8. Theorem 2.21 of the lecture notes (Kronecker’s approximation theorem) can be refined as follows:

Let \( \xi_1, \ldots, \xi_n \) be reals such that \( 1, \xi_1, \ldots, \xi_n \) are linearly independent over \( \mathbb{Q} \). Then for every \( \varepsilon > 0 \), \( \theta_1, \ldots, \theta_n \in \mathbb{R} \), \( y_0 > 0 \), there are \( x_1, \ldots, x_n, y \in \mathbb{Z} \) such that

\[
|\xi_1 y - x_1 - \theta_1| < \varepsilon, \ldots, |\xi_n y - x_n - \theta_n| < \varepsilon, \quad y > y_0
\]

(the condition that \( y > y_0 \) is new). To get this, the proof of Theorem 2.21 has to be modified as follows. On page 31, choose \( b = (\varepsilon^{-1} \theta_1, \ldots, \varepsilon^{-1} \theta_n, 2M\varepsilon)^T \). Then if one is able to show that there is \( x = (x_1, \ldots, x_n, y)^T \in \mathbb{Z}^{n+1} \) with

\[
\|Ax - b\|_2 \leq \varepsilon,
\]

it follows that

\[
\sum_{i=1}^n (x_i - \xi_i y - \theta_i)^2 + M^{-2}(y - 2M\varepsilon)^2 < \varepsilon^2.
\]

This shows that \( |\xi_i y - x_i - \theta_i| < \varepsilon \) for \( i = 1, \ldots, n \) and \( |y - 2M\varepsilon| < M\varepsilon \), hence \( y > M\varepsilon \). Now the proof of Theorem 2.21 can be followed without any changes, except that on page 32, one has to choose \( M > \max(R, R/\mu, y_0/\varepsilon) \), where \( R = (n + 1) \cdot c(n + 1)/2\varepsilon \).

4 a) Deduce the following result from the above refinement of Kronecker’s Theorem:

Let \( \xi_1, \ldots, \xi_n \) be real numbers, linearly independent over \( \mathbb{Q} \). Then for every \( t_0 > 0, \varepsilon > 0, \theta_1, \ldots, \theta_n \in \mathbb{R} \), there are \( t \in \mathbb{R} \) with \( t > t_0 \), and \( x_1, \ldots, x_n \in \mathbb{Z} \), such that

\[
|\xi_1 t - x_1 - \theta_1| < \varepsilon, \ldots, |\xi_n t - x_n - \theta_n| < \varepsilon
\]

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so compared with Kronecker’s Theorem, we have weakened the condition that \(\{1, \xi_1, \ldots, \xi_n\}\) is linearly independent over \(\mathbb{Q}\) to \(\{\xi_1, \ldots, \xi_n\}\) linearly independent over \(\mathbb{Q}\), but instead of an unknown \(y\) assuming integer values we have an unknown \(t\) assuming real values).

6b) A star has \(n\) planets, all whose orbits are circular with the star in the center and lie in the same plane. Each planet has a constant angular velocity with which it traverses its orbit. Prove that the planets are in almost the same direction infinitely often (i.e., for every \(\varepsilon > 0\) there are arbitrarily large \(t\) such that at time \(t\), seen from the star the directions of the planets are within an angle \(\varepsilon > 0\) from each other) in each of the following two cases:

(i) they once have been in the same direction;
(ii) their angular velocities are linearly independent over \(\mathbb{Q}\).

Let \(\alpha \in \overline{\mathbb{Q}}\) be an algebraic number of degree \(d\).

The denominator of \(\alpha\) is the smallest positive \(m \in \mathbb{Z}\) such that \(m\alpha\) is an algebraic integer, notation \(\text{den}(\alpha)\).

The house of \(\alpha\) is defined by

\[
\overline{\alpha} := \max(|\alpha^{(1)}|, \ldots, |\alpha^{(d)}|)
\]

where \(d = \deg \alpha\) and \(\alpha^{(1)}, \ldots, \alpha^{(d)}\) denote the conjugates of \(\alpha\).

In the next exercises you are asked to prove some properties of the house.

19.a) Let \(\alpha\) be a non-zero algebraic integer. Prove that \(\overline{\alpha} \geq 1\).

3b) Let \(\alpha, \beta\) be algebraic integers. Prove that

\[
|\alpha + \beta| \leq |\alpha| + |\beta|, \quad |\alpha \beta| \leq |\alpha| \cdot |\beta|, \quad |\alpha^n| = |\alpha|^n \text{ for } n \in \mathbb{Z}_{>0}.
\]

c) Let \(\alpha\) be a non-zero algebraic integer of degree \(d\). Prove that \(H(\alpha) \leq (2|\overline{\alpha}|)^d\) (consider the minimal polynomial of \(\alpha\)).

d) Compute an explicit expression \(f(C, d)\) depending only on \(C\) and \(d\), such that the number of algebraic integers \(\alpha \in \mathbb{C}\) with \(|\overline{\alpha}| \leq C\), \(\deg \alpha \leq d\) is at most \(f(C, d)\).
e) Let $\alpha$ be a non-zero algebraic integer. Prove that $|\alpha| = 1 \iff \alpha$ is a root of unity.

f) Let $\alpha$ be a non-zero algebraic integer of degree $d$ which is not a root of unity. Compute an explicit expression $c(d) > 1$ depending only on $d$ such that $|\alpha| \geq c(d)$.

**Hint.** Consider the set $\{\alpha^n : 0 \leq n \leq n_0\}$ where $n_0$ is the largest integer $n$ such that $|\alpha|^n \leq 2$.

**Remark.** The Schinzel-Zassenhaus conjecture asserts that there is a constant $c > 0$ independent of $d$, such that $|\alpha| \geq 1 + c/d$ for every non-zero algebraic integer $\alpha$ of degree $d$ which is not a root of unity. Apart from the value of $c$ this is best possible, since $\left\lfloor \sqrt[d]{2} \right\rfloor = 2^{1/d}$ which is about $1 + (\log 2)/d$ for $d$ large. In 1979, Dobrowolski proved that there is a constant $c > 0$ such that

$$|\alpha| \geq 1 + \frac{c}{d} \cdot \left(\frac{\log \log 3d}{\log 3d}\right)^3.$$  

This had not been improved since. But very recently (last September!), Verger-Gaugry posted a manuscript of 164 pages on arXiv in which he claimed a proof of the Schinzel-Zassenhaus conjecture. Presumably, Verger-Gaugry has submitted his paper to a journal, and one or more referees are now checking it for correctness. This may take some time. To my knowledge, this has not been finished yet.

arXiv is a freely accessible preprint server on which researchers in mathematics and natural sciences can also freely post preprints of their papers, prior to their publication in a journal. arXiv does not require that preprints posted on it are refereed, so it can not be excluded that they contain errors. For mathematical preprints, go to the website xxx.lanl.gov, then scroll and click on mathematics. For those interested: if in the box 'Search or Artice-id' you type 'Verger-Gaugry' you will find the manuscript mentioned above.
10. Let $\alpha \in \overline{\mathbb{Q}}$ be a non-zero algebraic number of degree $d$ and denote by $\alpha^{(1)}, \ldots, \alpha^{(d)}$ the conjugates of $\alpha$. Prove that
\[
den(\alpha)^d \cdot \alpha^{(1)} \cdots \alpha^{(d)} \in \mathbb{Z}, \quad |\alpha| \geq den(\alpha)^{-d} \cdot |\alpha|^{1-d}.
\]

6 b) Using a), give a proof of the following inequality of Liouville (1844):
let $\alpha$ be an algebraic number in $\mathbb{R}$ of degree $d \geq 2$. Then there is a constant $c(\alpha) > 0$ such that
\[
|\alpha - \frac{x}{y}| \geq c(\alpha)y^{-d} \text{ for all } x, y \in \mathbb{Z} \text{ with } y > 0.
\]

c) Using b), prove that $\sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

11. The Lindemann-Weierstrass Theorem asserts that if $\beta_1, \ldots, \beta_n$ are non-zero algebraic numbers and $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, all in $\mathbb{C}$, then
$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0$. Deduce the following consequences. The functions
$\sin z$, $\cos z$ and $\tan z$ are defined on $\mathbb{C}$ by $e^{iz} = \cos z + i \sin z$, $e^{-iz} = \cos z - i \sin z$ and $\tan z = \sin z/\cos z$ whenever $\cos z \neq 0$.

1 a) For an algebraic number $\alpha \in \mathbb{C}$ with $\alpha \notin \{0, 1\}$, prove that $\log \alpha$ is transcendental, where $\log \alpha$ is any solution of $e^z = \alpha$.

4 b) For a non-zero algebraic number $\alpha \in \mathbb{C}$, prove that $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ are transcendental.

5 c) Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ that are linearly independent over $\mathbb{Q}$. Prove that $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

5 d) Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$. Denote by $\text{rank}_\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ the largest integer $m$ such that $\alpha_1, \ldots, \alpha_n$ contain $m$ elements that are linearly independent over $\mathbb{Q}$. Prove that
\[
\text{trdeg}(e^{\alpha_1}, \ldots, e^{\alpha_n}) = \text{rank}_\mathbb{Q}(\alpha_1, \ldots, \alpha_n).
\]