Results and open problems related to Schmidt’s Subspace Theorem

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Define

\[ H(\xi) = \max(|p|, |q|), \text{ where } \xi = p/q, \ p, q \in \mathbb{Z}, \ \gcd(p, q) = 1. \]

**Theorem (Roth, 1955)**

*Let \( \alpha \) be a real algebraic number and \( \delta > 0 \). Then the inequality*

\[ |\alpha - \xi| \leq H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q} \]

*has only finitely many solutions.*

Roth’s proof, and later proofs of his Theorem, are ineffective, i.e., they do not give a method to determine the solutions.
A semi-effective result

The minimal polynomial of an algebraic number $\alpha$ is the irreducible polynomial $F \in \mathbb{Z}[X]$ with coprime coefficients such that $F(\alpha) = 0$. We define the height $H(\alpha) := \max |\text{coeff. of } F|$.

**Theorem (Bombieri, van der Poorten, 1987)**

*Let $\delta > 0$, $K = \mathbb{Q}(\alpha)$, $[K : \mathbb{Q}] = d$. Then for the solutions $\xi \in \mathbb{Q}$ of*

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta}$$

*we have $H(\xi) \leq \max (B_{\text{ineff}}(\delta, K), H(\alpha)^{c_{\text{eff}}(\delta, d)})$.*

Here $c_{\text{eff}}$, $B_{\text{ineff}}$ are constants, effectively, resp. not effectively computable from the method of proof, depending on the parameters between the parentheses.
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*Let $\delta > 0$, $K = \mathbb{Q}(\alpha)$, $[K : \mathbb{Q}] = d$. Then for the solutions $\xi \in \mathbb{Q}$ of

$\mid \alpha - \xi \mid \leq H(\xi)^{-2-\delta}$

we have $H(\xi) \leq \max (B^{\text{ineff}}(\delta, K), H(\alpha)^{c\text{eff}}(\delta,d))$."

**Equivalent formulation, ’Roth’s theorem with moving targets’**

*Let $K$ be a number field of degree $d$ and $\delta > 0$. Then there are only finitely many pairs $(\xi, \alpha) \in \mathbb{Q} \times K$ such that

$\mid \alpha - \xi \mid \leq H(\xi)^{-2-\delta}$, $H(\xi) > H(\alpha)^{c\text{eff}}(\delta,d)$."
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The minimal polynomial of an algebraic number $\alpha$ is the irreducible polynomial $F \in \mathbb{Z}[X]$ with coprime coefficients such that $F(\alpha) = 0$.

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Let $\delta > 0$, $K = \mathbb{Q}(\alpha)$, $[K : \mathbb{Q}] = d$. Then for the solutions $\xi \in \mathbb{Q}$ of

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we have $H(\xi) \leq \max \left( B^{\text{ineff}}(\delta, K), H(\alpha)^{\text{eff}(\delta,d)} \right)$.

Similar results follow from work of Vojta (1995), Corvaja (1997), McQuillan ($c^{\text{eff}}(\delta, d) = O(d(1 + \delta^{-2}))$, published ?).
Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let

$$L_i(X) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \ (i = 1, \ldots, n)$$

be linearly independent linear forms with coefficients $\alpha_{ij} \in \overline{\mathbb{Q}}$.

For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, put $\|x\| := \max_i |x_i|$.

**Theorem (W.M. Schmidt, 1972)**

Let $\delta > 0$. Then the set of solutions of

$$|L_1(x) \cdots L_n(x)| \leq \|x\|^{-\delta} \text{ in } x \in \mathbb{Z}^n$$

is contained in finitely many proper linear subspaces of $\mathbb{Q}^n$.

There are generalizations where the unknowns are taken from a number field and various archimedean and non-archimedean absolute values are involved. (Schmidt, Schlickewei)
By a combinatorial argument, the inequality (2) $|L_1(x) \cdots L_n(x)| \leq \|x\|^{-\delta}$ can be reduced to finitely many systems of inequalities of the shape

$$(3) \quad |L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{in} \ x \in \mathbb{Z}^n,$$

where $c_1 + \cdots + c_n < 0$. 
By a combinatorial argument, the inequality (2) $|L_1(x) \cdots L_n(x)| \leq \|x\|^{-\delta}$ can be reduced to finitely many systems of inequalities of the shape

$$\left| L_1(x) \right| \leq \|x\|^{c_1}, \ldots, \left| L_n(x) \right| \leq \|x\|^{c_n} \quad \text{in } x \in \mathbb{Z}^n,$$

where $c_1 + \cdots + c_n < 0$.

**Idea.**

Let $x \in \mathbb{Z}^n$ be a solution of (2). Then

$$\left| L_1(x) \right| \leq \|x\|^{c_1(x)}, \ldots, \left| L_n(x) \right| \leq \|x\|^{c_n(x)}$$

with

$$c(x) := (c_1(x), \ldots, c_n(x)) \in \text{bounded set } S.$$ Cover $S$ by a very fine, finite grid. Then $x$ satisfies (3) with $c = (c_1, \ldots, c_n)$ a grid point very close to $c(x)$.  

\[ \square \]
By a combinatorial argument, the inequality (2) \( |L_1(x) \cdots L_n(x)| \leq \|x\|^{-\delta} \) can be reduced to finitely many systems of inequalities of the shape

\[
(3) \quad |L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{in } x \in \mathbb{Z}^n,
\]

where \( c_1 + \cdots + c_n < 0 \).

Thus, the following is equivalent to the Subspace Theorem:

**Theorem**

*The solutions of (3) lie in finitely many proper linear subspaces of \( \mathbb{Q}^n \).*
A refinement of the Subspace Theorem

Let again $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ with coefficients in $\overline{\mathbb{Q}}$ and $c_1, \ldots, c_n$ reals with $c_1 + \cdots + c_n < 0$. Consider again

$$
(3) \quad |L_1(x)| \leq \|x\|^c_1, \ldots, |L_n(x)| \leq \|x\|^c_n \quad \text{in } x \in \mathbb{Z}^n.
$$

**Theorem (Vojta (1989), Schmidt (1993), Faltings-Wüstholz (1994))**

There is an effectively computable, proper linear subspace $T^{\text{exc}}$ of $\mathbb{Q}^n$ such that (3) has only finitely many solutions outside $T^{\text{exc}}$.

The space $T^{\text{exc}}$ belongs to a finite collection, depending only on $L_1, \ldots, L_n$ and independent of $c_1, \ldots, c_n$.

This refinement can be deduced from Schmidt’s basic Subspace Theorem, so it is in fact equivalent to Schmidt’s basic Subspace Theorem.
About the exceptional subspace

Assume for simplicity that $L_1, \ldots, L_n$ have real algebraic coefficients.

For a linear subspace $T$ of $\mathbb{Q}^n$, we say that a subset $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ is linearly independent on $T$ if no non-trivial $\mathbb{R}$-linear combination of $L_{i_1}, \ldots, L_{i_m}$ vanishes identically on $T$.

For a linear subspace $T$ of $\mathbb{Q}^n$, define $c(T)$ to be the minimum of the quantities $c_{i_1} + \cdots + c_{i_m}$, taken over all subsets $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ of cardinality $m = \dim T$ that are linearly independent on $T$.

$T^\text{exc}$ is the unique proper linear subspace $T$ of $\mathbb{Q}^n$ such that

$$\frac{c(\mathbb{Q}^n) - c(T)}{n - \dim T}$$

is minimal, subject to this condition, $\dim T$ is minimal.
About the exceptional subspace
Lemma (E., Ferretti, 2013)

Suppose that the coefficients of $L_1, \ldots, L_n$ have heights $\leq H$.

Then $T^{\text{exc}}$ has a basis $\{x_1, \ldots, x_m\} \subset \mathbb{Z}^n$ with

$$\|x_i\| \leq (\sqrt{n}H^n)^{4n} \quad (i = 1, \ldots, m).$$

Open problem

Is there an efficient method to determine $T^{\text{exc}}$ in general?

Easy combinatorial expression of $T^{\text{exc}}$ in terms of $L_1, \ldots, L_n$, $c_1, \ldots, c_n$?
Remarks

With the present methods of proof it is not possible to determine effectively the solutions of

\[(3) \quad |L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{in } x \in \mathbb{Z}^n \]

outside $T^{\text{exc}}$.

It is possible to give an explicit upper bound for the minimal number of proper linear subspaces of $\mathbb{Q}^n$ whose union contains all solutions of (3). This bound depends on $n$, $\delta := -(c_1 + \cdots + c_n)$ and on the heights and degrees of the coefficients of $L_1, \ldots, L_n$ (Schmidt (1989), \ldots, E. and Ferretti (2013)).

With the present methods it is not possible to estimate from above the number of solutions of (3) outside $T^{\text{exc}}$. 
A semi-effective version of the Subspace Theorem

Let $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ and $c_1, \ldots, c_n$ reals such that:

- the coefficients of $L_1, \ldots, L_n$ have heights $\leq H$ and generate a number field $K$ of degree $d$;
- $c_1 + \cdots + c_n = -\delta < 0$.

**Theorem**

*For every solution $x$ of*

\begin{equation}
|L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{with} \quad x \in \mathbb{Z}^n \setminus T^{exc}
\end{equation}

*we have $\|x\| \leq \max \left( B^{ineff}(n, \delta, K), H^{eff}(n, \delta, d) \right)$.*

**Proof.**

Small modification in the proof of the Subspace Theorem. \qed
A semi-effective version of the Subspace Theorem

Let $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ and $c_1, \ldots, c_n$ reals such that:

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*For every solution $x$ of*

\[
|L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{with} \quad x \in \mathbb{Z}^n \setminus T^{\text{exc}}
\]

*we have* \[
\|x\| \leq \max \left( B^{\text{ineff}}(n, \delta, K), H^{\text{eff}}(n, \delta, d) \right).
\]

*We may take* \[
c^{\text{eff}}(n, \delta, d) = \exp \left( 10^6 n (1 + \delta^{-3}) \log 4d \log \log 4d \right).
\]
A semi-effective version of the Subspace Theorem

Let $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ and $c_1, \ldots, c_n$ reals such that:

- the coefficients of $L_1, \ldots, L_n$ have heights $\leq H$ and generate a number field $K$ of degree $d$;
- $c_1 + \cdots + c_n = -\delta < 0$.

**Theorem**

*For every solution $x$ of*

\begin{equation}
|L_1(x)| \leq ||x||^{c_1}, \ldots, |L_n(x)| \leq ||x||^{c_n} \quad \text{with } x \in \mathbb{Z}^n \setminus T^{\text{exc}}
\end{equation}

*we have* $||x|| \leq \max \left( B^{\text{ineff}}(n, \delta, K), H^{\text{eff}}(n, \delta, d) \right)$.

This may be viewed as a version of the Subspace Theorem with moving targets, where we have only finitely many tuples $(x, L_1, \ldots, L_n)$ with (4), such that the coefficients of $L_1, \ldots, L_n$ lie in a given number field $K$ and have small heights with respect to $x$.

(How to compare this with a result of Ru and Vojta?)
Keep the assumptions

- $L_1, \ldots, L_n$ are linearly independent linear forms, whose coefficients have heights $\leq H$ and generate a number field of degree $d$;
- $c_1 + \cdots + c_n = -\delta < 0$.

**Conjecture 1**

*There are an effectively computable constant $c_{\text{eff}}^{\text{eff}}(n, \delta, d) > 0$ and a constant $B'(n, \delta, d) > 0$ such that for every solution $x$ of

\[(4) \quad |L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{with} \quad x \in \mathbb{Z}^n \setminus T^{\text{exc}}\]

we have $\|x\| \leq \max \left( B'(n, \delta, d), H^{c_{\text{eff}}^{\text{eff}}(n, \delta, d)} \right)$.*

(In moving targets terms: there are only finitely many tuples $(x, L_1, \ldots, L_n)$ with (4) such that the coefficients of $L_1, \ldots, L_n$ have bounded degree and have heights small compared with $x$).
A conjectural improvement

Keep the assumptions

• $L_1, \ldots, L_n$ are linearly independent linear forms, whose coefficients have heights $\leq H$ and generate a number field of degree $d$;

• $c_1 + \cdots + c_n = -\delta < 0$.

**Conjecture 1**

There are an effectively computable constant $c^\text{eff}(n, \delta, d) > 0$ and a constant $B'(n, \delta, d) > 0$ such that for every solution $x$ of

\[
|L_1(x)| \leq \|x\|^{c_1}, \ldots, |L_n(x)| \leq \|x\|^{c_n} \quad \text{with } x \in \mathbb{Z}^n \setminus T^{\text{exc}}
\]

we have $\|x\| \leq \max \left( B'(n, \delta, d), H^{c\text{eff}}(n, \delta, d) \right)$.

This is hopeless with $B'$ effective. But what if we allow $B'$ to be ineffective?
Let $K$ be an algebraic number field of degree $d$ and discriminant $D_K$. Let $a, b, c$ be non-zero elements of $O_K$ with $a + b = c$. Put $H_K(a, b, c) := \prod_{\sigma:K \hookrightarrow \mathbb{C}} \max(|\sigma(a)|, |\sigma(b)|, |\sigma(c)|)$.

**Theorem 1 (Effective abc-inequality, Győry, 1978)**

We have $H_K(a, b, c) \leq (2|N_{K/\mathbb{Q}}(abc)|)^{c_1(d)}|D_K|^{c_2(d)}$ with $c_1(d), c_2(d)$ effectively computable in terms of $d$.

**Proof.**

Baker-type logarithmic forms estimates.
abc-type inequalities

Let $K$ be an algebraic number field of degree $d$ and discriminant $D_K$. Let $a, b, c$ be non-zero elements of $O_K$ with $a + b = c$. Put $H_K(a, b, c) := \prod\limits_{\sigma: K \hookrightarrow \mathbb{C}} \max(|\sigma(a)|, |\sigma(b)|, |\sigma(c)|)$.

**Theorem 1 (Effective abc-inequality, Győry, 1978)**

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**Proof.**

Baker-type logarithmic forms estimates.

**Theorem 2 (Semi-effective abc-inequality, well-known)**

For every $\delta > 0$ we have $H_K(a, b, c) \leq C_{\text{ineff}}(K, \delta)|N_{K/\mathbb{Q}}(abc)|^{1+\delta}$.

**Proof.**

Roth’s Theorem over number fields.
A very weak abc-conjecture

Let again $K$ be a number field of degree $d$ and discriminant $D_K$.

**Conjecture 2 (Very weak abc-conjecture)**

There are a constant $C(d, \delta) > 0$ and an effectively computable constant $c_{\text{eff}}(d, \delta) > 0$ with the following property: for every non-zero $a, b, c \in O_K$ with $a + b = c$ and every $\delta > 0$ we have

$$H_K(a, b, c) \leq C(d, \delta)|D_K|^{c_{\text{eff}}(d, \delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$
A very weak abc-conjecture

Let again $K$ be a number field of degree $d$ and discriminant $D_K$.

**Conjecture 2 (Very weak abc-conjecture)**

There are a constant $C(d, \delta) > 0$ and an effectively computable constant $c^\text{eff}(d, \delta) > 0$ with the following property:

for every non-zero $a, b, c \in O_K$ with $a + b = c$ and every $\delta > 0$ we have

$$H_K(a, b, c) \leq C(d, \delta)|D_K|^{c^\text{eff}(d, \delta)}|N_K/\mathbb{Q}(abc)|^{1+\delta}.$$

**Conjecture 1 $\Rightarrow$ Conjecture 2 (idea).**

Choose a $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ of $O_K$ with conjugates bounded from above in terms of $D_K$. Write $a = \sum_{i=1}^d x_i \omega_i$, $b = \sum_{i=1}^d y_i \omega_i$ with $x_i, y_i \in \mathbb{Z}$. Then $x = (x_1, \ldots, y_d)$ satisfies one of finitely many systems of inequalities of the type

$$|L_i(x)| \leq \|x\|^{c_i} \quad (i = 1, \ldots, 2d)$$

where the $L_i$ are linear forms whose coefficients lie in the Galois closure of $K$ and have heights bounded above in terms of $|D_K|$.
Discriminants of binary forms

**Definition**

The discriminant of a binary form

\[ F = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n = \prod_{i=1}^{n} (\alpha_i X - \beta_i Y) \]

is given by

\[ D(F) = \prod_{1 \leq i < j \leq n} (\alpha_i \beta_j - \alpha_j \beta_i)^2. \]

This is a homogeneous polynomial in \( \mathbb{Z}[a_0, \ldots, a_n] \) of degree \( 2n - 2 \).
Discriminants of binary forms

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This is a homogeneous polynomial in \( \mathbb{Z}[a_0, \ldots, a_n] \) of degree \( 2n - 2 \).

For a matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we define \( F_A(X, Y) = F(aX + bY, cX + dY) \).

Two binary forms \( F, G \in \mathbb{Z}[X, Y] \) are called equivalent if \( G = \pm F_A \) for some \( A \in \text{GL}(2, \mathbb{Z}) \).

Equivalent binary forms have the same discriminant.
A finiteness result for binary forms of given discriminant

Theorem (Lagrange \((n = 2, 1773)\), Hermite \((n = 3, 1851)\), Birch and Merriman \((n \geq 4, 1972)\))

For every \(n \geq 2\) and \(D \neq 0\), there are only finitely many equivalence classes of binary forms \(F \in \mathbb{Z}[X, Y]\) of degree \(n\) and discriminant \(D\).

The proofs of Lagrange and Hermite are effective (in that they allow to compute a full system of representatives for the equivalence classes), that of Birch and Merriman is ineffective.
An effective finiteness result

Define the height of \( F = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n \in \mathbb{Z}[X, Y] \) by \( H(F) := \max_i |a_i| \).

**Theorem 3 (E., Győry, recent improvement of result from 1991)**

Let \( F \in \mathbb{Z}[X, Y] \) be a binary form of degree \( n \geq 4 \) and discriminant \( D \neq 0 \). Then \( F \) is equivalent to a binary form \( G \) for which

\[
H(G) \leq \exp \left( (16n^3)^{25n^2} |D|^{5n-3} \right).
\]
An effective finiteness result

Define the height of $F = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n \in \mathbb{Z}[X, Y]$ by $H(F) := \max_i |a_i|$.

**Theorem 3 (E., Győry, recent improvement of result from 1991)**

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then $F$ is equivalent to a binary form $G$ for which

$$H(G) \leq \exp \left( (16n^3)^{25n^2}|D|^{5n-3} \right).$$

More precise versions of the arguments of Lagrange and Hermite give a bound $H(G) \leq \text{constant} \cdot |D|$ in case that $F$ has degree $\leq 3$. 
An effective finiteness result

Define the height of $F = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n \in \mathbb{Z}[X, Y]$ by $H(F) := \max_i |a_i|.$

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$$H(G) \leq \exp \left( (16n^3)^{25n^2} |D|^{5n-3} \right).$$

**Proof (idea).**

Let $L$ be the splitting field of $F.$ Assume for convenience that $F = \prod_{i=1}^n (\alpha_i X - \beta_i Y)$ with $\alpha_i, \beta_i \in O_L \ \forall i.$ Put $\Delta_{ij} := \alpha_i \beta_j - \alpha_j \beta_i$ and apply an explicit version of the effective abc-inequality (Theorem 1) to the identities

$$\Delta_{ij} \Delta_{kl} + \Delta_{jk} \Delta_{il} = \Delta_{ik} \Delta_{jl} \ (1 \leq i, j, k, l \leq n).$$
A semi-effective finiteness result

**Theorem 4 (E., 1993)**

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 4$, discriminant $D \neq 0$ and splitting field $L$. Then $F$ is equivalent to a binary form $G$ for which

$$H(G) \leq C^{\text{ineff}}(n, L) \cdot |D|^{21/(n-1)}.$$

**Proof (idea).**

Apply the semi-effective abc-inequality Theorem 2 to the identities

$$\Delta_{ij}\Delta_{kl} + \Delta_{jk}\Delta_{il} = \Delta_{ik}\Delta_{jl} \quad (1 \leq i, j, k, l \leq n).$$
Conjecture 3

Let \( F \in \mathbb{Z}[X, Y] \) be a binary form of degree \( n \geq 4 \) and discriminant \( D \neq 0 \). Then \( F \) is equivalent to a binary form \( G \) for which

\[
H(G) \leq C_1(n)|D|^{c_2^{\text{eff}}(n)}.
\]

Conjecture 2 \( \implies \) Conjecture 3.

Let \( L \) be the splitting field of \( F \). Following the proof of Theorem 4 and using the very weak abc-conjecture, one obtains that there is \( G \) equivalent to \( F \) such that

\[
H(G) \leq C_3(n)|D_L|^{c_4^{\text{eff}}(n)}|D|^{21/(n-1)}.
\]

Use that \( D_L \) divides \( D^n \).
Conjecture 3

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then $F$ is equivalent to a binary form $G$ for which

$$H(G) \leq C_1(n)|D|^{c_2^{\text{eff}}(n)}.$$ 

Problem

What is the right value of the exponent on $|D|$?
A function field analogue

Let $k$ be an algebraically closed field of characteristic 0, $K = k(t)$, $R = k[t]$.
Define $|·|$ on $k(t)$ by $|f/g| := e^{\text{deg} f - \text{deg} g}$ for $f, g \in R$.

Define the height of $F = a_0 X^n + a_1 X^{n-1} Y + \cdots + a_n Y^n \in R[X, Y]$ by $H(F) := \max_i |a_i|$.

Call two binary forms $F, G \in R[X, Y]$ equivalent if $G = uF_A$ for some $u \in k^*$, $A \in \text{GL}(2, R)$.

**Theorem 5 (W. Zhuang)**

Let $F \in R[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then $F$ is equivalent to a binary form $G$ for which

$$H(G) \leq e^{n^2 + 4n + 14} |D|^{20 + 7/(n-2)}.$$

**Proof.**

Follow the proof over $\mathbb{Z}$ and apply Mason’s abc-theorem for function fields.
Thank you for your attention!