Probabilistic and Game Theoretic solutions to The Three Doors Problem

Richard Gill*

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Abstract

This short paper was inspired by discussions on the Monty Hall problem at a work-shop on legal and scientific proof. I react to Fred Vos' insistence that the choice to be made by the quiz-master is not necessarily random. I allow the quiz-master to make a deterministic choice, or a randomized choice with any conditional probabilities whatever, given the course of the game so far. We now have a game between the player and the quiz-team consisting of the people who set up the stage, in advance, and which includes the quiz-master, who knows the location of the car. I assume only that the player wants to win the car and the quiz-team wants the player to win the goat. The solution is that all choices should be made uniformly at random, both by quiz-team and player, and the player should always switch. I also take account of Gerhard Ris' point, that the quiz-master might sometimes wish a particularly sympathetic player to win. Unfortunately this change to the game does not change the minimax solution.

The basic problem, probabilistic solution

I assume that the quiz-master (who knows the location of the car) *always* opens a door revealing a goat; and I suppose that the player prefers to win a car to a goat. Let's name the three doors A, B and C, from left to right as seen by the audience. The quiz-master knows the location of the car and may name the doors α, β, γ where the car stands behind door number α, and goats stand behind the doors numbered β and γ, being the leftmost and the rightmost of the two doors concealing a goat, as seen from the audience.

*Mathematical Institute, Leiden University; `http://www.math.leidenuniv.nl/~gill`

I suppose that the player has a chance of $\frac{1}{3}$ of initially picking the correct door. This would be realized if the quiz-team fixes the door with the car behind it by a secret toss, in advance, of an unbiased three-sided die. Then assuming the player can't for instance *smell* where is the car and where are the goats, and has no powers of extra-sensory perception either, then even if he does have some systematic bias to choose a particular door A or B or C, the chance is still $\frac{1}{3}$ that he picks the door named α by the quiz-master.

If the player chooses doors β or γ, then the quiz-master has no choice but to open the other of these two doors with goats behind them. On the other hand, if the player chooses door α, then the quiz-master does have a choice; he can open door β or door γ. His choice might or might not be random, and it might depend on the true labeling of the doors (e.g., maybe he has a bias to A, or B, or C). For what I am going to calculate I only need to specify the probabilities, *given* that the player has chosen α, that the quiz-master opens β or γ. In fact, we will see that what I am actually going to calculate does not even depend on the values of these two conditional probabilities.

Given that the player has chosen door α, then when the quiz-master opens γ he leaves β closed, and vice-versa. Let me define p_β as the conditional probability that he leaves door β closed when the player chooses door α, and define p_γ as the conditional probability that he leaves door γ closed when the player chooses door α. All we know about these conditional probabilities is that $p_\beta + p_\gamma = 1$.

In combination therefore, the probability that the player chooses α *and* door β is left closed by the quiz-master equals $\frac{1}{3}p_\beta$; the probability that the player chooses door α and door γ is left closed by the quizmaster equals $\frac{1}{3}p_\gamma$. If on the other hand the player chooses door β or γ then the quiz-master opens the other, and leaves door α closed. Therefore the probability that the player chooses door β *or* γ and door α is left closed by the quizmaster equals $\frac{2}{3}$.

The probability that the door left closed by the quiz-master conceals the car is therefore $\frac{2}{3}$, and the probability that the door left closed by the quiz-master conceals a goat is $\frac{1}{3}p_\beta + \frac{1}{3}p_\gamma \ = \ \frac{1}{3}$.

The game-theoretic point of view

The player could let his choice of door depend on their names A, B, C, and the quiz-master could let his choice of door to leave closed (when he has a choice) also depend on A, B and C. Moreover, the quiz-team could use other probabilities than uniform-at-random for hiding the car in the first place. These choices create a zero-sum game (in the game-theoretic sense), played by the quiz-team (including the quiz-master) on the one hand, and the player on the other hand. The game has a saddle-point and the minimax solution is the solution where all possible choices are taken uniform at random out of the available choices, and where the player always switches.

This is because when the quiz-team (including the quiz-master) uses the strategy of uniform random choices, the player has *at most* a chance of $\frac{2}{3}$ of winning (and this is achieved by choosing his door uniformly at random and always switching doors). Moreover, *whatever* strategy is chosen by quiz-team, if the player chooses a door uniformly at random and subsequently *always* switches doors, then he always does win with probability $\frac{2}{3}$. These strategies are therefore minimax strategies and the game has a saddle-point and hence a value, which is $\frac{2}{3}$. (Von Neumann's theorem tells us indeed that this zero-sum game has a value: that is because the player and the quiz-team are allowed randomized strategies and because the number of deterministic strategies for each side is finite).

We could even give the quiz-master the option of opening a door revealing a car. This addition to his strategy space naturally does not do him any good. However it could be that sometimes the quiz-master would like a particularly sympathetic player to win. If we rule out any kind of communication between the quiz-master and the player, then the only way for the quiz-master to improve a sympathetic player's chance of winning is to actually open the right door (we assume the player would then switch to *that* door!). This is because the player does not know whether or not the quiz-master is "on his side", therefore he has to stick to his minimax strategy. It might be more effective for the quiz-master to give a sympathetic player a course in probability theory, in advance of the show.

For completeness, here is a complete list of the possible deterministic strategies of the two players. I give the doors here numbers, 1, 2 and 3.

Quiz-team's deterministic strategies

Car behind door 1 and QM will open door 2 when player chooses door 1
Car behind door 1 and QM will open door 3 when player chooses door 1
Car behind door 2 and QM will open door 1 when player chooses door 2
Car behind door 2 and QM will open door 3 when player chooses door 2
Car behind door 3 and QM will open door 1 when player chooses door 3
Car behind door 3 and QM will open door 2 when player chooses door 3

The player's deterministic strategies

Choose door 1 & switch if QM opens 2, switch if QM opens 3
Choose door 1 & switch if QM opens 2, don't switch if QM opens 3
Choose door 1 & don't switch if QM opens 2, switch if QM opens 3
Choose door 1 & don't switch if QM opens 2, don't switch if QM opens 3
Choose door 2 & switch if QM opens 1, switch if QM opens 3
Choose door 2 & switch if QM opens 1, don't switch if QM opens 3
Choose door 2 & don't switch if QM opens 1, switch if QM opens 3
Choose door 2 & don't switch if QM opens 1, don't switch if QM opens 3
Choose door 3 & switch if QM opens 1, switch if QM opens 2
Choose door 3 & switch if QM opens 1, don't switch if QM opens 2
Choose door 3 & don't switch if QM opens 1, switch if QM opens 2
Choose door 3 & don't switch if QM opens 1, don't switch if QM opens 2

The saddle point is that the quiz-team uses the uniform randomized strategy over all 6 deterministic strategies; the player uses the uniform randomized strategy over those 3 strategies of his possible total of 12, in which he always switches.

As remarked before, if the quiz-team uses the just defined randomized strategy, then the player's chance of winning cannot be larger than $\frac{2}{3}$; when the player uses his just defined randomized strategy, he wins with probability $\frac{2}{3}$, independently of the quiz-team's strategy. Since by the theorem of von Neumann a finite zero-sum game does have a value when randomized strategies are allowed, these two strategies must define the saddle-point of the game and its value is $\frac{2}{3}$.