UITWERKING TENTAMEN CONTINUE WISKUNDE

16-1-2015 14:00-17:00

Deel 2

1 a) $\int x\ln x\,dx = (\ln x)\frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \ln' x\,dx$

$f(x) = \ln x$
$g'(x) = x$
$g(x) = \frac{1}{2}x^2$

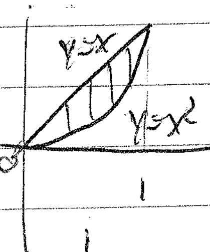
$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$

$= \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$

b) $\int_0^\infty \frac{2x\,dx}{(x^2+1)^2} = \lim_{A\to\infty} \int_0^A \frac{2x\,dx}{(x^2+1)^2} = \lim_{A\to\infty} \int_1^{A^2+1} \frac{du}{u^2}$

$u = x^2+1$

$= \lim_{A\to\infty} \left[-\frac{1}{u}\right]_1^{A^2+1} = \lim_{A\to\infty} -\frac{1}{A^2+1} - (-1) = \boxed{1}$

c) 

De grafieken van $y=x$ en $y=x^2$ snijden in $x=0$, $x=1$. De oppervlakte van het gebied wordt dan:

$\int_0^1 x\,dx - \int_0^1 x^2\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

2 a) $(x^2-1)^2 + 2(x-y)^2 - 1 = x^4 - 2x^2 + 1 + 2x^2 - 4xy + 2y^2 - 1$
$= x^4 - 4xy + 2y^2 = f(x,y)$

Kwadraten zijn ≥ 0 dus $f(x,y) \geq -1$ voor alle x,y.

b) $\frac{\partial f}{\partial x} = 4x^3 - 4y$, $\frac{\partial f}{\partial y} = -4x + 4y$

We vinden de stationaire punten door $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ op te lossen

$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0 \Leftrightarrow 4x^3=4y,\ 4x=4y \Leftrightarrow x^3=x,\ x=y$

$\Leftrightarrow x(x^2-1)=0,\ x=y$

$\Leftrightarrow \cancel{x(x^2}\ x=0$ of $x^2=1,\ x=y$

$\Leftrightarrow x=0, 1$ of $-1,\ x=y \Leftrightarrow (x,y)=(0,0),(1,1),(-1,-1).$

c) $A=\frac{\partial^2 f}{\partial x^2}=12x^2,\ B=\frac{\partial^2 f}{\partial y \partial x}=-4,\ C=\frac{\partial^2 f}{\partial y^2}=4,\ H=AC-B^2$

	A	B	C	H	
(0,0)	0	−4	4	−16<0	zadelpunt
(1,1)	12>0	−4	4	32>0	minimum
(−1,−1)	12>0	−4	4	32>0	minimum

Uit a) volgt:

$f(x,y)=(x^2-1)^2+2(x-y)^2-1 \geq -1$ voor alle x, y,

$f(1,1)=-1,\ f(-1,-1)=-1$

Dus de minima zijn $\boxed{\text{absoluut}}$

d) Vergelijking raakvlak

$$z=f(2,1)+\frac{\partial f}{\partial x}(2,1)(x-2)+\frac{\partial f}{\partial y}(2,1)(y-1)$$

$f(2,1)=2^4-4\times 2+2=10$

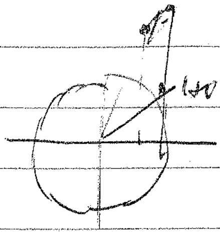
$\frac{\partial f}{\partial x}(2,1)=4\times 2^3-4=28,\quad \frac{\partial f}{\partial y}(2,1)=-4\times 2+4\times 1=-4$

Dit geeft: $\boxed{z=10+28(x-2)-4(y-1)=28x-4y-42}$

3

a) $\frac{z}{w} = \frac{1+\sqrt{3}\,i}{1+i} = \frac{(1+\sqrt{3}\,i)(1-i)}{(1+i)(1-i)} = \frac{1+\sqrt{3}\,i - i - \sqrt{3}\,i^2}{1^2+1^2}$

$= \frac{1+\sqrt{3}+(\sqrt{3}-1)i}{2} = \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}\,i$



$|z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$ $\quad \text{Mod}\left(\frac{z}{w}\right) = \frac{|z|}{|w|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$|w| = \sqrt{1^2+1^2} = \sqrt{2}$

$z = r(\cos\varphi + i\sin\varphi)$, $r=2$, $\cos\varphi = \frac{1}{2}$, $\sin\varphi = \frac{\sqrt{3}}{2}$

$\varphi = \frac{1}{3}\pi$, $\text{Arg}\, z = \frac{1}{3}\pi$

$w = s(\cos\psi + i\sin\psi)$, $s = \sqrt{2}$, $\cos\psi = \frac{1}{\sqrt{2}}$, $\sin\psi = \frac{1}{\sqrt{2}}$

~~Arg~~ $\psi = \frac{1}{4}\pi$, $\text{Arg}\, w = \frac{1}{4}\pi$

$\text{Arg}\,\frac{z}{w} = \frac{1}{3}\pi - \frac{1}{4}\pi = \boxed{\frac{1}{12}\pi}$

b) Schrijf $2-2i = r(\cos\varphi + i\sin\varphi)$, $r = |2-2i|$, $\varphi = \arg(2-2i)$

$r = \sqrt{2^2+(-2)^2} = \sqrt{8}$

$\cos\varphi = \frac{2}{\sqrt{8}} = \frac{2}{2\sqrt{2}} = \frac{1}{2}\sqrt{2}$, $\sin\varphi = \frac{-2}{\sqrt{8}} = -\frac{1}{2}\sqrt{2}$,

$\varphi = -\frac{1}{4}\pi$, $(2-2i)^{20} = (\sqrt{8})^{20}\left(\cos\left(-\frac{20}{4}\pi\right) + i\sin\left(-\frac{20}{4}\pi\right)\right)$

$= 8^{20}(\cos(-5\pi) + i\sin(-5\pi)) = \boxed{-8^{20}}$

c) Schrijf $q = \frac{81}{2}(1-\sqrt{3}\,i)$ in de vorm $r(\cos\varphi + i\sin\varphi)$

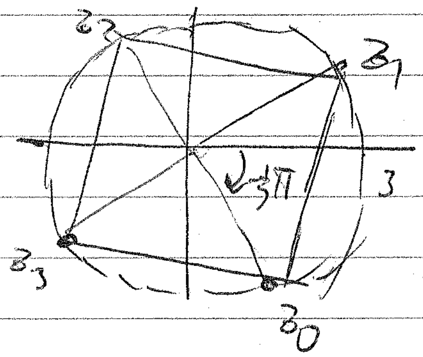
$r = \sqrt{\left(\frac{81}{2}\right)^2 + \left(\frac{81}{2}\right)^2 3} = \sqrt{81^2} = 81$

$\cos\varphi = \frac{1}{2}$, $\sin\varphi = -\frac{1}{2}\sqrt{3}$ $\Rightarrow$ $\varphi = -\frac{1}{3}\pi$

Oplossingen:

$$z_k = \sqrt[4]{81}\left(\cos\left(-\tfrac{1}{3}\pi + \frac{2k\pi}{4}\right) + i\sin\left(-\tfrac{1}{3}\pi + \frac{2k\pi}{4}\right)\right)$$

$$= \boxed{3\left(\cos\left(-\tfrac{1}{3}\pi + \tfrac{1}{2}k\pi\right) + i\sin\left(-\tfrac{1}{3}\pi + \tfrac{1}{2}k\pi\right)\right) \quad k=0,1,2,3}$$



d) $D = 12^2 - 4\times 2 \cdot 19 = 144 - 8\times 19 = -8.$

oplossingen: $z_{1,2} = \dfrac{-12 \pm i\sqrt{8}}{4} = -3 \pm \dfrac{2\sqrt{2}}{4} i = \boxed{-3 \pm \tfrac{1}{2}\sqrt{2}\, i}$

4) a) $0{,}09\,09\,09\,09 \ldots = \dfrac{9}{10^2} + \dfrac{9}{10^4} + \dfrac{9}{10^6} + \ldots$

$$= \frac{9}{10^2}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right) = \frac{9}{10^2}\,\frac{1}{1-\frac{1}{10^2}} = \frac{9}{10^2}\,\frac{1}{99/10^2}$$

$$= \frac{9}{99} = \boxed{\frac{1}{11}}$$

b) $\dfrac{\sqrt{n}+2}{n^2+1}$ is voor n groot ongeveer $\dfrac{\sqrt{n}}{n^2} = n^{-3/2}$

We vergelijken met $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{\sqrt{n}+2}{n^2+1} \Big/ \frac{1}{n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+2)}{n^2+1} = \lim_{n\to\infty} \frac{n^2 + n^{3/2}}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1 + n^{-1/2}}{1 + \frac{1}{n^2}} = 1.$$

4b) (vervolg) De reeks $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

Dus $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{n^2+1}$ is $\boxed{\text{convergent}}$

<u>HELE STOF</u>

1 a) $$\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} \overset{0}{\underset{0}{=}} \lim_{x\to 0} \frac{-\sin x - 2x e^{x^2}}{x^2}$$

$$\overset{0}{\underset{0}{=}} \lim_{x\to 0} \frac{-\cos x - 2e^{x^2} - 4x^2 \cdot e^{x^2}}{2} = \boxed{-\frac{3}{2}}$$

b) $$\lim_{x\to\infty} \frac{4^x+3^x}{4^x+2^x} = \lim_{x\to\infty} \frac{1+\overbrace{(3/4)^x}^{\to 0}}{1+\underbrace{(2/4)^x}_{\to 0}} = \boxed{1}$$

2 a) $\lim_{x\downarrow 0} f(x) = \lim_{x\downarrow 0} x^3+2 = 2$

$\lim_{x\uparrow 0} f(x) = \lim_{x\uparrow 0} \frac{\ln(x^2+c)}{\ln(x^2+2)} = \frac{\ln c}{\ln 2}$

$\lim_{x\to 0} f(x)$ bestaat $\Leftrightarrow \frac{\ln c}{\ln 2} = 2 \Leftrightarrow \ln c = 2\ln 2$

$\Leftrightarrow \boxed{c=4}$

b) $f(x)$ is continu in $x=0 \Leftrightarrow \lim_{x\to 0} f(x) = f(0) = d$

Dus f is continu in $x=0 \Leftrightarrow \boxed{d=2}$

3 a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\ln(1+x)$	0	0
1	$\frac{1}{1+x}$	1	1
2	$-\frac{1}{(1+x)^2}$	-1	$-\frac{1}{2!} = -\frac{1}{2}$
3	$\frac{2}{(1+x)^3}$	2	$\frac{2}{3!} = \frac{1}{3}$

$$P_3(x) = P_3(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$= \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$$

$$f^{(4)}(x) = \frac{-6}{(1+4)^4}$$

b) $E_3(x) = \frac{f^{(4)}(\xi)}{4!}x^4 = \frac{-6}{24(1+\xi)^4}x^4$, ξ tussen 0 en x

c) $|E_3(0{,}01)| = \frac{1}{4(1+\xi)^4}(0{,}01)^4 < \frac{1}{4}\times 10^{-8}$. ξ tussen 0 en 0,01

4) $f(x) = \frac{x^4}{x^4-1}$

a) Er zijn verticale asymptoten waar de noemer 0 is en de teller $\neq 0$. Dit is het geval voor $x=1$, $x=-1$
Dus $x=1$, $x=-1$ zijn de verticale asymptoten van f.

$\lim_{x \downarrow 1} f(x) = \lim_{x \downarrow 1} \frac{x^4}{x^4 - 1} = \infty$ (noemer > 0, teller > 0)

$\lim_{x \uparrow 1} f(x) = -\infty$ (noemer < 0, teller > 0)

$\lim_{x \downarrow -1} f(x) = -\infty$ (noemer < 0, teller > 0)

$\lim_{x \uparrow -1} f(x) = \infty$ (noemer > 0, teller > 0)

b) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1 - \frac{1}{x^4}} = 1$, $\lim_{x \to -\infty} f(x) = 1$ (idem).

Dus $y = 1$ is een horizontale asymptoot zowel voor $x \to \infty$ als $x \to -\infty$

c) $f'(x) = \frac{(x^4 - 1) \cdot 4x^3 - x^4 \cdot 4x^3}{(x^4 - 1)^2} = \frac{4x^7 - 4x^3 - 4x^7}{(x^4 - 1)^2}$

$= \frac{-4x^3}{(x^4 - 1)^2}$

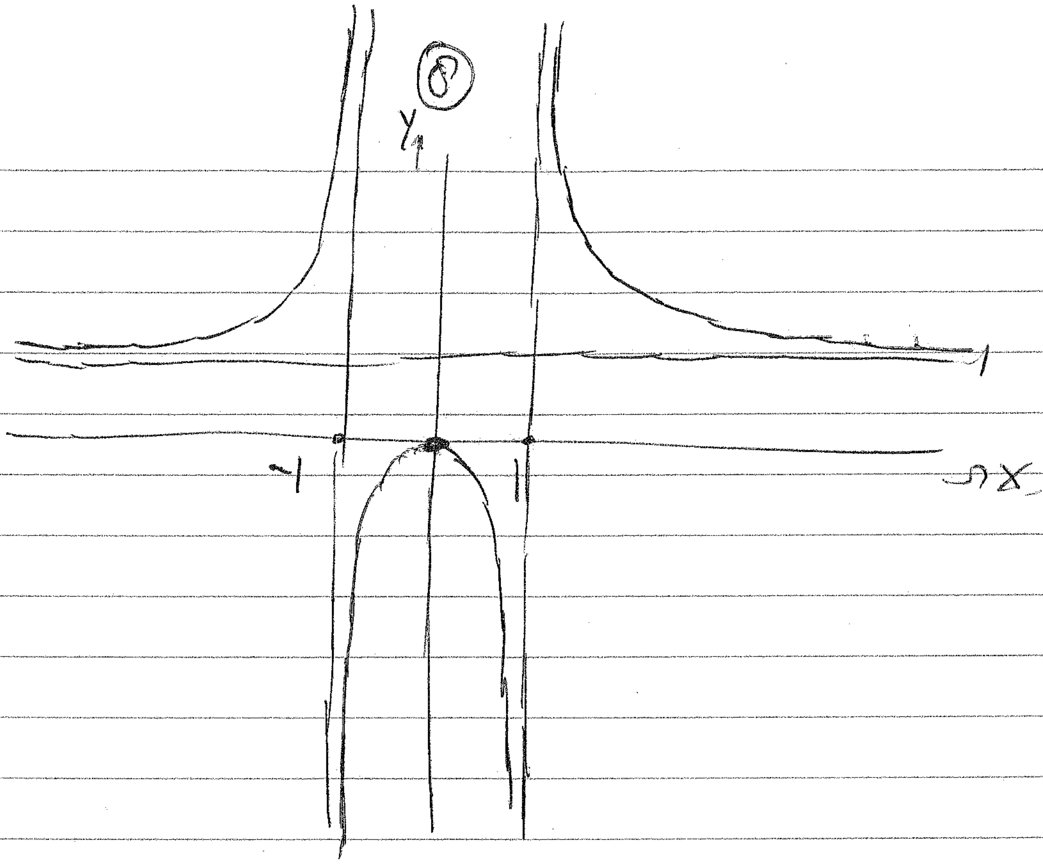
Tekenoverzicht van f'

f'	+	NG	+	0	−	NG	−
f	$\nearrow$	−1	$\nearrow$	0	$\searrow$	1	$\searrow$

f heeft in $x = 0$ een maximum, grootte $f(0) = 0$

Dit is relatief want $\lim_{x \downarrow 1} f(x) = \infty$

4d)



5: zie ded 2 1a) 1b)

6 zie ded 2 2a), 2b) 2c)

7 zie ded 2 3a), 3b), 3c)

8 zie ded 2 4a), 4b)