This note is a result of a discussion with Yann Bugeaud.

Denote by $H(\xi)$ the naive height, that is the maximum of the absolute values of the coefficients of the minimal polynomial of an algebraic number $\xi$. Schmidt proved that for every real algebraic number $\alpha \in \mathbb{R}$ and every $\varepsilon > 0$ there are only finitely many algebraic numbers $\xi$ of degree $d$ such that $|\alpha - \xi| < H(\xi)^{-d-1-\varepsilon}$. For algebraic numbers $\alpha \in \mathbb{C}\setminus\mathbb{R}$ one expects a similar result but with exponent $-\frac{1}{2}(d+1)-\varepsilon$. In this note we prove such a type of result, but unfortunately we have to impose some technical condition on $\alpha$.

We start with an auxiliary result. Given a linear form $L(X) = \alpha_1 X_1 + \cdots + \alpha_n X_n$ with algebraic coefficients in $\mathbb{C}$, define the complex conjugate linear form $\overline{L}(X) = \overline{\alpha_1} X_1 + \cdots + \overline{\alpha_n} X_n$. Further, we define the norm of $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ by $\|x\| := \max(|x_1|, \ldots, |x_n|)$.

**Theorem 1.** Let $n \geq 2$. Let $L(X) = \alpha_1 X_1 + \cdots + \alpha_n X_n$ be a linear form with algebraic coefficients in $\mathbb{C}$ satisfying the following technical hypothesis:

(0.1) For any $\mathbb{Q}$-linear subspace $T$ of $\mathbb{Q}^n$ of dimension $> n/2$, the restrictions of $L, \overline{L}$ to $T$ are linearly independent.

Then for any $\varepsilon > 0$, the inequality

(0.2) $0 < |L(x)| < \|x\|^{1-(n/2)-\varepsilon}$ in $x \in \mathbb{Z}^n$

has only finitely many solutions.

**Proof.** Write $L(x) = L_1(x) + iL_2(x)$, where $L_1$ consists of the real parts of the coefficients of $L$, and $L_2$ of the imaginary parts. We apply Theorem 2A on p. 157 of [W.M. Schmidt, Diophantine approximation, Springer Verlag LNM 785, 1980] to $L_1, L_2$. Thus in Schmidt’s notation, $u = 2, v = n - 2$. Our assumption on $L$ implies that for every $d$-dimensional $\mathbb{Q}$-linear subspace $T$ of $\mathbb{Q}^n$, the restrictions of $L_1, L_2$ to $T$ have rank $\geq d \cdot 2/n$. This is precisely the condition to be satisfied in Schmidt’s theorem. Thus, it follows that for every
\( \varepsilon > 0 \), the system of inequalities
\[
|L_1(x)| < |x|^{-\frac{n^2}{2} - \varepsilon}, \quad |L_2(x)| < |x|^{-\frac{n^2}{2} - \varepsilon}
\]
has only finitely many solutions in \( x \in \mathbb{Z}^n \). It follows that (0.2) has only finitely many solutions. \( \square \)

Denote by \( V_d \) the vector space of polynomials in \( \mathbb{Q}[X] \) of degree \( \leq d \).

**Theorem 2.** Let \( \varepsilon > 0 \). Let \( \alpha \) be an algebraic number in \( \mathbb{C} \setminus \mathbb{R} \) satisfying the following technical hypothesis:
\[
(h_1(\alpha)h_2(\bar{\alpha})) \in \mathbb{R} \quad \text{for each pair of polynomials } h_1, h_2 \in T,
\]
then \( \dim T \leq (d + 1)/2 \).

Then the inequality
\[
(0.4) \quad |\alpha - \xi| < H(\xi)^{-\frac{1}{2}(d+1)-\varepsilon}
\]
has only finitely many solutions in algebraic numbers \( \xi \) of degree \( d \).

**Proof.** Denote by \( f \) the minimal polynomial of \( \xi \) (with coefficients in \( \mathbb{Z} \) having \( \gcd 1 \) and with positive leading coefficient). Let \( \xi \) be a solution of (0.4). Then \( |f(\alpha)| \ll H(\xi)|\alpha - \xi| \) and so
\[
(0.5) \quad |f(\alpha)| \ll H(f)^{1-((d+1)/2)-\varepsilon}.
\]

We may view \( f(\alpha) \) as a linear form on \( V_d \) in \( d + 1 \) variables with algebraic coefficients in \( \mathbb{C} \). We claim that if \( T \) is a \( \mathbb{Q} \)-linear subspace of \( V_d \) of dimension \( > (d + 1)/2 \), then the restrictions of \( f(\alpha), f(\bar{\alpha}) \) to \( T \) are linearly independent. Then by Theorem 1, inequality (0.5) has only finitely many solutions \( f \), and this gives only finitely many possibilities for \( \xi \).

So it remains to prove our claim. Choose a basis \( \{g_1, \ldots, g_t\} \) of \( T \). We have to show that the vectors \( (g_1(\alpha), \ldots, g_t(\alpha)), (g_1(\bar{\alpha}), \ldots, g_t(\bar{\alpha})) \) are linearly independent. But if this is not the case, then each of the determinants \( g_i(\alpha)g_j(\bar{\alpha}) - g_j(\alpha)g_i(\bar{\alpha}) = 0 \), i.e., \( g_i(\alpha)g_j(\bar{\alpha}) \in \mathbb{R} \) for each pair \( i, j \). But then by \( \mathbb{Q} \)-linearity, \( h_1(\alpha)h_2(\bar{\alpha}) \in \mathbb{R} \) for each \( h_1, h_2 \in T \). By assumption (0.3) this is possible only if \( t \leq (d + 1)/2 \). This proves our claim, hence Theorem 2. \( \square \)
Corollary. Let \( \alpha \) be an algebraic number in \( \mathbb{C} \setminus \mathbb{R} \) such that either

\[
(0.6) \quad [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha) \cap \mathbb{R}] \geq \left\lceil \frac{1}{2}(d + 3) \right\rceil
\]

or

\[
(0.7) \quad [\mathbb{Q}(\alpha) \cap \mathbb{R} : \mathbb{Q}] \leq \left\lfloor \frac{1}{2}(d + 1) \right\rfloor.
\]

Then for any \( \epsilon > 0 \), (0.4) has only finitely many solutions in algebraic numbers \( \xi \) of degree \( d \).

Proof. We first show that there is no loss of generality to assume \([\mathbb{Q}(\alpha) : \mathbb{Q}] > d + 1\). Suppose that \( \alpha \) has degree \( r \leq d + 1 \), and let \( \xi \) be a non-real algebraic number of degree \( d \). Let \( h, f \) denote the minimal polynomials of \( h, f \), respectively. Let \( \alpha_1 = \alpha, \alpha_2 = \overline{\alpha}, \alpha_3, \ldots, \alpha_r \) denote the conjugates of \( \alpha \) and \( \xi_1 = \xi, \xi_2, \xi_3, \ldots, \xi_d \) those of \( \xi \). Suppose that \( \xi \) is not equal to a conjugate of \( \alpha \). Then, using some basic facts about the resultant \( R(h, f) \) of \( h, f \),

\[
1 \leq |R(h, f)| = M(h)^d M(f)^r \cdot \prod_{i=1}^{r} \prod_{j=1}^{d} \frac{|\alpha_i - \xi_j|}{\max(1, |\alpha_i|) \max(1, |\xi_j|)}
\]

\[
\ll H(\alpha)^d H(\xi)^r |\alpha - \xi| \cdot |\overline{\alpha} - \overline{\xi}| = H(\alpha)^d H(\xi)^r |\alpha - \xi|^2,
\]

where \( M(h), M(f) \) denote the Mahler measures of \( h, f \), respectively. Therefore,

\[
|\alpha - \xi| \gg H(\xi)^{-r/2}
\]

where the constant implied by \( \gg \) depends only on \( \alpha \). Since \( r \leq d + 1 \), this trivially implies that (0.4) has only finitely many solutions in algebraic numbers \( \xi \) of degree \( d \).

Now assume that \([\mathbb{Q}(\alpha) : \mathbb{Q}] > d + 1 \) and that either (0.6), or (0.7) is satisfied. We have to verify (0.3). Let \( T \) be a \( \mathbb{Q} \)-linear subspace of \( V_d \) such that \( h_1(\alpha)h_2(\overline{\alpha}) \in \mathbb{R} \) for each \( h_1, h_2 \in T \). Suppose \( T \) has dimension \( t \) and choose a basis \( \{g_1, \ldots, g_t\} \) of \( T \). Then \( g_i(\alpha)/g_1(\alpha) = g_i(\alpha)g_1(\overline{\alpha})/|g_1(\overline{\alpha})|^2 \in \mathbb{R} \) for \( i = 1, \ldots, t \); we know that \( g_1(\alpha) \neq 0 \) since \( \alpha \) has degree \( > d + 1 \). Further, since \( \alpha \) has degree \( > d + 1 \), the numbers \( 1, g_2(\alpha)/g_1(\alpha), \ldots, g_t(\alpha)/g_1(\alpha) \) are \( \mathbb{Q} \)-linearly independent elements of \( \mathbb{Q}(\alpha) \cap \mathbb{R} \). Therefore, \( t \leq [\mathbb{Q}(\alpha) \cap \mathbb{R} : \mathbb{Q}] \).

So if (0.7) holds, then (0.3) is satisfied.
After applying Gauss elimination or the like to a given basis of $T$, we obtain a basis \( \{g_1, \ldots, g_t\} \) with \( \deg g_1 < \deg g_2 < \cdots < \deg g_t \). Thus, \( \deg g_i \leq d - t + i \) for \( i = 1, \ldots, t \). Then similarly as above, \( g_2(\alpha)/g_1(\alpha) \in \mathbb{R} \), i.e., there is a \( \lambda \in \mathbb{Q}(\alpha) \cap \mathbb{R} \) such that \( g_2(\alpha) - \lambda g_1(\alpha) = 0 \), i.e., \( h(\alpha) = 0 \) where \( h \) is a non-zero polynomial of degree \( \leq d - t + 2 \) with coefficients in \( \mathbb{Q}(\alpha) \cap \mathbb{R} \). Now if (0.6) holds, then \( d - t + 2 \geq \lceil \frac{1}{2}(d + 3) \rceil \), i.e., \( t \leq d + 2 - \lceil \frac{1}{2}(d + 3) \rceil = \lfloor \frac{1}{2}(d + 1) \rfloor \), which again implies (0.3). Our Corollary follows.