Morgan, J. P., Chaganty, N. R., Dahiya, R. C., and Doviak, M. J. (1991), "Let's Make a Deal: The Player's Dilemma," *The American Statistician*, 45 (4), 284–287: Comment by Hogbin and Nijdam and Response

In the referred paper the well-known Monty Hall Problem is extensively analyzed by the authors. In the calculation of the probability of winning the car by switching in the Bayesian approach, Morgan et al. calculate an average using the uniform prior resulting in a value of log(2). By using the posterior probability after the host has revealed a goat behind door No. 3, we arrive at a different result of 2/3.

For a clear understanding the Monty Hall Problem will be presented in a nutshell.

A player is given the choice of one of three doors. Behind one of the doors is a car, placed randomly, and behind the other two are goats. In the presented situation the player has chosen door No. 1, and before opening the chosen door, Monty Hall, the host, opens door No. 3, revealing a goat. Monty then offers the player to alter her choice and switch to door No. 2.

The analysis normally goes along the lines of calculating the (conditional) probability of winning the car by switching in the given situation. For any other combination of chosen and opened doors the analysis is similar.

We adopt the notation: C_1 is the event of the car being behind door No. 1, and similar C_2 and C_3, and D_3 the event of door No. 3 opened by the host. The host will only open a door with a goat behind it, and given the choice between two doors with a goat, the host may in general show a preference for one of the doors, modeled by the parameter:

$$q = P(D_3|C_1). \tag{1}$$

The player will base her decision on the probability in her situation, that is, the conditional probability given door No. 3 opened, of the car being behind door No. 1 or equivalently on the complementary probability for door No. 3:

$$\begin{aligned} P(C_1|D_3) &= 1 - P(C_2|D_3) \\ &= \frac{P(D_3|C_1)P(C_1)}{P(D_3|C_1)P(C_1) + P(D_3|C_2)P(C_2)} \\ &= \frac{q}{1+q}. \end{aligned} \tag{2}$$

This comes down to 1/3 in the case $q = 1/2$, showing that no information is revealed about the car being behind door No. 1. For door No. 2 it is different. Originally the car is found there with probability 1/3, but opening door No. 3 increases this probability to 2/3.

In a Bayesian approach to the problem the authors model the preference of the host as the value q of a random parameter Q, with prior density f_Q, independent of the position of the car. Given this value, the relevant probabilities are written:

$$P(D_3|C_1, Q=q) = q \qquad \text{and} \qquad P(D_3|C_2, Q=q) = 1. \tag{3}$$

Hence

$$\begin{aligned} &P(C_2|D_3, Q=q) \\ &= \frac{P(D_3|C_2, Q=q)P(C_2|Q=q)}{P(D_3|C_2, Q=q)P(C_2|Q=q) + P(D_3|C_1, Q=q)P(C_1|Q=q)} \\ &= \frac{1}{1+q}. \end{aligned} \tag{4}$$

As

$$\begin{aligned} P(D_3|C_1) &= \int P(D_3|C_1, Q=q) f_Q(q|C_1)\,dq \\ &= \int q f_Q(q|C_1)\,dq \\ &= \int q f_Q(q)\,dq = E(Q), \end{aligned} \tag{5}$$

the posterior probability of winning the car by switching is then

$$P(C_2|D_3) = \frac{P(D_3|C_2)P(C_2)}{P(D_3|C_2)P(C_2) + P(D_3|C_1)P(C_1)} = \frac{1}{1+E(Q)} \tag{6}$$

showing a value 2/3 for all priors with expectation 1/2.

The main merit of the Bayesian approach seems to be the replacement of the (unknown) value q of the parameter Q by its (unknown) expectation.

The above result can also be obtained by the following calculation:

$$\begin{aligned} P(C_2|D_3) &= \int P(C_2|D_3, Q=q) f_Q(q|D_3)\,dq \\ &= \int \frac{1}{1+q}\,\frac{1+q}{1+E(Q)} f_Q(q)\,dq, \end{aligned} \tag{7}$$

in which the posterior density of Q, given the opened door No. 3, is

$$\begin{aligned} f_Q(q|D_3) &= \frac{P(D_3|Q=q) f_Q(q)}{P(D_3)} \\ &= const \cdot f_Q(q)\{P(D_3|C_1, Q=q) + P(D_3|C_2, Q=q)\} \\ &= const \cdot f_Q(q)(q+1) = \frac{1+q}{1+E(Q)} f_Q(q), \end{aligned} \tag{8}$$

showing how information is obtained about Q by the opening of door No. 3.

On page 286 of the referred article this calculation of the probability of winning by switching given a uniform prior distribution of q by Morgan et al. reads (the event of winning by switching is called W_s):

$$\begin{aligned} \Pr(W_s|D_3) \equiv P(C_2|D_3) &= \int P(C_2|D_3, Q=q) f_Q(q)\,dq \\ &= \int \frac{1}{1+q}\,dq = \log(2). \end{aligned} \tag{9}$$

Morgan et al. integrate over the prior density instead of over the posterior, and thus do not take into account the information revealed by the opening of door No. 3. As the prior distribution is uniform, the revelation of the goat changes the information on Q, resulting in a posterior distribution given D_3 not being uniform.

The use of the proper distribution leads to the value of 2/3, rather than the surprising value of log(2) in Morgan et al.'s analysis, due to omitting the dependency of Q and D_3.

Martin HOGBIN and W. NIJDAM

Response

Our kind thanks to Mr. Hogbin and Dr. Nijdam for correcting our mistake. We will add that should the player have observed any previous plays of the game, those data, too, will modify the prior, and can produce posterior calculations other than 2/3 even with a symmetric prior. This, of course, is something else that we should have pursued. In any case, it should not distract from the essential fact that $1/(1+q) \geq 1/2$ regardless of q. Simply put, *if the host must show a goat, the player should switch.*

We take this opportunity to address another issue related to our article, one that arose in vos Savant's (1991) reply and in Bell's (1992) letter, and has come up many times since. To wit, had we adopted conditions implicit in the problem, the answer is 2/3, period. We maintained in our article and rejoinder that we simply wanted to answer the reader's question as posed without enforcing unstated conditions. Shortly thereafter we discovered that the reader's question as it appeared in *Ask Marilyn* had been edited, and that the actual question allowed much more leeway than vos Savant was willing to admit. We reproduce here, for the first time, the key section from Craig F. Whitaker's letter to vos Savant, exactly as he wrote it (including grammatical errors; a copy of the original letter was obtained from Mr. Whitaker, and provided to us, by Mike D'Orso, then of the *Virginian Pilot*):

> "I've worked out two different situations (based on Monty's prior behavior i.e. weather or not he knows what's behind the doors) in one situation it is to your advantage to switch, in the other there is no advantage to switch."
>
> "What do you think?"

In the context of the calculations here, which themselves are for a restricted version of the problem, there is no advantage to switch when $q = 1$ and hence $1/(1+q) = 1/2$. More pertinent, if the host randomly opens one of the two unselected doors (Monty does not know where the car is), then the more general problem as presented in our article shows the probability of winning by switch-

 DOI: 10.1198/tast.2010.09227

ing is again 1/2. Mr. Whitaker, if after twenty years you are still interested, this is your answer.

J. P. MORGAN
Virginia Tech
N. R. CHAGANTY
Old Dominion University
R. C. DAHIYA
Old Dominion University
M. J. DOVIAK
Old Dominion University

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