LINEAR FORMS IN LOGARITHMS I: COMPLEX AND p-ADIC

Abstract. We collect some useful and relatively recent results from the theory of linear forms in logarithms.

1. Linear forms in complex logarithms

In 1900, The 7th of Hilbert's 23 problems for the International Congress of Mathematics was the following : If $\alpha \neq 0, 1$ and β are algebraic numbers with β irrational, prove that α^β is transcendental. This was proved, independently, in 1934 by Gelfond and Schneider and is equivalent to showing, for a given choice of branch of the logarithm and algebraic nonzero γ, the nonvanishing of

$$|\beta \log \alpha - \log \gamma| .$$

The extension of this result to several logarithms of algebraic numbers is due to Baker.

For the purposes of applications, however, it is more important to know more than that a linear form in logarithms is nonvanishing. Indeed, we would like to have a lower bound upon its modulus. Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers distinct from 0 and 1, and take $\log \alpha_1, \dots, \log \alpha_n$ to be any determination of their logarithms. Let $b_1, \dots, b_n$ be non-zero integers such that

$$\Lambda_a := |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n|$$

is non-zero. Instead of making an historical survey of lower bounds, we will just quote a corollary of the, at present time, best estimate, due to Matveev [2]. Let D be the degree of a number field $\mathbb{K}$ containing the α_i, let $E \geq e$ and $A_1, \dots, A_n$ be real numbers > 1 with

$$\log A_i \geq \max\big\{D\mathrm{h}(\alpha_i), |\log \alpha_i|, 0.16\big\}, \qquad 1 \leq i \leq n.$$

Set

$$B = \max\{|b_1|, \dots, |b_n|\}.$$

Next results are corollaries of Theorem 2 of Matveev [2].

Theorem 1. *We have*

$$\log |\Lambda_a| > -2 \times 30^{n+4}\,(n+1)^6\,D^2\,\log(eD) \log A_1 \dots \log A_n \log(eB). \quad (6)$$

Date: May 4, 2007.

Set

$$B' = \max_{1 \le j < n} \left\{ \frac{|b_n|}{\log A_j} + \frac{|b_j|}{\log A_n} \right\}$$

Theorem 2. *We have*

$$\log|\Lambda_a| > -2\times 30^{n+4}\,(n+1)^6\,D^2\,\log(eD)\log A_1 \ldots \log A_n \log(eB). \quad (6)$$

To give an example of an estimate involving an auxiliary parameter E, we display a consequence of Corollaire 3 of Laurent, Mignotte and Nesterenko [1].

Our notation is the following. Let x_1/y_1 and x_2/y_2 be multiplicatively independent rational numbers, both > 1. Let b_1 and b_2 be positive rational integers and consider the linear form

$$\Lambda = b_2 \log(x_2/y_2) - b_1 \log(x_1/y_1).$$

Let A_1 and A_2 be real numbers such that

$$\log A_i \ge \max\{\log x_i, 1\}, \quad (i = 1, 2).$$

Theorem 3. *Keep the above notation. Let $E \ge 3$ be a real number such that*

$$E \le 1 + \min\left\{ \frac{\log A_1}{\log(x_1/y_1)}, \frac{\log A_2}{\log(x_2/y_2)} \right\},$$

and set

$$\log B = \max\left\{ \log\left(\frac{b_1}{\log A_1} + \frac{b_2}{\log A_2} \right) + \log\log E + 0.47, 10\log E \right\}.$$

Assuming that $E \le \min\{A_1^{3/2}, A_2^{3/2}\}$, we have

$$\log \Lambda \ge -35.1\,(\log A_1)(\log A_2)(\log B)^2(\log E)^{-3}. \quad (4)$$

2. Linear forms in p-adic logarithms

The p-adic analogue of Baker's theory has been studied by many authors, and we choose to quote only a recent result of Yu [3].

Keep the above notation. Let $\mathcal{P}$ be a prime ideal in $\mathbb{K}$, lying above the prime number p. Denote by v_p the extension of the usual p-adic absolute value to $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. We identify the completion of $\mathbb{K}$ with respect to $\mathcal{P}$, denoted by $\mathbb{K}_{\mathcal{P}}$, with a sub-field of $\mathbb{C}_p$ and we wish to state an upper bound for

$$\Lambda_u := v_p\big(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1\big),$$

which is finite, since we have assumed that $\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1$ is non-zero. Let $A_1, \ldots, A_n$ be real numbers with

$$\log A_i \geq \max\{\mathrm{h}(\alpha_i), \log p\}, \qquad 1 \leq i \leq n.$$

Theorem 4. *With the above notation, we have*

$$\Lambda_u < 12 \left(\frac{6(n+1)D}{\sqrt{\log p}} \right)^{2(n+1)} p^D \log(e^5 nD) \log A_1 \ldots \log A_n \log B. \quad (7)$$

This result has subsequently been improved by Yu, essentially eliminating the n^n term.

References

[1] M. Laurent, M. Mignotte and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.

[2] E.M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II, *Izvestiya Math.* **64** (2000), 1217–1269.

[3] K. Yu, p-adic logarithmic forms and group varieties I, *J. Reine Angew. Math.* **502** (1998), 29–92.