Chapter 5

The Riemann zeta function and L-functions

We prove some results that will be used in the proof of the Prime Number Theorem (for arithmetic progressions). The L-function of a Dirichlet character $\chi$ modulo $q$ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$  

We view $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ as the L-function of the principal character modulo 1, more precisely, $\zeta(s) = L(s, \chi_{(1)})$, where $\chi_{(1)}(n) = 1$ for all $n \in \mathbb{Z}$.

We first prove that $\zeta(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \text{Re } s > 0\} \setminus \{1\}$. We use an important summation formula, due to Euler.

**Lemma 5.1** (Euler’s summation formula). Let $a, b$ be integers with $a < b$ and $f : [a, b] \to \mathbb{C}$ a continuously differentiable function. Then

$$\sum_{n=a}^{b} f(n) = \int_{a}^{b} f(x)dx + f(a) + \int_{a}^{b} (x - [x]) f'(x)dx.$$  

**Remark.** This result often occurs in the more symmetric form

$$\sum_{n=a}^{b} f(n) = \int_{a}^{b} f(x)dx + \frac{1}{2}(f(a) + f(b)) + \int_{a}^{b} (x - [x] - \frac{1}{2}) f'(x)dx.$$  

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Proof. Let $n \in \{a, a + 1, \ldots, b - 1\}$. Then
\[
\int_n^{n+1} (x - \lfloor x \rfloor) f'(x) \, dx = \int_n^{n+1} (x - n) f'(x) \, dx
\]
\[= \left((x - n) f(x)\right)_n^{n+1} - \int_n^{n+1} f(x) \, dx = f(n + 1) - \int_n^{n+1} f(x) \, dx.\]
By summing over $n$ we get
\[
\int_a^b (x - \lfloor x \rfloor) f'(x) \, dx = \sum_{n=a+1}^{b} f(n) - \int_a^b f(x) \, dx,
\]
which implies at once Lemma 5.1.

\begin{theorem}
\(\zeta(s)\) has a unique analytic continuation to the set
\(\{s \in \mathbb{C} : \text{Re } s > 0, s \neq 1\}\), with a simple pole with residue 1 at $s = 1$.
\end{theorem}

Proof. By Corollary 2.21 we know that an analytic continuation of \(\zeta(s)\), if such exists, is unique.

For the moment, let $s \in \mathbb{C}$ with $\text{Re } s > 1$. Then by Lemma 5.1, with $f(x) = x^{-s}$,
\[
\sum_{n=1}^{N} n^{-s} = \int_{1}^{N} x^{-s} \, dx + 1 + \int_{1}^{N} (x - \lfloor x \rfloor)(-sx^{-s}) \, dx
\]
\[= \frac{1 - N^{1-s}}{s - 1} + 1 - s \int_{1}^{N} (x - \lfloor x \rfloor)x^{-s} \, dx.\]
If we let $N \to \infty$ then the left-hand side converges, and also the first term on the right-hand side, since $|N^{-1-s}| = N^{-1-\text{Re } s} \to 0$. Hence the integral on the right-hand side must converge as well. Thus, letting $N \to \infty$, we get for $\text{Re } s > 1$,
\[
(5.1) \quad \zeta(s) = \frac{1}{s - 1} + 1 - s \int_{1}^{\infty} (x - \lfloor x \rfloor)x^{-s} \, dx.
\]

We now show that the integral on the right-hand side defines an analytic function on $U := \{s \in \mathbb{C} : \text{Re } s > 0\}$, by means of Theorem 2.23.

The function $F(x, s) := (x - \lfloor x \rfloor)x^{-1-s}$ is measurable on $[1, \infty) \times U$ (by, e.g., the fact that its set of discontinuities has Lebesgue measure 0) and for every fixed $x$ it is analytic in $s$. 

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Let $K$ be a compact subset of $U$. Then there is $\sigma > 0$ such that $\text{Re } s \geq \sigma$ for all $s \in K$. Now for $x \geq 1$ and $s \in K$ we have

$$|(x - \lfloor x \rfloor)x^{-1-s}| \leq x^{-1-\sigma}$$

and $\int_1^\infty x^{-1-\sigma}dx < \infty$. Hence all conditions of Theorem 2.23 are satisfied, and we may indeed conclude that the integral on the right-hand side of (5.1) defines an analytic function on $U$.

Consequently, the right-hand of (5.1) is analytic on $\{s \in \mathbb{C} : \text{Re } s > 0, s \neq 1\}$ and it has a simple pole at $s = 1$ with residue 1. We may take this as our analytic continuation of $\zeta(s)$.

**Theorem 5.3.** Let $q \in \mathbb{Z}_{>2}$, and let $\chi$ be a Dirichlet character mod $q$.

(i) $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$ for $s \in \mathbb{C}$, $\text{Re } s > 1$.

(ii) If $\chi \neq \chi_0^{(q)}$, then $L(s, \chi)$ converges, and is analytic on $\{s \in \mathbb{C} : \text{Re } s > 0\}$.

(iii) $L(s, \chi_0^{(q)})$ can be continued to an analytic function on $\{s \in \mathbb{C} : \text{Re } s > 0, s \neq 1\}$, and for $s$ in this set we have

$$L(s, \chi_0^{(q)}) = \zeta(s) \prod_{p \nmid q} (1 - p^{-s}).$$

Hence $L(s, \chi_0^{(q)})$ has a simple pole at $s = 1$.

**Proof.** (i) $\chi$ is a strongly multiplicative function, and $L(s, \chi)$ converges absolutely for $\text{Re } s > 1$. Apply Corollary 3.14.

(ii) Let $N$ be any positive integer. Then $N = tq + r$ for certain integers $t, r$ with $t \geq 0$ and $0 \leq r < q$. By one of the orthogonality relations for characters (see Theorem 4.9), we have $\sum_{m=1}^{q} \chi(m) = 0$, $\sum_{m=q+1}^{2q} \chi(m) = 0$, etc. Hence

$$\left| \sum_{n=1}^{N} \chi(n) \right| = \left| \chi(tq + 1) + \cdots + \chi(tq + r) \right| \leq r < q.$$

This last upper bound is independent of $N$. Now Theorem 3.2 implies that the L-series $L(s, \chi)$ converges and is analytic on $\text{Re } s > 0$.

(iii) By (i) we have for $\text{Re } s > 1$,

$$L(s, \chi_0^{(q)}) = \prod_{p \nmid q} (1 - p^{-s})^{-1} = \zeta(s) \prod_{p \nmid q} (1 - p^{-s}).$$
The right-hand side is defined and analytic on \( \{ s \in \mathbb{C} : \text{Re} \, s > 0, s \neq 1 \} \), and so it can be taken as an analytic continuation of \( L(s, \chi_0^{(q)}) \) on this set. \( \square \)

**Corollary 5.4.** Both \( \zeta(s) \) and \( L(s, \chi) \) for any character \( \chi \) modulo an integer \( q \geq 2 \) are \( \neq 0 \) on \( \{ s \in \mathbb{C} : \text{Re} \, s > 1 \} \).

**Proof.** Use part (i) of the above theorem, together with Corollary 3.14. \( \square \)

The remainder of this section is dedicated to the proof that \( \zeta(s) \neq 0 \) if \( \text{Re} \, s = 1 \) and \( s \neq 1 \), and \( L(s, \chi) \neq 0 \) for any \( s \in \mathbb{C} \) with \( \text{Re} \, s = 1 \) and any non-principal character \( \chi \) modulo an integer \( q \geq 2 \). We have to distinguish two cases, which are treated quite differently. We interpret \( \zeta(s) \) as \( L(s, \chi_0^{(1)}) \).

**Theorem 5.5.** Let \( q \in \mathbb{Z}_{\geq 1} \), \( \chi \) a character mod \( q \), and \( t \) a real. Assume that either \( t \neq 0 \), or \( t = 0 \) but \( \chi^2 \neq \chi^{(q)} \). Then \( L(1 + it, \chi) \neq 0 \).

**Proof.** We use a famous idea, due to Hadamard. It is based on the inequality

\[
3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0 \quad \text{for} \quad \theta \in \mathbb{R}.
\]

Suppose that \( L(1 + it, \chi) = 0 \). Consider the function

\[
F(s) := L(s, \chi_0^{(q)})^3 \cdot L(s + it, \chi)^4 \cdot L(s + 2it, \chi^2).
\]

By our assumption on \( \chi \) and \( t \), \( L(s + 2it, \chi^2) \) is analytic around \( s = 1 \). Further, \( L(s, \chi_0^{(q)}) \) has a simple pole at \( s = 1 \), while \( L(s + it, \chi) \) has by assumption a zero at \( s = 1 \). Hence

\[
\text{ord}_{s=1}(F) = 3 \cdot \text{ord}_{s=1}(L(s, \chi_0^{(q)})) + 4 \cdot \text{ord}_{s=1}(L(s + it, \chi)) + \text{ord}_{s=1}(L(s + 2it, \chi^2))
\]

\[
\geq -3 + 4 = 1.
\]

This shows that \( F \) is analytic around \( s = 1 \), and has a zero at \( s = 1 \). We now prove that \( |F(\sigma)| \geq 1 \) (or rather, \( \log |F(\sigma)| \geq 0 \)). This gives a contradiction since by continuity, \( \lim_{s \downarrow 1} |F(\sigma)| \) should be 0. So our assumption that \( L(1 + it, \chi) = 0 \) must be false.

From the definition of the function \( F \) we obtain

\[
\log |F(\sigma)| = \log \prod_p \left( \frac{1}{1 - \chi_0^{(q)}(p)p^{-\sigma}} \right)^3 \cdot \left( \frac{1}{1 - \chi(p)p^{-\sigma-it}} \right)^4 \cdot \left( \frac{1}{1 - \chi(p)^2p^{-\sigma-2it}} \right)
\]

\[
= \sum_{p \neq q} \left( 3 \log \left| \frac{1}{1 - p^{-\sigma}} \right| + 4 \log \left| \frac{1}{1 - \chi(p)p^{-\sigma-it}} \right| + \log \left| \frac{1}{1 - \chi(p)^2p^{-\sigma-2it}} \right| \right).
\]

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Note that if $p \nmid q$ then $\chi(p)$ is a root of unity. Hence $|\chi(p)p^{-it}| = |\chi(p)e^{-it\log p}| = 1$. So we have $\chi(p)p^{-it} = e^{i\varphi_p}$ with $\varphi_p \in \mathbb{R}$. Hence
\[
\log |F(\sigma)| = \sum_{p\nmid q} \left( 3 \log \left| \frac{1}{1 - p^{-\sigma}} \right| + 4 \log \left| \frac{1}{1 - p^{-\sigma}e^{i\varphi_p}} \right| + \log \left| \frac{1}{1 - p^{-\sigma}e^{2i\varphi_p}} \right| \right).
\]
Recall that
\[
\log \frac{1}{1 - z} = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \log \left| \frac{1}{1 - z} \right| = \text{Re} \log \frac{1}{1 - z} \quad \text{for } z \in \mathbb{C} \text{ with } |z| < 1.
\]
Hence for $r, \varphi \in \mathbb{R}$ with $0 < r < 1$,
\[
\log \left| \frac{1}{1 - re^{i\varphi}} \right| = \text{Re} \left( \log \frac{1}{1 - re^{i\varphi}} \right) = \text{Re} \left( \sum_{n=1}^{\infty} \frac{(re^{i\varphi})^n}{n} \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{r^n}{n} \text{Re} (e^{in\varphi}) = \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos n\varphi.
\]
This leads to
\[
\log |F(\sigma)| = \sum_{p\nmid q} \left( 3 \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} + 4 \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \cdot \cos n\varphi_p + \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \cos 2n\varphi_p \right)
\]
\[
= \sum_{p\nmid q} \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} (3 + 4 \cos n\varphi_p + \cos 2n\varphi_p) \geq 0,
\]
using (5.2). This shows that indeed, $|F(\sigma)| \geq 1$ for $\sigma > 1$, giving us the contradiction we want. \qed

It remains to prove that $L(1, \chi) \neq 0$ for any character $\chi \mod q$ such that $\chi \neq \chi_0^{(q)}$, $\chi^2 = \chi_0^{(q)}$, i.e., for any real character $\chi$ not equal to the principal character. Dirichlet needed this fact already in his proof that for every pair of integers $q, a$ with $q \geq 3$ and $\gcd(a, q) = 1$ there are infinitely many primes $p$ with $p \equiv a \pmod{q}$. Dirichlet had a rather complicated proof that $L(1, \chi) \neq 0$, based on Dirichlet series associated with quadratic forms (in modern language: Dedekind zeta functions for quadratic number fields) and class number formulas.

Landau found a much more direct proof, which we give here, based on a simple result for Dirichlet series, which more or less asserts that a Dirichlet series with non-negative real coefficients cannot be continued analytically beyond the boundary of its half plane of convergence.
Lemma 5.6 (Landau). Let \( f : \mathbb{Z}_{>0} \to \mathbb{R} \) be an arithmetical function with \( f(n) \geq 0 \) for all \( n \). Suppose that \( L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \) has abscissa of convergence \( \sigma_0 \). Then \( L_f(s) \) cannot be continued analytically to any open set containing \( \{ s \in \mathbb{C} : \Re s > \sigma_0 \} \cup \{ \sigma_0 \} \).

Proof.

Suppose \( L_f(s) \) can be continued to an analytic function \( g(s) \) on an open set containing \( \{ s \in \mathbb{C} : \Re s > \sigma_0 \} \cup \{ \sigma_0 \} \). Then there is \( \delta > 0 \) such that \( g(s) \) is analytic on the open disk \( D(\sigma_0, \delta) \) with center \( \sigma_0 \) and radius \( \delta \). Let \( \sigma_1 := \sigma_0 + \delta/3 \). Then \( D(\sigma_1, 2\delta/3) \subset D(\sigma_0, \delta) \), so \( g(s) \) is analytic and has a Taylor series expansion around \( \sigma_1 \) converging on \( D(\sigma_1, 2\delta/3) \). Now let \( \sigma_0 - \delta/3 < \sigma < \sigma_0 \), so that \( \sigma \in D(\sigma_1, 2\delta/3) \). Using the Taylor series expansion of \( g(s) \) around \( \sigma_1 \), we get

\[
g(\sigma) = \sum_{k=0}^{\infty} \frac{g^{(k)}(\sigma_1)}{k!} \cdot (\sigma - \sigma_1)^k.
\]

Since \( \sigma_1 \) is larger than the abscissa of convergence \( \sigma_0 \) of \( L_f(s) \), we have

\[
g^{(k)}(\sigma_1) = L_f^{(k)}(\sigma_1) = \sum_{n=1}^{\infty} f(n)(-\log n)^{k}n^{-\sigma_1} \quad \text{for} \; k \geq 0.
\]

Hence

\[
g(\sigma) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{n=1}^{\infty} f(n)(-\log n)^{k}n^{-\sigma_1} \right) (\sigma - \sigma_1)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \sum_{n=1}^{\infty} f(n)(\log n)^{k}n^{-\sigma_1} \right) (\sigma_1 - \sigma)^k.
\]
Now all terms are non-negative, hence it is allowed to interchange the summations. Thus,
\[ g(\sigma) = \sum_{n=1}^{\infty} f(n)n^{-\sigma} \left( \sum_{k=0}^{\infty} \frac{1}{k!} (\log n)^k (\sigma_1 - \sigma)^k \right) \]
\[ = \sum_{n=1}^{\infty} f(n)n^{-\sigma} e^{(\log n)(\sigma_1 - \sigma)} = \sum_{n=1}^{\infty} f(n)n^{-\sigma_1 n^{\sigma_1 - \sigma}} = \sum_{n=1}^{\infty} f(n)n^{-\sigma}. \]

We see that \( L_f(s) \) converges for \( s = \sigma \). But this is impossible, since \( \sigma \) is smaller than the abscissa of convergence \( \sigma_0 \) of \( L_f(s) \). So our initial assumption that \( L_f(s) \) has an analytic continuation to an open set containing \( \{ s \in \mathbb{C} : \text{Re} s > \sigma_0 \} \cup \{ \sigma_0 \} \) must have been false. \( \square \)

Remark. Lemma 5.6 becomes false if we drop the condition that \( f(n) \geq 0 \) for all \( n \). For instance, if \( \chi \) is a non-principal character mod \( q \), then \( L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} \) diverges if \( \text{Re} s < 0 \), but one can show that \( L(s, \chi) \) has an analytic continuation to the whole of \( \mathbb{C} \).

Theorem 5.7. Let \( q \in \mathbb{Z}_{\geq 2} \), and let \( \chi \) be a character mod \( q \) with \( \chi \neq \chi_0^{(q)} \) and \( \chi^2 = \chi_0^{(q)} \). Then \( L(1, \chi) \neq 0 \).

Proof. Assume that \( L(1, \chi) = 0 \). Consider the function

\[ F(s) := L(s, \chi)\zeta(s). \]

By Theorems 5.2, 5.3, this function is analytic at least on \( \{ s \in \mathbb{C} : \text{Re} s > 0, s \neq 1 \} \). But the simple pole of \( \zeta(s) \) at \( s = 1 \) is cancelled by the zero of \( L(s, \chi) \). Hence \( F(s) \) is analytic for all \( s \) with \( \text{Re} s > 0 \). We show that for \( s \in \mathbb{C} \) with \( \text{Re} s > 1 \), \( F(s) \) is expressable as a Dirichlet series with non-negative coefficients. By Lemma 5.6, this Dirichlet series should have abscissa of convergence \( \leq 0 \). But we show that the abscissa of convergence of this series is \( \geq \frac{1}{2} \) and derive a contradiction.

The series \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) and \( \sum_{n=1}^{\infty} \chi(n)n^{-s} \) converge absolutely if \( \text{Re} s > 1 \). So by Theorem 3.12,

\[ F(s) = L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{for } s \in \mathbb{C}, \text{ Re } s > 1, \]
where \( f = E \ast \chi \), i.e.,
\[
f(n) = \sum_{d|n} \chi(d) \quad \text{for } n \in \mathbb{Z}_{>0}.
\]

Hence \( f \) is a multiplicative function. We compute \( f \) in the prime powers. Since \( \chi^2 = \chi_0^{(q)} \), we have \( \chi(n) = \pm 1 \) for all \( n \in \mathbb{Z} \) with \( \gcd(n,q) = 1 \), while \( \chi(n) = 0 \) if \( \gcd(n,q) > 1 \). Hence, if \( p \) is a prime and \( k \) a non-negative integer, we have
\[
f(p^k) = \sum_{j=0}^{k} \chi(p)^j = \begin{cases} 1 & \text{if } p|q, \\ k+1 & \text{if } p \nmid q, \quad \chi(p) = 1, \\ 1 & \text{if } p \nmid q, \quad \chi(p) = -1, \quad k \text{ even,} \\ 0 & \text{if } p \nmid q, \quad \chi(p) = -1, \quad k \text{ odd.} \end{cases}
\]

Therefore, \( f(p^k) \geq 0 \) for all prime powers \( p^k \). Since \( f \) is multiplicative, it follows that \( f(n) \geq 0 \) for all \( n \in \mathbb{Z}_{>0} \).

The series \( L_f(s) \) has an analytic continuation to \( \{ s \in \mathbb{C} : \Re s > 0 \} \), that is, \( F(s) \). So by Lemma 5.6, \( L_f(s) \) has abscissa of convergence \( \sigma_0(f) \leq 0 \). On the other hand, from the above table and from the fact that \( f \) is multiplicative, it follows that if \( n = m^2 \) is a square, then \( f(n) \geq 1 \). Hence
\[
L_f(\sigma) = \sum_{n=1}^{\infty} f(n) n^{-\sigma} \geq \sum_{m=1}^{\infty} m^{-2\sigma} = \infty \quad \text{if } \sigma \leq \frac{1}{2}.
\]

So \( \sigma_0(f) \geq \frac{1}{2} \). This gives a contradiction, and so our assumption that \( L(1, \chi) = 0 \) has to be false. \( \square \)