Supplement to my solution of the three doors problem

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In the three doors problem we have the choice of the quiz-Team (if you like, the True door), the choice of the Player, and the choice of Quizmaster. I will denote these by T, P, Q. The player must choose between keeping with his own door and switching to the Other closed door, which I will denote O. These variables take values in the set $\{1, 2, 3\}$. I'll assume that T and P are statistically independent. We know that $Q \neq T$ and that (P, Q, O) is a permutation of $\{1, 2, 3\}$.

One may be interested in $\Pr(O = T)$, or one may be interested in $\Pr(O = T|P = 1, Q = 2)$: these are the unconditional and the conditional probabilities (in a particular situation) that a player ought to switch doors.

I consider the game built up in three stages, namely the determination of T, P, Q in turn by the quizteam, the player and the quizmaster. I consider the joint probability law of the triple as correspondingly built up from law(T), law$(P|T)$, and law$(Q|T, P)$. By assumption, for the second component we have law$(P|T) =$ law(P).

Concerning the unconditional probability, one sees immediately that

$$\Pr(O = T) \;=\; 1 - \Pr(P \neq T) \;=\; \frac{2}{3}$$

if either $P \sim \text{Unif}$ or $Q \sim \text{Unif}$.

Regarding the conditional probability, we have

$$\Pr(O = T|P = 1, Q = 2) \;=\; \Pr(T = 3|P = 1, Q = 2) \;=\; \frac{\Pr(T = 3, P = 1, Q = 2)}{\Pr(P = 1, Q = 2)}$$

$$= \frac{\Pr(T = 3)\Pr(P = 1)\Pr(Q = 2|T = 3, P = 1)}{\Pr(T = 3)\Pr(P = 1)\Pr(Q = 2|T = 3, P = 1) + \Pr(T = 1)\Pr(P = 1)\Pr(Q = 2|T = 1, P = 1)}$$

$$= \frac{1}{1 + \Pr(Q = 2|T = 1, P = 1)}$$

where the last step results from first canceling out $\Pr(P = 1)$, then using $\Pr(T = 1) = \Pr(T = 3)$ and canceling, and finally using $\Pr(Q = 2|T = 3, P = 1) = 1$.

Since $\Pr(Q = 2|T = 1, P = 1)$ is at most 1 and at least 0, $\Pr(O = T|P = 1, Q = 2)$ is at least $\frac{1}{2}$ and at most 1.

Conclusions. As long at $T \sim \text{Unif}$ is never hurts to switch and it does sometimes pay (it certainly pays on average). As long as $P \sim \text{Unif}$ it pays on average to switch. Regarding the conditional probability however, we only obtain in this case $\Pr(O = T|P = 1, Q = 2) = \Pr(T = 3)/(\Pr(T = 3) + \Pr(T = 1)\Pr(Q = 2|T = 1, P = 1))$ which can be as low as 0, namely when $\Pr(T = 3) = 0$. This is where the game theoretic approach offers consolation. Whatever strategy is chosen by the quizteam and quizmaster, the player who chooses uniformly at random and always switches is guaranteed a win-chance of 2/3, and there is no way to guarantee a better win-chance. Too bad that he then loses when the car happens to be behind door 1 and he chooses door 1 (so that the quizmaster chooses 2 and he switches). It's all part of his game. Of course he always loses if the quizteam never chooses 3, on those occasions when he chooses 1 and the quizmaster chooses 2.

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