Buffon's needle problem

Chapter 21

A French nobleman, Georges Louis Leclerc, Comte de Buffon, posed the following problem in 1777:

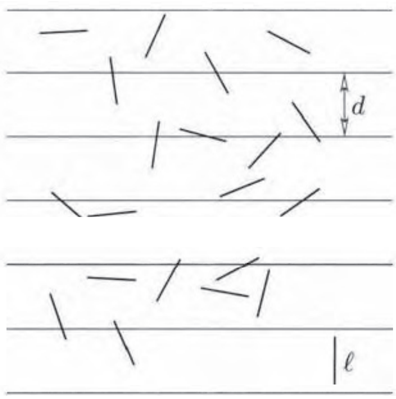
> ***Suppose that you drop a short needle on ruled paper — what is then the probability that the needle comes to lie in a position where it crosses one of the lines?***

Le Comte de Buffon

The probability depends on the distance d between the lines of the ruled paper, and it depends on the length ℓ of the needle that we drop — or rather it depends only on the ratio $\frac{\ell}{d}$. A *short* needle for our purpose is one of length $\ell \leq d$. In other words, a short needle is one that cannot cross two lines at the same time (and will come to touch two lines only with probability zero). The answer to Buffon's problem may come as a surprise: It involves the number π.

Theorem ("Buffon's needle problem")
If a short needle, of length ℓ, is dropped on paper that is ruled with equally spaced lines of distance $d \geq \ell$, then the probability that the needle comes to lie in a position where it crosses one of the lines is exactly

$$p = \frac{2}{\pi}\frac{\ell}{d}.$$



The result means that from an experiment one can get approximate values for π: If you drop a needle N times, and get a positive answer (an intersection) in P cases, then $\frac{P}{N}$ should be approximately $\frac{2}{\pi}\frac{\ell}{d}$, that is, π should be approximated by $\frac{2\ell N}{dP}$. The most extensive (and exhaustive) test was perhaps done by Lazzarini in 1901, who allegedly even built a machine in order to drop a stick 3408 times (with $\frac{\ell}{d} = \frac{5}{6}$). He found that it came to cross a line 1808 times, which yields the approximation $\pi \approx 2 \cdot \frac{5}{6}\frac{3408}{1808} = 3.1415929....$, which is correct to six digits of π, and much too good to be true! (The values that Lazzarini chose lead directly to the well-known approximation $\pi \approx \frac{355}{113}$; see page 31. This explains the more than suspicious choices of 3408 and $\frac{5}{6}$, where $\frac{5}{6}\,3408$ is a multiple of 355. See [5] for a discussion of Lazzarini's hoax.)

The needle problem can be solved by evaluating an integral. We will do that below, and by this method we will also solve the problem for a long needle. But the Book Proof, presented by E. Barbier in 1860, needs no integrals. It just drops a different needle ...

If you drop *any* needle, short or long, then the expected number of crossings will be

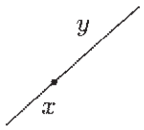
$$E = p_1 + 2p_2 + 3p_3 + \dots,$$

where p_1 is the probability that the needle will come to lie with exactly one crossing, p_2 is the probability that we get exactly two crossings, p_3 is the probability for three crossings, etc. The probability that we get at least one crossing, which Buffon's problem asks for, is thus

$$p = p_1 + p_2 + p_3 + \dots$$

(Events where the needle comes to lie exactly on a line, or with an end-point on one of the lines, have probability zero — so they can be ignored throughout our discussion.)

On the other hand, if the needle is *short* then the probability of more than one crossing is zero, $p_2 = p_3 = \dots = 0$, and thus we get $E = p$: The probability that we are looking for is just the expected number of crossings. This reformulation is extremely useful, because now we can use linearity of expectation (cf. page 84). Indeed, let us write $E(\ell)$ for the expected number of crossings that will be produced by dropping a straight needle of length ℓ. If this length is $\ell = x + y$, and we consider the "front part" of length x and the "back part" of length y of the needle separately, then we get



$$E(x + y) = E(x) + E(y),$$

since the crossings produced are always just those produced by the front part, plus those of the back part.

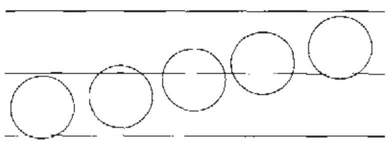
By induction on n this "functional equation" implies that $E(nx) = nE(x)$ for all $n \in \mathbb{N}$, and then that $mE(\frac{n}{m}x) = E(m\frac{n}{m}x) = E(nx) = nE(x)$, so that $E(rx) = rE(x)$ holds for all *rational* $r \in \mathbb{Q}$. Furthermore, $E(x)$ is clearly monotone in $x \geq 0$, from which we get that $E(x) = cx$ for all $x \geq 0$, where $c = E(1)$ is some constant.

But what is the constant?

For that we use needles of different shape. Indeed, let's drop a "polygonal" needle of total length ℓ, which consists of straight pieces. Then the number of crossings it produces is (with probability 1) the sum of the numbers of crossings produced by its straight pieces. Hence, the expected number of crossings is again

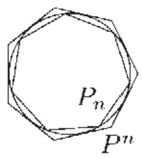
$$E = c\,\ell,$$

by linearity of expectation. (For that it is not even important whether the straight pieces are joined together in a rigid or in a flexible way!)



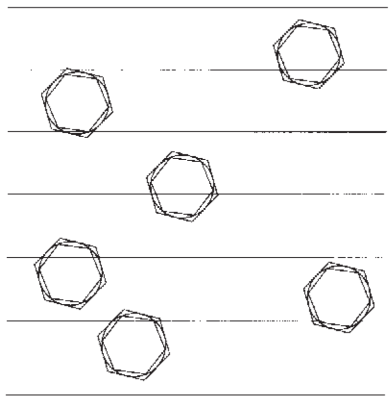
The key to Barbier's solution of Buffon's needle problem is to consider a needle that is a perfect circle C of diameter d, which has length $x = d\pi$. Such a needle, if dropped onto ruled paper, produces exactly two intersections, always!

The circle can be approximated by polygons. Just imagine that together with the circular needle C we are dropping an inscribed polygon P_n, as well as a circumscribed polygon P^n. Every line that intersects P_n will also intersect C, and if a line intersects C then it also hits P^n. Thus the expected numbers of intersections satisfy



$$E(P_n) \ \leq \ E(C) \ \leq \ E(P^n).$$

Now both P_n and P^n are polygons, so the number of crossings that we may expect is "c times length" for both of them, while for C it is 2, whence



$$c\,\ell(P_n) \ \leq \ 2 \ \leq \ c\,\ell(P^n). \qquad (1)$$

Both P_n and P^n approximate C for $n \longrightarrow \infty$. In particular,

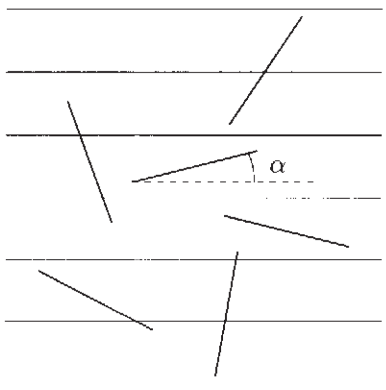
$$\lim_{n\to\infty} \ell(P_n) \ = \ d\pi \ = \ \lim_{n\to\infty} \ell(P^n),$$

and thus for $n \longrightarrow \infty$ we infer from (1) that

$$c\,d\pi \ \leq \ 2 \ \leq c\,d\pi,$$

which gives $c = \frac{2}{\pi}\frac{1}{d}$. □

But we *could* also have done it by calculus! The trick to obtain an "easy" integral is to first consider the slope of the needle; let's say it drops to lie with an angle of α away from horizontal, where α will be in the range $0 \leq \alpha \leq \frac{\pi}{2}$. (We will ignore the case where the needle comes to lie with negative slope, since that case is symmetric to the case of positive slope, and produces the same probability.) A needle that lies with angle α has height $\ell \sin\alpha$, and the probability that such a needle crosses one of the horizontal lines of distance d is $\frac{\ell\sin\alpha}{d}$. Thus we get the probability by averaging over the possible angles α, as



$$p \ = \ \frac{2}{\pi}\int_0^{\pi/2} \frac{\ell\sin\alpha}{d}\,d\alpha \ = \ \frac{2}{\pi}\frac{\ell}{d}\Big[-\cos\alpha\Big]_0^{\pi/2} \ = \ \frac{2}{\pi}\frac{\ell}{d}.$$

For a long needle, we get the same probability $\frac{\ell\sin\alpha}{d}$ as long as $\ell\sin\alpha \leq d$, that is, in the range $0 \leq \alpha \leq \arcsin\frac{d}{\ell}$. However, for larger angles α the needle *must* cross a line, so the probability is 1. Hence we compute

$$\begin{aligned} p \ &= \ \frac{2}{\pi}\Big(\int_0^{\arcsin(d/\ell)} \frac{\ell\sin\alpha}{d}\,d\alpha \ + \int_{\arcsin(d/\ell)}^{\pi/2} 1\,d\alpha\Big) \\ &= \ \frac{2}{\pi}\Big(\frac{\ell}{d}\Big[-\cos\alpha\Big]_0^{\arcsin(d/\ell)} + \Big(\frac{\pi}{2} - \arcsin\frac{d}{\ell}\Big)\Big) \\ &= \ 1 + \frac{2}{\pi}\Big(\frac{\ell}{d}\Big(1 - \sqrt{1-\frac{d^2}{\ell^2}}\Big) - \arcsin\frac{d}{l}\Big) \end{aligned}$$

for $\ell \geq d$.

So the answer isn't that pretty for a longer needle, but it provides us with a nice exercise: Show ("just for safety") that the formula yields $\frac{2}{\pi}$ for $\ell = d$, that it is strictly increasing in ℓ, and that it tends to 1 for $\ell \longrightarrow \infty$.