# EXERCISES FOR THE COURSE 'ANALYTIC NUMBER THEORY 2018'

Exercise sheet 1

Thursday 8th November 2018

1) Formulate the twin prime conjecture as a sieve problem. What is the minimal value of $z$ that one has to take (notation as in the lecture or the book by Iwaniec and Kowalski)?

2) Let $n \in \mathbb{N}$. Give a proof of the identity

$$\sum_{d^2|n} \mu(d) = \begin{cases} 1 & \text{if} \quad \text{n is square-free} \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Can you similarly build a function which identifies $m$-free numbers for some $m \geq 2$? A natural number $n$ is called $m$-free if there is no prime number $p$ with $p^m \mid n$. Now go back to Exercise 2.9 from the 2nd lecture of the first half of the course. Generalize this to give an asymptotic formula for the number of $m$-free numbers smaller than $x$ (with $x \to \infty$).

3) For a positive real number $x$, let $\pi(x)$ be the number of primes less than or equal to $x$. Let $10 \leq z \leq x$. Show with methods that we covered so far in the lectures, that

$$\pi(x+z) - \pi(x) \ll \frac{z}{\log \log z}.$$

4) Consider the arithmetic function $\rho$ given by

$$\rho(d) = \sharp\{\nu \bmod d : \nu^2 + 1 \equiv 0 \bmod d\}.$$

Compute $\rho(d)$ for all positive integers $d$. Can you also compute it for more general binary quadratic forms?

5) In the lecture we ran into a (negative) Pell's equation of the form $x^2 - ky^2 = -1$. If you haven't seen these type of equations before, then read for example section 7.3 in Rose's book 'A course in number theory', 2nd edition to familiarize yourself with Pell's equation (or choose another book on elementary number theory where you find something on the structure of the solutions to Pell's equation. If you would like a Dutch book, you could look into chapter 16 in Frits Beukers book 'Getaltheorie, een inleiding').

6) In the lecture we gave an asymptotic formula for the number of positive integers $m$ such that $m^2+1$ is squarefree and $m^2+1 \leq x$. Let $g(m)$ be a quadratic polynomial in $m$ with integer coefficients and positive leading coefficient and consider the counting function

$$N_g(X) := \sharp\{m \in \mathbb{N} : g(m) \leq X,\ g(m) \text{ is square-free}\}.$$

For what quadratic polynomials $g(m)$ can you give an asymptotic formula for $N_g(X)$ as $X \to \infty$?