SOME OPEN PROBLEMS ABOUT DIOPHANTINE EQUATIONS

We have collected some open problems which were posed by participants of an instructional conference (May 7-11, 2007) and a subsequent more advanced workshop (May 14-16, 2007) on solvability of Diophantine equations, both held at the Lorentz Center of Leiden University, The Netherlands.

Problems posed by Mike Bennett, Nils Bruin, Yann Bugeaud and Samir Siksek during the instructional conference.

1. Find all integer solutions to the equation

$$x^2 - x = y^5 - y.$$

2. Do the same for the equation

$$\binom{x}{2} = \binom{y}{5}.$$

3. Extend Ellenberg's approach to solve

$$x^2 + y^6 = z^n \quad (n \geq 3).$$

4. Can one solve

$$x^2 - 2 = y^p$$

for $p \geq 3$?

5. Do there exist primes q for which we can "solve"

$$x^2 + y^3 = qz^p$$

for all large enough primes p?

6. Can one solve Kraus' equation

$$x^3 + y^3 = z^p$$

for all primes $p \geq 3$?

7. The curve

$$y^2 = -3x^6 - x^5 + 2x^4 + 2x^2 - 3x - 3$$

has no rational point under BSD. Can this be proved unconditionally?

8. If we denote by $||x||$ the distance from a real number x to the nearest integer, does there exist a positive absolute constant c such that

$$|| \log \ n|| > n^{-c}$$

for all $n \geq 2$ integral?

Problems posed at the workshop

Two problems posed by Yann Bugeaud

9. Let D and k be positive integers and p be a prime number such that $\gcd(D, kp) = 1$. Prove that there is an absolute constant C such that the Diophantine equation $x^2 + D = kp^n$ has at most C solutions (x, n).
Note that Stiller proved that the equation $x^2 + 119 = 15 \cdot 2^n$ has exactly six solutions.

10. For any positive real number x and any positive integer n, let $\Xi(n, x)$ denote the number of perfect powers in $[n, n + x]$ and set

$$\Xi(x) = \limsup_{n \to +\infty} \Xi(n, x).$$

Give an upper bound for $\Xi(x)$.
Using sieve methods, it is possible to prove that $\Xi(x) \ll x/(\log x)$ Presumably, this upper estimate is very far from the true order of magnitude of Ξ. It is even likely that $\Xi(x) = 1$ for any $x \geq 1$.

Problem posed by Lajos Hajdu and Szabolcs Tengely

11. Determine all arithemetic progressions of the form a^2, b^2, c^2, d^5 where a, b, c, d are integers with $\gcd(a, b) = 1$.

Three problems posed by Gary Walsh

12. (cf. Problem 17.) Show that for all $n \geq 5$,

$$\frac{x^n - y^n}{x - y} = z^2$$

has only trivial solutions in integers x, y, z.

13. Are there infinitely many positive integer solutions to

$$\frac{x^3 - 1}{y^3 - 1} = z^2?$$

14. Find all integer solutions to $x^4 + x^2 + y^4 + y^2 = z^4 + z^2$.

Three problems posed by Wilfrid Ivorra

15. Consider the equation

$$ax^p + by^p = cz^2$$

where p is a prime number and a, b, c are pairwise coprime integers. Let $S_p(a,b,c)$ be the set of proper non trival solutions $(x,y,z) \in \mathbb{Z}^3$ of equation (1) that is solutions with

$$xyz \neq 0 \quad (\mathrm{non-trivial}),$$

$$\gcd(x,y,z) = 1 \quad (\mathrm{proper}).$$

Prove the following conjectures:

Suppose that the integers $a+b$, $a-b$ and $b-a$ do not belong to $c\mathbb{Z}^2$, then there exists a constant $f(a,b,c)$ such as we have

$$p > f(a,b,c) \Longrightarrow S_p(a,b,c) \text{ is empty.}$$

Suppose that one of the integers $a+b$, $a-b$ and $b-a$ belongs to $c\mathbb{Z}^2$, then there exists a constant $g(a,b,c)$ such as, for all $p > g(a,b,c)$ we have

$$(x,y,z) \in S_p(a,b,c) \Longrightarrow xy = \pm 1.$$

16. Let $p \geq 7$ be a prime number. Find the triples (x,y,z) in $\mathbb{Z}$ such as $xyz \neq 0$, $\gcd(x,y,z) = 1$ and

$$x^p + 2y^p = z^2.$$

17. (cf. Problem 12.) Let p be a prime number ≥ 7, Φ_p the p-th cyclotomic polynomial and $C_p/\mathbb{Q}$ and $D_p/\mathbb{Q}$ the hyperelliptic curves :

$$C_p : y^2 = \Phi_p(x) \quad \text{et} \quad D_p : py^2 = \Phi_p(x).$$

Do we have

$$C_p(\mathbb{Q}) = \Big\{(-1,-1),(-1,1),(0,-1),(0,1)\Big\},$$

$$D_p(\mathbb{Q}) = \Big\{(1,-1),(1,1)\Big\}$$

for all $p \geq 7$?
This is true if $p \in \{7, 11, 13, 17\}$.

Problem posed by Johnny Edwards

18. Consider $AX^2 + BY^3 = CZ^5$ with A, B, C fixed non-zero integers and the variables X, Y, Z required to be co-prime integers. Is the existence of such a rational integer solution equivalent to the existence of co-prime p-adic solutions for all primes p. I.e. is there a surprising Hasse Principle working?

(Open question due to Darmon and Granville, 1995)

Problem posed by Szabolcs Tengely

19. Are there infinitely many positive integers n such that the sum of the first n primes is a square (perfect power)?
Four solutions are given by
$S_9 = 10^2, \quad S_{2474} = 5063^2, \quad S_{6694} = 14573^2, \quad S_{7785} = 17098^2.$

Three problems posed by Florian Luca

20. Show that the Diophantine equation $x^{2n} - q^{2n} = py^m$ has only finitely many integer solutions (x, y, p, q, m, n) with $n \geq 2,\ m \geq 3$, p and q primes and $q \nmid x$.

F. Luca and A. Togbé have recently shown that this equation has no solutions when $(n, m) = (2, 3)$.

21. Let G be a finitely generated multiplicative subgroup of $\mathbb{Q}^*$ and m an integer ≥ 4. Show that if

$$x_1 + x_2 + \cdots + x_m = n! \quad \text{with } x_i \in G \cap \mathbb{Z}_+ \text{ for } i = 1, \ldots, m,$$

then n is bounded by some constant depending only on G.

Regarding this problem, M. Cipu, F. Luca and M. Mignotte have recently computed all the solutions of the equations

$$p_1^{y_1} + \cdots + p_m^{y_m} = n!$$

in nonegative integers $(y_1, \ldots, y_m, n)$ when $(p_1, p_2, \ldots, p_m) = (2, 3, 5, 7),\ (3, 5, 7, 11)$ and $(2, 3, 5, 7, 11)$.

22. Show that the equation $F_n = \binom{m}{k}$ has only finitely many integer solutions (n, m, k), with $2 \leq k \leq m/2$. Here, F_n is the nth Fibonacci number.