrm{expl}}(U^{(1)}) - f_{\mathrm{apx};\mathrm{expl}}(U^{(2)}) \right\|_{\ell^2_h} & \leq & K \left\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \right\|_{\ell^2_h} \\ & \leq & K S_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big), \\ \left\| f_{\mathrm{apx};\mathrm{expl}}(U^{(1)}) - f_{\mathrm{apx};\mathrm{expl}}(U^{(2)}) \right\|_{\ell^\infty_h} & \leq & K \left\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \right\|_{\ell^\infty_h} \\ & \leq & K S_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big). \end{array} \tag{11.40}$$

Proof. These estimates follow by inspection. □

Lemma 11.9. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{Y}_1, \mathcal{X}_A\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\begin{array}{lcl} \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^2_h} & \leq & K \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h} \|\partial^+ V\|_{\ell^2_h} \\ & \leq & K \mathcal{E}_{\mathrm{rem};U}(V). \end{array} \tag{11.41}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bounds

$$\begin{array}{lcl} \left\| f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V] \right\|_{\ell^2_h} & = & 0, \\ \left\| f_{\mathrm{lin};U^{(1)};\mathrm{rem}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{rem}}[V] \right\|_{\ell^2_h} & \leq & K \|\partial^+ V\|_{\ell^2_h} \Big[\left\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \right\|_{\ell^2_h} \\ & & \qquad + \left\| \partial^0 \partial U^{(1)} - \partial^0 \partial U^{(2)} \right\|_{\ell^2_h} \Big] \\ & \leq & K \mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{11.42}$$

Proof. Recalling the Lipschitz bound (7.65), the estimates follow by inspection. □

11.3 Decomposition for $\mathcal{Y}_2$

Substituting the relevant expressions from §10 into (11.3)-(11.4), we compute

$$\begin{array}{rcl} \mathcal{Y}_{2;\mathrm{apx}}(U) & = & 2\gamma_U^{-4}\partial^0\partial U + g(U), \\ \mathcal{Y}_{2;\mathrm{lin}}[V] & = & 8\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0 V + 2\gamma_U^{-4}\partial^0\partial V + g'(U)[V] \\ & = & \gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V. \end{array} \tag{11.43}$$

Recalling the function $\mathcal{E}_{\mathrm{tw}}$ introduced in (7.61), we realize the splittings (11.7) by writing

$$\begin{array}{rclcrcl} \mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U) & = & c_*\gamma_U^{-1}\partial^0 U, & & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V] & = & \gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V, \\ \mathcal{Y}_{2;\mathrm{apx};\mathrm{sh}}(U) & = & 0, & & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{sh}}[V] & = & 0, \\ \mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U) & = & \mathcal{E}_{\mathrm{tw}}(U), & & \mathcal{Y}_{2;\mathrm{lin};U;\mathrm{rem}}[V] & = & 0. \end{array} \tag{11.44}$$

In addition, we introduce the sets

$$Q_{\mathcal{Y}_2;\mathrm{pref}} = \overline{Q}_{\mathcal{Y}_2;\mathrm{pref}} = \{2\}, \tag{11.45}$$

together with

$$Q_{\mathcal{Y}_2} = \{2, \infty\}, \qquad Q^A_{\mathcal{Y}_2;\mathrm{lin}} = Q^B_{\mathcal{Y}_2;\mathrm{lin}} = \{2\}. \tag{11.46}$$

Lemma 11.10. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}_2$. Then the conditions in $(hf)_{\mathrm{lin}}$ are satisfied with $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}} = 0$ and there exists a constant $K > 0$ so that the bound*

$$\left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} \leq K \|V\|_{\ell^{2;2}_h} \leq K T_{\mathrm{safe}}(V) \tag{11.47}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. This follows from Proposition 7.15. □

Lemma 11.11. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^2_h} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\|_{\ell^\infty_h} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell^2_h} + \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell^\infty_h} \leq K. \tag{11.48}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{rcccl} \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell^2_h} & \leq & \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} & \leq & S_{\mathrm{rem};\mathrm{full}}(U), \\ \|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U)\|_{\ell^\infty_h} & \leq & \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h} & \leq & S_{\mathrm{rem};\mathrm{full}}(U). \end{array} \tag{11.49}$$

(iii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{rcl} \left\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}) - \mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)})\right\|_{\ell^2_h} & \leq & K \left\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\right\|_{\ell^2_h} \\ & \leq & K \min\{S_{\mathrm{diff};2;\mathrm{fix}}\big(U^{(1)}, U^{(2)}\big), \\ & & \qquad S_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big)\}, \\ \left\|\mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U^{(1)}) - \mathcal{Y}_{2;\mathrm{apx};\mathrm{rem}}(U^{(2)})\right\|_{\ell^2_h} & \leq & K \left\|U^{(1)} - U^{(2)}\right\|_{\ell^{2;2}_h} \\ & \leq & K \min\{S_{\mathrm{diff};2;\mathrm{fix}}\big(U^{(1)}, U^{(2)}\big), \\ & & \qquad S_{\mathrm{diff};\mathrm{full}}\big(U^{(1)}, U^{(2)}\big)\}. \end{array} \tag{11.50}$$

Proof. Recalling (7.65), these bounds follow by inspection. □

Lemma 11.12. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bound*

$$\begin{aligned} \big\| \mathcal{Y}_{2;\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - \mathcal{Y}_{2;\mathrm{lin};U^{(2)};\mathrm{expl}}[V] \big\|_{\ell_h^2} &\leq K \big\| U^{(1)} - U^{(2)} \big\|_{\ell_h^{2;2}} \| V \|_{\ell_h^{\infty;1}} \\ &\quad + K \big\| U^{(1)} - U^{(2)} \big\|_{\ell_h^{\infty;1}} \| V \|_{\ell_h^{2;2}} \\ &\leq K \mathcal{E}_{\mathrm{prod}}(U^{(1)} - U^{(2)}, V) \end{aligned} \tag{11.51}$$

for any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. This bound follows directly from Corollary 7.2 and Proposition 7.15. □

11.4 Decomposition for $\mathcal{F}^{\diamond_0;+}$ and $\mathcal{F}^{\diamond_-;+}$

For both functions $f \in \{\mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}\}$ we write $f_{\mathrm{apx;sh}}(U) = f_{\mathrm{apx;rem}}(U) = 0$ and $f_{\mathrm{lin;sh}}(U) = f_{\mathrm{lin;rem}}(U) = 0$. Besides the Lipschitz estimates below, all the estimates we require here can be found in Proposition 10.3.

Lemma 11.13. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{F}^{\diamond_0;+}, \mathcal{F}^{\diamond_-;+}\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{aligned} \big\| f_{\mathrm{apx;expl}}(U^{(1)}) - f_{\mathrm{apx;expl}}(U^{(2)}) \big\|_{\ell_h^2} &\leq K \big[\big\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \big\|_{\ell_h^2} + \big\| \partial^+ \partial^+ U^{(1)} - \partial^+ \partial^+ U^{(2)} \big\|_{\ell_h^2} \big] \\ &\leq K \min\{ S_{\mathrm{diff;2;fix}}\big(U^{(1)}, U^{(2)}\big), \\ &\qquad\qquad S_{\mathrm{diff;full}}\big(U^{(1)}, U^{(2)}\big) \}. \end{aligned} \tag{11.52}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\begin{aligned} \big\| f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V] \big\|_{\ell_h^2} &\leq K \| \partial^+ V \|_{\ell_h^\infty} \big\| \partial^+ \partial^+ U^{(1)} - \partial^+ \partial^+ U^{(2)} \big\|_{\ell_h^2} \\ &\quad + K \big[\| \partial^+ V \|_{\ell_h^2} + \| \partial^+ \partial^+ V \|_{\ell_h^2} \big] \big\| \partial^+ U^{(1)} - \partial^+ U^{(2)} \big\|_{\ell_h^\infty} \\ &\leq K \mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{aligned} \tag{11.53}$$

Proof. These bounds follow by inspecting the definitions (10.22). □

11.5 Decompositions for $\mathcal{X}_B$, $\mathcal{X}_C$ and $\mathcal{X}_D$

Identities for $\mathcal{X}_B$ Substituting the relevant expressions from §10 into (11.5)-(11.6), we compute

$$\begin{aligned} \mathcal{X}_{B;\mathrm{apx}}(U) &= S^+[\gamma_U^{-1}]\gamma_U^4, \\ \mathcal{X}_{B;\mathrm{lin};U}[V] &= S^+\Big[\gamma_U^{-3}\partial^0 U \partial^0 V + \gamma_U^{-1}\big[h \textstyle\sum_{-;h} \mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\big]\Big]\gamma_U^4 \\ &\quad + S^+[\gamma_U^{-1}](-4\gamma_U^2)\partial^0 U \partial^0 V. \end{aligned} \tag{11.54}$$

We realize the splittings (11.7) by writing

$$\begin{array}{rcl}\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U) & = & 2T^+[\gamma_U^3],\\ \mathcal{X}_{B;\mathrm{apx};\mathrm{sh}}(U) & = & -hS^+[\gamma_U^{-1}]\partial^+[\gamma_U^4] - h\partial^+[\gamma_U^{-1}]T^+[\gamma_U^4],\\ \mathcal{X}_{B;\mathrm{apx};\mathrm{rem}}(U) & = & 0,\end{array} \tag{11.55}$$

together with

$$\begin{array}{rcl}\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V] & = & -6T^+\Big[\gamma_U\partial^0U\partial^0V\Big],\\ \mathcal{X}_{B;\mathrm{lin};U;\mathrm{sh}}[V] & = & -h\partial^+\Big[\gamma_U^{-3}\partial^0U\partial^0V\Big]\gamma_U^4\\ & & \quad -hS^+[\gamma_U^{-1}]\partial^+[-4\gamma_U^2\partial^0V] - h\partial^+[\gamma_U^{-1}]T^+\big[-4\gamma_U^2\partial^0V\big],\\ \mathcal{X}_{B;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+\Big[\gamma_U^{-1}\big[h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\big]\Big]\gamma_U^4.\end{array} \tag{11.56}$$

Identities for $\mathcal{X}_C$ Substituting the relevant expressions from §10 into (11.5)-(11.6), we compute

$$\begin{array}{rcl}\mathcal{X}_{C;\mathrm{apx}}(U) & = & S^+[\gamma_U^{-1}](\gamma_U^4-\gamma_U^2),\\ \mathcal{X}_{C;\mathrm{lin};U}[V] & = & S^+\Big[\gamma_U^{-3}\partial^0U\partial^0V + \gamma_U^{-1}h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\Big](\gamma_U^4-\gamma_U^2)\\ & & \quad +S^+[\gamma_U^{-1}][2-4\gamma_U^2]\partial^0U\partial^0V.\end{array} \tag{11.57}$$

We realize the splittings (11.7) by writing

$$\begin{array}{rcl}\mathcal{X}_{C;\mathrm{apx};\mathrm{expl}}(U) & = & 2\gamma_U(\gamma_U^2-1),\\ \mathcal{X}_{C;\mathrm{apx};\mathrm{sh}}(U) & = & h\partial^+[\gamma_U^{-1}]\gamma_U^2(\gamma_U^2-1),\\ \mathcal{X}_{C;\mathrm{apx};\mathrm{rem}}(U) & = & 0,\end{array} \tag{11.58}$$

together with

$$\begin{array}{rcl}\mathcal{X}_{C;\mathrm{lin};U;\mathrm{expl}}[V] & = & 2\gamma_U^{-1}(1-3\gamma_U^2)\partial^0U\partial^0V,\\ \mathcal{X}_{C;\mathrm{lin};U;\mathrm{sh}}[V] & = & h\partial^+\big[\gamma_U^{-3}\partial^0U\partial^0V\big]\gamma_U^2(\gamma_U^2-1)\\ & & \quad +h\partial^+\big[\gamma_U^{-1}\big][2-4\gamma_U^2]\partial^0U\partial^0V,\\ \mathcal{X}_{C;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+\big[\gamma_U^{-1}h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\big](\gamma_U^4-\gamma_U^2).\end{array} \tag{11.59}$$

Identities for $\mathcal{X}_D$ Substituting the relevant expressions from §10 into (11.5)-(11.6), we compute

$$\begin{array}{rcl}\mathcal{X}_{D;\mathrm{apx}}(U) & = & S^+[\gamma_U\partial^0U]\gamma_U^{-1}\gamma_U\partial^0U\\ & = & S^+[\gamma_U\partial^0U]\partial^0U,\\ \mathcal{X}_{D;\mathrm{lin};U}[V] & = & S^+[\gamma_U^{-1}(2\gamma_U^2-1)\partial^0V]\gamma_U^{-1}\gamma_U\partial^0U\\ & & \quad +S^+[\gamma_U\partial^0U]\Big[\gamma_U^{-3}\partial^0U\partial^0V+\gamma_U^{-1}h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\Big]\gamma_U\partial^0U\\ & & \quad +S^+[\gamma_U\partial^0U]\gamma_U^{-1}\gamma_U^{-1}(2\gamma_U^2-1)\partial^0V\\ & = & S^+[\gamma_U^{-1}(2\gamma_U^2-1)\partial^0V]\partial^0U + S^+[\gamma_U\partial^0U]\partial^0V\\ & & \quad +S^+[\gamma_U\partial^0U]\partial^0Uh\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V.\end{array} \tag{11.60}$$

We realize the splittings (11.7) by writing

$$\begin{array}{rcl} \mathcal{X}_{D;\mathrm{apx;expl}}(U) & = & 2\gamma_U(1-\gamma_U^2), \\ \mathcal{X}_{D;\mathrm{apx;sh}}(U) & = & h\partial^+[\gamma_U\partial^0 U]\partial^0 U, \\ \mathcal{X}_{D;\mathrm{apx;rem}}(U) & = & 0, \end{array} \tag{11.61}$$

together with

$$\begin{array}{rcl} \mathcal{X}_{D;\mathrm{lin};U;\mathrm{expl}}[V] & = & 2\gamma_U^{-1}(3\gamma_U^2-1)\partial^0 U\partial^0 V, \\ \mathcal{X}_{D;\mathrm{lin};U;\mathrm{sh}}[V] & = & h\partial^+\big[\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V\big]\partial^0 U + h\partial^+[\gamma_U\partial^0 U]\partial^0 V, \\ \mathcal{X}_{D;\mathrm{lin};U;\mathrm{rem}}[V] & = & S^+[\gamma_U\partial^0 U]\partial^0 U h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V. \end{array} \tag{11.62}$$

Exponent sets For any $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$, we introduce the sets

$$Q_{f;\mathrm{pref}} = \overline{Q}_{f;\mathrm{pref}} = \{\infty\}, \tag{11.63}$$

together with

$$Q_f = \{\infty\}, \qquad Q^A_{f;\mathrm{lin}} = \{\infty\}, \qquad Q^B_{f;\mathrm{lin}} = \{2,\infty\}. \tag{11.64}$$

Estimates

Lemma 11.14. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. Then the conditions in $(hf)_{\mathrm{lin}}$ are satisfied with $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}}$ and there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{rcl} \left\| f^A_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq \; K\left\|\partial^+ V\right\|_{\ell^2_h} & \leq \; K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} & \leq \; K\left\|\partial^+ V\right\|_{\ell^2_h} & \leq \; K T_{\mathrm{safe}}(V), \\ \left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} & \leq \; K\left\|\partial^+ V\right\|_{\ell^\infty_h} & \leq \; K T_{\infty;\mathrm{opt}}(V) \end{array} \tag{11.65}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. Writing $f^B_{\mathrm{lin};U}[V] = f_{\mathrm{lin};U;\mathrm{expl}}[V] + f_{\mathrm{lin};U;\mathrm{sh}}[V]$, the bounds follow by inspection. □

Lemma 11.15. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\|f_{\mathrm{apx;expl}}(U)\|_{\ell^\infty_h} + \|f_{\mathrm{apx;sh}}(U)\|_{\ell^\infty_h} \;\leq\; K. \tag{11.66}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\|f_{\mathrm{apx;sh}}(U)\|_{\ell^\infty_h} \;\leq\; Kh \;\leq\; K S_{\mathrm{sh;full}}(U). \tag{11.67}$$

(iii) For any $h > 0$ and any pair $(U^{(1)}, U^{(2)}) \in \Omega_{h;\kappa}$, we have the bounds

$$\begin{array}{rcl} \left\|f_{\mathrm{apx;expl}}(U^{(1)}) - f_{\mathrm{apx;expl}}(U^{(2)})\right\|_{\ell^\infty_h} & \leq & K\left\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\right\|_{\ell^\infty_h} \\ & \leq & K S_{\mathrm{diff;full}}\big(U^{(1)}, U^{(2)}\big), \\ \left\|f_{\mathrm{apx;sh}}(U^{(1)}) - f_{\mathrm{apx;sh}}(U^{(2)})\right\|_{\ell^\infty_h} & \leq & K\left\|\partial^+ U^{(1)} - \partial^+ U^{(2)}\right\|_{\ell^\infty_h} \\ & \leq & K S_{\mathrm{diff;full}}\big(U^{(1)}, U^{(2)}\big). \end{array} \tag{11.68}$$

Proof. These bounds follow from the discrete derivative expressions (4.2) and the Lipschitz bounds for γ_U in Corollary 7.2. $\square$

Lemma 11.16. *Fix $0 < \kappa < \frac{1}{12}$ and pick $f \in \{\mathcal{X}_B, \mathcal{X}_C, \mathcal{X}_D\}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bounds

$$\begin{array}{lcl} \|f_{\mathrm{lin};U;\mathrm{sh}}[V]\|_{\ell^2_h} & \leq & Kh\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h}\,\big] \\ & \leq & K\mathcal{E}_{\mathrm{sh};U}(V), \\ \|f_{\mathrm{lin};U;\mathrm{rem}}[V]\|_{\ell^\infty_h} & \leq & K\, \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h}\, \|\partial^+ V\|_{\ell^2_h} \\ & \leq & K\mathcal{E}_{\mathrm{rem};U}(V). \end{array} \tag{11.69}$$

(ii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bounds

$$\begin{array}{lcl} \big\|f_{\mathrm{lin};U^{(1)};\mathrm{expl}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{expl}}[V]\big\|_{\ell^2_h} & \leq & K\,\big\|\partial^+U^{(2)} - \partial^+U^{(1)}\big\|_{\ell^\infty_h}\, \|\partial^+V\|_{\ell^2_h} \\ & \leq & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \\ \big\|f_{\mathrm{lin};U^{(1)};\mathrm{sh}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{sh}}[V]\big\|_{\ell^2_h} & \leq & K\,\big\|\partial^+U^{(2)} - \partial^+U^{(1)}\big\|_{\ell^\infty_h}\, \|\partial^+V\|_{\ell^2_h} \\ & \leq & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \\ \big\|f_{\mathrm{lin};U^{(1)};\mathrm{rem}}[V] - f_{\mathrm{lin};U^{(2)};\mathrm{rem}}[V]\big\|_{\ell^\infty_h} & \leq & K\,\|\partial^+V\|_{\ell^2_h}\,\Big[\,\big\|U^{(1)} - U^{(2)}\big\|_{\ell^{2,2}_h} \\ & & \qquad + \big\|\partial^+U^{(1)} - \partial^+U^{(2)}\big\|_{\ell^\infty_h}\Big] \\ & \leq & K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{11.70}$$

Proof. Recalling the Lipschitz bounds (7.65), these estimates follow from inspection. $\square$

11.6 Decomposition for $\mathcal{Y}_1^+$

Substituting the relevant expressions from §10 into (11.3)-(11.4), we compute

$$\begin{array}{lcl} \mathcal{Y}^+_{1;\mathrm{apx}}(U) & = & \gamma_U^2\gamma_U^{-3}S^+[\partial^0\partial U]T^+\gamma_U \\ & = & \gamma_U^{-1}S^+[\partial^0\partial U]T^+\gamma_U, \\ \mathcal{Y}^+_{1;\mathrm{lin};U}[V] & = & -2\partial^0U\partial^0V\gamma_U^{-3}S^+[\partial^0\partial U]T^+\gamma_U \\ & & \quad +\gamma_U^2\big[3\gamma_U^{-5}\partial^0U[S^+\partial^0\partial U]\partial^0V + \gamma_U^{-3}S^+\partial^0\partial V\big]T^+\gamma_U \\ & & \quad -\gamma_U^2\gamma_U^{-3}S^+[\partial^0\partial U]T^+\Big[\gamma_U^{-1}\partial^0U\partial^0V + \gamma_U h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\Big] \\ & = & \big[\gamma_U^{-3}\partial^0U[S^+\partial^0\partial U]\partial^0V + \gamma_U^{-1}S^+\partial^0\partial V\big]T^+\gamma_U \\ & & \quad -\gamma_U^{-1}S^+[\partial^0\partial U]T^+\Big[\gamma_U^{-1}\partial^0U\partial^0V + \gamma_U h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0V\Big]. \end{array} \tag{11.71}$$

We realize the splittings (11.7) by writing

$$\begin{array}{lcl} \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U) & = & 2\partial^0\partial U, \\ \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U) & = & h\partial^+\partial^0\partial U + h\partial^+[\gamma_U]\gamma_U^{-1}S^+[\partial^0\partial U], \\ \mathcal{Y}^+_{1;\mathrm{apx};\mathrm{rem}}(U) & = & 0, \end{array} \tag{11.72}$$

together with

$$\begin{array}{rcl}\mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{expl}}[V] & = & S^+[\partial^0\partial V],\\ \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{sh}}[V] & = & h\gamma_U^{-3}\partial^+[\gamma_U]\partial^0 U S^+[\partial^0\partial U]\partial^0 V\\ & & +h\gamma_U^{-1}\partial^+[\gamma_U]S^+[\partial^0\partial V]\\ & & -h\gamma_U^{-1}S^+[\partial^0\partial U]\partial^+\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big],\\ \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{rem}}[V] & = & \gamma_U^{-1}S^+[\partial^0\partial U]T^+\Big[\gamma_U h\sum_{-;h}\mathcal{E}_{\mathrm{sm}}(U)\partial^0 V\Big].\end{array} \tag{11.73}$$

Notice that we have eliminated the $T^+[\partial^0\partial U]$ term in the explicit expressions, while keeping the $T^+[\partial^0\partial V]$ dependency. This inconsistency is deliberate as it will help us to make a useful substitution in the sequel.

In addition, we introduce the sets

$$\overline{Q}_{\mathcal{Y}_1^+;\mathrm{pref}} = \{2,\infty\}, \tag{11.74}$$

together with

$$Q_{\mathcal{Y}_1^+} = \{2,\infty\}, \qquad Q^A_{\mathcal{Y}_1^+;\mathrm{lin}} = Q^B_{\mathcal{Y}_1^+;\mathrm{lin}} = \{2,\infty\}. \tag{11.75}$$

Lemma 11.17. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}_1^+$. Then the conditions in $(hf)_{\mathrm{lin}}$ are satisfied with $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}}$ and there exists a constant $K > 0$ so that the bounds*

$$\begin{array}{rclcl}\left\|f^A_{\mathrm{lin};U}[V]\right\|_{\ell^2_h} & \le & K\,\|\partial^+V\|_{\ell^2_h} & \le & K\overline{T}_{\mathrm{safe}}(V),\\ \left\|f^A_{\mathrm{lin};U}[V]\right\|_{\ell^\infty_h} & \le & K\,\|\partial^+V\|_{\ell^2_h} & \le & K\overline{T}_{\mathrm{safe}}(V),\\ \left\|f^B_{\mathrm{lin};U}[V]\right\|_{\ell^2_h} & \le & K\big[\,\|\partial^+V\|_{\ell^2_h}+\|\partial^+\partial^+V\|_{\ell^2_h}\,\big] & \le & K\overline{T}_{\mathrm{safe}}(V),\\ \left\|f^B_{\mathrm{lin};U}[V]\right\|_{\ell^\infty_h} & \le & K\big[\,\|\partial^+V\|_{\ell^\infty_h}+\|\partial^+\partial^+V\|_{\ell^\infty_h}\,\big] & \le & K\overline{T}_{\infty;\mathrm{opt}}(V)\end{array} \tag{11.76}$$

hold for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. Writing $f^B_{\mathrm{lin};U}[V] = f_{\mathrm{lin};U;\mathrm{expl}}[V] + f_{\mathrm{lin};U;\mathrm{sh}}[V]$, the bounds follow by inspection. □

Lemma 11.18. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\right\|_{\ell^2_h} + \left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\right\|_{\ell^\infty_h} + \left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\right\|_{\ell^2_h} + \left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\right\|_{\ell^\infty_h} \le K. \tag{11.77}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\begin{array}{rclcl}\left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\right\|_{\ell^2_h} & \le & Kh[1+\|\partial^+\partial^+\partial^+U\|_{\ell^2_h}] & \le & K\overline{S}_{\mathrm{sh};\mathrm{full}}(U),\\ \left\|\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{sh}}(U)\right\|_{\ell^\infty_h} & \le & Kh[1+\|\partial^+\partial^+\partial^+U\|_{\ell^\infty_h}] & \le & K\overline{S}_{\mathrm{sh};\mathrm{full}}(U).\end{array} \tag{11.78}$$

Proof. These bounds follow from the discrete derivative identities in Lemma 4.2. □

Lemma 11.19. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bounds*

$$\begin{array}{lcl} \left\| \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{sh}}[V] \right\|_{\ell^2_h} & \leq & Kh\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \, \big] \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\| \mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{rem}}[V] \right\|_{\ell^2_h} & \leq & K \, \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h} \, \|\partial^+ V\|_{\ell^2_h} \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V) \end{array} \tag{11.79}$$

for any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. These estimates follow by inspection. □

11.7 Decomposition for $\mathcal{Y}^+_{2b}$

Substituting the relevant expressions from §10 into (11.5) and recalling (7.62), we compute

$$\begin{array}{lcl} \mathcal{Y}^+_{2b;\mathrm{apx}}(U) & = & 8\gamma_U^{-6}\partial^0 U S^+[\partial^0\partial U]T^+[\partial^0\partial U] + g'(U)\partial^0 U \\ & = & \Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U\Big] + c_*\gamma_U^{-3}S^+[\partial^0\partial U]. \end{array} \tag{11.80}$$

We can hence realize the first splitting in (11.7) by writing

$$\begin{array}{lcl} \mathcal{Y}^+_{2b;\mathrm{apx;expl}}(U) & = & \Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U\Big] + 2c_*\gamma_U^{-3}\partial^0\partial U, \\ \mathcal{Y}^+_{2b;\mathrm{apx;sh}}(U) & = & c_* h\gamma_U^{-3}\partial^+[\partial^0\partial U], \\ \mathcal{Y}^+_{2b;\mathrm{apx;rem}}(U) & = & 0. \end{array} \tag{11.81}$$

Substituting the relevant expressions from §10 into (11.4), we compute

$$\begin{array}{lcl} \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] & = & 8[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^0\partial U]T^+[\partial^0\partial U]\partial^0 V \\ & & +8\gamma_U^{-6}\partial^0 U\Big[T^+[\partial^0\partial U]S^+[\partial^0 V] + S^+[\partial^0\partial U]T^+[\partial^0\partial V]\Big] \\ & & +g''(U)[\partial^0 U]V + g'(U)\partial^0 V. \end{array} \tag{11.82}$$

We realize the second splitting (11.7) implicitly by writing

$$\begin{array}{lcl} \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V] & = & \gamma_U^{-2}\partial^+\big[M_U[V]\big] + 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U]M_U[V] - \widetilde{M}_{U;E}[V] \\ & & \quad +c_*\Big[6\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V + \gamma_U^{-3}S^+[\partial^0\partial V]\Big], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh}}[V] & = & \mathcal{Y}^+_{2b;\mathrm{lin};U}[V] - \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V], \\ \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{rem}}[V] & = & 0. \end{array} \tag{11.83}$$

In addition, we introduce the sets

$$\overline{Q}_{\mathcal{Y}^+_{2b};\mathrm{pref}} = \{2\}, \tag{11.84}$$

together with

$$Q_{\mathcal{Y}^+_{2b}} = \{2, \infty\}, \qquad Q^A_{\mathcal{Y}^+_{2b};\mathrm{lin}} = Q^B_{\mathcal{Y}^+_{2b};\mathrm{lin}} = \{2\}. \tag{11.85}$$

Lemma 11.20. *Fix $0 < \kappa < \frac{1}{12}$ and write $f = \mathcal{Y}^+_{2b}$. Then the conditions in $(hf)_{\mathrm{lin}}$ are satisfied with $f^A_{\mathrm{lin};U} = f_{\mathrm{lin};U;\mathrm{rem}} = 0$ and there exists a constant $K > 0$ so that the bound*

$$\left\| f^B_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} \quad \leq \quad K\, \|V\|_{\ell^{2;2}_h} \quad \leq \quad K\overline{T}_{\mathrm{safe}}(V) \tag{11.86}$$

holds for all $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$.

Proof. The bound follows by inspection. □

Lemma 11.21. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the following properties are true.*

(i) For any $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bound

$$\left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U) \right\|_{\ell^2_h} + \left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^2_h} + \left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{expl}}(U) \right\|_{\ell^\infty_h} + \left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^\infty_h} \leq K. \tag{11.87}$$

(ii) For any $h > 0$ and $U \in \Omega_{h;\kappa}$, we have the bound

$$\begin{array}{lllll} \left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^2_h} & \leq & Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}] & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\| \mathcal{Y}^+_{2b;\mathrm{apx};\mathrm{sh}}(U) \right\|_{\ell^\infty_h} & \leq & Kh[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}] & \leq & K\overline{S}_{\mathrm{sh;full}}(U). \end{array} \tag{11.88}$$

Proof. Recalling (7.62), the bounds follow by inspection. □

Lemma 11.22. *Fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that we have the bound*

$$\begin{array}{lll} \left\| \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh}}[V] \right\|_{\ell^2_h} & \leq & Kh\Big[\|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ V\|_{\ell^2_h} \Big] \\ & & \quad +Kh \, \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \Big[\|V\|_{\ell^2_h} + \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \Big] \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{sm};U}(V) \end{array} \tag{11.89}$$

for any $h > 0$, any pair $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$.

Proof. We proceed by obtaining the decomposition

$$\mathcal{Y}^+_{2b;\mathrm{lin};U}[V] = \mathcal{Y}^+_{2b;\mathrm{lin};U;I}[V] + \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};a}[V], \tag{11.90}$$

in which we have introduced the function

$$\begin{array}{lll} \mathcal{Y}^+_{2b;\mathrm{lin};U;I}[V] & = & 16[6\gamma_U^{-8} - 5\gamma_U^{-6}][\partial^0\partial U]^2 \partial^0 V \\ & & \quad +32\gamma_U^{-6}\partial^0 U[\partial^0\partial U]\partial^0\partial V \\ & & \quad +g''(U)[\partial^0 U]V + g'(U)\partial^0 V, \end{array} \tag{11.91}$$

together with

$$\begin{array}{lll} \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};a}[V] & = & 8h[6\gamma_U^{-8} - 5\gamma_U^{-6}][\partial^0\partial U]\partial^+[\partial^0\partial U]\partial^0 V \\ & & \quad +8h[6\gamma_U^{-8} - 5\gamma_U^{-6}]S^+[\partial^0\partial U]\partial^+[\partial^0\partial U]\partial^0 V \\ & & \quad +8h\gamma_U^{-6}\partial^0 U\Big[\partial^+[\partial^0\partial U]\partial^0\partial V + [\partial^0\partial U]\partial^+[\partial^0\partial V]\Big] \\ & & \quad +16h\gamma_U^{-6}\partial^0 U\Big[\partial^+[\partial^0\partial U]T^+[\partial^0\partial V] + [\partial^0\partial U]\partial^+[\partial^0\partial V]\Big]. \end{array} \tag{11.92}$$

Using Proposition 7.16, we see that

$$\mathcal{Y}^+_{2b;\mathrm{lin};U;I}[V] = \widetilde{M}_{U;A}[V] + \widetilde{M}_{U;B}[V] + \widetilde{M}_{U;C}[V] - \widetilde{M}_{U;E}[V]. \tag{11.93}$$

Introducing the function

$$\mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};b}[V] = -\gamma_U^{-2}\Big[\partial^+\big[M_U[V]\big] - M^+_{U;\mathrm{apx}}[V]\Big] - hc_*\gamma_U^{-3}\partial^+[\partial^0\partial V], \tag{11.94}$$

we see that

$$\begin{array}{rcl}\mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{expl}}[V] + \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};b}[V] & = & \gamma_U^{-2} M^+_{U;\mathrm{apx}}[V] + 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U]M_U[V] - \widetilde{M}_{U;E}[V] \\ & & +c_*\Big[6\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V + 2\gamma_U^{-3}[\partial^0\partial V]\Big] \\ & = & \widetilde{M}_{U;A}[V] + \widetilde{M}_{U;B}[V] + \widetilde{M}_{U;C}[V] + \widetilde{M}_{U;D}[V] \\ & & -\widetilde{M}_{U;E}[V] - \widetilde{M}_{U;D}[V] \\ & = & \widetilde{M}_{U;A}[V] + \widetilde{M}_{U;B}[V] + \widetilde{M}_{U;C}[V] - \widetilde{M}_{U;E}[V] \\ & = & \mathcal{Y}^+_{2b;\mathrm{lin};U} - \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};a}[V].\end{array} \tag{11.95}$$

In particular, we obtain

$$\mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh}}[V] = \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{Y}^+_{2b;\mathrm{lin};U;\mathrm{sh};b}[V]. \tag{11.96}$$

Recalling Proposition 7.15, the desired bound now follows by inspection. □

Proof of Propositions 11.1-11.6. The statements can be readily verified by inspecting the results in §11.2-§11.7. □

12 Estimates for $\mathcal{G}$

In this section we exploit the component estimates from §10 to analyze the function $\mathcal{G}$ discussed in §9. In particular, we introduce the approximants

$$\begin{array}{rcl}\mathcal{G}_{\mathrm{apx}}(U) & = & c_*\partial^0 U, \\ \mathcal{G}_{\mathrm{lin};U}[V] & = & c_*\partial^0 V + M_U[V] + 2\partial^0 U h \sum_{-;h} \gamma_U^{-2}[\partial^0\partial U]M_U[V]\end{array} \tag{12.1}$$

and write

$$\mathcal{G}_{\mathrm{nl};U}(V) = \mathcal{G}(U+V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U}(V). \tag{12.2}$$

Our main result quantifies the approximation errors in terms of the quantities

$$\begin{array}{rcl}\mathcal{E}_{\mathrm{sh};U}(V) & = & h\,\|V\|_{\ell_h^{2;2}}\,, \\ \mathcal{E}_{\mathrm{rem};U}(V) & = & \|V\|_{\ell_h^{2;2}}\Big[\,\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell_h^2}\,\Big], \\ \mathcal{E}_{\mathrm{prod}}(W^{(1)},W^{(2)}) & = & \big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}} \\ & & +\big\|W^{(1)}\big\|_{\ell_h^{2;2}}\big\|W^{(2)}\big\|_{\ell_h^{\infty;1}} + \big\|W^{(1)}\big\|_{\ell_h^{\infty;1}}\big\|W^{(2)}\big\|_{\ell_h^{2;2}}\end{array} \tag{12.3}$$

that were originally introduced in §11.1.

Proposition 12.1. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the following properties hold.*

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have

$$\|\mathcal{G}(U) - \mathcal{G}_{\mathrm{apx}}(U)\| \quad \leq \quad K\big[h + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty}\big]. \tag{12.4}$$

(ii) For any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ for which $U + V \in \Omega_{h;\kappa}$, we have the estimate

$$\|\mathcal{G}_{\mathrm{nl};U}(V)\|_{\ell_h^2} \quad \leq \quad K\mathcal{E}_{\mathrm{prod}}(V,V) + K\mathcal{E}_{\mathrm{sh};U}(V) + K\mathcal{E}_{\mathrm{rem};U}(V) \tag{12.5}$$

(iii) Consider any $h > 0$, $U \in \Omega_{h;\kappa}$ and any pair $(V^{(1)}, V^{(2)}) \in \ell_h^2 \times \ell_h^2$ for which the inclusions $U + V^{(1)} \in \Omega_{h;\kappa}$ and $U + V^{(2)} \in \Omega_{h;\kappa}$ both hold. Then we have the Lipschitz estimate

$$\begin{aligned} \big\|\mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)})\big\|_{\ell_h^2} \quad \leq \quad & K\mathcal{E}_{\mathrm{prod}}(V^{(1)}, V^{(2)} - V^{(1)}) + K\mathcal{E}_{\mathrm{prod}}(V^{(2)}, V^{(2)} - V^{(1)}) \\ & + K\mathcal{E}_{\mathrm{sh};U}(V^{(2)} - V^{(1)}) + K\mathcal{E}_{\mathrm{rem};U}(V^{(2)} - V^{(1)}). \end{aligned} \tag{12.6}$$

Proof of Proposition 2.4. On account of Proposition 6.2, it is possible to pick constants $0 < \kappa < \frac{1}{12}$ and ϵ_0 such that for any $V \in \ell_h^2$ with $\|V\|_{\ell_h^2} < \epsilon_0$, we have $U_0 + V \in \Omega_{h;\kappa}$. Recall the continuous embedding $\ell_h^\infty \subset \ell_h^2$. Inspecting $\mathcal{G}_{\mathrm{lin};U}$ using (7.82), we see that item (iii) of Proposition 12.1 implies that the map

$$V \mapsto \mathcal{G}(U_0 + V) \in \ell_h^2 \tag{12.7}$$

is Lipschitz smooth on the set $\{V \in \ell_h^2 : \|V\|_{\ell_h^2} < \epsilon_0\}$. The result now follows from standard ODE theory. □

In §12.1 we apply the theory developed in §6.2 to estimate the nonlinear component of our error, exploiting the structural decomposition of $\mathcal{G}(U)$ described in §9.2 and the estimates obtained in §10. In the remainder of the section we discuss the linear terms. Considerable effort will be required to reduce the expressions (6.67) to our relatively simple approximants (12.1).

12.1 Nonlinear estimates

Applying the expressions (6.67) to the terms (9.9), we obtain the initial expressions

$$\begin{aligned} \mathcal{G}_{A;\mathrm{apx};I}(U) &= \Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U) T^- \big[\mathcal{X}_{A;\mathrm{apx}}(U)\big]\Big] \mathcal{Y}_{2;\mathrm{apx}}(U), \\ \mathcal{G}_{B;\mathrm{apx};I}(U) &= \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big] \mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U), \\ \mathcal{G}_{C;\mathrm{apx};I}(U) &= \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{C;\mathrm{apx}}(U) \mathcal{F}_{\mathrm{apx}}^{\diamond_0;+}(U)\Big], \\ \mathcal{G}_{D;\mathrm{apx};I}(U) &= \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{D;apx}(U) \mathcal{F}_{\mathrm{apx}}^{\diamond_0;+}(U)\Big], \end{aligned} \tag{12.8}$$

together with

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;I}[V] \quad = \quad & -\mathcal{Y}_{1;\mathrm{lin};U}[V] T^- \big[\mathcal{X}_{A;\mathrm{apx}}(U)\big] \mathcal{Y}_{2;\mathrm{apx}}(U) \\ & -\mathcal{Y}_{1;\mathrm{apx}}(U) T^- \big[\mathcal{X}_{A;\mathrm{lin};U}[V]\big] \mathcal{Y}_{2;\mathrm{apx}}(U) \\ & +\Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U) T^- \big[\mathcal{X}_{A;\mathrm{apx}}(U)\big]\Big] \mathcal{Y}_{2;\mathrm{lin};U}[V], \\ \mathcal{G}_{B;\mathrm{lin};U;I}[V] \quad = \quad & \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{lin};U}[V] h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big] \mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{lin};U}[V] T^- \Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big] \mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{B;\mathrm{lin};U}[V]\Big] \mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U) h \textstyle\sum_{-;h} \mathcal{Y}_{2;\mathrm{apx}}(U) T^- \Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big] \mathcal{F}_{\mathrm{lin};U}^{\diamond_-;+}[V] \end{aligned} \tag{12.9}$$

and finally

$$\begin{aligned}\mathcal{G}_{\#;\mathrm{lin};U;I}[V] &= \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{lin};U}[V]h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ &\quad +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{lin};U}[V]T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ &\quad +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{lin};U}[V]\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ &\quad +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V]\Big]\end{aligned} \tag{12.10}$$

for $\# \in \{C, D\}$. Combining these expressions, we introduce the initial approximants

$$\begin{aligned}\mathcal{G}_{\mathrm{apx};I}(U) &= \mathcal{G}_{A;\mathrm{apx};I}(U) + \mathcal{G}_{B;\mathrm{apx};I}(U) + \mathcal{G}_{C;\mathrm{apx};I}(U) + \mathcal{G}_{D;\mathrm{apx};I}(U), \\ \mathcal{G}_{\mathrm{lin};U;I}[V] &= \mathcal{G}_{A;\mathrm{lin};U;I}[V] + \mathcal{G}_{B;\mathrm{lin};U;I}[V] + \mathcal{G}_{C;\mathrm{lin};U;I}[V] + \mathcal{G}_{D;\mathrm{lin};U;I}[V]\end{aligned} \tag{12.11}$$

and write

$$\mathcal{G}_{\mathrm{nl};U;I}(V) = \mathcal{G}(U+V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U;I}[V]. \tag{12.12}$$

Lemma 12.2. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the approximation estimate*

$$\|\mathcal{G}(U) - \mathcal{G}_{\mathrm{apx};I}(U)\|_{\ell^2_h} \quad \leq \quad Kh \tag{12.13}$$

and the residual bound

$$\|\mathcal{G}_{\mathrm{nl};U;I}(V)\|_{\ell^2_h} \quad \leq \quad K\mathcal{E}_{\mathrm{prod}}(V,V) + K\mathcal{E}_{\mathrm{sh};U}(V) \tag{12.14}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Our strategy is to apply Lemma 6.6 to each of the products in the decomposition of $\mathcal{G}(U)$ obtained in Lemma 9.7. Let us therefore consider a single element of the sum (9.42), which we characterize by the set $(\pi, \mathbf{q}_\pi, \mathbf{f}, k)$.

We first claim that

$$\mathcal{J}_{\mathrm{nl};U}(V) \leq C_1'\mathcal{E}_{\mathrm{nl}}(V) \leq C_2'\mathcal{E}_{\mathrm{prod}}(V,V) + C_2'h\,\|V\|_{\ell^{2;2}_h}. \tag{12.15}$$

Indeed, consider any $1 \leq i \leq k$ and any $\# \in \{A, B\}$. If $q_{\pi;i} = 2$, then certainly $2 \in Q_{f_i;\mathrm{pref}}$ by item (iii) of Lemma 9.7, which allows us to take

$$\mathbf{q}^{\#}_{i,\mathrm{nl}} = \mathbf{q}_\pi \tag{12.16}$$

for the sequences in item (c) of Lemma 6.6. This allows us to apply (10.47), as desired. Suppose therefore that $q_{\pi;i} = \infty$ and consider the integer q defined in Corollary 10.9. If $q = \infty$, then we can again take $\mathbf{q}^{\#}_{i,\mathrm{nl}} = \mathbf{q}_\pi$ and apply (10.48). If $q = 2$, then we choose $\mathbf{q}^{\#}_{i,\mathrm{nl}}$ to be the admissable sequence defined by the swapping Lemma 9.9, which has

$$\mathbf{q}^{\#}_{i,\mathrm{nl};i} = 2, \qquad \mathbf{q}^{\#}_{i,\mathrm{nl};j_*[i]} = \infty. \tag{12.17}$$

Corollary 10.6 shows that $\infty \in Q_{f_{j_*[i]}}$, which now again allows us to apply (10.48).

Our second claim is that

$$\begin{aligned}\mathcal{J}_{\mathrm{cross};U}(V) &\leq C_3'\Big[T_{\mathrm{safe}}(V)S_{\mathrm{full}}(V) + T_{\infty;\mathrm{opt}}(V)S_{2;\mathrm{fix}}(V)\Big] \\ &\leq C_4'\mathcal{E}_{\mathrm{prod}}(V,V).\end{aligned} \tag{12.18}$$

Indeed, consider any $\# \in \{A, B\}$ and any pair $(i, j) \in \{1, \ldots, k\}^2$ with $i \neq j$. If item (a) in Corollary 10.8 holds for f_i and $q = q_{\pi;i}$, then the claim follows from (10.42). Suppose therefore that item (b) in Corollary 10.8 holds for f_i and $q = q_{\pi;i} = \infty$.

Write $\mathbf{q}_{\mathrm{sw}}$ for the admissable sequence defined by the swapping Lemma 9.9. If $j_*[i] \neq j$, then we have $q_{\mathrm{sw};j} = q_{\pi;j}$. Writing $\mathbf{q}^{\#}_{ij,\mathrm{lin}} = \mathbf{q}_{\mathrm{sw}}$ for the sequence in item (d) of Lemma 6.6, the contribution from the pair (i, j) can be absorbed by $T_{\mathrm{safe}}(V)S_{\mathrm{full}}(V)$. On the other hand, if $j_*[i] = j$, then $q_{\mathrm{sw};j} = \infty$ and $q_{\pi;j} = 2$. If item (b) of Corollary 10.7 holds, then we again write $\mathbf{q}^{\#}_{ij,\mathrm{lin}} = \mathbf{q}_{\mathrm{sw}}$, noting that the contribution can be bounded by $T_{\mathrm{safe}}(V)S_{\mathrm{full}}(V)$. However, we write $\mathbf{q}^{\#}_{ij,\mathrm{lin}} = \mathbf{q}_{\pi}$ if item (a) of Corollary 10.7 holds. In this case the contribution from the pair (i, j) can be bounded by $T_{\infty;\mathrm{opt}}(V)S_{2;\mathrm{fix}}(V)$.

Our final claim is that

$$\mathcal{J}_{\mathrm{apx};U}(V) \leq C_5' h T_{\mathrm{safe}}(V) = C_5' h \, \|V\|_{\ell_h^{2;2}} . \tag{12.19}$$

This follows directly from the fact that $\|f(U) - f_{\mathrm{apx}}(U)\|_{\ell_h^q} \leq Kh$ for every $f \in \mathcal{S}_{\mathrm{nl}}$ and $q \in Q_f$, together with the swapping technique described above. We note that this observation also implies the bound (12.13). □

12.2 Error terms

We now introduce the expressions

$$\{\mathcal{G}_{A;\mathrm{apx};II}(U), \mathcal{G}_{B;\mathrm{apx};II}(U), \mathcal{G}_{C;\mathrm{apx};II}(U), \mathcal{G}_{D;\mathrm{apx};II}(U)\} \tag{12.20}$$

together with

$$\{\mathcal{G}_{A;\mathrm{lin};U;II}[V], \mathcal{G}_{B;\mathrm{lin};U;II}[V], \mathcal{G}_{C;\mathrm{lin};U;II}[V], \mathcal{G}_{D;\mathrm{lin};U;II}[V]\} \tag{12.21}$$

by inspecting the definitions (12.8)-(12.9) and making the replacements

$$f_{\mathrm{apx}}(U) \mapsto f_{\mathrm{apx};\mathrm{expl}}(U), \qquad f_{\mathrm{lin};U}[V] \mapsto f_{\mathrm{lin};U;\mathrm{expl}}[V] \tag{12.22}$$

for each $f \in \mathcal{S}_{\mathrm{nl;short}}$.

Lemma 12.3. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with sequences*

$$\mathcal{G}_{\mathrm{apx;sh};a}(U) \in \ell_h^2, \qquad \mathcal{G}_{\mathrm{apx;rem};a}(U) \in \ell_h^2, \tag{12.23}$$

defined for every $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the identity

$$\mathcal{G}_{\mathrm{apx};I}(U) \quad = \quad \mathcal{G}_{\mathrm{apx};II}(U) + \mathcal{G}_{\mathrm{apx;sh};a}(U) + \mathcal{G}_{\mathrm{apx;rem};a}(U). \tag{12.24}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rclcl} \|\mathcal{G}_{\mathrm{apx;sh};a}(U)\|_{\ell_h^2} & \leq & K S_{\mathrm{sh;full}}(U) & = & Kh, \\ \|\mathcal{G}_{\mathrm{apx;rem};a}(U)\|_{\ell_h^2} & \leq & K S_{\mathrm{rem;full}}(U) & = & K\big[\, \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^2} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell_h^\infty} \,\big]. \end{array} \tag{12.25}$$

Proof. We consider a single element of the sum (9.42) obtained by using $\mathcal{S}_{\mathrm{nl;short}}$ instead of $\mathcal{S}_{\mathrm{nl}}$. We characterize this element by the set $(\pi, \mathbf{q}_\pi, \mathbf{f}, k)$, taking $\mathbf{f} \subset \mathcal{S}_{\mathrm{nl;short}}$.

We introduce the expression

$$\mathcal{I}_{\pi}(U) \quad = \quad \pi\big[f_{1;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] - \pi\big[f_{1;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \tag{12.26}$$

and note that $\mathcal{G}_{\mathrm{apx};I}(U) - \mathcal{G}_{\mathrm{apx};II}(U)$ can be written as a sum expressions of the form $\mathcal{I}_{\pi}$.

Recalling the general identity

$$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) - a_1 a_2 a_3 = b_1(a_2 + b_2)(a_3 + b_3) + a_1 b_2 (a_3 + b_3) + a_1 a_2 b_3 \tag{12.27}$$

and its extensions, we write

$$\begin{aligned} \mathcal{I}_{\pi;\#}(U) \quad = \quad & \pi\big[f_{1;\mathrm{apx};\#}(U), f_{2;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & + \pi\big[f_{1;\mathrm{apx};\mathrm{expl}}(U), f_{2;\mathrm{apx};\#}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & + \ldots + \pi\big[f_{1;\mathrm{apx};\mathrm{expl}}(U), f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\#}(U)\big] \end{aligned} \tag{12.28}$$

for $\# \in \{\mathrm{sh}, \mathrm{rem}\}$ and observe that

$$\mathcal{I}_{\pi}(U) \quad = \quad \mathcal{I}_{\pi;\mathrm{sh}}(U) + \mathcal{I}_{\pi;\mathrm{rem}}(U). \tag{12.29}$$

We now use (11.17) together with Proposition 11.3 to derive the bound

$$\|\mathcal{I}_{\pi;\#}(U)\|_{\ell^2_h} \quad \leq \quad C_1' S_{\#;\mathrm{full}}(U), \tag{12.30}$$

from which the desired estimates follow.

□

Lemma 12.4. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with linear maps*

$$\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a} \in \mathcal{L}(\ell^2_h, \ell^2_h), \qquad \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a} \in \mathcal{L}(\ell^2_h, \ell^2_h), \tag{12.31}$$

defined for all $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the identity

$$\mathcal{G}_{\mathrm{lin};U;I}[V] \quad = \quad \mathcal{G}_{\mathrm{lin};U;II}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V]. \tag{12.32}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{aligned} \|\mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V]\|_{\ell^2_h} \quad & \leq \quad K\mathcal{E}_{\mathrm{sh};U}(V), \\ \|\mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V]\|_{\ell^2_h} \quad & \leq \quad K\mathcal{E}_{\mathrm{rem};U}(V). \end{aligned} \tag{12.33}$$

(iii) For every $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\big\|\mathcal{G}_{\mathrm{lin};U^{(2)};\mathrm{rem};a}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)};\mathrm{rem};a}[V]\big\|_{\ell^2_h} \quad \leq \quad K\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \tag{12.34}$$

Proof. Reconsider the set $(\pi, \mathbf{q}_\pi, \mathbf{f}, k)$ discussed in the proof of Lemma 12.3. We introduce the two expressions

$$\begin{aligned} \mathcal{I}_{\pi;a;U}[V] \quad = \quad & \pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx}}(U), \ldots, f_{k;\mathrm{apx}}(U)\big] \\ & - \pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big], \\ \mathcal{I}_{\pi;b;U}[V] \quad = \quad & \pi\big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \\ & - \pi\big[f_{1;\mathrm{lin};U;\mathrm{expl}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \end{aligned} \tag{12.35}$$

and note that $\mathcal{G}_{\mathrm{lin};U;I}[V] - \mathcal{G}_{\mathrm{lin};U;II}[V]$ can be written as a sum of expressions of the form $\mathcal{I}_{\pi;a} + \mathcal{I}_{\pi;b}$, together with their obvious permutations.

Writing

$$\begin{array}{lcl} \mathcal{I}_{\pi;a;U;\#}[V] & = & \pi\Big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\#}(U), \ldots, f_{k;\mathrm{apx}}(U)\Big] \\ & & \qquad + \ldots + \pi\Big[f_{1;\mathrm{lin};U}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\#}(U)\Big], \\ \mathcal{I}_{\pi;b;U;\#}[V] & = & \pi\big[f_{1;\mathrm{lin};U;\#}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U)\big] \end{array} \tag{12.36}$$

for $\# \in \{\mathrm{sh}, \mathrm{rem}\}$, we see that

$$\begin{array}{lcl} \mathcal{I}_{\pi;a;U}[V] & = & \mathcal{I}_{\pi;a;U;\mathrm{sh}}[V] + \mathcal{I}_{\pi;a;U;\mathrm{rem}}[V], \\ \mathcal{I}_{\pi;b;U}[V] & = & \mathcal{I}_{\pi;b;U;\mathrm{sh}}[V] + \mathcal{I}_{\pi;b;U;\mathrm{rem}}[V]. \end{array} \tag{12.37}$$

Following the same reasoning used above to obtain (12.18), we may use Propositions 11.2 and 11.3 to derive the bound

$$\begin{array}{lcl} \|\mathcal{I}_{\pi;a;\#}\|_{\ell^2_h} & \leq & C_1'\Big[T_{\mathrm{safe}}(V)S_{\#;\mathrm{full}}(V) + T_{\infty;\mathrm{opt}}(V)S_{\#;2;\mathrm{fix}}(V)\Big] \\ & \leq & C_2'\mathcal{E}_{\#;U}(V). \end{array} \tag{12.38}$$

In addition, following the arguments used above to derive (12.15), we may use Proposition 11.5 to obtain the bound

$$\|\mathcal{I}_{\pi;a;\#}\|_{\ell^2_h} \quad \leq \quad C_3'\mathcal{E}_{\#;U}(V). \tag{12.39}$$

Writing

$$\begin{array}{lcl} \Delta_{b;i} & = & \pi\Big[f_{1;\mathrm{lin};U^{(2)};\mathrm{rem}}[V] - f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)})\Big], \\ \Delta_{b;ii} & = & \pi\Big[f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(2)}) - f_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)})\Big] \\ & & \quad + \ldots \\ & & \quad + \pi\Big[f_{1;\mathrm{lin};U^{(1)};\mathrm{rem}}[V], f_{2;\mathrm{apx};\mathrm{expl}}(U^{(1)}), \ldots, f_{k;\mathrm{apx};\mathrm{expl}}(U^{(2)}) - f_{k;\mathrm{apx};\mathrm{expl}}(U^{(1)})\Big], \end{array} \tag{12.40}$$

we easily see that

$$\Delta_{b;i} + \Delta_{b;ii} = \mathcal{I}_{\pi;b;U^{(2)};\mathrm{rem}} - \mathcal{I}_{\pi;b;U^{(1)};\mathrm{rem}}. \tag{12.41}$$

Arguing as above, Proposition 11.6 yields

$$\|\Delta_{b;i}\|_{\ell^2_h} \quad \leq \quad C_1'\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V), \tag{12.42}$$

while Propositions 11.2 and 11.4 imply

$$\begin{array}{lcl} \|\Delta_{b;ii}\|_{\ell^2_h} & \leq & C_2'\Big[T_{\mathrm{safe}}(V)S_{\mathrm{diff};\mathrm{full}}(V) + T_{\infty;\mathrm{opt}}(V)S_{\mathrm{diff};2;\mathrm{fix}}(V)\Big] \\ & \leq & C_3'\mathcal{E}_{\mathrm{prod}}(U^{(2)} - U^{(1)}, V). \end{array} \tag{12.43}$$

Finally, we write

$$\Delta_a = \mathcal{I}_{\pi;a;U^{(2)};\mathrm{rem}}[V] - \mathcal{I}_{\pi;a;U^{(1)};\mathrm{rem}}[V]. \tag{12.44}$$

We note that Δ_a consists of sums of expressions that arise from $\Delta_{b;i}$ and $\Delta_{b;ii}$ after replacing $f_{1;\mathrm{lin};U^{(i)};\mathrm{rem}}$ by $f_{1;\mathrm{lin};U^{(i)}}$ and each occurrence of $f_{j;\mathrm{apx};\mathrm{expl}}$ by an element of the set

$$\{f_{j;\mathrm{apx}}, f_{j;\mathrm{apx};\mathrm{expl}}, f_{j;\mathrm{apx};\mathrm{rem}}\}. \tag{12.45}$$

We can hence again use Propositions 11.2, 11.4 and 11.6 to conclude that $\|\Delta_a\|_{\ell^2_h}$ can be bounded by terms that have already appeared above.

□

12.3 Simplifications for $\mathcal{G}_A$

We recall the definition

$$\mathcal{G}_{A;\mathrm{apx};II}(U) \quad = \quad -\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)T^{-}\big[\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U). \tag{12.46}$$

Substituting the relevant expressions from §11, we find

$$\mathcal{G}_{A;\mathrm{apx};II}(U) \quad = \quad \Big[1 - \partial^0 U T^{-}\big[\partial^0 U\big]\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big). \tag{12.47}$$

We now make the decomposition

$$\mathcal{G}_{A;\mathrm{apx};II}(U) = \mathcal{G}_{A;\mathrm{apx};III}(U) + \mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U), \tag{12.48}$$

by introducing

$$\begin{aligned} \mathcal{G}_{A;\mathrm{apx};III}(U) \quad &= \quad \Big[1 - (\partial^0 U)^2\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ &= \quad c_*\gamma_U \partial^0 U \end{aligned} \tag{12.49}$$

together with

$$\mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U) \quad = \quad -h\partial^0 U \partial^{-}\big[\partial^0 U\big]\big(c_*\gamma_U^{-1}\partial^0 U\big). \tag{12.50}$$

We also recall the definition

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;II}[V] \quad = \quad & -\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]T^{-1}\big[\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U) \\ & -\mathcal{Y}_{1;\mathrm{apx}}(U)T^{-1}\big[\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U) \\ & +\Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U)T^{-1}\big[\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\big]\Big]\mathcal{Y}_{2;\mathrm{lin};U}[V]. \end{aligned} \tag{12.51}$$

Substituting the relevant expressions from §11, we find

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;II}[V] \quad = \quad & -\partial^0 V T^{-1}\Big[\partial^0 U\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & -\partial^0 U T^{-1}\Big[\partial^0 V\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & +\Big[1 - \partial^0 U T^{-1}\big[\partial^0 U\big]\Big]\big(\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\big). \end{aligned} \tag{12.52}$$

We now make the decomposition

$$\mathcal{G}_{A;\mathrm{lin};U;II}[V] = \mathcal{G}_{A;\mathrm{lin};U;III}[V] + \mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] \tag{12.53}$$

by introducing

$$\begin{aligned} \mathcal{G}_{A;\mathrm{lin};U;III}[V] \quad = \quad & -\partial^0 V \partial^0 U\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & -\partial^0 U \partial^0 V\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & +\Big[1 - \partial^0 U \partial^0 U\Big]\big(\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\big) \\ = \quad & c_*\gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V + M_U[V], \end{aligned} \tag{12.54}$$

together with

$$\begin{array}{rcl}\mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] & = & -h\partial^0 V\partial^-\Big[\partial^0 U\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & & -h\partial^0 U\partial^-\Big[\partial^0 V\Big]\big(c_*\gamma_U^{-1}\partial^0 U\big) \\ & & -h\partial^0 U\partial^-\big[\partial^0 U\big]\big(\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\big).\end{array} \tag{12.55}$$

We summarize our results by writing

$$\begin{array}{rcl}\mathcal{G}_{A;\mathrm{apx}}(U) & = & \mathcal{G}_{A;\mathrm{apx};III}(U) \\ & = & c_*\gamma_U\partial^0 U, \\ \mathcal{G}_{A;\mathrm{lin};U}[V] & = & \mathcal{G}_{A;\mathrm{lin};U;III}[V] \\ & = & c_*\gamma_U^{-1}(2\gamma_U^2-1)\partial^0 V+M_U[V]\end{array} \tag{12.56}$$

and obtaining the following bound.

Lemma 12.5. *Assume that (Hg) is satisfied and pick $0<\kappa<\frac{1}{12}$. Then there exists a constant $K>0$ so that the following properties hold true.*

(i) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell_h^2$, we have the identities

$$\begin{array}{rcl}\mathcal{G}_{A;\mathrm{apx};II}(U) & = & \mathcal{G}_{A;\mathrm{apx}}(U)+\mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U), \\ \mathcal{G}_{A;\mathrm{lin};U;II}[V] & = & \mathcal{G}_{A;\mathrm{lin};U}[V]+\mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{12.57}$$

(ii) For every $h>0$, $U\in\Omega_{h;\kappa}$ and $V\in\ell_h^2$ we have the bounds

$$\begin{array}{rcccl}\|\mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U)\|_{\ell_h^2} & \leq & Kh & = & KS_{\mathrm{sh};\mathrm{full}}(U), \\ \|\mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V]\|_{\ell_h^2} & \leq & Kh\,\|V\|_{\ell_h^{2;2}} & \leq & K\mathcal{E}_{\mathrm{sh};U}(V).\end{array} \tag{12.58}$$

Proof. Recalling Proposition 7.15, the bounds follow by inspection. □

12.4 Simplifications for $\mathcal{G}_B$

We recall the definition

$$\mathcal{G}_{B;\mathrm{apx};II}(U) \;=\; \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U). \tag{12.59}$$

Substituting the relevant expressions from §11, we find

$$\mathcal{G}_{B;\mathrm{apx};II}(U) \;=\; 2\partial^0 Uh\textstyle\sum_{-;h}\big[c_*\gamma_U^{-1}\partial^0 U\big]\partial^0\partial U. \tag{12.60}$$

In view of Lemma 7.11, we introduce the expressions

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{apx};III}(U) & = & c_*\partial^0 U(1-\gamma_U), \\ \mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U) & = & \mathcal{G}_{B;\mathrm{apx};II}(U)-\mathcal{G}_{B;\mathrm{apx};III}(U).\end{array} \tag{12.61}$$

We also recall the definition

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^-\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{F}_{\mathrm{apx}}^{\diamond_-;+}(U) \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{F}_{\mathrm{lin};U}^{\diamond_-;+}[V].\end{array} \tag{12.62}$$

Substituting the relevant expressions from §11, we find

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & +2\partial^0 V h \sum_{-;h} \big[c_* \gamma_U^{-1} \partial^0 U\big] \partial^0 \partial U \\ & & +2\partial^0 U h \sum_{-;h} \big[\gamma_U^{-2} M_U[V] + c_* \gamma_U^{-3} \partial^0 V\big] \partial^0 \partial U \\ & & -6\partial^0 U h \sum_{-;h} \big[c_* \gamma_U^{-1} \partial^0 U\big] \big[\partial^0 U \partial^0 V\big] \gamma_U^{-2} \partial^0 \partial U \\ & & +\partial^0 U h \sum_{-;h} \big[c_* \gamma_U^{-1} \partial^0 U\big] \big(6\gamma_U^{-2} \partial^0 U [\partial^0 \partial U] \partial^0 V + 2\partial^0 \partial V\big).\end{array} \tag{12.63}$$

A little algebra yields

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & 2\partial^0 V h \sum_{-;h} \big[c_* \gamma_U^{-1} \partial^0 U\big] \partial^0 \partial U \\ & & +2\partial^0 U h \sum_{-;h} \big[\gamma_U^{-2} M_U[V] + c_* \gamma_U^{-3} \partial^0 V\big] \partial^0 \partial U \\ & & +2c_* \partial^0 U h \sum_{-;h} \gamma_U^{-1} [\partial^0 U] \partial^0 \partial V.\end{array} \tag{12.64}$$

In view of Lemma's 7.10 and 7.11, we introduce the expressions

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;III}[V] & = & c_* \partial^0 V (1 - \gamma_U) \\ & & +2\partial^0 U h \sum_{-;h} \big[\gamma_U^{-2} M_U[V] + c_* \gamma_U^{-3} \partial^0 V\big] \partial^0 \partial U \\ & & +c_* \partial^0 U \gamma_U^{-1} \partial^0 U \partial^0 V - 2c_* \partial^0 U h \sum_{-;h} \Big[\gamma_U^{-3} [\partial^0 \partial U] \partial^0 V\Big], \\ \mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V] & = & \mathcal{G}_{B;\mathrm{lin};U;II}[V] - \mathcal{G}_{B;\mathrm{lin};U;III}[V].\end{array} \tag{12.65}$$

After a short computation, we find

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{lin};U;III}[V] & = & c_* \partial^0 V (1 + \gamma_U^{-1} - 2\gamma_U) \\ & & +2\partial^0 U h \sum_{-;h} \gamma_U^{-2} [\partial^0 \partial U] M_U[V].\end{array} \tag{12.66}$$

We summarize our results by writing

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{apx}}(U) & = & \mathcal{G}_{B;\mathrm{apx};III}(U) \\ & = & c_* \partial^0 U (1 - \gamma_U), \\ \mathcal{G}_{B;\mathrm{lin};U}[V] & = & \mathcal{G}_{B;\mathrm{lin};U;III}[V] \\ & = & c_* \partial^0 V (1 + \gamma_U^{-1} - 2\gamma_U) \\ & & +2\partial^0 U h \sum_{-;h} \gamma_U^{-2} [\partial^0 \partial U] M_U[V]\end{array} \tag{12.67}$$

and obtaining the following bounds.

Lemma 12.6. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$, we have the identities

$$\begin{array}{rcl}\mathcal{G}_{B;\mathrm{apx};II}(U) & = & \mathcal{G}_{B;\mathrm{apx}}(U) + \mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U), \\ \mathcal{G}_{B;\mathrm{lin};U;II}[V] & = & \mathcal{G}_{B;\mathrm{lin};U}[V] + \mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{12.68}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell_h^2$ we have the bounds

$$\begin{array}{rclcl}\|\mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U)\|_{\ell_h^2} & \leq & Kh & = & K S_{\mathrm{sh};\mathrm{full}}(U), \\ \|\mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V]\|_{\ell_h^2} & \leq & Kh\big[\, \|\partial^+ V\|_{\ell_h^2} + \|\partial^+ \partial^+ V\|_{\ell_h^2} \,\big] & \leq & K \mathcal{E}_{\mathrm{sh};U}(V).\end{array} \tag{12.69}$$

Proof. The estimates follow from Lemma's 7.10 and 7.11. □

12.5 Simplifications for $\mathcal{G}_C$ and $\mathcal{G}_D$

We recall the definition

$$\mathcal{G}_{\#;\mathrm{apx};II}(U) \quad = \quad \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \tag{12.70}$$

for $\# \in \{C, D\}$. Inspecting (11.58) and (11.61), we see that

$$\mathcal{G}_{C;\mathrm{apx};II}(U) = -\mathcal{G}_{D;\mathrm{apx};II}(U). \tag{12.71}$$

We also recall the definition

$$\begin{array}{lcl}\mathcal{G}_{\#;\mathrm{lin};U;II}[V] & = & \tfrac{1}{2}\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & +\tfrac{1}{2}\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)h\sum\nolimits_{-;h}\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)T^{-}\Big[\mathcal{X}_{\#;\mathrm{apx};\mathrm{expl}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V]\Big]\end{array} \tag{12.72}$$

for $\# \in \{C, D\}$. Using (11.59) and (11.62) we hence see

$$\mathcal{G}_{C;\mathrm{lin};U;II}[V] = -\mathcal{G}_{D;\mathrm{lin};U;II}[V]. \tag{12.73}$$

12.6 Summary

Recalling the definitions (12.1), we observe that

$$\begin{array}{lcl}\mathcal{G}_{A;\mathrm{apx}}(U) + \mathcal{G}_{B;\mathrm{apx}}(U) & = & c_*\gamma_U\partial^0 U + c_*\partial^0 U(1-\gamma_U) \\ & = & c_*\partial^0 U \\ & = & \mathcal{G}_{\mathrm{apx}}(U),\end{array} \tag{12.74}$$

together with

$$\begin{array}{lcl}\mathcal{G}_{A;\mathrm{lin};U}[V] + \mathcal{G}_{B;\mathrm{lin};U}[V] & = & c_*\gamma_U^{-1}(2\gamma_U^2 - 1)\partial^0 V + M_U[V] \\ & & \quad +c_*\partial^0 V(1 + \gamma_U^{-1} - 2\gamma_U) + 2\partial^0 U h\sum\nolimits_{-;h}\gamma_U^{-2}[\partial^0\partial U]M_U[V] \\ & = & c_*\partial^0 V + M_U[V] + 2\partial^0 U h\sum\nolimits_{-;h}\gamma_U^{-2}[\partial^0\partial U]M_U[V] \\ & = & \mathcal{G}_{\mathrm{lin};U}[V].\end{array} \tag{12.75}$$

We define the error terms

$$\begin{array}{lcl}\mathcal{G}_{\mathrm{apx};\mathrm{rem}}(U) & = & \mathcal{G}_{\mathrm{apx};\mathrm{rem};a}(U), \\ \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}[V] & = & \mathcal{G}_{\mathrm{lin};U;\mathrm{rem};a}[V],\end{array} \tag{12.76}$$

together with

$$\begin{array}{lcl}\mathcal{G}_{\mathrm{apx};\mathrm{sh}}(U) & = & \mathcal{G}_{\mathrm{apx};\mathrm{sh};a}(U) + \mathcal{G}_{A;\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}_{B;\mathrm{apx};\mathrm{sh};b}(U), \\ \mathcal{G}_{\mathrm{lin};U;\mathrm{sh}}[V] & = & \mathcal{G}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}_{A;\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}_{B;\mathrm{lin};U;\mathrm{sh};b}[V].\end{array} \tag{12.77}$$

The computations above show that

$$\begin{array}{rcl}\mathcal{G}_{\mathrm{apx};I}(U) & = & \mathcal{G}_{\mathrm{apx}}(U) + \mathcal{G}_{\mathrm{apx;rem}}(U) + \mathcal{G}_{\mathrm{apx;sh}}(U), \\ \mathcal{G}_{\mathrm{lin};U;I}[V] & = & \mathcal{G}_{\mathrm{lin};U}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh}}[V].\end{array} \quad (12.78)$$

Recalling the definition (12.12), this implies that

$$\mathcal{G}_{\mathrm{nl};U}(V) \quad = \quad \mathcal{G}_{\mathrm{nl};U;I}(V) + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}[V] + \mathcal{G}_{\mathrm{lin};U;\mathrm{sh}}[V]. \quad (12.79)$$

Corollary 12.7. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rcl}\mathcal{G}_{\mathrm{apx;sh}}(U) & \leq & Kh, \\ \mathcal{G}_{\mathrm{apx;rem}}(U) & \leq & K\big[\, \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h} \,\big].\end{array} \quad (12.80)$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the bounds

$$\begin{array}{rcl}\mathcal{G}_{\mathrm{lin};U;\mathrm{sh}}[V] & \leq & K\mathcal{E}_{\mathrm{sh};U}(V), \\ \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}[V] & \leq & K\mathcal{E}_{\mathrm{rem};U}[V].\end{array} \quad (12.81)$$

(iii) For any $h > 0$, any pair $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$ and any $V \in \ell^2_h$, we have the bound

$$\big\|\mathcal{G}_{\mathrm{lin};U^{(2)};\mathrm{rem}}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)};\mathrm{rem}}[V]\big\|_{\ell^2_h} \quad \leq \quad K\mathcal{E}_{\mathrm{prod}}\big(U^{(2)} - U^{(1)}, V\big). \quad (12.82)$$

Proof. These estimates follow directly from Lemma's 12.3, 12.4, 12.5 and 12.6. □

Lemma 12.8. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the estimate*

$$\big\|\mathcal{G}_{\mathrm{lin};U^{(2)}}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)}}[V]\big\|_{\ell^2_h} \quad \leq \quad K\mathcal{E}_{\mathrm{prod}}\big(U^{(2)} - U^{(1)}, V\big) \quad (12.83)$$

holds for all $h > 0$, all $V \in \ell^2_h$ and all pairs $(U^{(1)}, U^{(2)}) \in \Omega^2_{h;\kappa}$.

Proof. We compute

$$\begin{array}{rcl}\big\|\mathcal{G}_{\mathrm{lin};U^{(2)}}[V] - \mathcal{G}_{\mathrm{lin};U^{(1)}}[V]\big\|_{\ell^2_h} & \leq & \|M_{U^{(2)}}[V] - M_{U^{(1)}}[V]\|_{\ell^2_h} + C_1' \big\|\partial^0 U^{(2)} - \partial^0 U^{(1)}\big\|_{\ell^2_h} \|M_{U_2}[V]\|_{\ell^2_h} \\ & & +C_1' \big\|\gamma_{U^{(2)}}\partial^0\partial U^{(2)} - \gamma_{U^{(1)}}\partial^0\partial U^{(1)}\big\|_{\ell^2_h} \|M_{U_2}[V]\|_{\ell^2_h} \\ & & +C_1' \|M_{U^{(2)}}[V] - M_{U^{(1)}}[V]\|_{\ell^2_h} .\end{array} \quad (12.84)$$

Exploiting the a-priori bound (7.82) together with the Lipschitz bounds (7.3) and (7.84), this yields the desired estimate. □

Lemma 12.9. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ so that the estimate*

$$\begin{array}{rcl}\big\|\mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)})\big\|_{\ell^2_h} & \leq & K\mathcal{E}_{\mathrm{prod}}\big(V^{(2)} - V^{(1)}, V^{(2)} - V^{(1)}\big) + Kh \big\|V^{(2)} - V^{(1)}\big\|_{\ell^{2;2}_h} \\ & & +K\mathcal{E}_{\mathrm{rem};U}\big(V^{(2)} - V^{(1)}\big) \\ & & +K\mathcal{E}_{\mathrm{prod}}\big(V^{(1)}, V^{(2)} - V^{(1)}\big)\end{array} \quad (12.85)$$

holds for all $h > 0$, all $U \in \Omega_{h;\kappa}$ and all pairs $(V^{(1)}, V^{(2)}) \in \ell^2_h \times \ell^2_h$ for which the inclusions $U + V^{(1)} \in \Omega_{h;\kappa}$ and $U + V^{(2)} \in \Omega_{h;\kappa}$ both hold.

Proof. By definition, we have

$$\mathcal{G}_{\mathrm{nl};U}(V) \quad = \quad \mathcal{G}(U+V) - \mathcal{G}(U) - \mathcal{G}_{\mathrm{lin};U}[V]. \tag{12.86}$$

In particular, we get

$$\begin{aligned} \mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)}) &= \mathcal{G}(U+V^{(2)}) - \mathcal{G}_{\mathrm{lin};U}[V^{(2)}] + \mathcal{G}_{\mathrm{lin};U}[V^{(1)}] - \mathcal{G}(U+V^{(1)}) \\ &= \mathcal{G}\big(U+V^{(1)} + (V^{(2)} - V^{(1)})\big) - \mathcal{G}(U+V^{(1)}) \\ &\qquad - \mathcal{G}_{\mathrm{lin};U}[V^{(2)} - V^{(1)}] \\ &= \mathcal{G}_{\mathrm{lin};U+V^{(1)}}[V^{(2)} - V^{(1)}] + \mathcal{G}_{\mathrm{nl};U+V^{(1)}}(V^{(2)} - V^{(1)}) \\ &\qquad - \mathcal{G}_{\mathrm{lin};U}[V^{(2)} - V^{(1)}] \\ &= \mathcal{G}_{\mathrm{nl};U+V^{(1)}}(V^{(2)} - V^{(1)}) \\ &\qquad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}]. \end{aligned} \tag{12.87}$$

Substituting (12.79), we find

$$\begin{aligned} \mathcal{G}_{\mathrm{nl};U}(V^{(2)}) - \mathcal{G}_{\mathrm{nl};U}(V^{(1)}) &= \mathcal{G}_{\mathrm{nl};U+V^{(1)};I}(V^{(2)} - V^{(1)}) \\ &\qquad + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{rem}}(V^{(2)} - V^{(1)}) + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{sh}}(V^{(2)} - V^{(1)}) \\ &\qquad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}] \\ &= \mathcal{G}_{\mathrm{nl};U+V^{(1)};I}(V^{(2)} - V^{(1)}) \\ &\qquad + \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}(V^{(2)} - V^{(1)}) + \mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{sh}}(V^{(2)} - V^{(1)}) \\ &\qquad + \Big[\mathcal{G}_{\mathrm{lin};U+V^{(1)};\mathrm{rem}} - \mathcal{G}_{\mathrm{lin};U;\mathrm{rem}}\Big](V^{(2)} - V^{(1)}) \\ &\qquad + \big[\mathcal{G}_{\mathrm{lin};U+V^{(1)}} - \mathcal{G}_{\mathrm{lin};U}\big][V^{(2)} - V^{(1)}]. \end{aligned} \tag{12.88}$$

The desired bound now follows from Lemma 12.2 and Corollary 12.7. □

Proof of Proposition 12.1. In view of the expression (12.79), the statements follow from Lemma 12.2, Corollary 12.7 and Lemma 12.9. □

13 Estimates for $\mathcal{G}^+$

In this section we exploit the component estimates from §10-§11 to analyze the function $\mathcal{G}^+$ discussed in §9. In particular, we introduce the approximants

$$\begin{aligned} \mathcal{G}^+_{\mathrm{apx}}(U) &= c_* S^+[\partial^0 \partial U], \\ \mathcal{G}^+_{\mathrm{lin};U}[V] &= c_* S^+[\partial^0 \partial V] + \partial^+\big[M_U[V]\big] + 2\gamma_U^{-2}\partial^0 U[\partial^0\partial U] M_U[V] \\ &\qquad + 2S^+[\partial^0\partial U] T^+ h \textstyle\sum_{-;h} \gamma_U^{-2} \partial^0 \partial U M_U[V] \end{aligned} \tag{13.1}$$

and write

$$\mathcal{G}^+_{\mathrm{nl};U}(V) = \mathcal{G}^+(U+V) - \mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{lin};U}(V). \tag{13.2}$$

Using (4.4), (4.5) and (4.11) one may readily verify the identities

$$\begin{aligned} \mathcal{G}^+_{\mathrm{apx}}(U) &= \partial^+\big[\mathcal{G}_{\mathrm{apx}}(U)\big], \\ \mathcal{G}^+_{\mathrm{lin};U}[V] &= \partial^+\big[\mathcal{G}_{\mathrm{lin};U}[V]\big], \end{aligned} \tag{13.3}$$

which implies that also

$$\mathcal{G}^+_{\mathrm{nl};U}(V) = \partial^+\big[\mathcal{G}_{\mathrm{nl};U}(V)\big]. \tag{13.4}$$

Our main result quantifies the approximation errors in terms of the quantities

$$\begin{array}{rcl} \overline{\mathcal{E}}_{\mathrm{sh};U}(V) & = & \|V\|_{\ell^{2;3}_h} + \big[\, \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\,\big]\, \|V\|_{\ell^{2;2}_h}\,, \\ \overline{\mathcal{E}}_{\mathrm{rem};U}(V) & = & K\, \|V\|_{\ell^{2;2}_h} \Big[\, \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h} + \|\mathcal{E}_{\mathrm{sm}}(U)\|_{\ell^2_h}\,\Big] \\ & & \quad + K\, \|V\|_{\ell^{2;1}_h}\, \|\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\|_{\ell^\infty_h} \end{array} \tag{13.5}$$

that were originally introduced in §11.1, together with

$$\begin{array}{rcl} \overline{\mathcal{E}}_{\mathrm{prod};U}(W^{(1)},W^{(2)}) & = & \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} \Big[\, \big\|W^{(1)}\big\|_{\ell^{2;1}_h} \big\|W^{(2)}\big\|_{\ell^{\infty;1}_h} + \big\|W^{(1)}\big\|_{\ell^{\infty;1}_h} \big\|W^{(2)}\big\|_{\ell^{2;1}_h}\,\Big] \\ & & + \big\|W^{(1)}\big\|_{\ell^{2;2}_h} \big\|W^{(2)}\big\|_{\ell^{2;2}_h} \\ & & + \big\|W^{(1)}\big\|_{\ell^{2;2}_h} \big\|W^{(2)}\big\|_{\ell^{\infty;2}_h} + \big\|W^{(1)}\big\|_{\ell^{\infty;2}_h} \big\|W^{(2)}\big\|_{\ell^{2;2}_h} \\ & & + \big\|W^{(1)}\big\|_{\ell^{2;3}_h} \big\|W^{(2)}\big\|_{\ell^{\infty;1}_h} + \big\|W^{(1)}\big\|_{\ell^{\infty;1}_h} \big\|W^{(2)}\big\|_{\ell^{2;3}_h}\,. \end{array} \tag{13.6}$$

Proposition 13.1. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the following properties hold.*

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$, we have

$$\begin{array}{rcl} \big\|\mathcal{G}^+(U) - \mathcal{G}^+_{\mathrm{apx}}(U)\big\|_{\ell^2_h} & \le & Kh\big[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\,\big] \\ & & + K\big[\, \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h} + \|\partial^+\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h}\,\big]. \end{array} \tag{13.7}$$

(ii) For any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$, we have the estimate

$$\Big\|\mathcal{G}^+_{\mathrm{nl};U}(V)\Big\|_{\ell^2_h} \le K\overline{\mathcal{E}}_{\mathrm{prod};U}(V,V) + Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V) + K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \tag{13.8}$$

(iii) For any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bound

$$\Big\|\mathcal{G}^+_{\mathrm{lin};U}[V] - \mathcal{G}_{\mathrm{lin};U}[\partial^+ V]\Big\|_{\ell^2_h} \le K\Big[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}\Big]\, \|V\|_{\ell^{2;2}_h}\,. \tag{13.9}$$

13.1 Nonlinear estimates

Applying the expressions (6.67) to the term

$$\mathcal{G}^+_{A'a}(U) = 2\widetilde{p}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+}(U)\partial^+\partial^0\partial U \tag{13.10}$$

defined in (9.39), we obtain the initial approximants

$$\begin{array}{rcl} \mathcal{G}^+_{A'a;\mathrm{apx};I}(U) & = & 2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial U, \\ \mathcal{G}^+_{A'a;\mathrm{lin};U;I}[V] & = & 2\widetilde{p}_{\mathrm{lin};U}[V]\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial U + 2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V]\partial^+\partial^0\partial U \\ & & + 2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial V \end{array} \tag{13.11}$$

and write

$$\mathcal{G}^+_{A'a;\mathrm{nl};U;I}(V) = \mathcal{G}^+_{A'a}(U+V) - \mathcal{G}^+_{A'a}(U) - \mathcal{G}^+_{A'a;\mathrm{lin};U;I}[V]. \tag{13.12}$$

Lemma 13.2. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the approximation estimate*

$$\left\| \mathcal{G}^{+}_{A'a}(U) - \mathcal{G}^{+}_{A'a;\mathrm{apx};I}(U) \right\|_{\ell^2_h} \leq Kh \left\| \partial^+ \partial^+ \partial^+ U \right\|_{\ell^2_h} \tag{13.13}$$

and the residual bound

$$\begin{aligned} \left\| \mathcal{G}^{+}_{A'a;\mathrm{nl};U;I}(V) \right\|_{\ell^2_h} &\leq K\big[\left\| \partial^+ V \right\|_{\infty} + h \big] \left\| \partial^+ \partial^+ \partial^+ V \right\|_{\ell^2_h} \\ &\quad + K \left\| \partial^+ V \right\|_{\ell^\infty_h} \left\| \partial^+ V \right\|_{\ell^2_h} \left\| \partial^+ \partial^+ \partial^+ U \right\|_{\ell^\infty_h} \\ &\quad + Kh \Big[\left\| \partial^+ V \right\|_{\ell^2_h} + \left\| \partial^+ \partial^+ V \right\|_{\ell^2_h} \Big] \left\| \partial^+ \partial^+ \partial^+ U \right\|_{\ell^\infty_h} \\ &\leq K \overline{\mathcal{E}}_{\mathrm{prod};U}(V,V) + K \overline{\mathcal{E}}_{\mathrm{sh};U}(V) \end{aligned} \tag{13.14}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. The first estimate follows immediately from Proposition 10.2. To obtain the second estimate, we observe that the uniform bound in item (i) of this proposition shows that

$$\begin{aligned} \left\| \mathcal{G}^{+}_{A'a;\mathrm{nl};U;I}(V) \right\|_{\ell^2_h} &\leq C_1' \left\| \partial^+ \partial^+ \partial^+ V \right\|_{\ell^2_h} \Big[\left\| \widetilde{p}_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} + \left\| \widetilde{p}_{\mathrm{nl};U}(V) \right\|_{\ell^\infty_h} \\ &\qquad + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V) \right\|_{\ell^\infty_h} \Big] \\ &\quad + C_1' \left\| \partial^+ \partial^+ \partial^+ U \right\|_{\ell^\infty_h} \Big[\left\| \widetilde{p}_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V) \right\|_{\ell^2_h} \\ &\qquad + \left\| \widetilde{p}_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} \Big]. \end{aligned} \tag{13.15}$$

We note that Lemma's 10.15 and 10.17 yield the preliminary estimates

$$\begin{aligned} \left\| \widetilde{p}_{\mathrm{nl};U}(V) \right\|_{\infty} + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V) \right\|_{\infty} &\leq C_2' \left\| \partial^+ V \right\|^2_{\ell^\infty_h} + C_2' h \big[\left\| \partial^+ V \right\|^2_{\ell^\infty_h} + \left\| \partial^+ \partial^+ V \right\|_{\ell^\infty_h} \big] \\ &\leq C_3' \big[\left\| \partial^+ V \right\|_{\ell^\infty_h} + h \big]. \end{aligned} \tag{13.16}$$

In addition, Proposition 10.2 yields the bounds

$$\begin{aligned} \left\| \widetilde{p}_{\mathrm{lin};U}[V] \right\|_{\ell^2_h} &\leq C_4' \left\| \partial^+ V \right\|_{\ell^2_h}, \\ \left\| \widetilde{p}_{\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V] \right\|_{\ell^\infty_h} &\leq C_4' \left\| \partial^+ V \right\|_{\ell^\infty_h}, \end{aligned} \tag{13.17}$$

together with

$$\left\| \widetilde{p}_{\mathrm{nl};U}(V) \right\|_{\ell^2_h} + \left\| \mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{nl};U}(V) \right\|_{\ell^2_h} \leq C_6' \left\| \partial^+ V \right\|_{\ell^\infty_h} \left\| \partial^+ V \right\|_{\ell^2_h} + C_6' h \big[\left\| \partial^+ V \right\|_{\ell^2_h} + \left\| \partial^+ \partial^+ V \right\|_{\ell^2_h} \big]. \tag{13.18}$$

Substituting these bounds into (13.15) yields the desired estimate. □

We now apply (6.67) to the terms (9.14) to obtain the initial approximants

$$\begin{aligned} \mathcal{G}^{+}_{A'b;\mathrm{apx};I}(U) &= \Big[1 - \mathcal{Y}_{1;\mathrm{apx}}(U) \mathcal{X}_{A;\mathrm{apx}}(U) \Big] \mathcal{Y}^{+}_{2b;\mathrm{apx}}(U), \\ \mathcal{G}^{+}_{A'c;\mathrm{apx};I}(U) &= -\mathcal{Y}^{+}_{1;\mathrm{apx}}(U) \mathcal{X}_{A;\mathrm{apx}}(U) T^{+} \big[\mathcal{Y}_{2;\mathrm{apx}}(U) \big], \end{aligned} \tag{13.19}$$

together with

$$\begin{array}{lcl}\mathcal{G}^+_{A'b;\mathrm{lin};U;I}[V] & = & -\mathcal{Y}^+_{1;\mathrm{lin};U}[V]\mathcal{X}_{A;\mathrm{apx}}(U)\mathcal{Y}^+_{2b;\mathrm{apx}}(U)] \\ & & \quad -\mathcal{Y}^+_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{lin};U}[V]\mathcal{Y}^+_{2b;\mathrm{apx}}(U)\Big] \\ & & \quad \big[1-\mathcal{Y}_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{apx}}(U)\big]\mathcal{Y}^+_{2b;\mathrm{lin};U}[V]\big], \\ \mathcal{G}^+_{A'c;\mathrm{lin};U;I}[V] & = & -\mathcal{Y}^+_{1;\mathrm{lin};U}[V]\mathcal{X}_{A;\mathrm{apx}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx}}(U)\big] \\ & & \quad -\mathcal{Y}^+_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{lin};U}[V]T^+\big[\mathcal{Y}_{2;\mathrm{apx}}(U)\big] \\ & & \quad -\mathcal{Y}^+_{1;\mathrm{apx}}(U)\mathcal{X}_{A;\mathrm{apx}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{lin};U}[V]\big].\end{array} \tag{13.20}$$

Applying the expressions (6.67) one final time to the terms (9.15), we also obtain

$$\begin{array}{lcl}\mathcal{G}^+_{B';\mathrm{apx};I}(U) & = & \frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)hT^+\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U)\Big], \\ \mathcal{G}^+_{B';\mathrm{lin};U;I}[V] & = & \frac{1}{2}\mathcal{Y}^+_{1;\mathrm{lin};U}[V]T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U}[V]T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{lin};U}[V]\Big]\mathcal{F}^{\diamond_-;+}_{\mathrm{apx}}(U) \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx}}(U)\Big]\mathcal{F}^{\diamond_-;+}_{\mathrm{lin};U}[V]\end{array} \tag{13.21}$$

together with

$$\begin{array}{lcl}\mathcal{G}^+_{\#';\mathrm{apx};I}(U) & = & \frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#;\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big], \\ \mathcal{G}^+_{\#';\mathrm{lin};U;I}[V] & = & \frac{1}{2}\mathcal{Y}^+_{1;\mathrm{lin};U}[V]T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#';\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U}[V]T^-\Big[\mathcal{X}_{\#';\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#';\mathrm{lin};U}[V]\mathcal{F}^{\diamond_0;+}_{\mathrm{apx}}(U)\Big] \\ & & \quad +\frac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx}}(U)T^+h\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx}}(U)T^-\Big[\mathcal{X}_{\#';\mathrm{apx}}(U)\mathcal{F}^{\diamond_0;+}_{\mathrm{lin};U}[V]\Big]\end{array} \tag{13.22}$$

for $\# \in \{C, D\}$.

Writing

$$\mathcal{G}^+_{\mathrm{low}}(U) = \mathcal{G}^+(U) - \mathcal{G}^+_{A'a}(U) \tag{13.23}$$

we use the expressions above to introduce the initial approximants

$$\begin{array}{lcl}\mathcal{G}^+_{\mathrm{low};\mathrm{apx};I}(U) & = & \mathcal{G}^+_{A'b;\mathrm{apx};I}(U) + \mathcal{G}^+_{A'c;\mathrm{apx};I}(U) \\ & & \quad +\mathcal{G}^+_{B';\mathrm{apx};I}(U) + \mathcal{G}^+_{C';\mathrm{apx};I}(U) + \mathcal{G}^+_{D';\mathrm{apx};I}(U), \\ \mathcal{G}^+_{\mathrm{low};\mathrm{lin};U;I}[V] & = & \mathcal{G}^+_{A'b;\mathrm{lin};U;I}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U;I}[V] \\ & & \quad +\mathcal{G}^+_{B';\mathrm{lin};U;I}[V] + \mathcal{G}^+_{C';\mathrm{lin};U;I}[V] + \mathcal{G}^+_{D';\mathrm{lin};U;I}[V]\end{array} \tag{13.24}$$

and write

$$\mathcal{G}^+_{\mathrm{low};\mathrm{nl};U;I}(V) = \mathcal{G}^+_{\mathrm{low}}(U+V) - \mathcal{G}^+_{\mathrm{low}}(U) - \mathcal{G}^+_{\mathrm{low};\mathrm{lin};U;I}[V]. \tag{13.25}$$

Lemma 13.3. *Suppose that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. Then there exists $K > 0$ so that the approximation estimate*

$$\Big\|\mathcal{G}^+_{\mathrm{low}}(U) - \mathcal{G}^+_{\mathrm{low};\mathrm{apx};I}(U)\Big\|_{\ell^2_h} \quad \leq \quad Kh \tag{13.26}$$

and the residual bound

$$\begin{aligned} \left\| \mathcal{G}^{+}_{\mathrm{low;nl};I}(V) \right\|_{\ell^2_h} &\leq K \left\| V \right\|_{\ell^{2;2}_h} \left[\left\| V \right\|_{\ell^{2;2}_h} + \left\| V \right\|_{\ell^{\infty;2}_h} \right] \\ &\qquad + Kh \left\| V \right\|_{\ell^{2;2}_h} \\ &\leq K \overline{\mathcal{E}}_{\mathrm{prod};U}(V,V) + K \overline{\mathcal{E}}_{\mathrm{sh};U}(V) \end{aligned} \tag{13.27}$$

both hold for any $h > 0$, any $U \in \Omega_{h;\kappa}$ and any $V \in \ell^2_h$ for which $U + V \in \Omega_{h;\kappa}$.

Proof. Following the strategy developed in the proof of Lemma 12.2, the error terms in Lemma 6.6 can be controlled by

$$\begin{aligned} \mathcal{J}_{\mathrm{nl};U}(V) &\leq C_1' \overline{\mathcal{E}}_{\mathrm{nl}}(V) \\ &\leq C_2' \left\| V \right\|_{\ell^{2;2}_h} \left[\left\| V \right\|_{\ell^{2;2}_h} + \left\| V \right\|_{\ell^{2;\infty}_h} + h \right], \\ \mathcal{J}_{\mathrm{cross};U}(V) &\leq C_1' \Big[\overline{T}_{\mathrm{safe}}(V) \overline{S}_{\mathrm{full}}(V) + \overline{T}_{\infty;\mathrm{opt}}(V) \overline{S}_{2;\mathrm{fix}}(V) \Big] \\ &\leq C_2' \left\| V \right\|_{\ell^{2;2}_h} \left[\left\| V \right\|_{\ell^{2;2}_h} + \left\| V \right\|_{\ell^{2;\infty}_h} \right], \\ \mathcal{J}_{\mathrm{apx};U}(V) &\leq C_1' h \overline{T}_{\mathrm{safe}}(V) \\ &= C_1' h \left\| V \right\|_{\ell^{2;2}_h}, \end{aligned} \tag{13.28}$$

which yields the desired bounds. □

13.2 Error terms

We now introduce the set of expressions

$$\{\mathcal{G}^{+}_{A'b;\mathrm{apx};II}(U), \mathcal{G}^{+}_{A'c;\mathrm{apx};II}(U), \mathcal{G}^{+}_{B';\mathrm{apx};II}(U), \mathcal{G}^{+}_{C';\mathrm{apx};II}(U), \mathcal{G}^{+}_{D';\mathrm{apx};II}(U) \tag{13.29}$$

together with

$$\{\mathcal{G}^{+}_{A'b;\mathrm{lin};U;II}[V], \mathcal{G}^{+}_{A'c;\mathrm{lin};U;II}[V], \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V], \mathcal{G}^{+}_{C';\mathrm{lin};U;II}[V], \mathcal{G}^{+}_{D';\mathrm{lin};U;II}[V]\} \tag{13.30}$$

by inspecting the definitions (13.19), (13.20), (13.21) and (13.22) and making the replacements

$$f_{\mathrm{apx}}(U) \mapsto f_{\mathrm{apx;expl}}(U), \qquad f_{\mathrm{lin};U}[V] \mapsto f_{\mathrm{lin};U;\mathrm{expl}}[V] \tag{13.31}$$

for each $f \in \overline{\mathcal{S}}_{\mathrm{nl;short}}$.

In addition, we simply write

$$\begin{aligned} \mathcal{G}^{+}_{A'a;\mathrm{apx};II}(U) &= \mathcal{G}^{+}_{A'a;\mathrm{apx};I}(U) \\ \mathcal{G}^{+}_{A'a;\mathrm{lin};U;II}[V] &= \mathcal{G}^{+}_{A'a;\mathrm{lin};U;I}[V]. \end{aligned} \tag{13.32}$$

We now define

$$\begin{aligned} \mathcal{G}^{+}_{\mathrm{apx};II}(U) &= \mathcal{G}^{+}_{A'a;\mathrm{apx};II}(U) + \mathcal{G}^{+}_{A'b;\mathrm{apx};II}(U) + \mathcal{G}^{+}_{A'c;\mathrm{apx};II}(U) \\ &\qquad + \mathcal{G}^{+}_{B';\mathrm{apx};II}(U) + \mathcal{G}^{+}_{C';\mathrm{apx};II}(U) + \mathcal{G}^{+}_{D';\mathrm{apx};II}(U), \\ \mathcal{G}^{+}_{\mathrm{lin};U;II}[V] &= \mathcal{G}^{+}_{A'a;\mathrm{lin};U;II}[V] + \mathcal{G}^{+}_{A'b;\mathrm{lin};U;II}[V] + \mathcal{G}^{+}_{A'c;\mathrm{lin};U;II}[V] \\ &\qquad + \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] + \mathcal{G}^{+}_{C';\mathrm{lin};U;II}[V] + \mathcal{G}^{+}_{D';\mathrm{lin};U;II}[V]. \end{aligned} \tag{13.33}$$

Lemma 13.4. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with quantities*

$$\mathcal{G}^{+}_{\mathrm{apx;sh};a}(U) \in \ell^2_h, \qquad \mathcal{G}^{+}_{\mathrm{apx;rem};a}(U) \in \ell^2_h, \tag{13.34}$$

defined for every $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the identity

$$\mathcal{G}^{+}_{A'a;\mathrm{apx};I}(U) + \mathcal{G}^{+}_{\mathrm{low};\mathrm{apx};I}(U) \quad = \quad \mathcal{G}^{+}_{\mathrm{apx};II}(U) + \mathcal{G}^{+}_{\mathrm{apx};\mathrm{sh};a}(U) + \mathcal{G}^{+}_{\mathrm{apx};\mathrm{rem};a}(U). \tag{13.35}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{lllll} \left\| \mathcal{G}^{+}_{\mathrm{apx};\mathrm{sh};a}(U) \right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{sh};\mathrm{full}}(U) & = & Kh\big[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h}\,\big], \\ \left\| \mathcal{G}^{+}_{\mathrm{apx};\mathrm{rem};a}(U) \right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{rem};\mathrm{full}}(U) & = & K\big[\,\|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\mathrm{tw}}(U)\|_{\ell^\infty_h}\,\big]. \end{array} \tag{13.36}$$

Proof. The arguments in the proof of Lemma 12.3 also work in the current setting. □

Lemma 13.5. *Assume that (Hg) is satisfied and fix $0 < \kappa < \frac{1}{12}$. There exists a constant $K > 0$ together with linear maps*

$$\mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{sh};a} \in \mathcal{L}(\ell^2_h, \ell^2_h), \qquad \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{rem};a} \in \mathcal{L}(\ell^2_h, \ell^2_h), \tag{13.37}$$

defined for all $h > 0$ and $U \in \Omega_{h;\kappa}$, so that the following properties hold true.

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the identity

$$\mathcal{G}^{+}_{A'a;\mathrm{lin};U;I}[V] + \mathcal{G}^{+}_{\mathrm{low};\mathrm{lin};U;I}[V] \quad = \quad \mathcal{G}^{+}_{\mathrm{lin};U;II}[V] + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{rem};a}[V]. \tag{13.38}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{lll} \left\| \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{sh};a}[V] \right\|_{\ell^2_h} & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\| \mathcal{G}^{+}_{\mathrm{lin};U;\mathrm{rem};a}[V] \right\|_{\ell^2_h} & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{13.39}$$

Proof. The arguments in Lemma 12.4 show that for $\# \in \{\mathrm{sh}, \mathrm{rem}\}$ we have

$$\begin{array}{lll} \|\mathcal{G}_{\mathrm{lin};U;\#;a}[V]\|_{\ell^2_h} & \leq & C_1'\Big[\overline{T}_{\mathrm{safe}}(V)\overline{S}_{\#;\mathrm{full}}(V) + \overline{T}_{\infty;\mathrm{opt}}(V)\overline{S}_{\#;2;\mathrm{fix}}(V)\Big] \\ & & +C_1'\overline{\mathcal{E}}_{\#;U}(V), \end{array} \tag{13.40}$$

from which the desired bounds can be read off. □

13.3 Simplifications for $\mathcal{G}^{+}_{A'a}$

We recall the definition

$$\mathcal{G}^{+}_{A'a;\mathrm{apx};II}(U) \quad = \quad 2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial U. \tag{13.41}$$

Substituting the relevant expressions from §10, we find

$$\mathcal{G}^{+}_{A'a;\mathrm{apx};II}(U) \quad = \quad 2\gamma_U^{-2}\partial^+\partial^0\partial U. \tag{13.42}$$

We also recall the definition

$$\begin{array}{lll} \mathcal{G}^{+}_{A'a;\mathrm{lin};U;II}[V] & = & 2\widetilde{p}_{\mathrm{lin};U}[V]\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial U + 2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{lin};U}[V]\partial^+\partial^0\partial U \\ & & +2\widetilde{p}_{\mathrm{apx}}(U)\mathcal{I}^{\diamond\diamond_0;+}_{+;\mathrm{apx}}(U)\partial^+\partial^0\partial V. \end{array} \tag{13.43}$$

Substituting the relevant expressions from §10 and recalling (7.86), we find

$$\begin{array}{lcl} \mathcal{G}^{+}_{A'a;\mathrm{lin};U;II}[V] & = & -4\gamma_U^{-4}[\partial^+\partial^0\partial U]\partial^0 U\partial^0 V \\ & & +8\gamma_U^{-4}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V \\ & & +2\gamma_U^{-2}\partial^0\partial V \\ & = & \gamma_U^2\widetilde{M}_{U;E}[V] - 4\gamma_U^{-4}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V. \end{array} \tag{13.44}$$

We conclude by writing

$$\begin{array}{lcl} \mathcal{G}^{+}_{A'a;\mathrm{apx}}(U) & = & \mathcal{G}^{+}_{A'a;\mathrm{apx};II}(U), \\ \mathcal{G}^{+}_{A'a;\mathrm{lin};U}[V] & = & \mathcal{G}^{+}_{A'a;\mathrm{lin};U;II}[V]. \end{array} \tag{13.45}$$

13.4 Simplifications for $\mathcal{G}^{+}_{A'b}$

We recall the definition

$$\mathcal{G}^{+}_{A'b;\mathrm{apx};II}(U) = \Big[1 - \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\Big]\mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{expl}}(U). \tag{13.46}$$

Substituting the relevant expressions from §11, we find

$$\begin{array}{lcl} \mathcal{G}^{+}_{A'b;\mathrm{apx};II}(U) & = & \gamma_U^2[\mathcal{E}^{+}_{\mathrm{tw};\mathrm{apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U + 2c_*\gamma_U^{-3}\partial^0\partial U] \\ & = & \gamma_U^2\mathcal{E}^{+}_{\mathrm{tw};\mathrm{apx}}(U) - 2\gamma_U^{-2}\partial^+\partial^0\partial U + 2c_*\gamma_U^{-1}\partial^0\partial U. \end{array} \tag{13.47}$$

We now make the decomposition

$$\mathcal{G}^{+}_{A'b;\mathrm{apx};II}(U) = \mathcal{G}^{+}_{A'b;\mathrm{apx};III}(U) + \mathcal{G}^{+}_{A'b;\mathrm{apx};\mathrm{sh};b}(U) \tag{13.48}$$

by introducing

$$\mathcal{G}^{+}_{A'b;\mathrm{apx};III}(U) = \gamma_U^2\mathcal{E}^{+}_{\mathrm{tw};\mathrm{apx}}(U) - 2\gamma_U^{-2}\partial^+\partial^0\partial U + c_*\gamma_U^{-1}S^+[\partial^0\partial U], \tag{13.49}$$

together with

$$\mathcal{G}^{+}_{A'b;\mathrm{apx};\mathrm{sh};b}(U) = -c_* h\gamma_U^{-1}\partial^+[\partial^0\partial U]. \tag{13.50}$$

We also recall the definition

$$\begin{array}{lcl} \mathcal{G}^{+}_{A'b;\mathrm{lin};U;II}[V] & = & -\mathcal{Y}_{1;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{expl}}(U) \\ & & -\mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{Y}^{+}_{2b;\mathrm{apx};\mathrm{expl}}(U) \\ & & \big[1 - \mathcal{Y}_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)\big]\mathcal{Y}^{+}_{2b;\mathrm{lin};U;\mathrm{expl}}[V]. \end{array} \tag{13.51}$$

Substituting the relevant expressions from §11, we find

$$\begin{array}{lcl} \mathcal{G}^{+}_{A'b;\mathrm{lin};U;II}[V] & = & -2\partial^0 U\partial^0 V\Big[\mathcal{E}^{+}_{\mathrm{tw};\mathrm{apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U + 2c_*\gamma_U^{-3}\partial^0\partial U\Big] \\ & & +\gamma_U^2\Big[\gamma_U^{-2}\partial^+\big[M_U[V]\big] + 4\gamma_U^{-4}\partial^0 U[\partial^0\partial U]M_U[V] - \widetilde{M}_{U;E}[V]\Big] \\ & & +\gamma_U^2 c_*\Big[6\gamma_U^{-5}\partial^0 U[\partial^0\partial U]\partial^0 V + \gamma_U^{-3}S^+[\partial^0\partial V]\Big] \\ & = & 2c_*\gamma_U^{-3}\partial^0 U[\partial^0\partial U]\partial^0 V + c_*\gamma_U^{-1}S^+[\partial^0\partial V] \\ & & +\partial^+\big[M_U[V]\big] + 4\gamma_U^{-2}\partial^0 U[\partial^0\partial U]M_U[V] - \gamma_U^2\widetilde{M}_{U;E}[V] \\ & & -2\partial^0 U\Big[\mathcal{E}^{+}_{\mathrm{tw};\mathrm{apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U\Big]\partial^0 V. \end{array} \tag{13.52}$$

We conclude by writing

$$\begin{array}{rcl} \mathcal{G}^+_{A'b;\mathrm{apx}}(U) & = & \mathcal{G}^+_{A'b;\mathrm{apx};III}(U) \\ & = & \gamma_U^2 \mathcal{E}^+_{\mathrm{tw;apx}}(U) - 2\gamma_U^{-2}\partial^+\partial^0\partial U + c_*\gamma_U^{-1} S^+[\partial^0\partial U], \\ \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] & = & \mathcal{G}^+_{A'b;\mathrm{lin};U;II}[V] \end{array} \quad (13.53)$$

and obtaining the following bound.

Lemma 13.6. *Assume that (Hg) is satisfied and pick* $0 < \kappa < \frac{1}{12}$. *Then there exists a constant* $K > 0$ *so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$, *we have the identity*

$$\mathcal{G}^+_{A'b;\mathrm{apx};II}(U) \quad = \quad \mathcal{G}^+_{A'b;\mathrm{apx}}(U) + \mathcal{G}^+_{A'b;\mathrm{apx;sh};b}(U). \quad (13.54)$$

(ii) For every $h > 0$ *and* $U \in \Omega_{h;\kappa}$ *we have the bound*

$$\left\| \mathcal{G}^+_{A'b;\mathrm{apx;sh};b}(U) \right\|_{\ell^2_h} \quad \leq \quad Kh \left\| \partial^+\partial^+\partial^+ U \right\|_{\ell^2_h} \quad \leq \quad K\overline{S}_{\mathrm{sh;full}}(U). \quad (13.55)$$

Proof. The results follow by inspection. □

13.5 Simplifications for $\mathcal{G}^+_{A'c}$

We recall the definition

$$\mathcal{G}^+_{A'c;\mathrm{apx};II}(U) \quad = \quad -\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)\mathcal{X}_{A;\mathrm{apx;expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx;expl}}(U)\big]. \quad (13.56)$$

Substituting the relevant expressions from §11, we find

$$\mathcal{G}^+_{A'c;\mathrm{apx};II}(U) \quad = \quad -2[\partial^0\partial U]\partial^0 U T^+\big[c_*\gamma_U^{-1}\partial^0 U\big]. \quad (13.57)$$

We now make the decomposition

$$\mathcal{G}_{A'c;\mathrm{apx};II}(U) = \mathcal{G}_{A'c;\mathrm{apx};III}(U) + \mathcal{G}_{A'c;\mathrm{apx;sh};b}(U) \quad (13.58)$$

by introducing

$$\begin{array}{rcl} \mathcal{G}^+_{A'c;\mathrm{apx};III}(U) & = & -2c_*[\partial^0\partial U]\partial^0 U\big[\gamma_U^{-1}\partial^0 U\big] \\ & = & -2c_*\gamma_U^{-1}(1-\gamma_U^2)[\partial^0\partial U], \end{array} \quad (13.59)$$

together with

$$\mathcal{G}^+_{A'c;\mathrm{apx;sh};b}(U) \quad = \quad -2h[\partial^0\partial U]\partial^0 U\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big]. \quad (13.60)$$

In addition, we make the splitting

$$\mathcal{G}_{A'c;\mathrm{apx};III}(U) = \mathcal{G}_{A'c;\mathrm{apx};IV}(U) + \mathcal{G}_{A'c;\mathrm{apx;sh};c}(U) \quad (13.61)$$

by writing

$$\mathcal{G}^+_{A'c;\mathrm{apx};IV}(U) \quad = \quad -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial U], \quad (13.62)$$

together with

$$\mathcal{G}^+_{A'c;\mathrm{apx};\mathrm{sh};c}(U) \quad = \quad hc_*\gamma_U^{-1}(1-\gamma_U^2)\partial^+[\partial^0\partial U]. \tag{13.63}$$

We also recall the definition

$$\begin{array}{lcl}\mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] & = & -\mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{expl}}[V]\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\big] \\ & & -\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{lin};U;\mathrm{expl}}[V]T^+\big[\mathcal{Y}_{2;\mathrm{apx};\mathrm{expl}}(U)\big] \\ & & -\mathcal{Y}^+_{1;\mathrm{apx};\mathrm{expl}}(U)\mathcal{X}_{A;\mathrm{apx};\mathrm{expl}}(U)T^+\big[\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]\big].\end{array} \tag{13.64}$$

Substituting the relevant expressions from §11, we find

$$\begin{array}{lcl}\mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] & = & -S^+[\partial^0\partial V]\partial^0 U T^+\Big[c_*\gamma_U^{-1}\partial^0 U\Big] \\ & & -2[\partial^0\partial U]\partial^0 V T^+\Big[c_*\gamma_U^{-1}\partial^0 U\Big] \\ & & -2[\partial^0\partial U]\partial^0 U T^+\Big[\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\Big].\end{array} \tag{13.65}$$

We now make the decomposition

$$\mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] = \mathcal{G}^+_{A'c;\mathrm{lin};U;III}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V] \tag{13.66}$$

by introducing

$$\begin{array}{lcl}\mathcal{G}^+_{A'c;\mathrm{lin};U;III}[V] & = & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial V] \\ & & -2c_*\gamma_U^{-3}(1+\gamma_U^2)[\partial^0\partial U]\partial^0 U\partial^0 V \\ & & -2[\partial^0\partial U]\partial^0 U\Big[\gamma_U^{-2}M_U[V]\Big],\end{array} \tag{13.67}$$

together with

$$\begin{array}{lcl}\mathcal{G}^+_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V] & = & -hS^+[\partial^0\partial V]\partial^0 U\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big] \\ & & -2h[\partial^0\partial U]\partial^0 V\partial^+\big[c_*\gamma_U^{-1}\partial^0 U\big] \\ & & -2h[\partial^0\partial U]\partial^0 U\partial^+\Big[\gamma_U^{-2}M_U[V] + c_*\gamma_U^{-3}\partial^0 V\Big].\end{array} \tag{13.68}$$

We summarize our results by writing

$$\begin{array}{lcl}\mathcal{G}^+_{A'c;\mathrm{apx}}(U) & = & \mathcal{G}^+_{A'c;\mathrm{apx};IV}(U) \\ & = & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial U], \\ \mathcal{G}^+_{A'c;\mathrm{lin};U}[V] & = & \mathcal{G}^+_{A'c;\mathrm{lin};U;III}[V] \\ & = & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial V] \\ & & \quad -2c_*\gamma_U^{-3}(1+\gamma_U^2)[\partial^0\partial U]\partial^0 U\partial^0 V \\ & & \quad -2[\partial^0\partial U]\partial^0 U\Big[\gamma_U^{-2}M_U[V]\Big]\end{array} \tag{13.69}$$

and obtaining the following bounds.

Lemma 13.7. *Assume that (Hg) is satisfied and pick* $0 < \kappa < \frac{1}{12}$*. Then there exists a constant* $K > 0$ *so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the identities

$$\begin{aligned} \mathcal{G}^+_{A'c;\mathrm{apx};II}(U) &= \mathcal{G}^+_{A'c;\mathrm{apx}}(U) + \mathcal{G}^+_{A'c;\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}^+_{A'c;\mathrm{apx};\mathrm{sh};c}(U), \\ \mathcal{G}^+_{A'c;\mathrm{lin};U;II}[V] &= \mathcal{G}^+_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^+_{A;\mathrm{lin};U;\mathrm{sh};b}[V]. \end{aligned} \tag{13.70}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{aligned} \left\|\mathcal{G}^+_{A'c;\mathrm{apx};\mathrm{sh};b}(U)\right\|_{\ell^2_h} &\leq Kh && \leq K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\|\mathcal{G}^+_{A'c;\mathrm{apx};\mathrm{sh};c}(U)\right\|_{\ell^2_h} &\leq Kh \left\|\partial^+\partial^0\partial U\right\|_{\ell^2_h} && \leq K\overline{S}_{\mathrm{sh;full}}(U), \end{aligned} \tag{13.71}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{aligned} \left\|\mathcal{G}^+_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V]\right\|_{\ell^2_h} &\leq Kh\left\|V\right\|_{\ell^{2;3}_h} + Kh\left\|\partial^+\partial^+\partial^+U\right\|_{\ell^\infty_h}\left\|\partial^+V\right\|_{\ell^2_h} \\ &\leq Kh\overline{\mathcal{E}}_{\mathrm{sh};U}[V] \end{aligned} \tag{13.72}$$

Proof. Recalling Proposition 7.15, the bounds follow by inspection. □

13.6 Simplifications for $\mathcal{G}^+_{B'}$

We recall the definition

$$\mathcal{G}^+_{B';\mathrm{apx};II}(U) = \tfrac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)T^+h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx;expl}}(U)T^-\Big[\mathcal{X}_{B;\mathrm{apx;expl}}(U)\Big]\mathcal{F}^{\diamond-;+}_{\mathrm{apx}}(U) \tag{13.73}$$

Substituting the relevant expressions from §11, we find

$$\begin{aligned} \mathcal{G}^+_{B';\mathrm{apx};II}(U) &= [\partial^0\partial U]T^+h\textstyle\sum_{-;h}c_*\gamma_U^{-1}\partial^0U\Big[2\gamma_U^3[2\gamma_U^{-3}\partial^0\partial U]\Big] \\ &= 4[\partial^0\partial U]T^+h\textstyle\sum_{-;h}c_*\gamma_U^{-1}\partial^0U\Big[\partial^0\partial U\Big]. \end{aligned} \tag{13.74}$$

In view of Lemma 7.11, we introduce the expressions

$$\begin{aligned} \mathcal{G}^+_{B';\mathrm{apx};III}(U) &= 2c_*[\partial^0\partial U]T^+(1-\gamma_U), \\ \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};b}(U) &= \mathcal{G}^+_{B';\mathrm{apx};II}(U) - \mathcal{G}^+_{B';\mathrm{apx};III}(U). \end{aligned} \tag{13.75}$$

In addition, we make the splitting

$$\mathcal{G}^+_{B';\mathrm{apx};III}(U) = \mathcal{G}^+_{B';\mathrm{apx};IV}(U) + \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U) \tag{13.76}$$

by writing

$$\mathcal{G}^+_{B';\mathrm{apx};IV}(U) = c_*S^+[\partial^0\partial U](1-\gamma_U), \tag{13.77}$$

together with

$$\begin{aligned} \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U) &= -c_*h\partial^+[\partial^0\partial U]T^+(1-\gamma_U) \\ &\quad +c_*hS^+[\partial^0\partial U]\partial^+(1-\gamma_U). \end{aligned} \tag{13.78}$$

We also recall the definition

$$\begin{aligned} \mathcal{G}^+_{B';\mathrm{lin};U;II}[V] = \; & \tfrac{1}{2}\mathcal{Y}^+_{1;\mathrm{lin};U;\mathrm{expl}}[V]T^+h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx;expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx;expl}}(U)\big]\mathcal{F}^{\diamond-;+}_{\mathrm{apx}}(U) \\ & +\tfrac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)T^+h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{lin};U;\mathrm{expl}}[V]T^-\big[\mathcal{X}_{B;\mathrm{apx;expl}}(U)\big]\mathcal{F}^{\diamond-;+}_{\mathrm{apx}}(U) \\ & +\tfrac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)T^+h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx;expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{lin};U;\mathrm{expl}}[V]\big]\mathcal{F}^{\diamond-;+}_{\mathrm{apx}}(U) \\ & +\tfrac{1}{2}\mathcal{Y}^+_{1;\mathrm{apx;expl}}(U)T^+h\textstyle\sum_{-;h}\mathcal{Y}_{2;\mathrm{apx;expl}}(U)T^-\big[\mathcal{X}_{B;\mathrm{apx;expl}}(U)\big]\mathcal{F}^{\diamond-;+}_{\mathrm{lin};U}[V]. \end{aligned} \tag{13.79}$$

Substituting the relevant expressions from §11, we find

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] & = & \frac{1}{2}S^{+}[\partial^0\partial V]T^{+}h\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[2\gamma_U^3[2\gamma_U^{-3}\partial^0\partial U]\Big] \\ & & +[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\Big[2\gamma_U^3[2\gamma_U^{-3}\partial^0\partial U]\Big] \\ & & +[\partial^0\partial U]T^{+}h\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[(-6)\gamma_U\partial^0 U\partial^0 V[2\gamma_U^{-3}\partial^0\partial U]\Big] \\ & & +[\partial^0\partial U]T^{+}h\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\Big[2\gamma_U^3[6\gamma_U^{-5}\partial^0 U\partial^0\partial U\partial^0 V+2\gamma_U^{-3}\partial^0\partial V]\Big].\end{array} \tag{13.80}$$

A little algebra yields

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V] & = & 2S^{+}[\partial^0\partial V]T^{+}h\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U\partial^0\partial U \\ & & +4[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\partial^0\partial U \\ & & +4[\partial^0\partial U]T^{+}h\sum_{-;h}c_*\gamma_U^{-1}\partial^0 U[\partial^0\partial V].\end{array} \tag{13.81}$$

In view of Lemma's 7.11 and 7.10, we introduce the expressions

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V] & = & c_*S^{+}[\partial^0\partial V]T^{+}(1-\gamma_U) \\ & & +4[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}M_U[V]+c_*\gamma_U^{-3}\partial^0 V\Big]\partial^0\partial U \\ & & +4c_*[\partial^0\partial U]T^{+}\Big[\frac{1}{2}\gamma_U^{-1}\partial^0 U\partial^0 V-h\sum_{-;h}\big[\gamma_U^{-3}\partial^0\partial U\partial^0 V\big]\Big], \\ \mathcal{G}^{+}_{B';\mathrm{lin};U;\mathrm{sh};b}[V] & = & \mathcal{G}^{+}_{B';\mathrm{lin};U;II}[V]-\mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V].\end{array} \tag{13.82}$$

A short computation yields

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V] & = & c_*S^{+}[\partial^0\partial V]T^{+}(1-\gamma_U)+2c_*[\partial^0\partial U]T^{+}\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\ & & +4[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big].\end{array} \tag{13.83}$$

We now make the decomposition

$$\mathcal{G}^{+}_{B';\mathrm{lin};U;III}[V]=\mathcal{G}^{+}_{B';\mathrm{lin};U;IV}[V]+\mathcal{G}^{+}_{B';\mathrm{lin};U;\mathrm{sh};c}[V] \tag{13.84}$$

by writing

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;IV}[V] & = & c_*S^{+}[\partial^0\partial V](1-\gamma_U)+2c_*[\partial^0\partial U]\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\ & & +2S^{+}[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big],\end{array} \tag{13.85}$$

together with

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{lin};U;\mathrm{sh};c}[V] & = & c_*hS^{+}[\partial^0\partial V]\partial^{+}[1-\gamma_U]+2hc_*[\partial^0\partial U]\partial^{+}\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\ & & -2h\partial^{+}[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big].\end{array} \tag{13.86}$$

We summarize our results by writing

$$\begin{array}{lcl}\mathcal{G}^{+}_{B';\mathrm{apx}}(U) & = & \mathcal{G}^{+}_{B';\mathrm{apx};IV}(U) \\ & = & c_*S^{+}[\partial^0\partial U](1-\gamma_U), \\ \mathcal{G}^{+}_{B';\mathrm{lin};U}[V] & = & \mathcal{G}^{+}_{B';\mathrm{lin};U;IV}[V] \\ & = & c_*S^{+}[\partial^0\partial V](1-\gamma_U)+2c_*[\partial^0\partial U]\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\ & & +2S^{+}[\partial^0\partial U]T^{+}h\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big]\end{array} \tag{13.87}$$

and obtaining the following bounds.

Lemma 13.8. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(a) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the identities

$$\begin{array}{rcl} \mathcal{G}^+_{B';\mathrm{apx};II}(U) & = & \mathcal{G}^+_{B';\mathrm{apx}}(U) + \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};b}(U) + \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U), \\ \mathcal{G}^+_{B';\mathrm{lin};U;II}[V] & = & \mathcal{G}^+_{B';\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V]. \end{array} \tag{13.88}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{rcccl} \left\| \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};b}(U) \right\|_{\ell^2_h} & \leq & Kh & \leq & K\overline{S}_{\mathrm{sh;full}}(U), \\ \left\| \mathcal{G}^+_{B';\mathrm{apx};\mathrm{sh};c}(U) \right\|_{\ell^2_h} & \leq & Kh \left\| \partial^+ \partial^0 \partial U \right\|_{\ell^2_h} & \leq & K\overline{S}_{\mathrm{sh;full}}(U). \end{array} \tag{13.89}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{rcl} \left\| \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};b}[V] \right\|_{\ell^2_h} & \leq & Kh\big[\, \|\partial^+ V\|_{\ell^2_h} + \|\partial^+\partial^+ V\|_{\ell^2_h} \, \big] \\ & \leq & Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\| \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V] \right\|_{\ell^2_h} & \leq & Kh \, \|V\|_{\ell^{2;2}_h} + Kh \, \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h} \, \|V\|_{\ell^{2;2}_h} \\ & \leq & Kh\overline{\mathcal{E}}_{\mathrm{sh};U}(V). \end{array} \tag{13.90}$$

Proof. Recalling Lemma's 7.10 and 7.11, the bounds in (ii) and the first bound in (iii) follow by inspection. The final bound in (iii) follows from Proposition 7.15. □

13.7 Simplifications for $\mathcal{G}^+_{C'}$ and $\mathcal{G}^+_{D'}$

Arguing as in §12.5 we see that

$$\begin{array}{rcl} \mathcal{G}^+_{C;\mathrm{apx};II}(U) & = & -\mathcal{G}^+_{D;\mathrm{apx};II}(U), \\ \mathcal{G}^+_{C;\mathrm{lin};U;II}[V] & = & -\mathcal{G}^+_{D;\mathrm{lin};U;II}[V]. \end{array} \tag{13.91}$$

13.8 Intermediate total

We now define the total

$$\mathcal{G}^+_{\mathrm{apx};III}(U) \quad = \quad \mathcal{G}^+_{A'a;\mathrm{apx}}(U) + \mathcal{G}^+_{A'b;\mathrm{apx}}(U) + \mathcal{G}^+_{A'c;\mathrm{apx}}(U) + \mathcal{G}^+_{B';\mathrm{apx}}(U). \tag{13.92}$$

Substituting the relevant expressions from §13.3-13.6 we obtain

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx};III}(U) & = & 2\gamma_U^{-2}\partial^+\partial^0\partial U \\ & & \qquad \gamma_U^2\mathcal{E}^+_{\mathrm{tw;apx}}(U) - 2\gamma_U^{-2}\partial^+\partial^0\partial U + c_*\gamma_U^{-1}S^+[\partial^0\partial U] \\ & & \qquad -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial U] \\ & & \qquad c_*S^+[\partial^0\partial U](1-\gamma_U) \\ & = & c_*S^+[\partial^0\partial U] + \gamma_U^2\mathcal{E}^+_{\mathrm{tw;apx}}(U). \end{array} \tag{13.93}$$

In order to suppress the final term, we introduce the expressions

$$\begin{array}{rcl} \mathcal{G}^+_{\mathrm{apx};\mathrm{sh};d}(U) & = & \gamma_U^2\big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\big], \\ \mathcal{G}^+_{\mathrm{apx};\mathrm{rem};d}(U) & = & \gamma_U^2\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]. \end{array} \tag{13.94}$$

Moving on to the linear approximants, we define the function

$$\mathcal{G}^+_{\mathrm{lin};U;III}[V] \quad = \quad \mathcal{G}^+_{A'a;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U}[V]. \tag{13.95}$$

As a first step towards evaluating this expression, we substitute the relevant identities from §13.3-13.6 to compute

$$\begin{aligned}
\mathcal{G}^+_{A'a;\mathrm{lin};U}[V] + \mathcal{G}^+_{A'b;\mathrm{lin};U}[V] \quad = \quad & \gamma_U^2 \widetilde{M}_{U;E}[V] - 4\gamma_U^{-4}\partial^0 U[\partial^+\partial^0\partial U]\partial^0 V \\
& +2c_*\gamma_U^{-3}\partial^0 U[\partial^0\partial U]\partial^0 V + c_*\gamma_U^{-1} S^+[\partial^0\partial V] \\
& +\partial^+\big[M_U[V]\big] + 4\gamma_U^{-2}\partial^0 U[\partial^0\partial U]M_U[V] - \gamma_U^2\widetilde{M}_{U;E}[V] \\
& -2\partial^0 U\Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - 2\gamma_U^{-4}\partial^+\partial^0\partial U\Big]\partial^0 V \\
= \quad & 2c_*\gamma_U^{-3}\partial^0 U[\partial^0\partial U]\partial^0 V + c_*\gamma_U^{-1}S^+[\partial^0\partial V] \\
& +\partial^+\big[M_U[V]\big] + 4\gamma_U^{-2}\partial^0 U[\partial^0\partial U]M_U[V] \\
& -2\partial^0 U\Big[\mathcal{E}^+_{\mathrm{tw;apx}}(U)\Big]\partial^0 V.
\end{aligned} \tag{13.96}$$

In a similar fashion, we find

$$\begin{aligned}
\mathcal{G}^+_{A'c;\mathrm{lin};U}[V] + \mathcal{G}^+_{B';\mathrm{lin};U}[V] \quad = \quad & -c_*\gamma_U^{-1}(1-\gamma_U^2)S^+[\partial^0\partial V] \\
& -2c_*\gamma_U^{-3}(1+\gamma_U^2)[\partial^0\partial U]\partial^0 U\partial^0 V \\
& -2[\partial^0\partial U]\partial^0 U\Big[\gamma_U^{-2}M_U[V]\Big] \\
& +c_*S^+[\partial^0\partial V](1-\gamma_U) + 2c_*[\partial^0\partial U]\big[\gamma_U^{-1}\partial^0 U\partial^0 V\big] \\
& +2S^+[\partial^0\partial U]T^+h\textstyle\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big] \\
= \quad & c_*S^+[\partial^0\partial V] - c_*\gamma_U^{-1}S^+[\partial^0\partial V] - 2c_*\gamma_U^{-3}\partial^0 U[\partial^0\partial U]\partial^0 V \\
& -2\gamma_U^{-2}\partial^0 U[\partial^0\partial U]M_U[V] \\
& +2S^+[\partial^0\partial U]T^+h\textstyle\sum_{-;h}\Big[\gamma_U^{-2}\partial^0\partial U M_U[V]\Big].
\end{aligned} \tag{13.97}$$

In particular, we see that

$$\begin{aligned}
\mathcal{G}^+_{\mathrm{apx;lin};U;III}[V] \quad = \quad & c_*S^+[\partial^0\partial V] + \partial^+\big[M_U[V]\big] + 2\gamma_U^{-2}\partial^0 U[\partial^0\partial U]M_U[V] \\
& +2S^+[\partial^0\partial U]T^+h\textstyle\sum_{-;h}\big[\gamma_U^{-2}\partial^0\partial U M_U[V]\big] \\
& -2\partial^0 U[\mathcal{E}^+_{\mathrm{tw;apx}}(U)]\partial^0 V.
\end{aligned} \tag{13.98}$$

Comparing this expression with (13.1), we set out to suppress the final term by introducing the functions

$$\begin{aligned}
\mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};d}[V] \quad &= \quad -2\partial^0 U\big[\mathcal{E}^+_{\mathrm{tw;apx}}(U) - \partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\big]\partial^0 V, \\
\mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};d}[V] \quad &= \quad -2\partial^0 U\partial^+[\mathcal{E}_{\mathrm{tw}}(U)]\partial^0 V.
\end{aligned} \tag{13.99}$$

Lemma 13.9. *Assume that (Hg) is satisfied, pick* $0 < \kappa < \frac{1}{12}$ *and recall the definitions (13.1). There exists a constant* $K > 0$ *so that the following properties hold true.*

(i) For every $h > 0$, $U \in \Omega_{h;\kappa}$ *and* $V \in \ell^2_h$, *we have the identities*

$$\begin{aligned}
\mathcal{G}^+_{\mathrm{apx};III}(U) \quad &= \quad \mathcal{G}^+_{\mathrm{apx}}(U) + \mathcal{G}^+_{\mathrm{apx;sh};d}(U) + \mathcal{G}^+_{\mathrm{apx;rem};d}(U), \\
\mathcal{G}^+_{\mathrm{lin};U;III}[V] \quad &= \quad \mathcal{G}^+_{\mathrm{lin};U}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};d}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};d}[V].
\end{aligned} \tag{13.100}$$

(ii) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{lclcl} \left\| \mathcal{G}^+_{\mathrm{apx;sh};d}(U) \right\|_{\ell^2_h} & \leq & Kh & \leq & K\overline{\mathcal{S}}_{\mathrm{sh;full}}(U), \\ \left\| \mathcal{G}^+_{\mathrm{apx;rem};d}(U) \right\|_{\ell^2_h} & \leq & K \left\| \partial^+[\mathcal{E}_{\mathrm{tw}}(U)] \right\|_{\ell^2_h} & \leq & K\overline{\mathcal{S}}_{\mathrm{rem;full}}(U). \end{array} \tag{13.101}$$

(iii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the bounds

$$\begin{array}{lcl} \left\| \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};d}[V] \right\|_{\ell^2_h} & \leq & Kh \left\| \partial^+ V \right\|_{\ell^2_h} \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \left\| \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};d}[V] \right\|_{\ell^2_h} & \leq & K \left\| \partial^+[\mathcal{E}_{\mathrm{tw}}(U)] \right\|_{\ell^\infty_h} \left\| \partial^+ V \right\|_{\ell^2_h} \\ & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{13.102}$$

Proof. Recalling (7.64), the bounds follow by inspection. □

13.9 Summary

We define the final error terms

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{apx;rem}}(U) & = & \mathcal{G}^+_{\mathrm{apx;rem};a}(U) + \mathcal{G}^+_{\mathrm{apx;rem};d}(U), \\ \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem}}[V] & = & \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};a}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem};d}[V], \end{array} \tag{13.103}$$

together with

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{apx;sh}}(U) & = & \mathcal{G}^+_{\mathrm{apx;sh};a}(U) + \mathcal{G}^+_{A'b;\mathrm{apx;sh};b}(U) + \mathcal{G}^+_{A'c;\mathrm{apx;sh};b}(U) + \mathcal{G}^+_{A'c;\mathrm{apx;sh};c}(U) \\ & & \quad + \mathcal{G}^+_{B';\mathrm{apx;sh};b}(U) + \mathcal{G}^+_{B';\mathrm{apx;sh};c}(U) + \mathcal{G}^+_{\mathrm{apx;sh};d}(U), \\ \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh}}[V] & = & \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};a}[V] + \mathcal{G}^+_{A'c;\mathrm{lin};U;\mathrm{sh};b}[V] \\ & & \quad + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};b}[V] + \mathcal{G}^+_{B';\mathrm{lin};U;\mathrm{sh};c}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh};d}[V]. \end{array} \tag{13.104}$$

The computations above show that

$$\begin{array}{lcl} \mathcal{G}^+_{A'a;\mathrm{apx};I}(U) + \mathcal{G}^+_{\mathrm{low;apx};I}(U) & = & \mathcal{G}^+_{\mathrm{apx}}(U) + \mathcal{G}^+_{\mathrm{apx;sh}}(U) + \mathcal{G}^+_{\mathrm{apx;rem}}(U), \\ \mathcal{G}^+_{A'a;\mathrm{lin};U;I}[V] + \mathcal{G}^+_{\mathrm{low;lin};U;I}[V] & = & \mathcal{G}^+_{\mathrm{lin};U}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh}}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem}}[V]. \end{array} \tag{13.105}$$

Recalling the definitions (13.12) and (13.25), this implies that

$$\mathcal{G}^+_{\mathrm{nl};U}(V) = \mathcal{G}^+_{A'a;\mathrm{nl};U;I}(V) + \mathcal{G}^+_{\mathrm{low;nl};U;I}(V) + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem}}[V] + \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh}}[V]. \tag{13.106}$$

Corollary 13.10. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that the following properties hold true.*

(i) For every $h > 0$ and $U \in \Omega_{h;\kappa}$ we have the bounds

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{apx;sh}}(U) & \leq & K\overline{\mathcal{S}}_{\mathrm{sh;full}}(U), \\ \mathcal{G}^+_{\mathrm{apx;rem}}(U) & \leq & K\overline{\mathcal{S}}_{\mathrm{rem;full}}(U). \end{array} \tag{13.107}$$

(ii) For every $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$, we have the bounds

$$\begin{array}{lcl} \mathcal{G}^+_{\mathrm{lin};U;\mathrm{sh}}[V] & \leq & K\overline{\mathcal{E}}_{\mathrm{sh};U}(V), \\ \mathcal{G}^+_{\mathrm{lin};U;\mathrm{rem}}[V] & \leq & K\overline{\mathcal{E}}_{\mathrm{rem};U}(V). \end{array} \tag{13.108}$$

Proof. These estimates follow directly from Lemma's 13.4-13.9. □

Lemma 13.11. *Assume that (Hg) is satisfied and pick $0 < \kappa < \frac{1}{12}$. Then there exists a constant $K > 0$ so that for any $h > 0$, $U \in \Omega_{h;\kappa}$ and $V \in \ell^2_h$ we have the estimate*

$$\Big\|\partial^+\Big[\mathcal{G}_{\mathrm{lin};U}[V]\Big] - \mathcal{G}_{\mathrm{lin};U}[\partial^+ V]\Big\|_{\ell^2_h} \quad \leq \quad K\big[1 + \|\partial^+\partial^+\partial^+ U\|_{\ell^\infty_h} + \|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}\,\big]\,\|V\|_{\ell^{2;2}_h}\,. \tag{13.109}$$

Proof. Systematically applying (4.5), we compute

$$\begin{array}{rcl} \partial^+\big[\mathcal{G}_{\mathrm{lin};U}[V]\big] & = & c_* S^+[\partial^0\partial V] + \partial^+\big[M_U[V]\big] \\ & & +2S^+[\partial^0 U]T^+ h\sum_{-;h}\gamma_U^{-2}[\partial^0\partial U]M_U[V] \\ & & +2\partial^0 U h\sum_{-;h}\partial^+\big[\gamma_U^{-2}\big]T^+\Big[\partial^0\partial U M_U[V]\Big] \\ & & +2\partial^0 U h\sum_{-;h}\gamma_U^{-2}\partial^+\Big[\partial^0\partial U\Big]T^+\big[M_U[V]\big] \\ & & +2\partial^0 U h\sum_{-;h}\gamma_U^{-2}[\partial^0\partial U]\partial^+\Big[M_U[V]\Big]. \end{array} \tag{13.110}$$

On the other hand, a direct substitution yields

$$\mathcal{G}_{\mathrm{lin};U}\big[\partial^+ V\big] \quad = \quad c_* S^+[\partial^0\partial V] + M_U\big[\partial^+ V\big] + 2\partial^0 U h\sum_{-;h}\gamma_U^{-2}[\partial^0\partial U]M_U\Big[\partial^+ V\Big]. \tag{13.111}$$

Comparing these two expressions, we obtain the bound

$$\begin{array}{rcl} \Big\|\partial^+\Big[\mathcal{G}_{\mathrm{lin};U}[V]\Big] - \mathcal{G}_{\mathrm{lin};U}[\partial^+ V]\Big\|_{\ell^2_h} & \leq & C_1'\,\big\|\partial^+\big[M_U[V]\big] - M_U[\partial^+ V]\,\big\|_{\ell^2_h} + C_1'\,\|M_U[V]\|_{\ell^2_h} \\ & & +C_1'\,\|\partial^+\partial^+\partial^+ U\|_{\ell^2_h}\,\|M_U[V]\|_{\ell^2_h}\,. \end{array} \tag{13.112}$$

The desired estimate now follows from (7.82). □

Proof of Proposition 13.1. In view of the expression (13.106), the statements follow from Lemma's 13.2, 13.3 and 13.11 together with Corollary 13.10 . □

14 The full linear operator

In this section we study the linear operators $\mathcal{L}_h : H^1 \to L^2$ that act as

$$\mathcal{L}_h v = -c_* v' + \mathcal{G}_{\mathrm{lin};\Psi_*}[v]. \tag{14.1}$$

Strictly speaking, the equation $\mathcal{L}_h v = f$ with $v \in H^1$ and $f \in L^2$ needs to be interpreted as the statement that

$$\mathcal{G}_{\mathrm{lin};\mathrm{ev}_\vartheta\Psi_*}[\mathrm{ev}_\vartheta v] = \mathrm{ev}_\vartheta[c_* v' + f] \tag{14.2}$$

for almost all $\vartheta \in [0,h]$. We remark that the left-hand side is continuous in ℓ^2_h as a function of ϑ as a consequence of (5.13) and the continuity of the translation operator on H^1. Throughout the sequel we simply use the notation (14.1) and keep this interpretation in mind.

Our main result provides a quasi-inverse for $\mathcal{L}_h$ that bifurcates off a twisted version of the operator $\mathcal{L}_{\mathrm{cmp}}$ discussed in §3. This accounts for the presence in (iii) of the integral transform $\mathcal{T}_*$ that was defined in (3.14).

The crucial point in (i) is that we also obtain control on the L^2-norm of the second discrete derivative of v. This is slightly weaker than full H^2-control of v, but turns out to be sufficient to bound our nonlinear terms. In addition, item (ii) allows us to control an extra discrete derivative of v provided one is available for f.

Proposition 14.1. *Suppose that (Hg) and $(H\Phi_*)$ are satisfied. Then there exist constants $K > 0$ and $h_0 > 0$ together with linear maps*

$$\beta_h^* : L^2 \to \mathbb{R}, \qquad \mathcal{V}_h^* : L^2 \to H^1, \tag{14.3}$$

defined for all $h \in (0, h_0)$, so that the following properties hold true.

(i) For all $f \in L^2$ and $0 < h < h_0$, we have the bound

$$|\beta_h^* f| + \|\mathcal{V}_h^* f\|_{H^1} + \left\|\partial_h^+ \partial_h^+ \mathcal{V}_h^* f\right\|_{L^2} \leq K \|f\|_{L^2}. \tag{14.4}$$

(ii) For all $f \in L^2$ and $0 < h < h_0$, we have the bound

$$\left\|\partial_h^+ \mathcal{V}_h^* f\right\|_{H^1} + \left\|\partial_h^+ \partial_h^+ \partial_h^+ \mathcal{V}_h^* f\right\|_{L^2} \leq K\big[\, \|f\|_{L^2} + \left\|\partial_h^+ f\right\|_{L^2} \big]. \tag{14.5}$$

(iii) For all $f \in L^2$ and $0 < h < h_0$, the pair

$$(\beta, v) = \big(\beta_h^* f, \mathcal{V}_h^* f\big) \in \mathbb{R} \times H^1 \tag{14.6}$$

is the unique solution to the problem

$$\mathcal{L}_h v = f + \beta \Psi_*' \tag{14.7}$$

that satisfies the normalization condition

$$\langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* v \rangle_{L^2} = 0. \tag{14.8}$$

(iv) We have $\beta_h^* \Psi_*' = -1$ for all $h \in (0, h_0)$.

Our strategy is to apply the spirit of the ideas in [4] to our present more convoluted setting. In particular, in §14.1 we analyze the structure of the terms contained in the definition $\mathcal{L}_h$ and its adjoint and provide a decomposition that isolates the crucial expressions. In §14.2 we show how Proposition 14.1 can be established provided that a technical lower bound related to the sets $\{[\mathcal{L}_h - \delta]v\}_{\|v\|_{H^1}=1}$ can be obtained. We set out to derive this bound in §14.3, using a generalized version of the arguments in [4].

14.1 Structure

For any $v \in L^2$ and $h > 0$, we introduce the function

$$M_h[v] \;=\; -c_* \gamma_{\Psi_*}^{-1} \partial_h^0 v + 8\gamma_{\Psi_*}^{-4} \partial_h^0 \Psi_* \big[[\partial^0 \partial]_h \Psi_*\big] \partial_h^0 v + 2\gamma_{\Psi_*}^{-2} [\partial^0 \partial]_h v + \gamma_{\Psi_*}^2 g'(\Psi_*) v. \tag{14.9}$$

Here we recall the definition

$$\gamma_{\Psi_*} = \sqrt{1 - (\partial_h^0 \Psi_*)^2}, \tag{14.10}$$

which should not be confused with

$$\gamma_* = \sqrt{1 - (\Psi_*')^2}. \tag{14.11}$$

Upon writing

$$M_h^{\mathrm{adj}}[w] \;=\; c_* \partial_h^0 [\gamma_{\Psi_*}^{-1} w] - \partial_h^0 \Big[8\gamma_{\Psi_*}^{-4} \partial_h^0 \Psi_* \big[[\partial^0 \partial]_h \Psi_*\big] w \Big] + [\partial^0 \partial]_h \Big[2\gamma_{\Psi_*}^{-2} w \Big] + \gamma_{\Psi_*}^2 g'(\Psi_*) w, \tag{14.12}$$

one readily checks that for any pair $(v, w) \in L^2 \times L^2$ we have

$$\langle M_h[v], w \rangle_{L^2} \quad = \quad \langle v, M_h^{\mathrm{adj}}[w] \rangle_{L^2}. \tag{14.13}$$

From now on, we simply write $\partial^{\pm}$ and ∂^0 for the discrete derivatives if the value for h is clear from the context.

With this notation in hand, the operator $\mathcal{L}_h$ can be written as

$$\mathcal{L}_h v \quad = \quad -c_* v' + c_* \partial^0 v + M_h[v] + 2\partial^0 \Psi_* h \textstyle\sum_{-;h} \gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] M_h[v]. \tag{14.14}$$

We now introduce the formal adjoint $\mathcal{L}_h^{\mathrm{adj}} : H^1 \to L^2$ that acts as

$$\mathcal{L}_h^{\mathrm{adj}} w \quad = \quad c_* w' - c_* \partial^0 w + M_h^{\mathrm{adj}}[w] + M_h^{\mathrm{adj}} \Big[\gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] h \textstyle\sum_{+;h} 2 w \partial^0 \Psi_* \Big]. \tag{14.15}$$

Indeed, using the computation

$$\begin{array}{lcl} \langle 2\partial^0 \Psi_* h \sum_{-;h} \Big[\gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] M_h[v] \Big], w \rangle_{L^2} & = & \langle \gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] M_h[v], h \sum_{+;h} 2 w \partial^0 \Psi_* \rangle_{L^2} \\ & = & \langle M_h[v], \gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] h \sum_{+;h} 2 w \partial^0 \Psi_* \rangle_{L^2} \\ & = & \langle v, M_h^{\mathrm{adj}} \Big[\gamma_{\Psi_*}^{-2} [\partial^0 \partial \Psi_*] h \sum_{+;h} 2 w \partial^0 \Psi_* \Big] \rangle_{L^2}, \end{array} \tag{14.16}$$

one can verify that

$$\langle \mathcal{L}_h v, w \rangle_{L^2} = \langle v, \mathcal{L}_h^{\mathrm{adj}} w \rangle_{L^2} \tag{14.17}$$

for any pair $(v, w) \in H^1 \times H^1$.

Our goal here is to establish the following structural decomposition of $\mathcal{L}_h$ and $\mathcal{L}_h^{\mathrm{adj}}$. Roughly speaking, this decomposition isolates all the terms that cannot be exponentially localized. In addition, it explicitly describes how the formal $h \downarrow 0$ limit can be related to twisted versions of the operators $\mathcal{L}_{\mathrm{cmp}}$ and $\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}$ that were discussed in §3.

Proposition 14.2. *Suppose that* (Hg) *and* $(H\Phi_*)$ *are satisfied and pick* $\eta > 0$ *sufficiently small. There exists a constant* $K > 0$ *together with linear maps*

$$L_{c;h} : H^1 \to L^2, \qquad\qquad L_{c;h}^{\mathrm{adj}} : H^1 \to L^2, \tag{14.18}$$

defined for all $0 < h < 1$*, so that the following properties hold true.*

(i) For every $0 < h < 1$ *the identities*

$$\begin{array}{lcl} \mathcal{L}_h v & = & -c_* v' + 2\gamma_{\Psi_*}^{-2} \partial^0 \partial v + \gamma_{\Psi_*}^2 g'(\Psi_*) v + L_{c;h}[v], \\ \mathcal{L}_h^{\mathrm{adj}} w & = & c_* w' + 2\gamma_{\Psi_*}^{-2} \partial^0 \partial w + \gamma_{\Psi_*}^2 g'(\Psi_*) w + L_{c;h}^{\mathrm{adj}}[w] \end{array} \tag{14.19}$$

hold for all $v \in H^1$ *and* $w \in H^1$*.*

(ii) For any $0 < h < 1$ *we have the bounds*

$$\begin{array}{lcl} \left\| L_{c;h}[v] \right\|_{L^2} & \leq & K \left\| v \right\|_{H^1}, \\ \left\| L_{c;h}^{\mathrm{adj}}[w] \right\|_{L^2} & \leq & K \left\| w \right\|_{H^1} \end{array} \tag{14.20}$$

for all $v \in H^1$ *and* $w \in H^1$*.*

(iii) *For every* $0 < h < 1$ *we have the bounds*

$$\begin{array}{lll} \left\| e_{2\eta}^{-1} L_{c;h}[v] \right\|_{L^2_\eta} & \leq & K\big[\, \|v\|_{L^2_\eta} + \|\partial^+ v\|_{L^2_\eta} \,\big], \\ \left\| e_{2\eta}^{-1} L^{\mathrm{adj}}_{c;h}[w] \right\|_{L^2_\eta} & \leq & K\big[\, \|w\|_{L^2_\eta} + \|\partial^+ w\|_{L^2_\eta} \,\big] \end{array} \tag{14.21}$$

for all $v \in H^1$ *and* $w \in H^1$.

(iv) *Consider two sequences* $\{(h_j, v_j)\}$ *and* $\{(h_j, w_j)\}$ *that both satisfy the condition (hSeq) introduced in §5.3. Then there exist two pairs* $(V_*, W_*) \in H^2 \times H^2$ *and* $(F_*, F_*^{\mathrm{adj}}) \in L^2 \times L^2$ *for which the weak convergences*

$$(v_j, \mathcal{L}_{h_j}[v_j]) \rightharpoonup (V_*, F_*) \in H^1 \times L^2, \qquad (w_j, \mathcal{L}^{\mathrm{adj}}_{h_j}[w_j]) \rightharpoonup (W_*, F_*^{\mathrm{adj}}) \in H^1 \times L^2 \tag{14.22}$$

both hold, possibly after passing to a further subsequence. In addition, we have the identity

$$\mathcal{L}_{\mathrm{cmp}} V_* \;=\; \mathcal{T}_* F_* \tag{14.23}$$

and we have

$$W_* = \mathcal{T}_*^{\mathrm{adj}} H_* \tag{14.24}$$

for some $H_* \in H^2$ *that satisfies*

$$\mathcal{L}^{\mathrm{adj}}_{\mathrm{cmp}}[H_*] = F_*^{\mathrm{adj}}. \tag{14.25}$$

Decomposition for $\mathcal{L}_h$

We set out to identify all the terms in $\mathcal{L}_h$ that can be exponentially localized in the sense of (14.20). We start by analyzing the function $M_h[v]$, which can be treated by direct inspection.

Lemma 14.3. *Suppose that (Hg) and (HΦ_*) are satisfied and pick $\eta > 0$ sufficiently small. There exists a constant $K > 0$ together with functions $\alpha_{0;h} \in H^1$, defined for $0 < h < 1$, so that the following properties hold.*

(i) *For every* $0 < h < 1$ *and* $\tau \in \mathbb{R}$ *we have*

$$|\alpha_{0;h}(\tau)| \leq K e_{2\eta}(\tau). \tag{14.26}$$

(ii) *For any* $0 < h < 1$ *and* $v \in H^1$ *we have the identity*

$$c_* \partial^0 v + M_h[v] \;=\; 2\gamma_{\Psi_*}^{-2} \partial^0 \partial v + \gamma_{\Psi_*}^2 g'(\Psi_*) v + \alpha_{0;h} \partial^0 v. \tag{14.27}$$

(iii) *For any sequence* $\{(h_j, v_j)\}$ *that satisfies (hSeq), there exists* $V_* \in L^2$ *for which the weak convergences*

$$v_j \rightharpoonup V_*, \qquad M_{h_j}[v_j] \rightharpoonup \gamma_*^2 \mathcal{L}_{\mathrm{cmp}}[V_*] \in L^2 \tag{14.28}$$

both hold as $j \to \infty$, *possibly after passing to a subsequence.*

Proof. Writing

$$\alpha_{0;h} = c_*(1 - \gamma_{\Psi_*}^{-1}) + 8\gamma_{\Psi_*}^{-4} \partial^0 \Psi_* [\partial^0 \partial \Psi_*], \tag{14.29}$$

item (ii) follows by inspection. Item (i) follows from the exponential bounds (3.4) together with an application of the Lipschitz bound (4.23) with $U^{(1)} = 0$ and $\gamma_{U^{(1)}} = 1$.

Turning to (iii), we may exploit the fact that $\Psi_* \in H^4$ to reason as in the proof of Proposition 7.14 and obtain the strong limits

$$\gamma_{\Psi_*}^{-2} \to \gamma_*^{-2} \in H^1, \qquad \gamma_{\Psi_*}^2 g'(\Psi_*) \to \gamma_*^2 g'(\Psi_*) \in H^1, \tag{14.30}$$

together with

$$\alpha_{0;h_j} \to c_*(1-\gamma_*^{-1}) + 2\gamma_*^{-4}\Psi_*'\Psi_*'' \in H^1. \tag{14.31}$$

In particular, we may apply Lemma's 5.10 and 5.11 to obtain the weak convergence

$$M_{h_j}[v_j] \quad \rightharpoonup \quad -c_*\gamma_*^{-1}V_*' + 4\gamma_*^{-4}\Psi_*'\Psi_*''V_*' + \gamma_*^{-2}V_*'' + \gamma_*^2 g'(\Psi_*)V_* \in L^2. \tag{14.32}$$

Inspecting the definition (3.8) yields (iii). $\square$

It is convenient to introduce the notation

$$\Omega_h[v] \quad = \quad h\textstyle\sum_{-;h}\gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]M_h[v], \tag{14.33}$$

which in view of (14.27) allows us to obtain the expression (14.19) for $\mathcal{L}_h$ by writing

$$L_{c;h}[v] = \alpha_{0;h}\partial^0 v + 2[\partial^0\Psi_*]\Omega_h[v]. \tag{14.34}$$

Lemma 14.4. *Suppose that (Hg) and (HΦ_*) are satisfied and pick $\eta > 0$ sufficiently small. There exists a constant $K > 0$ so that the following properties hold.*

(i) For any $v \in H^1$ and $0 < h < 1$, we have the estimate

$$\|\Omega_h[v]\|_{L^2_\eta} \quad \le \quad K\Big[\|v_j\|_{L^2_\eta} + \|\partial^+ v_j\|_{L^2_\eta}\Big]. \tag{14.35}$$

(ii) For any sequence $\{(h_j, v_j)\}$ that satisfies (hSeq), there exists $V_* \in L^2$ for which the weak convergences

$$v_j \rightharpoonup V_*, \qquad\qquad 2[\partial^0\Psi_*]\Omega_{h_j}[v_j] \rightharpoonup \Psi_*'\int_- \Psi_*''\mathcal{L}_{\mathrm{cmp}}V_* \in L^2 \tag{14.36}$$

both hold as $j \to \infty$, possibly after passing to a subsequence.

Proof. We make the splitting $\Omega_h[v] = \Omega_{A;h}[v] + \Omega_{B;h}[v]$ by introducing the notation

$$\begin{aligned} \Omega_{A;h}[v] \quad &= \quad h\textstyle\sum_{-;h}\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\Big[M_h[v] - 2\gamma_{\Psi_*}^{-2}\partial^0\partial v\Big], \\ \Omega_{B;h}[v] \quad &= \quad 2h\textstyle\sum_{-;h}\gamma_{\Psi_*}^{-4}[\partial^0\partial\Psi_*]\partial^0\partial v. \end{aligned} \tag{14.37}$$

Applying Lemma 5.9 and inspecting (14.27), we see that

$$\begin{aligned} \|\Omega_{A;h}[v]\|_{L^2_\eta} \quad &\le \quad C_1'\left\|M_h[v] - 2\gamma_{\Psi_*}^{-2}\partial^0\partial v\right\|_{L^2_\eta} \\ &\le \quad C_2'\Big[\|v_j\|_{L^2_\eta} + \|\partial^+ v_j\|_{L^2_\eta}\Big]. \end{aligned} \tag{14.38}$$

Applying the summation-by-parts identity (4.13), we compute

$$\begin{aligned} \Omega_{B;h}[v] \quad &= \quad h\textstyle\sum_{-;h}\gamma_{\Psi_*}^{-4}[\partial^0\partial\Psi_*]\partial^+\partial^- v \\ &= \quad T^-\Big[\gamma_{\Psi_*}^{-4}\partial^0\partial\Psi_*\Big]\partial^- v \\ &\qquad - \textstyle\sum_{-;h}\partial^- v\partial^-\Big[\gamma_{\Psi_*}^{-4}\partial^0\partial\Psi_*\Big]. \end{aligned} \tag{14.39}$$

Item (i) now follows from a second application of Lemma 5.9.

To obtain (ii), we set out to apply Lemma 5.11 with $f_j = M_{h_j}[v_j]$, $\alpha_{2;j} = \gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*$ and $\alpha_{1;j} = 2\partial^0\Psi_*$. Exploiting the fact that $\Psi_* \in H^4$, we may reason as in the proof of Proposition 7.14 to obtain the strong limits

$$\alpha_{1;j} \to 2\Psi_*' \in H^1, \qquad \alpha_{2;j} \to \frac{1}{2}\gamma_*^{-2}\Psi_*'' \in H^1. \tag{14.40}$$

Item (iii) of Lemma 14.3 implies that

$$f_* = \gamma_*^2\mathcal{L}_{\mathrm{cmp}}[V_*], \tag{14.41}$$

from which the desired weak limit follows. □

Decomposition for $\mathcal{L}_h^{\mathrm{adj}}$

We set out to here to mimic the procedure above for $\mathcal{L}_h^{\mathrm{adj}}$, which has a more convoluted structure. Special care needs to be taken to handle the fact that M_h^{adj} acts on a discrete sum. The identities (4.11) play a crucial role here.

Lemma 14.5. *Suppose that (Hg) and (HΦ_*) are satisfied and pick $\eta > 0$ sufficiently small. There exists a constant $K > 0$ together with a set of functions*

$$(\alpha_{0;h}, \alpha_{0s;h}, \alpha_{+;h}, \alpha_{-;h}) \in H^1 \times H^1 \times H^1 \times H^1, \tag{14.42}$$

defined for $0 < h < 1$, so that the following properties hold.

(i) For every $0 < h < 1$ and $\tau \in \mathbb{R}$ we have

$$|\alpha_{0;h}(\tau)| + |\alpha_{0s;h}(\tau)| + |\alpha_{-;h}(\tau)| + |\alpha_{+;h}(\tau)| \leq Ke_{2\eta}(\tau). \tag{14.43}$$

(ii) For any $0 < h < 1$ and $w \in H^1$ we have the identity

$$\begin{aligned} -c_*\partial^0 w + M_h^{\mathrm{adj}}[w] \;\; &= \;\; 2\gamma_{\Psi_*}^{-2}\partial^0\partial w + \gamma_{\Psi_*}^2 g'(\Psi_*)w \\ &\quad + \alpha_{0;h}w + \alpha_{0s;h}T^+ w + \alpha_{+;h}\partial^+ w + \alpha_{-;h}\partial^- w. \end{aligned} \tag{14.44}$$

(iii) For any sequence $\{(h_j, w_j)\}$ that satisfies (hSeq), there exists $W_* \in L^2$ for which the weak convergences

$$w_j \rightharpoonup W_*, \qquad M_{h_j}^{\mathrm{adj}}[w_j] \rightharpoonup \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}[\gamma_*^2 W_*] \in L^2 \tag{14.45}$$

both hold as $j \to \infty$, possibly after passing to a subsequence.

Proof. Applying (4.5) and (4.6), we obtain

$$\begin{aligned} M_h^{\mathrm{adj}}[w] \;\; &= \;\; c_*\partial^0[\gamma_{\Psi_*}^{-1}]T^+[w] + c_* T^-[\gamma_{\Psi_*}^{-1}]\partial^0 w \\ &\quad -\partial^0\Big[8\gamma_{\Psi_*}^{-4}[\partial^0\Psi_*]\partial^0\partial\Psi_*\Big]T^+ w - T^-\Big[8\gamma_{\Psi_*}^{-4}\partial^0\Psi_*\partial^0\partial\Psi_*\Big]\partial^0 w \\ &\quad +2[\partial^0\partial\gamma_{\Psi_*}^{-2}]w + 2\gamma_{\Psi_*}^{-2}\partial^0\partial w + \partial^+[\gamma_{\Psi_*}^{-2}]\partial^+ w + \partial^-[\gamma_{\Psi_*}^{-2}]\partial^- w \\ &\quad +\gamma_{\Psi_*}^2 g'(\Psi_*)w, \end{aligned} \tag{14.46}$$

from which (i) and (ii) can be read off.

Turning to (iii), we note first that the identity

$$T^+ w_j = w_j + h_j\partial^+ w_j \tag{14.47}$$

shows that also $T^+ w_j \rightharpoonup W_* \in L^2$. Applying Lemma's 5.10 and 5.11 to the representation (14.46), we obtain the weak limit

$$\begin{array}{lcl} M^{\mathrm{adj}}_{h_j}[w_j] & \rightharpoonup & c_*[\gamma_{\Psi_*}^{-1}]'W_* + c_*[\gamma_{\Psi_*}^{-1}]W_*' \\ & & -\Big[4\gamma_*^{-4}\Psi_*'\Psi_*''\Big]'W_* - \Big[4\gamma_*^{-4}\Psi_*'\Psi_*''\Big]W_*' \\ & & +[\gamma_*^{-2}]''W_* + \gamma_*^{-2}W_*'' + 2[\gamma_*^{-2}]'W_*' \\ & & +\gamma_*^2 g'(\Psi_*)W_* \\ & = & c_*\partial_\tau[\gamma_*^{-1}W_*] - \partial_\tau\Big[4\gamma_*^{-4}\Psi_*'\Psi_*''W_*\Big] + \partial_{\tau\tau}\Big[\gamma_*^{-2}W_*\Big] + \gamma_*^2 g'(\Psi_*)W_*. \end{array} \tag{14.48}$$

Inspecting the definition (3.9) now yields the result. □

It is convenient to introduce the notation

$$\Omega_h^{\mathrm{adj}}[w] = h\sum_{+;h} 2w\partial^0\Psi_*, \tag{14.49}$$

which in view of (14.44) allows us to obtain the expression (14.19) for $\mathcal{L}_h^{\mathrm{adj}}$ by writing

$$\begin{array}{lcl} L^{\mathrm{adj}}_{c;h}[w] & = & \alpha_0 w + \alpha_{0,s}T^+ w + \alpha_+\partial^+ w + \alpha_-\partial^- w \\ & & \quad + M_h^{\mathrm{adj}}\Big[\gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\Omega_h^{\mathrm{adj}}[w]\Big]. \end{array} \tag{14.50}$$

Lemma 14.6. *Suppose that (Hg) and (HΦ_*) are satisfied and pick $\eta > 0$ sufficiently small. There exists a constant $K > 0$ together with a set of functions*

$$(\tilde{\alpha}_{0;h}, \tilde{\alpha}_{0s;h}, \tilde{\alpha}_{+;h}, \tilde{\alpha}_{\omega;h}, \tilde{\alpha}_{\omega s;h}) \in H^1 \times H^1 \times H^1 \times H^1 \times H^1, \tag{14.51}$$

defined for $0 < h < 1$, so that the following properties hold.

(i) For any $0 < h < 1$ and $w \in H^1$, we have the estimate

$$\left\|\Omega_h^{\mathrm{adj}}[w]\right\|_{L^2_\eta} \quad \leq \quad K\left\|w\right\|_{L^2_\eta}. \tag{14.52}$$

(ii) For every $0 < h < 1$ and $\tau \in \mathbb{R}$ we have

$$|\tilde{\alpha}_{0;h}(\tau)| + |\tilde{\alpha}_{0s;h}(\tau)| + |\tilde{\alpha}_{+;h}(\tau)| + |\tilde{\alpha}_{\omega;h}(\tau)| + |\tilde{\alpha}_{\omega s;h}(\tau)| \leq K e_{2\eta}(\tau). \tag{14.53}$$

(iii) For every $0 < h < 1$ and $w \in H^1$, we have the identity

$$\begin{array}{lcl} M_h^{\mathrm{adj}}\Big[\gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\Omega_h^{\mathrm{adj}}[w_j]\Big] & = & \tilde{\alpha}_{0;h}w + \tilde{\alpha}_{0s;h}T^+ w + \tilde{\alpha}_{+;h}\partial^+ w \\ & & \quad + \tilde{\alpha}_{\omega;h}\Omega_h^{\mathrm{adj}}[w] + \tilde{\alpha}_{\omega s;h}T^+\Omega_h^{\mathrm{adj}}[w]. \end{array} \tag{14.54}$$

(iv) For any sequence $\{(h_j, w_j)\}$ that satisfies (hSeq), there exists $W_* \in L^2$ for which the weak convergences

$$w_j \rightharpoonup W_*, \qquad M_{h_j}^{\mathrm{adj}}\Big[\gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\Omega_{h_j}^{\mathrm{adj}}[w_j]\Big] \rightharpoonup \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}\Big[\Psi_*''\int_+ \Psi_*'W_*\Big] \in L^2 \tag{14.55}$$

both hold as $j \to \infty$, possibly after passing to a subsequence.

Proof. Item (i) can be obtained in a similar fashion as item (i) of Lemma 14.4. Recalling the identities (4.5)-(4.6) and (4.11), we compute

$$\partial^- \big[\Omega_h^{\mathrm{adj}}[w]\big] \;=\; -2w\partial^0\Psi_* \tag{14.56}$$

and hence

$$\begin{aligned}\partial^0 \big[\Omega_h^{\mathrm{adj}}[w]\big] &= -S^+[w\partial^0\Psi_*] \\ \partial^0\partial\big[\Omega_h^{\mathrm{adj}}[w]\big] &= \tfrac{1}{2}\partial^+\partial^- \big[\Omega_h^{\mathrm{adj}}[w]\big] \\ &= -\partial^+[\partial^0\Psi_*]T^+w - [\partial^0\Psi_*]\partial^+ w.\end{aligned} \tag{14.57}$$

Writing

$$\mathcal{I}[w] = \gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\Omega_h^{\mathrm{adj}}[w], \tag{14.58}$$

this gives

$$\begin{aligned}\partial^0\Big[\mathcal{I}[w]\Big] &= \partial^0\big[\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\big]T^+\big[\Omega_h^{\mathrm{adj}}[w]\big] \\ &\quad -T^-\big[\gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\big]S^+[w\partial^0\Psi_*],\end{aligned} \tag{14.59}$$

together with

$$\begin{aligned}\partial^0\partial\Big[\mathcal{I}[w]\Big] &= \partial^0\partial\big[\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\big]\big[\Omega_h^{\mathrm{adj}}[w]\big] \\ &\quad +\big[\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\big]\Big[-\partial^+[\partial^0\Psi_*]T^+w - [\partial^0\Psi_*]\partial^+w\Big] \\ &\quad +\tfrac{1}{2}\partial^+\big[\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\big]T^+[-2w\partial^0\Psi_*] \\ &\quad +\tfrac{1}{2}\partial^-\big[\gamma_{\Psi_*}^{-2}\partial^0\partial\Psi_*\big][-2w\partial^0\Psi_*].\end{aligned} \tag{14.60}$$

Items (ii) and (iii) can now be read off from the representation (14.44) and the exponential bounds (3.4).

Suppose now that $\{(h_j, w_j)\}$ satisfies (hSeq) and write

$$\mathcal{I}_j = \gamma_{\Psi_*}^{-2}[\partial^0\partial\Psi_*]\Omega_{h_j}^{\mathrm{adj}}[w_j]. \tag{14.61}$$

Using the same arguments as in the proof of item (ii) of Lemma 14.4, we can apply Lemma 5.11 to obtain the weak convergence

$$\mathcal{I}_j \rightharpoonup \gamma_*^{-2}\Psi_*''\int_+ \Psi_* W_* \in L^2. \tag{14.62}$$

In addition, using the identity

$$\Big[\Omega_h^{\mathrm{adj}}[w]\Big]' = h\sum_{+;h}\big[2w\partial^0\Psi_*' + 2w'\partial^0\Psi_*\big] \tag{14.63}$$

together with Lemma 5.9, we see that $\|\mathcal{I}_j\|_{H^1}$ can be uniformly bounded. Finally, (14.60) together with the fact that $\Psi_* \in H^5$ implies that also $\|\partial^+\partial^+\mathcal{I}_j\|_{L^2}$ can be uniformly bounded. In particular, the sequence $\{(h_j, \mathcal{I}_j)\}$ also satisfies (hSeq). Applying item (iii) of Lemma 14.5 now yields (iv). □

Proof of Proposition 14.2. Items (i) and (ii) follow directly from Lemma's 14.3, 14.4, 14.5 and 14.6. Under the assumptions of (iii), the weak limits (14.22) follow from the fact that $\{\mathcal{L}_{h_j}[v_j]\}$ and $\{\mathcal{L}_{h_j}^{\mathrm{adj}}[w_j]\}$ are bounded sequences in L^2. Using Lemma's 14.3 and 14.4, we see that

$$F_* = \gamma_*^2\mathcal{L}_{\mathrm{cmp}}[V_*] + \Psi_*'\int_- \Psi_*''\mathcal{L}_{\mathrm{cmp}}[V_*]. \tag{14.64}$$

Applying (3.82) yields (14.23).

On the other hand, Lemma's 14.5 and 14.6 show that

$$F_*^{\mathrm{adj}} = \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}[\gamma_*^2 W_*] + \mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}[\Psi_*'' \int_+ \Psi_*' W_*]. \tag{14.65}$$

In particular, we can satisfy (14.25) by writing

$$H_* = \gamma_*^2 W_* + \Psi_*'' \int_+ \Psi_*' W_*. \tag{14.66}$$

Applying (3.83) we see that

$$W_* = \mathcal{T}_*^{\mathrm{adj}} H_*, \tag{14.67}$$

as desired. □

14.2 Strategy

In this subsection we show that Proposition 14.1 can be established by finding appropriate lower bounds for the quantities

$$\begin{array}{rcl} \mathcal{E}_h(\delta) & = & \inf_{\|v\|_{H^1}=1} \Big\{ \, \|\mathcal{L}_h v - \delta v\|_{L^2} + \delta^{-1} \Big| \langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* \big[\mathcal{L}_h v - \delta v\big] \rangle_{L^2} \Big| \, \Big\}, \\ \mathcal{E}_h^{\mathrm{adj}}(\delta) & = & \inf_{\|w\|_{H^1}=1} \Big\{ \, \Big\| \mathcal{L}_h^{\mathrm{adj}} w - \delta w \Big\|_{L^2} + \delta^{-1} \Big| \langle \Psi_*', \mathcal{L}_h^{\mathrm{adj}} w - \delta w \rangle_{L^2} \Big| \, \Big\}. \end{array} \tag{14.68}$$

In particular, the required bounds are formulated in the following result, which is analogous to [4, Lem. 6].

Proposition 14.7. *Suppose that (Hg) and (HΦ_*) are satisfied. Then there exists $\mu > 0$ and $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ we have*

$$\begin{array}{rcl} \mu(\delta) & := & \mathrm{liminf}_{h \downarrow 0} \, \mathcal{E}_h(\delta) \geq \mu, \\ \mu^{\mathrm{adj}}(\delta) & := & \mathrm{liminf}_{h \downarrow 0} \, \mathcal{E}_h^{\mathrm{adj}}(\delta) \geq \mu. \end{array} \tag{14.69}$$

We postpone the proof of this result to §14.3, but set out to explore the consequences here. In particular, it enables us to show that the operators $\mathcal{L}_h - \delta$ are invertible for small $h > 0$ and $\delta > 0$, providing us with the analogue of [4, Thm. 4].

Proposition 14.8. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists constants $K > 0$ and $\delta_0 > 0$ together with a map $h_0 : (0, \delta_0) \to (0, 1)$ so that the following holds true. For any $0 < \delta < \delta_0$ and any $0 < h < h_0(\delta)$, the operator $\mathcal{L}_h - \delta$ is invertible as a map from H^1 onto L^2 and satisfies the bound*

$$\big\| (\mathcal{L}_h - \delta)^{-1} f \big\|_{H^1} \leq K \Big[\, \|f\|_{L^2} + \delta^{-1} \big| \langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* f \rangle \big| \Big]. \tag{14.70}$$

Proof. Following the proof of [4, Thm. 4], we fix $0 < \delta < \delta_0$ and a sufficiently small $h > 0$. By Proposition 14.7, the operator $\mathcal{L}_h - \delta$ is an homeomorphism from H^1 onto its range

$$\mathcal{R} = (\mathcal{L}_h - \delta)\big(H^1\big) \subset L^2, \tag{14.71}$$

with a bounded inverse $\mathcal{I} : \mathcal{R} \to H^1$. The latter fact shows that $\mathcal{R}$ is a closed subset of L^2. If $\mathcal{R} \neq L^2$, there exists a non-zero $w \in L^2$ so that $\langle w, \mathcal{R} \rangle_{L^2} = 0$, i.e.,

$$\big\langle w, (\mathcal{L}_h - \delta) v \big\rangle_{L^2} = 0 \text{ for all } v \in H^1. \tag{14.72}$$

Restricting this identity to test functions $v \in C_c^\infty$ implies that in fact $w \in H^1$. In particular, we find

$$\big\langle (\mathcal{L}_h^{\mathrm{adj}} - \delta)w, v \big\rangle_{L^2} = 0 \text{ for all } v \in H^1, \tag{14.73}$$

which by the density of H^1 in L^2 means that $(\mathcal{L}_h^{\mathrm{adj}} - \delta)w = 0$. Applying Proposition 14.7 once more yields the contradiction $w = 0$ and establishes $\mathcal{R} = L^2$. The bound (14.70) with the δ-independent constant $K > 0$ now follows directly from the definition (14.68) of the quantities $\mathcal{E}_h(\delta)$ and the uniform lower bound (14.69). □

Following the ideas in [42, §3.3], we can take the $\delta \downarrow 0$ limit and establish our main result concerning $\mathcal{L}_h$. The bounds in (ii) rely heavily on the preliminary work in §13 related to the quantity

$$\mathcal{G}^+_{\mathrm{lin};U}[V] - \mathcal{G}_{\mathrm{lin};U}[\partial^+ V]. \tag{14.74}$$

Proof of Proposition 14.1. For convenience, we introduce the set

$$\mathcal{Z}_h = \{ v \in H^1 : \langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* v \rangle_{L^2} = 0 \}. \tag{14.75}$$

Our goal is to find, for any $f \in L^2$, a solution $(\beta, v) \in \mathbb{R} \times \mathcal{Z}_h$ to the problem

$$v = \mathcal{V}_{h;\delta}\big[f, v, \beta\big] := (\mathcal{L}_h - \delta)^{-1}\big[f + \beta \Psi_*' - \delta v\big]. \tag{14.76}$$

In order to ensure that the linear operator $\mathcal{V}_{h;\delta}$ indeed maps into $\mathcal{Z}_h$, it suffices to choose β in such a way that

$$\beta \langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_*(\mathcal{L}_h - \delta)^{-1}\Psi_*' \rangle_{L^2} = -\langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_*(\mathcal{L}_h - \delta)^{-1}(f - \delta v) \rangle_{L^2}. \tag{14.77}$$

Writing

$$(\mathcal{L}_h - \delta)^{-1}\Psi_*' = -\delta^{-1}\Psi_*' + \overline{v} \tag{14.78}$$

we see that

$$[\mathcal{L}_h - \delta]\overline{v} = \delta^{-1}\mathcal{L}_h \Psi_*', \tag{14.79}$$

which shows that

$$\|\overline{v}\|_{H^1} \le C_1' h \delta^{-2}. \tag{14.80}$$

Choosing $\delta < 1$ and recalling the normalization

$$\langle \Psi^{\mathrm{adj}}, \mathcal{T}_* \Psi_*' \rangle_{L^2} = 1, \tag{14.81}$$

we can impose a restriction $h \le [C_2']^{-1}\delta^2$ to ensure that

$$\big| \langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_*(\mathcal{L}_h - \delta)^{-1}\Psi_*' \rangle_{L^2} \big| \ge \frac{1}{2}\delta^{-1}. \tag{14.82}$$

In particular, we can find a unique solution $\beta = \beta_{h;\delta}[f, v]$ to (14.77) for every $v \in \mathcal{Z}_h$ and $f \in L^2$.

The definition of $\mathcal{Z}$ implies the bound

$$\big\| (\mathcal{L}_h - \delta)^{-1}(f - \delta v) \big\| \le C_3' \big[\delta^{-1} \|f\|_{L^2} + \delta \|v\|_{L^2} \big], \tag{14.83}$$

which allows us to obtain the estimate

$$|\beta_{h;\delta}[f, v]| \le C_4' \big[\|f\|_{L^2} + \delta^2 \|v\|_{L^2} \big]. \tag{14.84}$$

This in turn leads to the estimate

$$\big\|\mathcal{V}_{h;\delta}\big[f, v, \beta_{h;\delta}[f, v]\big]\big\|_{H^1} \le C_5'\big[\delta^{-1}\,\|f\|_{L^2} + \delta\,\|v\|_{L^2}\,\big]. \tag{14.85}$$

By choosing $\delta > 0$ to be sufficiently small, we hence see that the linear fixed point problem

$$v = \mathcal{V}_{h;\delta}\big[f, v, \beta_{h;\delta}[f, v]\big] \tag{14.86}$$

posed on $\mathcal{Z}_h$ has a unique solution for all $f \in L^2$. Writing $v = \mathcal{V}^*_{h;\delta} f$ for this solution together with

$$\beta^*_{h;\delta} f = \beta_{h;\delta}\big[f, \mathcal{V}^*_{h;\delta} f\big], \tag{14.87}$$

we obtain the estimates

$$\big\|\mathcal{V}^*_{h;\delta} f\big\|_{H^1} \le C_6'\delta^{-1}\,\|f\|_{L^2}\,, \qquad \big|\beta^*_{h;\delta} f\big| \le C_6'\,\|f\|_{L^2}\,. \tag{14.88}$$

The remarks above show that the problem (14.7)-(14.8) is equivalent to (14.86). We can hence fix a sufficiently small $\delta > 0$ and write $\beta^*_h = \beta^*_{h;\delta}$ and $\mathcal{V}^*_{h;\delta}$, which are well-defined for all sufficiently small $h > 0$. This establishes (iii). Item (iv) can be verified directly by noting that $(v, \beta) = (0, -1)$ is a solution to (14.7)-(14.8) for $f = \Psi_*'$.

Turning to (i) and (ii), let us pick $f \in L^2$ and write

$$(v, \beta) = (\mathcal{V}^*_h[f], \beta^*_h[f]). \tag{14.89}$$

Item (iii) implies that

$$\begin{array}{rcl} cv' + f + \beta\Psi_*' & = & \mathcal{G}_{\mathrm{lin};\Psi_*}[v] \\ & = & 2\gamma_{\Psi_*}^{-2}\partial^0\partial v + \gamma_{\Psi_*}^2 g'(\Psi_*)v + L_{c;h}[v]. \end{array} \tag{14.90}$$

The bound (i) follows from (14.88) and item (ii) of Proposition 14.2, which together provide L^2-bounds on all the terms in (14.90) that do not involve $\partial^0\partial v$. To see (ii), we compute

$$\begin{array}{rcl} c\partial^+[v]' + \partial^+[f] + \beta\partial^+[\Psi_*'] & = & \partial^+\big[\mathcal{G}_{\mathrm{lin};\Psi_*;\mathrm{expl}}[v]\big] \\ & = & \mathcal{G}_{\mathrm{lin};\Psi_*}[\partial^+ v] \\ & & \qquad +\partial^+\big[\mathcal{G}_{\mathrm{lin};\Psi_*}[v]\big] - \mathcal{G}_{\mathrm{lin};\Psi_*}[\partial^+ v]. \end{array} \tag{14.91}$$

In particular, we see that

$$\mathcal{L}_h[\partial^+ v] \;\; = \;\; \partial^+[f] + \beta\partial^+[\Psi_*'] + \mathcal{G}_{\mathrm{lin};\Psi_*}[\partial^+ v] - \partial^+\big[\mathcal{G}_{\mathrm{lin};\Psi_*}[v]\big]. \tag{14.92}$$

Using item (iii) of Proposition 13.1 together with item (iii) of Lemma 5.4, we obtain

$$\begin{array}{rcl} \big\|\mathcal{G}_{\mathrm{lin};\Psi_*}[\partial^+ v] - \partial^+\big[\mathcal{G}_{\mathrm{lin};\Psi_*}[v]\big]\big\|_{L^2} & \le & C_7'\big[\,\|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2}\,\big] \\ & \le & C_8'\,\|f\|_{L^2}\,. \end{array} \tag{14.93}$$

Using (i) we conclude that

$$\big\|\partial^+ v\big\|_{H^1} + \big\|\partial^+\partial^+[\partial^+ v]\big\|_{L^2} \le C_9'\big[\,\|f\|_{L^2} + \big\|\partial^+ f\big\|_{L^2}\,\big], \tag{14.94}$$

which establishes (ii). □

14.3 Proof of Proposition 14.7

We set out here to obtain lower bounds for the quantities (14.68). As a first step, we show that the limiting values can be approached via a sequence of realizations for which the weak limits described in (iv) of Proposition 14.2 hold and for which the full power of Lemma 5.10 is available.

Lemma 14.9. *Consider the setting of Proposition 14.7 and fix $0 < \delta < \delta_0$. Then there exist four functions*

$$(V_*, W_*) \in H^2 \times H^2, \qquad (Y_*, Z_*) \in L^2(\mathbb{R}), \tag{14.95}$$

together with a sequence

$$\{(h_j, v_j, y_j, w_j, z_j)\}_{j \in \mathbb{N}} \subset (0,1) \times H^1 \times L^2 \times H^1 \times L^2 \tag{14.96}$$

that satisfies the following properties.

(i) For any $j \in \mathbb{N}$, we have

$$\|v_j\|_{H^1} = \|w_j\|_{H^1} = 1, \tag{14.97}$$

together with

$$\begin{array}{rcl} \mathcal{L}_{h_j}[v_j] - \delta v_j & = & y_j, \\ \mathcal{L}^{\mathrm{adj}}_{h_j}[w_j] - \delta w_j & = & z_j. \end{array} \tag{14.98}$$

(ii) Recalling the constants $\big(\mu(\delta), \mu^{\mathrm{adj}}(\delta)\big)$ defined in (14.69), we have the limits

$$\begin{array}{rcl} \mu(\delta) & = & \lim_{j\to\infty}\{\|y_j\|_{L^2} + \delta^{-1}\big|\langle \Psi^{\mathrm{adj}}_*, \mathcal{T}_*[y_j]\rangle_{L^2}\big|\}, \\ \mu^{\mathrm{adj}}(\delta) & = & \lim_{j\to\infty}\{\|z_j\|_{L^2} + \delta^{-1}\big|\langle \Psi'_*, z_j\rangle_{L^2}\big|\}. \end{array} \tag{14.99}$$

(iii) As $j \to \infty$, we have the weak convergences

$$v_j \rightharpoonup V_* \in H^1, \qquad w_j \rightharpoonup W_* \in H^1, \tag{14.100}$$

together with

$$y_j \rightharpoonup Y_* \in L^2, \qquad z_j \rightharpoonup Z_* \in L^2. \tag{14.101}$$

(iv) The pairs $\{(h_j, v_j)\}$ and $\{(h_j, w_j)\}$ both satisfy (hSeq).

Proof. The existence of the sequences (14.96) that satisfy (i) and (ii) with $h_j \downarrow 0$ follows directly from the definitions (14.69). Notice that (14.99) implies that we can pick $C_1 > 0$ for which we have the uniform bound

$$\|y_j\|_{L^2} + \|z_j\|_{L^2} \le C_1 \tag{14.102}$$

for all $j \in \mathbb{N}$. In particular, after taking a subspace we obtain (iii). In addition, item (ii) of Proposition 14.2 implies that also

$$\big\|[\partial^0\partial]_{h_j} v_j\big\|_{L^2} + \big\|[\partial^0\partial]_{h_j} v_j\big\|_{L^2} \le C_2 \tag{14.103}$$

for some $C_2 > 0$ and all $j > 0$, which implies (iv). ∎

Lemma 14.10. *Consider the setting of Proposition 14.7. There exists a constant* $K_1 > 0$ *so that for any* $0 < \delta < \delta_0$*, the function* V_* *defined in Lemma 14.9 satisfies the bound*

$$\|V_*\|_{H^2} \le K_1 \mu(\delta). \tag{14.104}$$

Proof. Item (iv) of Proposition 14.2 implies that

$$\mathcal{L}_{\mathrm{cmp}}[V_*] = \mathcal{T}_*[Y_* + \delta V_*], \tag{14.105}$$

which we rewrite as

$$\mathcal{L}_{\mathrm{cmp}}[V_*] - \delta \mathcal{T}_*[V_*] = \mathcal{T}_* Y_*. \tag{14.106}$$

The lower-semicontinuity of the L^2-norm under weak limits implies that

$$\|Y_*\|_{L^2} + \delta^{-1} \big|\langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* Y_* \rangle\big| \le \mu(\delta), \tag{14.107}$$

while Lemma 3.13 implies that

$$\|\mathcal{T}_* Y_*\|_{L^2} + \delta^{-1} \big|\langle \Psi_*^{\mathrm{adj}}, \mathcal{T}_* Y_* \rangle\big| \le C_1' \mu(\delta). \tag{14.108}$$

The desired bound hence follows directly from Corollary 3.3. □

Lemma 14.11. *Consider the setting of Proposition 14.7. There exists a constant* $K_1 > 0$ *so that for any* $0 < \delta < \delta_0$*, the function* W_* *defined in Lemma 14.9 satisfies the bound*

$$\|W_*\|_{H^2} \le K_1 \mu^{\mathrm{adj}}(\delta). \tag{14.109}$$

Proof. Item (iv) of Proposition 14.2 implies that

$$W_* = \mathcal{T}_*^{\mathrm{adj}} H_* \tag{14.110}$$

for some $H_* \in H^2$ that satisfies the identity

$$\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}[H_*] = [Z_* + \delta W_*] = [Z_* + \delta \mathcal{T}_*^{\mathrm{adj}} H_*]. \tag{14.111}$$

In particular, we find

$$\mathcal{L}_{\mathrm{cmp}}^{\mathrm{adj}}[H_*] - \delta \mathcal{T}_*^{\mathrm{adj}}[H_*] = Z_*. \tag{14.112}$$

The lower-semicontinuity of the L^2-norm under weak limits implies that

$$\|Z_*\|_{L^2} + \delta^{-1} \big|\langle \Psi_*', Z_* \rangle\big| \le \mu(\delta). \tag{14.113}$$

Corollary 3.3 hence implies that

$$\|H_*\|_{H^2} \le C_1' \mu(\delta). \tag{14.114}$$

The desired bound hence follows from (14.110) and Lemma 3.13. □

The next result controls the size of the derivatives (v_j', w_j'), which is crucial to rule out the leaking of energy into oscillations that are not captured by the relevant weak limits. The key novel element here compared to the setting in [4] is that one needs to include $\partial^+ v_j$ in the bound. Our preparatory work enables us to measure this contribution in a weighted norm, which allows us to capture the bulk of the contribution on a compact interval.

Lemma 14.12. *Consider the setting of Lemma 14.9 and pick a sufficiently small $\eta > 0$. There exists a constant $K_2 > 1$ that does not depend on $0 < \delta < \delta_0$ so that the inequalities*

$$\begin{array}{lcl} \big\|v_j'\big\|_{L_2}^2 & \leq & K_2\Big[\, \|y_j\|_{L^2}^2 + \|v_j\|_{L^2}^2 + \|\partial^+ v_j\|_{L^2_\eta}^2 \,\Big], \\ \big\|w_j'\big\|_{L_2}^2 & \leq & K_2\Big[\, \|z_j\|_{L^2}^2 + \|w_j\|_{L^2}^2 + \|\partial^+ w_j\|_{L^2_\eta}^2 \,\Big] \end{array} \tag{14.115}$$

hold for all $j > 0$.

Proof. Using the representation in item (i) of Proposition 14.2, we expand the identity

$$\langle \mathcal{L}_{h_j} v_j - \delta v_j, \gamma_{\Psi_*}^2 v_j' \rangle_{L^2} = \langle y_j, \gamma_{\Psi_*}^2 v_j' \rangle_{L^2} \tag{14.116}$$

to obtain

$$\begin{array}{lcl} c_* \langle \gamma_{\Psi_*}^2 v_j', v_j' \rangle_{L^2} + \langle y_j, \gamma_{\Psi_*}^2 v_j' \rangle_{L^2} & = & -\delta \langle v_j, \gamma_{\Psi_*}^2 v_j' \rangle_{L^2} + \langle 2\partial^0 \partial v_j, v_j' \rangle_{L^2} + \langle \gamma_{\Psi_*}^4 g'(\Psi_*) v_j, v_j' \rangle_{L^2} \\ & & + \langle L_{c;h_j}[v_j], \gamma_{\Psi_*}^2 v_j' \rangle_{L^2}. \end{array} \tag{14.117}$$

Applying (5.55) together with item (iii) of Proposition 14.2, we note that

$$\begin{array}{lcl} \big|\langle L_{c;h_j}[v_j], \gamma_{\Psi_*}^2 v_j' \rangle_{L^2}\big| & = & \big|\langle e_{2\eta}^{-1} L_{c;h_j}[v_j], e_{2\eta} \gamma_{\Psi_*}^2 v_j' \rangle_{L^2}\big| \\ & \leq & \big\|e_{2\eta}^{-1} L_{c;h_j}[v_j]\big\|_{L^2_\eta} \big\|\gamma_{\Psi_*}^2 v_j'\big\|_{L^2_\eta} \\ & \leq & C_1' \big[\, \|v\|_{L^2_\eta} + \|\partial^+ v\|_{L^2_\eta} \,\big] \big\|v_j'\big\|_{L^2_\eta}. \end{array} \tag{14.118}$$

Using the identity $\langle \partial^0 \partial v_j, v_j' \rangle_{L^2} = 0$ together with the lower bound $\gamma_{\Psi_*}^2 \geq [C_2']^{-1}$ we may hence compute

$$\begin{array}{lcl} |c_*| \, \langle v_j', v_j' \rangle_{L^2} & \leq & |c_*| \, C_2' \langle \gamma_{\Psi_*}^2 v_j', v_j' \rangle_{L^2} \\ & \leq & C_3' \Big[\, \|v_j\|_{L^2} \big\|v_j'\big\|_{L^2} + \|y_j\|_{L^2} \big\|v_j'\big\|_{L^2} + \|v\|_{L^2_\eta} \big\|v_j'\big\|_{L^2_\eta} + \|\partial^+ v_j\|_{L^2_\eta} \big\|v_j'\big\|_{L^2_\eta} \,\Big]. \end{array} \tag{14.119}$$

Recalling the bound $\|a\|_{L^2_\eta} \leq \|a\|_{L^2}$ for $a \in L^2$ and using $c_* \neq 0$, we find

$$\big\|v_j'\big\|_{L^2}^2 \quad \leq \quad C_4' \Big[\, \|v_j\|_{L^2} + \|y_j\|_{L^2} + \|\partial^+ v_j\|_{L^2_\eta} \,\Big] \big\|v_j'\big\|_{L^2}. \tag{14.120}$$

Dividing through by $\big\|v_j'\big\|_{L^2}$ and squaring, we obtain

$$\big\|v_j'\big\|_{L_2}^2 \leq C_5' \Big[\, \|v_j\|_{L^2}^2 + \|y_j\|_{L^2}^2 + \|\partial^+ v_j\|_{L^2_\eta}^2 \,\Big]. \tag{14.121}$$

The same procedure works for w_j'. □

We are now almost ready to obtain lower bounds for $\|V_*\|_{H^1}$ and $\|W_*\|_{H^1}$, exploiting the fact that our nonlinearity is bistable. The next technical result is the analogue of the inequality $\langle \partial^0 \partial u, u \rangle_{L^2} \leq 0$ used in [4]. Due to the non-autonomous coefficient in front of the second difference, we obtain localized correction terms that need to be controlled.

Lemma 14.13. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists a constant $K > 0$ so that for any $v \in H^1$ and any $0 < h < 1$, we have the one-sided inequality*

$$2\langle \gamma_{\Psi_*}^{-2} \partial^0 \partial v_j, v_j \rangle_{L^2} \quad \leq \quad K\big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \,\big]. \tag{14.122}$$

Proof. Using (4.5) we compute

$$\begin{array}{rcl} -2\langle \gamma_{\Psi_*}^{-2}\partial^0\partial v_j, v_j\rangle_{L^2} & = & -\langle \gamma_{\Psi_*}^{-2}\partial^-\partial^+ v_j, v_j\rangle_{L^2} \\ & = & \langle \partial^+ v_j, \partial^+ \big[\gamma_{\Psi_*}^{-2} v_j]\rangle_{L^2} \\ & = & \langle \partial^+ v_j, \partial^+ \big[\gamma_{\Psi_*}^{-2}]T^+[v_j]\rangle_{L^2} + \langle \partial^+ v_j, \gamma_{\Psi_*}^{-2}\partial^+ v_j\rangle_{L^2} \\ & = & \langle \partial^+ v_j, \partial^+ \big[\gamma_{\Psi_*}^{-2}]T^+[v_j]\rangle_{L^2} + \langle \partial^+ v_j, (\gamma_{\Psi_*}^{-2} - 1)\partial^+ v_j\rangle_{L^2} \\ & & \quad +\langle \partial^+ v_j, \partial^+ v_j\rangle_{L^2} \\ & \geq & \langle \partial^+ v_j, \partial^+ \big[\gamma_{\Psi_*}^{-2}]T^+[v_j]\rangle_{L^2} + \langle \partial^+ v_j, (\gamma_{\Psi_*}^{-2} - 1)\partial^+ v_j\rangle_{L^2}. \end{array} \tag{14.123}$$

The result now follows from (5.55) together with the pointwise exponential bounds

$$\big|\gamma_{\Psi_*}^{-2} - 1\big| + \big|\partial^+ \big[\gamma_{\Psi_*}^{-2}]\big| \leq C_1' e_{2\eta}. \tag{14.124}$$

□

Lemma 14.14. *Consider the setting of Proposition 14.7. There exists constants $K_2 > 0$ and $K_3 > 0$ so that for any $0 < \delta < \delta_0$, the functions V_* and W_* defined in Lemma 14.9 satisfy the bounds*

$$\begin{array}{rcl} \|V_*\|_{H^1}^2 & \geq & K_3 - K_4\mu(\delta)^2, \\ \|W_*\|_{H^1}^2 & \geq & K_3 - K_4\mu^{\mathrm{adj}}(\delta)^2. \end{array} \tag{14.125}$$

Proof. Pick $m > 1$ and $\alpha > 0$ in such a way that

$$\gamma_{\Psi_*}^2(\tau) g'\big(\Psi_*(\tau)\big) \leq -\alpha \tag{14.126}$$

holds for all $|\tau| \geq m$. This is possible on account of the uniform lower bound $\gamma_{\Psi_*}^2 \geq [C_1']^{-1}$ and the fact that $g'(0) < 0$ and $g'(1) < 0$.

We now expand the identity

$$\langle \mathcal{L}_{h_j} v_j - \delta v_j, v_j\rangle_{L^2} = \langle y_j, v_j\rangle_{L^2} \tag{14.127}$$

to obtain the estimate

$$\begin{array}{rcl} \langle y_j, v_j\rangle_{L^2} & = & -c_*\langle v_j', v_j\rangle_{L^2} - \delta\langle v_j, v_j\rangle_{L^2} \\ & & \quad +2\langle \gamma_{\Psi_*}^{-2}\partial^0\partial v_j, v_j\rangle_{L^2} + \langle \gamma_{\Psi_*}^{-2} g'(\Psi_*) v_j, v_j\rangle_{L^2} \\ & & \quad +\langle L_{c;h}[v_j], v_j\rangle_{L^2}. \end{array} \tag{14.128}$$

Using $\langle v_j', v_j\rangle_{L^2} = 0$, Lemma 14.13 and item (iii) of Proposition 14.2, we find

$$\begin{array}{rcl} \langle y_j, v_j\rangle_{L^2} & \leq & C_2'\big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2\,\big] + \langle \gamma_{\Psi_*}^{-2} g'(\Psi_*) v_j, v_j\rangle_{L^2} \\ & \leq & C_2'\big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2\,\big] \\ & & \quad -\alpha\,\|v_j\|_{L^2}^2 + C_3'\int_{-m}^m |v_j(\tau)|^2\, d\tau. \end{array} \tag{14.129}$$

Using the basic inequality

$$xy = (\sqrt{\alpha}x)(y/\sqrt{\alpha}) \leq \frac{\alpha}{2}x^2 + \frac{1}{2\alpha}y^2, \tag{14.130}$$

we arrive at

$$
\begin{array}{rcl}
C_3' \int_{-m}^{m} |v_j(\tau)|^2 \, d\tau & \geq & \alpha \|v_j\|_{L^2}^2 - \langle y_j, v_j \rangle_{L^2} \\
& & \quad - C_2' \big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \, \big] \\
& \geq & \alpha \|v_j\|_{L^2}^2 - \|y_j\|_{L^2} \|v_j\|_{L^2} \\
& & \quad - C_2' \big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \, \big] \\
& \geq & \frac{\alpha}{2} \|v_j\|_{L^2}^2 - \frac{1}{2\alpha} \|y_j\|_{L^2}^2 \\
& & \quad - C_2' \big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \, \big].
\end{array}
\tag{14.131}
$$

Multiplying the first inequality in (14.115) by $\frac{\alpha}{2(1+K_2)}$, we find

$$
0 \geq \frac{\alpha}{2(1+K_2)} \left\|v_j'\right\|_{L^2}^2 - \frac{\alpha K_2}{2(1+K_2)} \|v_j\|_{L^2}^2 - C_4' \left\|\partial^+ v\right\|_{L^2_\eta}^2 - C_4' \|y_j\|_{L^2}^2 . \tag{14.132}
$$

Adding (14.131) and (14.132), we may use the identity

$$
\frac{\alpha}{2} - \frac{\alpha K_2}{2(1+K_2)} = \frac{\alpha}{2(1+K_2)}, \tag{14.133}
$$

to obtain

$$
\begin{array}{rcl}
C_3' \int_{-m}^{m} |v_j(\tau)|^2 \, d\tau & \geq & \frac{\alpha}{2(1+K_2)} \big[\, \|v_j\|_{L^2}^2 + \left\|v_j'\right\|_{L^2}^2 \, \big] - C_5' \|y_j\|_{L^2}^2 \\
& & \quad - C_5' \big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \, \big] \\
& = & \frac{\alpha}{2(1+K_2)} - C_5' \|y_j\|_{L^2}^2 \\
& & \quad - C_5' \big[\, \|\partial^+ v\|_{L^2_\eta}^2 + \|v\|_{L^2_\eta}^2 \, \big].
\end{array}
\tag{14.134}
$$

For any $M \geq 0$ and $a \in L^2$ we may compute

$$
\begin{array}{rcl}
\|a\|_{L^2_\eta}^2 & = & \int e^{-2\eta|\tau|} a(\tau)^2 \, d\tau \\
& \leq & e^{-2\eta M} \|a\|_{L^2}^2 + \int_{-M}^{M} e^{-2\eta|\tau|} a(\tau)^2 \, d\tau \\
& \leq & e^{-2\eta M} \|a\|_{L^2}^2 + \int_{-M}^{M} a(\tau)^2 \, d\tau.
\end{array}
\tag{14.135}
$$

Exploiting $\|\partial^+ v_j\|_{L^2} \leq \left\|v_j'\right\|_{L^2}$ and $\|v_j\|_{H^1} = 1$, we hence see

$$
\|v_j\|_{L^2_\eta} + \|\partial^+ v_j\|_{L^2_\eta}^2 \quad \leq \quad e^{-2\eta M} + \int_{-M}^{M} [v_j](\tau)^2 \, d\tau + \int_{-M}^{M} [\partial^+ v_j](\tau)^2 \, d\tau. \tag{14.136}
$$

In particular, by choosing $M \geq m$ to be sufficiently large, we find

$$
\begin{array}{rcl}
C_3' \int_{-M}^{M} |v_j(\tau)|^2 \, d\tau & \geq & C_3' \int_{-m}^{m} |v_j(\tau)|^2 \, d\tau \\
& \geq & \frac{\alpha}{4(1+K_2)} - C_5' \|f_j\|_{L^2}^2 \\
& & \quad - C_5' \Big[\int_{-M}^{M} [\partial^+ v_j](\tau)^2 \, d\tau + \int_{-M}^{M} v_j(\tau)^2 \, d\tau \Big].
\end{array}
\tag{14.137}
$$

We hence obtain

$$
C_6' \Big[\int_{-M}^{M} [\partial^+ v_j](\tau)^2 \, d\tau + \int_{-M}^{M} v_j(\tau)^2 \, d\tau \Big] \quad \geq \quad \frac{\alpha}{4(1+K_2)} - C_5' \|y_j\|_{L^2}^2 . \tag{14.138}
$$

In view of the bound

$$\limsup_{j\to\infty} \|y_j\|_{L^2}^2 \le \mu(\delta)^2, \tag{14.139}$$

the strong convergences $v_j \to V_* \in L^2([-M, M])$ and $\partial^+ v_j \to V'_* \in L^2([-M, M])$ imply that

$$\|V_*\|_{H^1}^2 \ge [C'_6]^{-1}\big[\frac{\alpha}{4(1+K_2)} - C'_5\mu(\delta)^2\big], \tag{14.140}$$

as desired. The bound for W_* follows in a very similar fashion. □

Proof of Proposition 14.7. For any $0 < \delta < 1$, Lemma's 14.10 and 14.14 show that the function V_* defined in Lemma 14.9 satisfies

$$K_1^2\mu(\delta)^2 \ge \|V_*\|_{H^1}^2 \ge K_3 - K_4\mu(\delta)^2, \tag{14.141}$$

which gives $\big(K_1^2 + K_4\big)\mu(\delta)^2 \ge K_3 > 0$, as desired. The same computation works for μ^{adj}, but now one uses Lemma's 14.11 and 14.14. □

15 Travelling waves

Formally substituting the travelling wave Ansatz (2.42) into the reduced system (2.29) leads to the nonlocal differential equation

$$c\Psi' = \mathcal{G}(\Psi). \tag{15.1}$$

In this section we set out to construct solutions to this equation for small $h > 0$ that can be written as

$$\Psi = \Psi_* + v, \qquad c = c_* + \tilde{c} \tag{15.2}$$

for pairs $(\tilde{c}, v)$ that tend to zero as $h \downarrow 0$. Care must be taken to ensure that the expression $\mathcal{G}(\Psi)$ is well-defined, but based on our preparations we are able to provide a relatively streamlined fixed-point argument here, which allows us to prove the results stated in §2.2.

In order to control the size of the perturbation $(\tilde{c}, v) \in \mathbb{R} \times H^1$, we introduce the norms

$$\|(\tilde{c}, v)\|_{\mathcal{Z}_h} = |\tilde{c}| + \|v\|_{H^1} + \big\|\partial_h^+\partial_h^+ v\big\|_{L^2} \tag{15.3}$$

for $h > 0$ and write $\mathcal{Z}_h$ for the set $\mathbb{R} \times H^1$ equipped with this new norm. Observe that for fixed h this norm is equivalent to the usual one on $\mathbb{R} \times H^1$.

We note that Proposition 6.3 allows us to fix $0 < \kappa < \frac{1}{12}$ and $\epsilon_0 > 0$ in such a way that the inclusion

$$\mathrm{ev}_\vartheta[\Psi_* + v] \in \Omega_{h;\kappa} \tag{15.4}$$

holds for all $0 < h < 1$, all $\vartheta \in [0, h]$ and all $v \in H^1$ that have

$$\|v\|_{H^1} + h^{-1/2}\big\|\partial_h^+ v\big\|_{H^1} < 2\epsilon_0. \tag{15.5}$$

In order to accommodate this, we pick two parameters $\delta > 0$ and $\delta_v^+ > 0$ and introduce the set

$$\begin{aligned} \mathcal{Z}_{h;\delta,\delta_v^+} \quad = \quad & \big\{(\tilde{c}, v) \in \mathcal{Z}_h : \|(\tilde{c}, v)\|_{\mathcal{Z}_h} \le \min\{\delta, \epsilon_0\} \\ & \qquad \text{and } \|(0, \partial^+ v)\|_{\mathcal{Z}_h} \le \min\{\delta_v^+, h^{1/2}\epsilon_0\}\big\}. \end{aligned} \tag{15.6}$$

Since ∂_h^+ is bounded on H^1 and L^2 for each fixed h, we note that this is a closed subset of $\mathcal{Z}_h$.

Substituting (15.2) into (15.1), we obtain

$$\begin{array}{rcl} c_*\Psi_*' + \tilde{c}\Psi_*' + \tilde{c}v' + c_*v' & = & \mathcal{G}(\Psi_* + v) \\ & = & \mathcal{G}(\Psi_*) + \mathcal{G}_{\mathrm{lin};\Psi_*}[v] + \mathcal{G}_{\mathrm{nl};\Psi_*}(v), \end{array} \tag{15.7}$$

which should be interpreted in the sense that was outlined at the start of §14.

Upon introducing the nonlinearity

$$\mathcal{H}_h(\tilde{c}, v) \quad = \quad \tilde{c}v' - \mathcal{G}_{\mathrm{nl};\Psi_*}(v), \tag{15.8}$$

we can rewrite (15.7) as

$$\mathcal{L}_h[v] \quad = \quad \tilde{c}\Psi_*' + \mathcal{H}_{\Psi_*}(\tilde{c}, v) + c_*\Psi_*' - \mathcal{G}(\Psi_*). \tag{15.9}$$

Recalling Proposition 14.1, we now introduce the map $\mathcal{W}_h : \mathcal{Z}_{h;\delta,\delta_v^+} \to \mathcal{Z}_h$ that acts as

$$\mathcal{W}_h(\tilde{c}, v) \quad = \quad [\beta_h^*, \mathcal{V}_h^*]\Big[\mathcal{H}_h(\tilde{c}, v) + c_*\Psi_*' - \mathcal{G}(\Psi_*)\Big], \tag{15.10}$$

which allows us to recast (15.9) as the fixed point problem

$$(\tilde{c}, v) = \mathcal{W}_h\big(\tilde{c}, v\big). \tag{15.11}$$

Lemma 15.1. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists $K > 0$ so that for any pair $(\delta, \delta_v^+) \in (0, 1)^2$ and any $0 < h < 1$ the estimates*

$$\begin{array}{rcl} \|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} & \leq & K\big[h\delta + \delta^2 + \delta\delta_v^+\big], \\ \|\partial^+\mathcal{H}_h(\tilde{c}, v)\|_{L^2} & \leq & K\big[[\delta + \delta_v^+]^2 + h^{-1/2}\delta[\delta + \delta_v^+] + h[\delta_v + \delta_v^+]\big] \end{array} \tag{15.12}$$

hold for each $(\tilde{c}, v) \in \mathcal{Z}_{h;\delta,\delta_v^+}$, while the estimate

$$\big\|\mathcal{H}_h(\tilde{c}^{(2)}, v^{(2)}) - \mathcal{H}_h(\tilde{c}^{(1)}, v^{(1)})\big\|_{L^2} \quad \leq \quad K\big[h^{-1/2}[\delta + \delta_v^+] + h\big]\,\big\|(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)})\big\|_{\mathcal{Z}_h} \tag{15.13}$$

holds for each set of pairs $(\tilde{c}^{(1)}, v^{(1)}) \in \mathcal{Z}_{h;\delta,\delta_v^+}$ and $(\tilde{c}^{(2)}, v^{(2)}) \in \mathcal{Z}_{h;\delta,\delta_v^+}$.

Proof. The first term in $\mathcal{H}_h$ can be handled by the elementary estimates

$$\begin{array}{rcl} \|\tilde{c}v'\|_{L^2} & \leq & \delta^2, \\ \|\tilde{c}\partial^+v'\|_{L^2} & \leq & \delta\,\|\partial^+v\|_{H^1} \\ & \leq & \delta\delta_v^+, \end{array} \tag{15.14}$$

together with

$$\begin{array}{rcl} \big\|\tilde{c}^{(2)}[v^{(2)}]' - \tilde{c}^{(1)}[v^{(1)}]'\big\|_{L^2} & \leq & \big|\tilde{c}^{(2)} - \tilde{c}^{(1)}\big|\,\big\|v^{(2)}\big\|_{H^1} + \big|\tilde{c}^{(1)}\big|\,\big\|v^{(1)} - v^{(2)}\big\|_{H^1} \\ & \leq & \delta\,\big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}. \end{array} \tag{15.15}$$

Corollary 5.3 yields the bound

$$\|v\|_{\ell_h^{2;2}} + \|v\|_{\ell_h^{\infty;1}} \leq C_1'[\delta + \delta_v^+] \tag{15.16}$$

for all $(\tilde{c}, v) \in \mathcal{Z}_{h;\delta,\delta^+}$. For any $\vartheta \in \mathbb{R}$, we may hence exploit Propositions 7.14 and 12.1 to obtain the estimate

$$\|\mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v)\|_{\ell^2_h} \leq C_2'\big[\delta + \delta_v^+ + h\big] \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}}, \tag{15.17}$$

together with

$$\begin{aligned} \big\|\mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v^{(2)}) - \mathcal{G}_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v^{(1)})\big\|_{\ell^2_h} &\leq C_2'\big[\delta + \delta_v^+ + h\big] \big\|\mathrm{ev}_\vartheta v^{(1)} - \mathrm{ev}_\vartheta v^{(2)}\big\|_{\ell_h^{2;2}} \\ &\quad + C_2'\big[\delta + \delta_v^+\big] \big\|\mathrm{ev}_\vartheta v^{(1)} - \mathrm{ev}_\vartheta v^{(2)}\big\|_{\ell_h^{\infty;1}}. \end{aligned} \tag{15.18}$$

A second application of Corollary 5.3 yields the bound

$$\|v\|_{\ell_h^{2;2}} + \|v\|_{\ell_h^{\infty;2}} \leq C_3' h^{-1/2}[\delta + \delta_v^+]. \tag{15.19}$$

For any $\vartheta \in \mathbb{R}$, we may hence use Propositions 7.14 and 13.1 to find

$$\begin{aligned} \Big\|\mathcal{G}^+_{\mathrm{nl};\Psi_*}(\mathrm{ev}_\vartheta v)\Big\|_{\ell^2_h} &\leq C_4'\big[\delta + \delta_v^+ + h\big] \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;3}} \\ &\quad + C_4' h^{-1/2}[\delta + \delta_v^+] \|\mathrm{ev}_\vartheta v\|_{\ell_h^{2;2}}. \end{aligned} \tag{15.20}$$

We now apply Lemma 5.4 to obtain

$$\begin{aligned} \|\mathcal{G}_{\mathrm{nl};\Psi_*}(v)\|_{L^2} &\leq C_2'\big[\delta + \delta_v^+ + h\big]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2} \,\big] \\ &\leq C_2'\big[\delta + \delta_v^+ + h\big]\delta, \\ \Big\|\mathcal{G}^+_{\mathrm{nl};\Psi_*}(v)\Big\|_{L^2} &\leq C_4'\big[\delta + \delta_v^+ + h\big]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2} + \|\partial^+ v\|_{H^1} + \|\partial^+\partial^+\partial^+ v\|_{L^2} \,\big] \\ &\quad + C_4' h^{-1/2}[\delta + \delta_v^+]\big[\, \|v\|_{H^1} + \|\partial^+\partial^+ v\|_{L^2} \,\big] \\ &\leq C_4'\big[\delta + \delta_v^+ + h\big]\big[\delta + \delta_v^+\big] \\ &\quad + C_4' h^{-1/2}[\delta + \delta_v^+]\delta. \end{aligned} \tag{15.21}$$

Using (5.13) we note that

$$\Big\|v^{(2)} - v^{(1)}\Big\|_{\ell_h^{\infty;1}} \leq 2h^{-1/2} \Big\|v^{(2)} - v^{(1)}\Big\|_{H^1}. \tag{15.22}$$

Applying Lemma 5.4 once more, we obtain

$$\begin{aligned} \big\|\mathcal{G}_{\mathrm{nl};\Psi_*}(v^{(2)}) - \mathcal{G}_{\mathrm{nl};\Psi_*}(v^{(1)})\big\|_{L^2} &\leq C_2'\big[\delta + \delta_v^+ + h\big]\Big[\, \big\|v^{(1)} - v^{(2)}\big\|_{H^1} + \big\|\partial^+\partial^+ v^{(1)} - \partial^+\partial^+ v^{(2)}\big\|_{L^2} \Big] \\ &\quad + 2C_2'\big[\delta + \delta_v^+\big] h^{-1/2} \big\|v^{(1)} - v^{(2)}\big\|_{H^1}. \end{aligned} \tag{15.23}$$

The desired bounds follow readily from these estimates. □

Lemma 15.2. *Suppose that (Hg) and (HΦ_*) are satisfied. There exists $K > 0$ so that for each $0 < h < 1$ we have the bounds*

$$\begin{aligned} \|c_* \Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} &\leq Kh, \\ \big\|\partial^+\big[c_* \Psi_*' - \mathcal{G}(\Psi_*)\big]\big\|_{L^2} &\leq Kh. \end{aligned} \tag{15.24}$$

Proof. Applying Lemma 5.4 together with Propositions 7.14, 12.1 and 13.1, we find

$$\|\mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*)\|_{L^2} + \big\|\mathcal{G}^+_{\mathrm{apx}}(\Psi_*) - \mathcal{G}^+(\Psi_*)\big\|_{L^2} \quad \le \quad C_1' h. \tag{15.25}$$

We now compute

$$\begin{array}{lcl} c_*\Psi_*' - \mathcal{G}(\Psi_*) & = & c_*\Psi_*' - \mathcal{G}_{\mathrm{apx}}(\Psi_*) + \mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*) \\ & = & c_*\Psi_*' - c_*\partial^0\Psi_* + \mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*), \end{array} \tag{15.26}$$

together with

$$\begin{array}{lcl} \partial^+\big[c_*\Psi_*' - \mathcal{G}(\Psi_*)\big] & = & \partial^+\big[c_*\Psi_*' - \mathcal{G}_{\mathrm{apx}}(\Psi_*)\big] + \partial^+\big[\mathcal{G}_{\mathrm{apx}}(\Psi_*) - \mathcal{G}(\Psi_*)\big] \\ & = & c_*\big[\partial^+\Psi_*\big]' - c_*\partial^0[\partial^+\Psi_*] \\ & & \quad + \mathcal{G}^+_{\mathrm{apx}}(\Psi_*) - \mathcal{G}^+(\Psi_*). \end{array} \tag{15.27}$$

Applying Lemma 5.5, we see that

$$\begin{array}{lcl} \|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} & \le & C_2' h \, \|\Psi_*''\|_{L^2} + C_1' h \\ & \le & C_3' h, \end{array} \tag{15.28}$$

together with

$$\begin{array}{lcl} \|\partial^+[c_*\Psi_*' - \mathcal{G}(\Psi_*)]\|_{L^2} & \le & C_2' h \, \|\partial^+\Psi_*''\|_{L^2} + C_1' h \\ & \le & C_3' h, \end{array} \tag{15.29}$$

as desired. □

Lemma 15.3. *Suppose that (Hg) and (HΦ_*) are satisfied. Then for each sufficiently small $h > 0$, the fixed point problem (15.11) posed on the set $\mathcal{Z}_{h;h^{3/4},h^{3/4}}$ has a unique solution.*

Proof. Using Proposition 14.1, together with the a-priori bounds $(h, \delta, \delta_v^+) \in (0,1)^3$, we obtain the estimates

$$\begin{array}{lcl} \|\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} & \le & C_1'\big[\, \|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} + \|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} \,\big] \\ & \le & C_2'\big[\delta^2 + \delta\delta_v^+ + h\big], \\ \|[0, \partial^+]\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} & \le & C_1'\big[\, \|\mathcal{H}_h(\tilde{c}, v)\|_{L^2} + \|\partial^+\mathcal{H}_h(\tilde{c}, v)\|_{L^2} \,\big] \\ & & \quad + C_1'\Big[\, \|c_*\Psi_*' - \mathcal{G}(\Psi_*)\|_{L^2} + \big\|\partial^+\big[c_*\Psi_*' - \mathcal{G}(\Psi_*)\big]\big\|_{L^2} \,\Big] \\ & \le & C_2'\Big[h^{-1/2}\delta[\delta + \delta_v^+] + (\delta + \delta_v^+)^2 + h\Big], \end{array} \tag{15.30}$$

together with

$$\begin{array}{lcl} \big\|\mathcal{W}_h\big(\tilde{c}^{(2)}, v^{(2)}\big) - \mathcal{W}_h\big(\tilde{c}^{(1)}, v^{(1)}\big)\big\|_{\mathcal{Z}_h} & \le & C_1' \, \big\|\mathcal{H}_h(\tilde{c}^{(2)}, v^{(2)}) - \mathcal{H}_h(\tilde{c}^{(1)}, v^{(1)})\big\|_{L^2} \\ & \le & C_2'\big[h^{-1/2}[\delta + \delta_v^+] + h\big] \, \big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}. \end{array} \tag{15.31}$$

Picking

$$\delta = \delta_v^+ = h^{3/4}, \tag{15.32}$$

we see that $\delta = \delta_v^+ \le h^{1/2}\epsilon_0$ for all sufficiently small $h > 0$. In addition, we find

$$\begin{aligned} \|\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_2'\big[2h^{3/4} + h^{1/4}\big]\delta, \\ \|[0, \partial^+]\mathcal{W}_h(\tilde{c}, v)\|_{\mathcal{Z}_h} &\le C_2'\Big[2h^{1/4} + 4h^{3/4} + h^{1/4}\Big]\delta, \end{aligned} \tag{15.33}$$

together with

$$\big\|\mathcal{W}_h\big(\tilde{c}^{(2)}, v^{(2)}\big) - \mathcal{W}_h\big(\tilde{c}^{(1)}, v^{(1)}\big)\big\|_{\mathcal{Z}_h} \le C_2'[2h^{1/4} + h]\,\big\|\big(\tilde{c}^{(2)} - \tilde{c}^{(1)}, v^{(2)} - v^{(1)}\big)\big\|_{\mathcal{Z}_h}. \tag{15.34}$$

The result hence follows from the contraction mapping theorem. □

Proof of Theorem 2.6. We write $(\tilde{c}_h, v_h)$ for the unique solution to the fixed point problem (15.11) that is provided by Lemma 15.3. This allows us to define

$$\Psi_h = \Psi_* + v_h, \qquad c_h = c_* + \tilde{c}_h. \tag{15.35}$$

For fixed $h > 0$, we claim that the map

$$\vartheta \mapsto \mathrm{ev}_\vartheta[\Psi_* + v_h] - \mathrm{ev}_0\Psi_* \in \ell_h^2 \tag{15.36}$$

is continous. Indeed, this follows from the smoothness of Ψ_* together with (5.13) and the fact that the translation operator is continuous on H^1. Since the map

$$V \mapsto \mathcal{G}(\Psi_* + V) \in \ell_h^2 \tag{15.37}$$

is continuous on a subset of ℓ_h^2 that contains $\mathrm{ev}_\vartheta v_h$ for all $\vartheta \in [0, h]$, we conclude that

$$\vartheta \mapsto \mathcal{G}\big(\mathrm{ev}_\vartheta\Psi_h\big) \in \ell_h^2 \tag{15.38}$$

is continuous. The travelling wave equation (15.1) now implies the inclusion (2.43).

In a similar fashion, the inclusion (2.47) follows from (5.13) and the continuity of the translation operator on H^1. The remaining statements are a direct consequence of Lemma 15.3. □

Proof of Corollary 2.7. Upon defining

$$\Psi_h^{(x)} = -h\sum_{-;h}\frac{(\partial^+\Psi_h)^2}{\sqrt{1 - (\partial^+\Psi_h)^2} + 1}, \tag{15.39}$$

Proposition 2.3 implies that (i) is satisfied. Using Proposition 8.2, we see that

$$\dot{x}_{jh}(t) = c_h[\Psi_h^{(x)}]'(jh + c_h t) = \big[\mathcal{Y}\big(\Psi_h(\cdot + c_h t)\big)\big]_{jh}. \tag{15.40}$$

Inspecting the computations in §9 and §12.3-12.4, we can recover an approximant for $\mathcal{Y}(U)$ by making the replacements

$$\begin{aligned} \mathcal{G}_{A;\mathrm{apx};III}(U) &\mapsto -\gamma_U\partial^0 U(c_*\gamma_U^{-1}\partial^0 U) \\ &= c_*(\gamma_U^2 - 1), \\ \mathcal{G}_{B;\mathrm{apx};III}(U) &\mapsto c_*\gamma_U(1 - \gamma_U). \end{aligned} \tag{15.41}$$

In particular, upon defining

$$\mathcal{Y}_{\mathrm{apx}}(U) = c_*(\gamma_U - 1), \tag{15.42}$$

we obtain the error bound

$$\|\mathcal{Y}(U) - \mathcal{Y}_{\rm apx}(U)\|_{\ell^\infty_h} \le C_1' \big[h + \|\mathcal{E}_{\rm tw}(U)\|_{\ell^2_h} + \|\mathcal{E}_{\rm tw}(U)\|_{\ell^\infty_h} \big]. \tag{15.43}$$

Substituting $U = \Psi_h$ and applying the Lipschitz bounds (7.65), we find

$$\begin{array}{lcl} \|\mathcal{E}_{\rm tw}(\Psi_h) - \mathcal{E}_{\rm tw}(\Psi_*)\|_{\ell^2_h} & \le & C_2' \, \|\Psi_h - \Psi_*\|_{\ell^{2;2}_h} \\ & \le & C_2' \big[\, \|\Psi_h - \Psi_*\|_{H^1} + \|\partial^+ [\Psi_h - \Psi_*]\|_{H^1} \, \big] \\ & \le & C_2' h^{3/4}. \end{array} \tag{15.44}$$

Using Proposition 7.14, we obtain

$$\|\mathcal{E}_{\rm tw}(\Psi_h)\|_{\ell^2_h} \le C_3' h^{3/4} \tag{15.45}$$

and hence

$$\|\mathcal{E}_{\rm tw}(\Psi_h)\|_{\ell^\infty_h} \le C_3' h^{1/4}. \tag{15.46}$$

In a similar fashion, we may exploit (7.3) to conclude

$$\|\gamma_{\Psi_h} - \gamma_{\Psi_*}\|_{\ell^2_h} \le C_2' h^{3/4} \tag{15.47}$$

and hence

$$\|\gamma_{\Psi_h} - \gamma_{\Psi_*}\|_{\ell^\infty_h} \le C_2' h^{1/4}. \tag{15.48}$$

Together, these observations yield the pointwise bound

$$|\mathcal{Y}(\Psi_h) - c_* (\gamma_{\Psi_*} - 1)| \le C_4' h^{1/4}. \tag{15.49}$$

Assuming for clarity that $c_* > 0$, this implies the pointwise inequality

$$\mathcal{Y}(\Psi_h) > c_* (\gamma_{\Psi_*} - 1) - C_4' h^{1/4}. \tag{15.50}$$

Since $|c_h - c_*| \le h^{3/4}$, we find

$$\begin{array}{lcl} c_h \Big[[\Psi_h^{(x)}]' + 1 \Big] & > & c_* (\gamma_{\Psi_*} - 1) + c_h - C_4' h^{1/4} \\ & > & c_* \gamma_{\Psi_*} - C_5' h^{1/4}. \end{array} \tag{15.51}$$

Since γ_{Ψ_*} is strictly bounded away from zero, uniformly in h, we conclude that

$$[\Psi_h^{(x)}]'(\tau) > -1 \tag{15.52}$$

for all sufficiently small $h > 0$ and all $\tau \in \mathbb{R}$. This shows that the coordinate transformation (2.53) is invertible, as desired. □

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