1) a) Since \( \deg \beta \) has degree 1, there are no \( \alpha \in \mathbb{Q} \), not all 0, with \( \alpha \beta + \beta = 0 \), i.e., \( \alpha \beta, \beta \) are linearly independent over \( \mathbb{Q} \). So \( \log \alpha, \log \beta \) are linearly independent over \( \mathbb{Q} \). Hence by Schanuel's conjecture:

\[
\text{irdeg} (\log \alpha, \alpha^d, \beta^d) = \text{irdeg} (\log \alpha, \log \beta, \alpha^d, \beta^d) \geq d.
\]

So \( \log \alpha, \alpha^d, \beta^d \) are algebraically independent. In particular, \( \alpha \beta, \beta \) are algebraically independent.

There are integers \( a, b \), with \( b \neq 0 \), such that:

\[ a \alpha + b \beta = 0. \]

Multiplying with \( \log \alpha \), and exponentiating gives:

\[ a \alpha^d + b \beta^d = 1. \]

Choose a positive integer \( m \) such that:

\[ m > a, m > b. \]

Then we see that:

\[ a \alpha^d, b \beta^d \]

is a zero of the polynomial \( f(x, y) = X_1 - a X_2 \cdots X_d \). Hence they are algebraically independent.

b) We have \( \text{irdeg} (e_1, e_2, e_3, e_4) = \text{irdeg} (1, 1, 1, 1) \geq 4 \), i.e., \( e_1, e_2, e_3, e_4 \) are algebraically independent, provided that \( 1, 1, 1, 1 \) are linearly independent over \( \mathbb{Q} \). To prove this, let \( a, b, c, d \) be rationals with \( a + b \pi + c \pi^2 + d \pi^3 \neq 0 \).

Then \( a + (b + c \pi + d \pi^2) \pi = 0. \) Since \( \pi \) is transcendental, we have \( b + c \pi + d \pi^2 = 0 \), hence \( a = 0. \) Since \( e \in \mathbb{Q} \), we have \( e = 0. \) Since \( e \in \mathbb{Q} \) we have \( e = 0. \) Hence \( b = 0. \)
2) a) We proceed by induction on \( r \). First let \( r = 1 \). Let \( x, y \in C \) and \( x, y \in \mathbb{R} \) with \( |x|, |y| \leq 1 \) but \( t = \frac{x+y}{2} \). Then \( x, y \in C \). Since \( C \) is a cscb, we have \( x, -y \in C \) and so \( t \cdot q(x - y, -y) = x, y \in C \).

Now let \( r \geq 2 \) and assume the assertion is true for fewer than \( r \) points. Let \( x_1, x_2, x_3, \ldots, x_r \in \mathbb{R} \) with \( |x_i| < 1 \) let \( \lambda := (x_1 + \cdots + x_r) \), assume w.l.o.g. \( \lambda > 0 \) (which we may since \( \lambda \in C \)), put \( \lambda' := |x_1|/\lambda \), and let \( t_i := \lambda i \) with the same sign as \( x_i \). Then \( x_1 + \cdots + x_r = \lambda (x_1 + \cdots + x_r) \).

where \( b_i, x_i \in C \), \( x_i > 0 \), \( x_1 + \cdots + x_r = 1 \). Assume \( x_1 + \cdots + x_r > 0 \); otherwise \( x_i = -x_i \), \( x_i \in C \) as we have seen above.

By the induction hypothesis, \( C \) contains \( \lambda(x_1/\lambda, \ldots, x_r/\lambda) \). and by the convexity of \( C \), \( \mu \in C \) for any \( \mu \in \mathbb{R} \).

From the case \( r = 1 \), we then infer that \( x_1 + \cdots + x_r \in C \).

b) Let \( C \) be a cscb in \( \mathbb{R}^n \) and \( L \) a lattice in \( \mathbb{R}^n \).

Then the successive minima \( \lambda_i \) of \( C \) with respect to \( L \) are defined as follows: \( \lambda_i \) is the minimum of all \( \lambda > 0 \) such that \( \lambda C \) contains at least \( i \) linearly independent points. Now Minkowski's theorem asserts that

\[
\frac{2^n}{n!} \cdot \text{vol}(L) = \lambda_1 \lambda_2 \cdots \lambda_n \cdot \text{det}(C),
\]

where \( \text{det}(C) \) is the determinant of \( C \).

c) We have \( x, y \in C \) for \( i = 1, \ldots, n \), so

\[
\text{dist}(x, C) = \sum_{i=1}^n \langle x_i, (A_i)^{-1} \rangle = \sum_{i=1}^n |x_i|,
\]

where \( A_i \) is the image of \( C_i \) under the linear map

\[
y \mapsto (y, y) \in \mathbb{R}^n \times \mathbb{R}^n \) which has determinant \( \det(A_i) = \prod_{i=1}^n \text{vol}(C_i). \) Therefore \( \text{vol}(L) = \frac{d(m)}{n!} \cdot \text{vol}(C) \),

where \( d(m) := d(m) \cdot \frac{\text{vol}(C)}{n!} \).

Combining this with Minkowski's theorem, we get

\[
\frac{d(m)}{n!} = \text{vol}(L) \cdot \text{vol}(C) \leq \frac{d(m)}{n!} \cdot 2^n. \quad \text{Hence } \left( \frac{d(m)}{n!} \right) \cdot \frac{d(m)}{d(L)} \leq n!.
\]
Many students had the following much easier solution, which I had overlooked. Let \( \mu_1, \ldots, \mu_n \) be the successive minors of \( \mathbf{M} \) with respect to \( \mathbf{M} \). Since \( \det \mathbf{M} \geq 0 \) for \( i \leq n \), on the other hand we have linearly independent points \( \mathbf{v}_1, \ldots, \mathbf{v}_n \), \( \mathbf{v}_i \in \mathbf{M} \). Hence \( \mu_i \neq 0 \) for \( i \leq n \).

\[
\frac{2^n d(M)}{n!} \leq \mu_1 \cdots \mu_n \leq \lambda - \lambda_0 \leq 2^n \frac{d(M)}{\det(C)},
\]

hence \( (2 \mathbf{M})_i \mathbf{M} \leq \frac{d(M)}{\det(C)} \leq n! \).

(3) a) We first determine \( \gamma, \delta \) from \( \gamma_0 = \delta + \delta, \gamma = \delta^2 + \delta \). This gives \( \gamma = \frac{3\lambda_0 - \lambda}{2}, \delta = \frac{\lambda_0 - \lambda}{2 - \lambda} \). Since \( \lambda \notin \mathbb{R} \) we have \( \gamma, \delta \neq 0 \).

We prove by induction on \( n \) that \( u_n = \delta^n \gamma^n \) for \( n \geq 0 \). For \( n = 0 \) we are done. Suppose \( n \geq 2 \) and \( u_n = \delta^n \gamma^n \) for \( 0 \leq k \leq n \). Since \( \delta^2 + \lambda - \lambda_0 \geq 0 \), \( \delta \neq 0 \) we have \( \delta^2 \lambda_0 \lambda + B \geq 0 \). Hence

\[
u_n = A u_n + B u_{n-1} = A (\delta^n \gamma^n + \delta^{n-1} \gamma) + B (\delta^{n-2} \gamma + \delta^{n-1} \gamma) = \delta^{n+1} \gamma + \delta^n \gamma.
\]

b) Suppose \( u_n = \delta^n + \delta^n \neq 0 \). Then by Balzer's Theorem,

\[
|u_n| = |\delta^n | + \delta^{n-1} \gamma| \geq |\delta^n | + (-\delta \gamma)\delta^n | > \delta^n | = 2^n | \epsilon(B)^n |,
\]

where \( \epsilon = \max(\lambda, n) \) and \( C \) is effectively computable in terms of \( \lambda, \delta, \epsilon, \delta \), hence \( |u_n| \) is bounded by \( 2^n e \). If \( \epsilon(2^n) \)

\[
\|u_n\| \leq 2^n (1 + e^{-n}) (\lambda | \epsilon(B)^{m-n} | + \lambda | \epsilon(B)^{n} | + \lambda | \epsilon(B)^{m-1} |) + \lambda | \epsilon(B)^{n+1} |,
\]

where \( \epsilon = \max(\lambda, n) \) and \( C \) is effectively computable in terms of \( \lambda, \delta, \epsilon(2^n) \).
Now suppose \( u_n = 0 \). Then \( \left( \frac{3}{2} \right)^n = -\frac{2}{3} \). This can be satisfied by at most one value of \( n \). For if there were two such values, \( n_1 \) and \( n_2 \), say then it would follow
\[
\left( \frac{3}{2} \right)^{n_1} = 1, \text{ hence } n_1 = n_2 \text{ since } \frac{3}{2} \text{ is not a root of unity.}
\]
Let \( n_1 \) be 1, if \( \left( \frac{3}{2} \right)^n = -\frac{2}{3} \) is unsolvable, and \( n_1 \) is the unique solution of \( \left( \frac{3}{2} \right)^n = -\frac{2}{3} \) otherwise. Then \( n_1 \) is effectively computable in terms of \( 2, 2, x, y, z \), hence in terms of \( 2, 2, y, z \), and \( u_\infty \) so for \( n > n_1 \), \( \Phi(t) \) holds for \( n > n_1 \).

(4) a) Let \( \ell_i = x_{i+1} x_i + \ldots + x_1 x_1 \) (i.e.,\(-n)) \) be linearly independent linear forms over \( \mathbb{C} \) with coefficients \( -\mathbb{C} \), and let \( C > 0 \).

Then the set of solutions of
\[
\left| f(x) - l_n(x) \right| \leq C \| x \|^{-\delta} \quad \text{in } x \in \mathbb{Z}^\ell
\]
is contained in a finite union \( T_1 \cup \ldots \cup T_l \) of proper linear subspaces of \( \mathbb{C}^n \). Here \( \| x \| = \max (|x_1|, |x_n|) \) for \( x = (x_1, \ldots, x_n) \).

b) Consider \( \| x \| = \max (|x_1|, |x_n|) \) for \( x = (x_1, \ldots, x_n) \),

where \( x_1, x_n \neq 0 \). By a), the solutions of \( (4) \) are in a union
\( T_1 \cup \ldots \cup T_l \) of one-dimensional linear subspaces of \( \mathbb{C}^2 \).

We have to prove that each of these subspaces contains only finitely many solutions of \( (4) \). Let \( T_i = \{ x_1, x_2 \} \),

then for a solution \( x \in \mathbb{Z}^2 \)

of \( (4) \), we have
\[
|c| < |A|^2 \cdot |l_1(x)| \leq C \cdot |A|^{-\delta} \| x \|^{-\delta},
\]
hence \( |A|^{2-\delta} \leq C \| x \|^{-\delta} \cdot |l_1(x)| \). \( \Rightarrow \) \( D \).

So \( \| x \| \leq D \cdot \| x \|^{-\delta} \). Then there are only finitely many \( x \in \mathbb{Z}^2 \)
with \( \|x\| \leq b \), so \( T \) contains indeed only finitely many solutions of \((++)\). 

By assumption, the numbers \( a_{ij} \) are linearly independent over \( \mathbb{Q} \). We prove by induction on \( n \) that \((++)\) has only finitely many solutions. For \( n = 0 \) we are done by \( b \).

Let \( n + 2 \). The linear forms \( L_1, L_2, \ldots, x_1, x_2 \) have determinant

\[
\begin{vmatrix}
L_1 & L_2 \\
x_1 & x_2
\end{vmatrix} = \pm \alpha_{n+2},
\]

so, hence are linearly independent. If

\( x \) is a solution of \((++)\). Then

\[
\begin{vmatrix}
L_1(x) & L_2(x) \\
x_1 & x_2
\end{vmatrix} = \pm \alpha_{n+2} \leq C. \|x\| \leq C. \|x\| \leq C. \|x\| \leq C. \|x\| \leq C. \|x\|
\]

By \( a \), no solution of \((++)\) lie \( n \) a limited union \( T_1 \cup \cdots \cup T_k \) of proper linear subspaces of \( \mathbb{Q}^n \). We have to prove that each of these spaces \( T \) contain a only finitely many solutions.

Suppose without loss of generality that \( T \) is given by

\[
x_1 = b_1 x_1 + \cdots + b_n x_n,
\]

for every \( x \in T \) satisfies

\[
x_1 = b_1 x_1 + \cdots + b_n x_n,
\]

with \( b_i = \frac{1}{4} \alpha_{n+2} \) for \( i = 1, \ldots, n \). The solutions of \((+)\) in \( T \) satisfy

\[
(++) \quad 0 < \| x_1 + \cdots + x_n \| \leq \alpha_{n+2} \| x_1 + \cdots + x_n \| \leq C. \| x_1 + \cdots + x_n \| \leq C. \| x_1 + \cdots + x_n \| \leq C. \| x_1 + \cdots + x_n \|
\]

The determinants

\[
0_j = (a_{1} + \cdots + a_{n}) (b_{j} + a_{n}) (c_{j} + a_{n}) (d_{j} + a_{n})
\]

\( (1 \leq j \leq n) \) are linearly independent over \( \mathbb{Q} \). Indeed, we have

\[
0_j - b_{j} - a_{j} + \alpha_{n+2} \quad \text{in } \mathbb{Q}.
\]

Assume that there are rational \( a_j \) such that

\[
\sum_{j=1}^{n} a_j \alpha_{n+2} = 0.
\]

Then
\[
\sum_{i} \sum_{j} c_{ij} (a_{ij} - q_{ij} y_{j} + q_{ij} x_{i} y_{j}) = 0,
\]
\[
\sum_{i} \sum_{j} c_{ij} a_{ij} - \sum_{i} (\sum_{j} c_{ij} y_{j}) a_{ij} + \sum_{j} (\sum_{i} c_{ij} y_{j}) a_{ij} = 0,
\]

implying \( c_{ij} = 0 \) for \( i, j \), since \( a_{ij} \) are linearly independent over \( \mathbb{Q} \).

We can now apply the induction hypothesis to (1.1) and conclude that (1.1) has only finitely many solutions. Consequently, (1.2) has only finitely many solutions in \( T \).