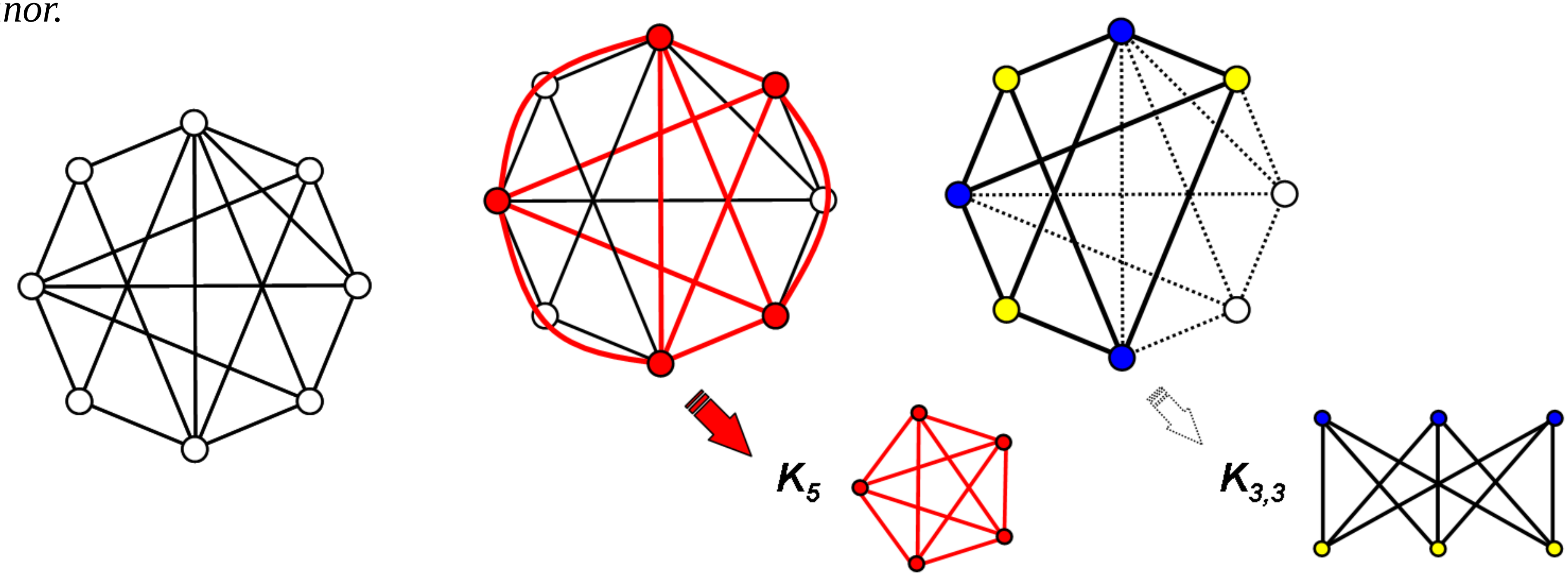
THEOREM OF THE DAY

Kuratowski's Theorem *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a topological minor.*



Is the graph on the left *planar*? That is, can it be redrawn so that edges only intersect each other at one of the eight nodes? Confirming a positive answer is potentially easy: just show the redrawing. But the negative seems elusive: we might try a million unsuccessful attempts but feel we *nearly* have it. Kuratowski's theorem says that nonplanarity is confirmed as soon as we exhibit either K_5 or $K_{3,3}$ as a *topological minor*. H is a topological minor of G if it appears as a subgraph of G but with its edges replaced by internally disjoint paths (edge-sequences which share only their end points). In the picture, three of the edges of K_5 appear as paths of length two; $K_{3,3}$ is actually a subgraph (all the disjoint paths are just the original edges). In this graph *both* of the forbidden topological minors are present but either one alone is enough to prevent planarity.

The Polish mathematician Kasimierz Kuratowski proved his theorem in 1930. Forty years later the dedication of Frank Harary's classic text on graph theory was:

> To KASIMIR KURATOWSKI,
> Who gave K_5 and $K_{3,3}$
> To those who thought planarity
> Was nothing but topology.

Web link: www.ams.org/featurecolumn/archive/gminor.html

Further reading: *Graph Theory* by Frank Harary, new paperback edition, Perseus Books, 1999.