Chapter 3

Dirichlet series and arithmetical functions

3.1 Dirichlet series

An arithmetical function is a function $f : \mathbb{Z}_{>0} \to \mathbb{C}$. To such a function we associate a Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$$

where $s$ is a complex variable. It is common practice (although this doesn’t make sense) to write $s = \sigma + it$, where $\sigma = \text{Re } s$ and $t = \text{Im } s$. We want to develop a theory for Dirichlet series similar to that for power series. Every power series $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R$ such that the series converges if $|z| < R$ and diverges if $|z| > R$. As we will see, a Dirichlet series $L_f(s)$ has an abscissa of convergence $\sigma_0(f)$ such that the series converges for all $s \in \mathbb{C}$ with $\text{Re } s > \sigma_0(f)$ and diverges for all $s \in \mathbb{C}$ with $\text{Re } s < \sigma_0(f)$. For instance, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has abscissa of convergence 1.

We start with an important summation result, which we shall use very frequently.

**Theorem 3.1** (Partial summation, summation by parts). Let $M, N$ be reals with $M < N$. Let $x_1, \ldots, x_r$ be real numbers with $M \leq x_1 < \cdots < x_r \leq N$, let $a(x_1), \ldots, a(x_r)$ be complex numbers, and put $A(t) := \sum_{x_k \leq t} a(x_k)$ for $t \in [M, N]$. 

57
Further, let \( g : [M, N] \to \mathbb{C} \) be a differentiable function. Then
\[
\sum_{k=1}^{r} a(x_k)g(x_k) = A(N)g(N) - \int_{M}^{N} A(t)g'(t)dt.
\]

Proof. Let \( x_0 < M \) and put \( A(x_0) := 0 \). Then
\[
\sum_{k=1}^{r} a(x_k)g(x_k) = \sum_{k=1}^{r} (A(x_k) - A(x_{k-1}))g(x_k)
\]
\[
= \sum_{k=1}^{r} A(x_k)g(x_k) - \sum_{k=1}^{r-1} A(x_k)g(x_{k+1})
\]
\[
= A(x_r)g(x_r) - \sum_{k=1}^{r-1} A(x_k)(g(x_{k+1}) - g(x_k)).
\]
Since \( A(t) = A(x_k) \) for \( x_k \leq t < x_{k+1} \) we have
\[
A(x_k)(g(x_{k+1}) - g(x_k)) = \int_{x_k}^{x_{k+1}} A(t)g'(t)dt.
\]
Hence
\[
\sum_{k=1}^{r} a(x_k)g(x_k) = A(x_r)g(x_r) - \sum_{k=1}^{r-1} \int_{x_k}^{x_{k+1}} A(t)g'(t)dt
\]
\[
= A(x_r)g(x_r) - \int_{x_1}^{x_r} A(t)g'(t)dt.
\]
In case that \( x_1 = M, x_r = N \) we are done. If \( x_1 > M \), then \( A(t) = 0 \) for \( M \leq t < x_1 \) and thus, \( \int_{M}^{x_1} A(t)g'(t)dt = 0 \). If \( x_r < N \), then \( A(t) = A(x_r) \) for \( x_r \leq t \leq N \), hence
\[
\int_{x_r}^{N} A(t)g'(t)dt = A(N)g(N) - A(x_r)g(x_r).
\]
Together with (3.1) this implies our Theorem.

\[\square\]

Theorem 3.2. Let \( f : \mathbb{Z}_{>0} \to \mathbb{C} \) be an arithmetical function with the property that there exists a constant \( C > 0 \) such that \( |\sum_{n=1}^{N} f(n)| \leq C \) for every \( N \geq 1 \). Then
\[ L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \] converges for every \( s \in \mathbb{C} \) with \( \text{Re} \, s > 0 \).
More precisely, on \( \{ s \in \mathbb{C} : \text{Re} \, s > 0 \} \) the function \( L_f \) is analytic, and for its \( k \)-th derivative we have
\[
L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^{k} n^{-s}.
\]
Proof. We prove that \( Lf \) converges uniformly on the rectangle

\[
R(\sigma_1, \sigma_2, T) := \{ s \in \mathbb{C} : \sigma_1 < \text{Re} s < \sigma_2, \ |\text{Im} s| < T \}
\]

for every \( \sigma_1, \sigma_2, T \) with \( \sigma_2 > \sigma_1 > 0 \) and \( T > 0 \), that is, the partial sums \( \sum_{n=1}^{N} f(n)n^{-s} \) converge uniformly to \( Lf(s) \), for \( s \) in the rectangle. Since these partial sums are all analytic, it follows from Theorem 2.24 from the previous chapter that for \( s \in R(\sigma_1, \sigma_2, T) \),

\[
Lf(s) = \lim_{N \to \infty} \sum_{n=1}^{N} f(n)n^{-s} \text{ is analytic,}
\]

\[
L_f^{(k)}(s) = \lim_{N \to \infty} \left( \sum_{n=1}^{N} f(n)n^{-s} \right)^{(k)} = \sum_{n=1}^{\infty} (-\log n)^k f(n)n^{-s}.
\]

Since every \( s \in \mathbb{C} \) with \( \text{Re} s > 0 \) lies in \( R(\sigma_1, \sigma_2, T) \) for some \( \sigma_2 > \sigma_1 > 0 \) and \( T > 0 \), this implies Theorem 3.2.

Fix \( \sigma_1, \sigma_2, T \) with \( \sigma_2 > \sigma_1 > 0 \) and \( T > 0 \). Recall that a sequence of functions \( \{f_n : U \to \mathbb{C}\}_{n=1}^{\infty} \) defined on a subset \( U \) of \( \mathbb{C} \) converges uniformly on \( U \) if and only if

\[
\lim_{M,N \to \infty} \sup_{s \in U} |f_N(s) - f_M(s)| = 0.
\]

Applying this to \( f_M(s) = \sum_{n=1}^{M} f(n)n^{-s} \), we see that we have to prove that

\[
\lim_{M,N \to \infty} \sup_{s \in R(\sigma_1, \sigma_2, T)} \left| \sum_{n=M+1}^{N} f(n)n^{-s} \right| = 0.
\]

We prove this using partial summation.

Let \( N > M > 0 \) and put \( F(t) := \sum_{M<n \leq t} f(n) \). Notice that by our assumption on \( F \),

\[
|F(t)| \leq \left| \sum_{n=1}^{M} f(n) \right| + \left| \sum_{n \leq t} f(n) \right| \leq 2C \quad \text{for all } t > M.
\]

By Theorem 3.1 (with \( \{x_1, \ldots, x_r\} = \{M+1, \ldots, N\} \) and \( g(t) := t^{-s} \)), we have

\[
\sum_{n=M+1}^{N} f(n)n^{-s} = F(N)N^{-s} - \int_{M}^{N} F(t)(-s)t^{-s-1}dt.
\]
We determine an upper bound for the absolute value of the right-hand side that is independent of \( s \in R(\sigma_1, \sigma_2, T) \). Let \( s \) be in this rectangle. Then

\[
|N^{-s}| = |e^{-s \log N}| = e^{-(\Re s) \log N} \leq N^{-\Re s} \leq N^{-\sigma_1},
\]

\[
|s| \leq \{(\Re s)^2 + (\Im s)^2\}^{1/2} \leq \{\sigma_2^2 + T^2\}^{1/2} =: B,
\]

and for \( t \geq 1 \),

\[
|t^{-s-1}| = t^{-\Re s-1} \leq t^{-\sigma_1-1}.
\]

Hence

\[
\left| \sum_{n=M}^{N} f(n)n^{-s} \right| \leq |F(N)| \cdot N^{-\sigma_1} + \int_{M}^{N} |F(t)| \cdot |s| \cdot t^{-\sigma_1-1} dt
\]

\[
\leq 2CN^{-\sigma_1} + 2CB \int_{M}^{N} t^{-\sigma_1-1} dt
\]

\[
= 2CN^{-\sigma_1} + 2CB\sigma_1^{-1}(M^{-\sigma_1} - N^{-\sigma_1}).
\]

This last bound is independent of \( s \), and tends to 0 as \( M, N \to \infty \). This proves the uniform convergence, hence our Theorem.

**Corollary 3.3.** Let \( f : \mathbb{Z}_{>0} \to \mathbb{C} \) be an arithmetical function and let \( s_0 \in \mathbb{C} \) be such that \( \sum_{n=1}^{\infty} f(n)n^{-s_0} \) converges. Then on \( \{s \in \mathbb{C} : \Re s > \Re s_0\} \) the function \( L_f \) is analytic, and

\[
L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s} \text{ for } k \geq 1.
\]

**Proof.** Write \( s = s' + s_0 \). Then \( \Re s' > 0 \) if \( \Re s > \Re s_0 \). There is \( C > 0 \) such that \( |\sum_{n=1}^{N} f(n)n^{-\Re s_0}| \leq C \) for all \( N \). Apply Theorem 3.2 to \( \sum_{n=1}^{\infty} (f(n)n^{-s_0})n^{-s'} \). \( \square \)

**Theorem 3.4.** There exists a number \( \sigma_0(f) \) with \( -\infty \leq \sigma_0(f) \leq \infty \) such that \( L_f(s) \) converges for all \( s \in \mathbb{C} \) with \( \Re s > \sigma_0(f) \) and diverges for all \( s \in \mathbb{C} \) with \( \Re s < \sigma_0(f) \).

Moreover, if \( \sigma_0(f) < \infty \), then on \( \{s \in \mathbb{C} : \Re s > \sigma_0(f)\} \) the function \( L_f \) is analytic, and

\[
L_f^{(k)}(s) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s} \text{ for } k \geq 1.
\]
Exercise 3.1. Show that there exist arithmetical functions \( \sigma \).

The number \( \sigma \) is called the \textit{abscissa of convergence} of \( L_f \).

There exists also a real number \( \sigma_a(f) \), called the \textit{abscissa of absolute convergence} of \( L_f \) such that \( L_f(s) \) converges absolutely if \( \Re s > \sigma_a(f) \), and does not converge absolutely if \( \Re s < \sigma_a(f) \).

In fact, we have \( \sigma_a(f) = \sigma_0(|f|) \) is the abscissa of convergence of \( L_{|f|}(s) = \sum_{n=1}^{\infty} |f(n)| n^{-s} \). Write \( \sigma = \Re s \). Then \( \sum_{n=1}^{\infty} |f(n)| n^{-s} = \sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \) converges if \( \Re s > \sigma_0(|f|) \) and diverges if \( \Re s < \sigma_0(|f|) \).

Theorem 3.5. For every arithmetical function \( f : \mathbb{Z}_0 \to \mathbb{C} \) we have \( \sigma_0(f) \leq \sigma_a(f) \leq \sigma_0(f) + 1 \).

Proof. It is clear that \( \sigma_0(f) \leq \sigma_a(f) \). To prove \( \sigma_a(f) \leq \sigma_0(f) + 1 \), we have to show that \( L_f(s) \) converges absolutely if \( \Re s > \sigma_0(f) + 1 \).

Take such \( s \); then \( \Re s = \sigma_0(f) + 1 + \varepsilon \) with \( \varepsilon > 0 \). Put \( \sigma := \sigma_0(f) + \varepsilon/2 \). The series \( \sum_{n=1}^{\infty} f(n) n^{-\sigma} \) converges, hence there is a constant \( C \) such that \( |f(n) n^{-\sigma}| \leq C \) for all \( n \). Therefore,

\[
|f(n) n^{-\sigma}| = |f(n)| \cdot n^{-\Re s} = |f(n) n^{-\sigma}| \cdot n^{-1-\varepsilon/2} \leq C n^{-1-\varepsilon/2}
\]

for \( n \geq 1 \). The series \( \sum_{n=1}^{\infty} n^{-1-\varepsilon/2} \) converges, hence \( \sum_{n=1}^{\infty} |f(n) n^{-s}| \) converges. \( \square \)

Exercise 3.1. Show that there exist arithmetical functions \( f \) such that \( \sigma_a(f) = \sigma_0(f) + 1 \).

The next theorem implies that an arithmetical function is uniquely determined by its Dirichlet series.

Theorem 3.6. Let \( f, g : \mathbb{Z}_0 \to \mathbb{C} \) be two arithmetical functions for which there is \( \sigma \in \mathbb{R} \) such that \( L_f(s) \), \( L_g(s) \) converge absolutely and \( L_f(s) = L_g(s) \) for all \( s \in \mathbb{C} \) with \( \Re s > \sigma \). Then \( f = g \).
Proof. Let $h := f - g$. Our assumptions imply that $L_h(s)$ converges absolutely, and $L_h(s) = 0$ for all $s \in \mathbb{C}$ with $\text{Re} \ s > \sigma$. We have to prove that $h = 0$.

Assume that there are positive integers $n$ with $h(n) \neq 0$, and let $m$ be the smallest such $n$. Then for all $s \in \mathbb{C}$ with $\text{Re} \ s > \sigma$ we have

$$h(m)m^{-s} = - \sum_{n=m+1}^{\infty} h(n)n^{-s}.$$ 

Let $\sigma_1 > \sigma$, and let $s \in \mathbb{C}$ with $\text{Re} \ s > \sigma_1$. Then

$$|h(m)| \leq \sum_{n=m+1}^{\infty} |h(n)|(m/n)^{\text{Re} \ s} = \sum_{n=m+1}^{\infty} |h(n)|(m/n)^{\sigma_1}(m/n)^{\text{Re} \ s - \sigma_1}$$

$$\leq m^{\sigma_1} \left( \sum_{n=m+1}^{\infty} |h(n)| \cdot n^{-\sigma_1} \right) \cdot (m/(m + 1))^{\text{Re} \ s - \sigma_1}.$$ 

The series between the parentheses is convergent, hence a finite number. So the right-hand side tends to 0 as $\text{Re} \ s \to \infty$. This contradicts that $h(m) \neq 0$. \qed

3.2 Arithmetical functions

A multiplicative function is an arithmetical function $f$ such that $f \neq 0$ and $f(mn) = f(m)f(n)$ for all positive integers $m,n$ with $\gcd(m,n) = 1$. A strongly multiplicative function is an arithmetical function $f$ with the property that $f \neq 0$ and $f(mn) = f(m)f(n)$ for all integers $m,n$.

Notation. In expressions $p_1^{k_1} \cdots p_t^{k_t}$ it is always assumed that the $p_i$ are distinct prime numbers, and the $k_i$ positive integers.

Remarks. 1) If $f$ is a multiplicative function, then $f(1) = 1$.

2) If $f$ is a multiplicative function and $n = p_1^{k_1} \cdots p_t^{k_t}$, then $f(n) = f(p_1^{k_1}) \cdots f(p_t^{k_t})$. That is, a multiplicative function is uniquely determined by its values in the prime powers or otherwise stated, two multiplicative functions coinciding on the prime powers are equal.

3) If $f$ is a strongly multiplicative function and $n = p_1^{k_1} \cdots p_t^{k_t}$, then $f(n) = f(p_1)^{k_1} \cdots f(p_t)^{k_t}$. Hence a strongly multiplicative function is uniquely determined by its values in the primes.
We define the convolution product \( f \ast g \) of two arithmetical functions \( f, g \) by

\[
(f \ast g)(n) := \sum_{d \mid n} f(n/d)g(d) \quad \text{for } n \in \mathbb{Z}_{>0},
\]

where \( 'd \mid n' \) means that the sum is taken over all positive divisors of \( n \).

**Examples.** Define the arithmetical functions \( e, E \) by

\[
e(1) = 1, \quad e(n) = 0 \quad \text{for all } n \in \mathbb{Z}_{>1},
\]

\[
E(n) = 1 \quad \text{for all } n \in \mathbb{Z}_{>0}.
\]

Clearly, \( e \) is multiplicative, and \( E \) is strongly multiplicative. If \( f \) is any arithmetical function, then \( e \ast f = f \), while

\[
(E \ast f)(n) = \sum_{d \mid n} f(d).
\]

**Lemma 3.7.** (i) For any two arithmetical functions \( f, g \) we have \( f \ast g = g \ast f \).

(ii) For any three arithmetical functions \( f, g, h \) we have \( (f \ast g) \ast h = f \ast (g \ast h) \).

**Proof.** Straightforward verification. \( \square \)

**Theorem 3.8.** (i) Let \( A \) be the set of arithmetical functions \( f \) with \( f(1) \neq 0 \). Then \( A \) with \( \ast \) is an abelian group with unit element \( e \).

(ii) Let \( M \) be the set of multiplicative functions. Then \( M \) with \( \ast \) is a subgroup of \( A \).

**Proof.** (i) We know already that \( \ast \) is commutative and associative and that \( e \) is the unit element of \( \ast \). It remains to verify that every element of \( A \) has an inverse with respect to \( \ast \). Let \( f \in A \), and define \( g \) recursively by

\[
g(1) := f(1)^{-1}, \quad g(n) := -f(1)^{-1} \sum_{d \mid n, d < n} f(n/d)g(d) \quad \text{for } n > 1.
\]

Then clearly, \( (f \ast g)(1) = 1 \) and \( (f \ast g)(n) = 0 \) for \( n > 1 \), i.e., \( f \ast g = e \). Hence \( f \) has an inverse. It should be observed here that the inverse of \( f \) is uniquely determined.

(ii) We first have to verify that the convolution product of two multiplicative functions is again multiplicative. Here we use that if \( m, n \) are two coprime integers
and $d$ is a positive divisor of $mn$, then $d$ has a unique decomposition $d = d_1d_2$ where $d_1$ is a positive divisor of $m$ and $d_2$ a positive divisor of $n$. Now let $f, g \in \mathcal{M}$ and let $m, n$ be two coprime positive integers. Then

$$(f \ast g)(mn) = \sum_{d \mid mn} f(mn/d)g(d) = \sum_{d \mid m, d \mid n} f(mn/d_1d_2)g(d_1d_2)$$

$$= \sum_{d_1 \mid m} \sum_{d_2 \mid n} f(m/d_1)f(n/d_2)g(d_1)g(d_2)$$

$$= \left( \sum_{d_1 \mid m} f(m/d_1)g(d_1) \right) \cdot \left( \sum_{d_2 \mid n} f(n/d_2)g(d_2) \right)$$

$$= (f \ast g)(m) \cdot (f \ast g)(n).$$

This shows that $f \ast g \in \mathcal{M}$.

It remains to show that the inverse of a multiplicative function is again multiplicative. Let $f \in \mathcal{M}$ and let $g$ be its inverse with respect to $\ast$. Define $h$ by

$$h(p^k) := g(p^k) \quad \text{for any prime power } p^k,$$

$$h(n) := h(p_1^{k_1}) \cdots h(p_t^{k_t}),$$

where $n = p_1^{k_1} \cdots p_t^{k_t}$. Then $h$ is multiplicative, and $(f \ast h)(p^k) = (f \ast g)(p^k) = e(p^k)$ for every prime power $p^k$. Both $f \ast h$ and $e$ are multiplicative, so in fact $f \ast h = e$. Since the inverse of $f$ is uniquely determined, this shows that $g = h$ is multiplicative.

Example. The Möbius function $\mu$ is the inverse under $\ast$ of $E$, where $E(n) = 1$ for all $n$.

Lemma 3.9. We have

$$\mu(n) = \begin{cases} (-1)^t & \text{if } n = p_1 \cdots p_t \text{ with } p_1, \ldots, p_t \text{ distinct primes}, \\ 0 & \text{if } n \text{ is divisible by the square of a prime}. \end{cases}$$

Proof. let $g$ denote the function defined by the right-hand side. Then clearly, $g$ is multiplicative, and $g(p) = -1$, $g(p^k) = 0$ for every prime $p$ and every $k \geq 2$. One verifies easily that $(E \ast g)(p^k) = 0$ for all primes $p$ and $k > 0$. So $E \ast g$ coincides with $e$ on the prime powers, hence $E \ast g = e$. But then it follows that $\mu = g$. 

64
Theorem 3.10 (Möbius’ Inversion Formula). Let \( f \) be an arithmetical function. Define \( F(n) := \sum_{d|n} f(n) \) for \( n \in \mathbb{Z}_{>0} \). Then

\[
f(n) = \sum_{d|n} \mu(n/d)F(d) \quad \text{for } n \in \mathbb{Z}_{>0}.
\]

Proof. We have \( F = E \ast f \). Hence

\[
\mu \ast F = \mu \ast (E \ast f) = (\mu \ast E) \ast f = e \ast f = f.
\]

\[\square\]

Examples. 1) Define \( \varphi(n) := \#\{k \in \mathbb{Z} : 1 \leq k \leq n, \gcd(k,n) = 1\} \). It is well-known that \( \sum_{d|n} \varphi(d) = n \) for \( n \in \mathbb{Z}_{>0} \). This implies that

\[
\varphi(n) = \sum_{d|n} \mu(n/d)d,
\]

or \( \varphi = \mu \ast I_1 \), where we define \( I_\alpha(n) = n^\alpha \) for \( n \in \mathbb{Z}_{>0} \), \( \alpha \in \mathbb{C} \). As a consequence, \( \varphi \) is multiplicative, and for \( n = p_1^{k_1} \cdots p_t^{k_t} \) we have

\[
\varphi(n) = \prod_{i=1}^{t} \varphi(p_i^{k_i}) = \prod_{i=1}^{t} (p_i^{k_i} - p_i^{k_i-1}).
\]

2) Let \( \alpha \in \mathbb{C} \) and define \( \sigma_\alpha(n) := \sum_{d|n} d^\alpha \) for \( n \in \mathbb{Z}_{>0} \). Then \( \sigma_\alpha = E \ast I_\alpha \), which implies that \( \sigma_\alpha \) is multiplicative. Hence for \( n = p_1^{k_1} \cdots p_t^{k_t} \) we have

\[
\sigma_\alpha(n) = \prod_{i=1}^{t} \sigma_\alpha(p_i^{k_i}) = \begin{cases} 
\prod_{i=1}^{t} \frac{p_i^{\alpha(k_i+1)-1}}{p_i^\alpha - 1} & \text{if } \alpha \neq 0, \\
\prod_{i=1}^{t} (k_i + 1) & \text{if } \alpha = 0.
\end{cases}
\]

We now give the relation between the convolution product of two arithmetical functions and their associated Dirichlet series.

Theorem 3.11. Let \( f, g \) be two arithmetical functions. Let \( s \in \mathbb{C} \) be such that \( L_f(s) \) and \( L_g(s) \) converge absolutely.
Then also \( L_{f \ast g}(s) \) converges absolutely, and \( L_{f \ast g}(s) = L_f(s)L_g(s) \).
Proof. Since both $L_f(s)$ and $L_g(s)$ are absolutely convergent we can rearrange their product as a double series and then rearrange the terms:

$$
\left( \sum_{m=1}^{\infty} f(m)m^{-s} \right) \left( \sum_{n=1}^{\infty} g(n)n^{-s} \right)
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m)g(n)(mn)^{-s} = \sum_{k=1}^{\infty} \left( \sum_{mn=k} f(m)g(n) \right) k^{-s}
= \sum_{k=1}^{\infty} (f * g)(k)k^{-s} = L_{f*g}(s).
$$

We now show that $L_{f*g}(s)$ converges absolutely:

$$
\sum_{k=1}^{\infty} |(f * g)(k)k^{-s}| \leq \sum_{k=1}^{\infty} \left( \sum_{mn=k} |f(m)| \cdot |g(n)| \right) \cdot |k^{-s}|
= \left( \sum_{m=1}^{\infty} |f(m)m^{-s}| \right) \left( \sum_{n=1}^{\infty} |g(n)n^{-s}| \right) < \infty
$$

by following the above reasoning in opposite direction and taking absolute values everywhere. This completes our proof. \(\square\)

We define $\sum_p (\cdots) = \lim_{N \to \infty} \sum_{p \leq N} (\cdots)$, and $\prod_p (\cdots) = \lim_{N \to \infty} \prod_{p \leq N} (\cdots)$ where the sums and products are taken over the primes.

**Theorem 3.12.** Let $f$ be a multiplicative function. Let $s \in \mathbb{C}$ be such that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely. Then

$$
L_f(s) = \prod_p \left( \sum_{j=0}^{\infty} f(p^j)p^{-js} \right)
$$

and the right-hand side converges absolutely.

Further, $L_f(s) \neq 0$ as soon as $\sum_{j=0}^{\infty} f(p^j)p^{-js} \neq 0$ for every prime $p$.

**Proof.** We first show the absolute convergence of the right-hand side of (3.3), i.e.,

$$
\prod_p \left( \sum_{j=0}^{\infty} |f(p^j)p^{-js}| \right)
$$

converges.
First notice that for every prime \( p \),

\[
A_p(s) := \sum_{j=0}^{\infty} |f(p^j)p^{-js}| \leq \sum_{n=1}^{\infty} |f(n)n^{-s}| < \infty.
\]

Recall that \( \prod_p A_p(s) \) converges if and only if \( \sum_p |A_p(s) - 1| \) converges (see the section on uniform convergence in the Prerequisites). But the latter holds, since

\[
\sum_p |A_p(s) - 1| = \sum_p \sum_{j=1}^{\infty} |f(p^j)p^{-js}| \leq \sum_{n=2}^{\infty} |f(n)n^{-s}| < \infty.
\]

This proves the absolute convergence of the right-hand side of (3.3).

Put \( L_p(s) := \sum_{j=0}^{\infty} f(p^j)p^{-js} \). We have seen that the series \( L_p(s) \) all converge absolutely. Further,

\[
\sum_p |L_p(s) - 1| \leq \sum_p \sum_{j=1}^{\infty} |f(p^j)p^{-js}| < \infty.
\]

Hence \( \prod_p L_p(s) \) converges, which implies that the product is 0 if and only if at least one of its factors is 0.

It remains to prove that \( L_f(s) = \prod_p L_p(s) \). Let \( N > 1 \) and let \( p_1, \ldots, p_t \) be the prime numbers \( \leq N \). Further, let \( S_N \) be the set of integers composed of prime numbers \( \leq N \) and \( T_N \) the set of remaining integers, i.e., divisible by at least one prime > \( N \). Since the series \( L_p(s) \) (\( p \) prime) converge absolutely, we have

\[
\prod_{p \leq N} L_p(s) = \sum_{j_1, \ldots, j_t \geq 0} f(p_1^{j_1}) \cdots f(p_t^{j_t})(p_1^{-j_1} \cdots p_t^{-j_t})^s = \sum_{n \in S_N} f(n)n^{-s}.
\]

Now clearly,

\[
\left| L_f(s) - \prod_{p \leq N} L_p(s) \right| = \left| \sum_{n=1}^{\infty} f(n)n^{-s} - \sum_{n \in S_N} f(n)n^{-s} \right| = \left| \sum_{n \in T_N} f(n)n^{-s} \right| \leq \sum_{n=N+1}^{\infty} |f(n)n^{-s}| \to 0 \text{ as } N \to \infty.
\]

This proves (3.3). \( \Box \)
Corollary 3.13. Let \( f \) be a strongly multiplicative function. Let \( s \in \mathbb{C} \) be such that \( L_f(s) \) converges absolutely. Then

\[
L_f(s) = \prod_p \frac{1}{1 - f(p)p^{-s}}
\]

and the right-hand side converges absolutely. Further, \( L_f(s) \neq 0 \).

Proof. Use that

\[
\sum_{j=0}^{\infty} f(p^j)p^{-js} = \sum_{j=0}^{\infty} (f(p)p^{-s})^j = \frac{1}{1 - f(p)p^{-s}} \quad \text{and} \quad \sum_{j=0}^{\infty} |f(p^j)p^{-js}| = \frac{1}{1 - |f(p)p^{-s}|}.
\]

Further, all factors \((1 - f(p)p^{-s})^{-1}\) are \( \neq 0 \), hence \( L_p(s) \neq 0 \).

Examples. 1) \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1} \) for \( s \in \mathbb{C} \) with \( \Re s > 1 \) (Euler).

2) \( L_\mu(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s} \) converges absolutely if \( \Re s > 1 \). So on \( \{ \Re s > 1 \} \) we have

\[
\zeta(s)L_\mu(s) = \sum_{n=1}^{\infty} (E*\mu)(n)n^{-s} = \sum_{n=1}^{\infty} e(n)n^{-s} = 1,
\]

that is, \( \zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s} \) for \( s \in \mathbb{C} \) with \( \Re s > 1 \). An alternative way to prove this is to observe that

\[
\zeta(s)^{-1} = \prod_p (1 - p^{-s}) = \prod_p \left( \sum_{j=0}^{\infty} \mu(p^j)p^{-js} \right) = \sum_{n=1}^{\infty} \mu(n)n^{-s}.
\]

3) Recall that \( \varphi = \mu * I_1 \). The series \( L_{I_1}(s) = \sum_{n=1}^{\infty} n/n^s = \zeta(s) - 1 \) converges absolutely on \( \{ \Re s > 2 \} \). Hence

\[
\sum_{n=1}^{\infty} \varphi(n)n^{-s} = L_{\varphi(s)} = L_\mu(s)L_{I_1}(s) = \zeta(s-1)/\zeta(s)
\]

and \( L_\varphi(s) \) converges absolutely if \( \Re s > 2 \).

4) The (very important) von Mangoldt function \( \Lambda \) is defined by

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ for some prime } p \text{ and some } k \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]
E.g., $\Lambda(1) = 0$, $\Lambda(2) = \log 2$, $\Lambda(3) = \log 3$, $\Lambda(4) = \log 2$, $\Lambda(5) = \log 5$, $\Lambda(6) = 0$, $\Lambda(7) = \log 7$, $\Lambda(8) = \log 2$, $\Lambda(9) = \log 3$, $\Lambda(10) = 0$.

For $n = p_1^{k_1} \cdots p_t^{k_t}$ (unique prime factorization) we have

$$
\sum_{d|n} \Lambda(n) = \sum_{i=1}^{t} \sum_{j=1}^{k_i} \log p_i = \sum_{i=1}^{t} k_i \log p_i = \log n.
$$

Hence $E \ast \Lambda = \log$, where log denotes the arithmetical function $n \mapsto \log n$. So $\Lambda = \mu \ast \log$.

**Lemma 3.14.** For $s \in \mathbb{C}$ with $\Re s > 1$, the series $\sum_{n=1}^{\infty} \Lambda(n)n^{-s}$ converges absolutely, and

$$
\sum_{n=1}^{\infty} \Lambda(n)n^{-s} = -\zeta'(s)/\zeta(s).
$$

**Proof.** We apply Theorem 3.11. First recall that $L_{\mu}(s)$ converges absolutely if $\Re s > 1$. Further, by Theorem 3.4, we have $\zeta'(s) = \sum_{n=1}^{\infty} (-\log n)n^{-s}$ for $\Re s > 1$. It follows that

$$
\sum_{n=1}^{\infty} \left| \log(n)n^{-s} \right| = \sum_{n=1}^{\infty} (\log n)n^{-\Re s} = -\zeta'(\Re s)
$$

converges if $\Re s > 1$. That is, $L_{\log}(s)$ converges absolutely if $\Re s > 1$. It follows that

$$
L_{\Lambda}(s) = L_{\mu}(s)L_{\log}(s) = -\zeta^{-1}(s)\zeta'(s)
$$

and $L_{\Lambda}(s)$ converges absolutely if $\Re s > 1$. \qed