Counting quickly the vectors with integer coordinates and with a given length

Bas Edixhoven
Universiteit Leiden
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with Jean-Marc Couveignes, Robin de Jong, Johan Bosman, Franz Merkl, Peter Bruin, Ila Varma
Computational Aspects of Modular Forms and Galois Representations

Edited by
Bas Edixhoven and Jean-Marc Couveignes
with contributions by Johan Bosman, Jean-Marc Couveignes, Bas Edixhoven, Robin de Jong, and Franz Merkl

ANNALS OF MATHEMATICS STUDIES
Modular forms are tremendously important in various areas of mathematics, from number theory and algebraic geometry to combinatorics and lattices.

Their Fourier coefficients, with Ramanujan's tau-function as a typical example, have deep arithmetic significance. Prior to this book, the fastest known algorithms for computing these Fourier coefficients took exponential time, except in some special cases. The case of elliptic curves (Schoof's algorithm) was at the birth of elliptic curve cryptography around 1985. This book gives an algorithm for computing coefficients of modular forms of level one in polynomial time. For example, Ramanujan's tau of a prime number $p$ can be computed in time bounded by a fixed power of the logarithm of $p$. . .
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To illustrate the progress made in the book and Peter Bruin’s PhD thesis, we consider the problem of computing quickly, for $d$ and $n$ in $\mathbb{Z}$:

$$r_d(n) := \#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 = n\}.$$
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Geometric interpretation (Pythagoras): count the number of lattice points in $\mathbb{Z}^d$ at a given distance $\sqrt{n}$ from the origin.
Sums of squares: some examples

\[ r_2(3) = 0. \]

\[ r_2(5) = 8: \]

\[ 5 = (\pm 2)^2 + (\pm 1)^2 \]

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Do not use the factorisation of $n$ into primes, because we do not know how to do that fast enough.
Diophantus of Alexandria ($\approx$ 3rd century):

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$
Pierre de Fermat (lawyer, Toulouse, 17th century), for \( n \geq 1: r_2(n) \neq 0 \) if and only if every prime factor of \( n \) that is 3 modulo 4, occurs an even number of times in the factorisation of \( n \).
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Carl Friedrich Gauss (1801) gave a general formula for $r_2(n)$, and a formula for $r_3(n)$ that shows that the $r_d(n)$ for odd $d$ are more complicated (involve class numbers).
Higher even dimensions: Jacobi

Carl Gustav Jacob Jacobi (1829) proved for $n > 1$:

$$r_2(n) = 4 \sum_{d \mid n} \chi(d), \quad \text{with} \quad \chi(d) = \begin{cases} 
0 & \text{if } d \text{ is even}, \\
1 & \text{if } d = 4r + 1, \\
-1 & \text{if } d = 4r + 3,
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and:

$$r_4(n) = 8 \sum_{2 \mid d \mid n} d + 16 \sum_{2 \mid d \mid (n/2)} d.$$
It follows from work of Jacobi, Ferdinand Eisenstein and Henry Smith that:

\[ r_6(n) = 16 \sum_{d \mid n} \chi(n/d)d^2 - 4 \sum_{d \mid n} \chi(d)d^2, \]

\[ r_8(n) = 16 \sum_{d \mid n} d^3 - 32 \sum_{d \mid (n/2)} d^3 + 256 \sum_{d \mid (n/4)} d^3. \]
For $d = 10$ Joseph Liouville (1865) found a formula in terms of the Gaussian integers $d = a + bi$ with $a$ and $b$ in $\mathbb{Z}$:

$$r_{10}(n) = \frac{4}{5} \sum_{d|n} \chi(d)d^4 + \frac{64}{5} \sum_{d|n} \chi(n/d)d^4 + \frac{8}{5} \sum_{d \in \mathbb{Z}[i], |d|^2=n} d^4.$$
James Whitbread Lee Glaisher, reinterpreted by Srinivasa Ramanujan in 1916, proved that:

$$r_{12}(n) = 8 \sum_{d|n} d^5 - 512 \sum_{d|(n/4)} d^5 + 16a_n$$
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where:

\[ \sum_{n \geq 1} a_n q^n = q \prod_{m \geq 1} (1 - q^{2m})^{12} \quad \text{in } \mathbb{Z}[[q]]. \]
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Note: unlike for \( d \leq 10 \), this formula does \textit{not} lead to computation of \( r_{12}(n) \) in time polynomial in \( \log n \), if \( n \) is given with its factorisation into primes.
Negative. Ila Varma (masters thesis, Leiden, June 2010): there is no even $d > 10$ for which there is an “elementary” formula for $r_d(n)$. 

Note: for $n = pq$ with $p$ and $q$ distinct odd primes: 

$$r_4(n) = 8(1 + p + q + n).$$
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Positive (book and Peter Bruin’s PhD thesis). For every even $d$ one can compute $r_d(n)$ in time polynomial in $\log n$, if $n \in \mathbb{N}$ is given with its factorisation into primes.

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Conclusion. From an algorithmic perspective this classical problem is now solved for all even $d$. The question for formulas has a negative answer, but for computing that negative answer does not matter and we now have a positive answer.
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Explanation: generating series

It is more than time to explain what is going on behind all these formulas. Generating series:

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\theta_d := \sum_{x \in \mathbb{Z}^d} q^{x_1^2 + \cdots + x_d^2} = \sum_{n \geq 0} r_d(n) q^n \quad \text{in } \mathbb{Z}[[q]].
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Let \( \theta := \theta_1 \) (Jacobi theta function at \( z = 0 \)). Then:

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\theta^d = \left( \sum_{x_1 \in \mathbb{Z}} q^{x_1^2} \right) \cdots \left( \sum_{x_d \in \mathbb{Z}} q^{x_d^2} \right) = \theta_d.
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Compute \( \theta^d \) in \( \mathbb{Z}[[q]]/(q^{n+1}) \): gives \( r_d(n) \) but takes time at least linear in \( nd \).
Theta functions are modular forms

Key idea: $q: \mathbb{H} = \{z \in \mathbb{C}: \Im(z) > 0\} \rightarrow \mathbb{C}, \quad z \mapsto e^{2\pi i z}$. 
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Then $\theta_d : \mathbb{H} \to \mathbb{C}$, and for $z \in \mathbb{H}$: $\theta_d(z + 1) = \theta_d(z)$, and Jacobi proved (Poisson summation formula):

$$\theta_d(-1/4z) = (2z/i)^{d/2} \theta_d(z).$$
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This implies: \( \theta_d \) is in the \( \mathbb{C} \)-vector space \( M_{d/2}(\Gamma_1(4)) \) of modular forms of weight \( d/2 \) on the subgroup \( \Gamma_1(4) \) of \( \text{SL}_2(\mathbb{Z}) \). Assume from now on that \( d \) is even. Then \( k = d/2 \) is in \( \mathbb{Z} \).
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To get further (SL$_2(\mathbb{Z})$ does not suffice, we need Galois symmetry), interpret $M_k(\Gamma)$ in terms of de Rham cohomology of the quotient $E^{k-2}$ of $\mathbb{C}^{k-2} \times \mathbb{H}$ by an action of $\mathbb{Z}^{2k-2} \rtimes \Gamma$:

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For $f \in M_k(\Gamma)$, $f \, dx_1 \cdot \cdots \cdot dx_{k-2} \, dz$ is a $\mathbb{Z}^{2k-2} \rtimes \Gamma$-invariant and closed holomorphic $(k-1)$-form.
Complex analytic geometry

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The coefficients $a_n(f)$ of the modular forms $f = \sum_{n \geq 0} a_n(f)q^n$ are closely related to Hecke operators $T_n$ coming from the $\text{GL}_2(\mathbb{Q})^+$-action on $\mathbb{H}$. 

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- We do \textit{not} know (yet?): $\exists \sigma \in \text{Aut}(\mathbb{C}), \sigma(\pi) = e \text{ and } \sigma(e) = \pi$. 
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To simplify, consider $M_{12}(\text{SL}_2(\mathbb{Z})) = \mathbb{C} \cdot E_{12} \oplus \mathbb{C} \cdot \Delta$, where

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E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \left( \sum_{d \mid n} d^{11} \right) q^n,
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\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n.
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For $p$ prime, the action of $T_p$ on $H^{k-1}(E^{k-2}, \mathbb{Z}/m\mathbb{Z})$ can be computed from the Galois action, as the trace of a Frobenius element at $p$.

To simplify, consider $M_{12}(\text{SL}_2(\mathbb{Z})) = \mathbb{C} \cdot E_{12} \oplus \mathbb{C} \cdot \Delta$, where

$$E_{12} = 1 + \frac{65520}{691} \sum_{n \geq 1} \left( \sum_{d|n} d^{11} \right) q^n,$$

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n.$$

Deligne: for every integer $m > 0$ there is $\rho_m : \text{Aut}(\mathbb{C}) \rightarrow \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$, such that for every prime $p \nmid m$, $\tau(p) = \text{trace}(\text{Frob}_p)$ in $\mathbb{Z}/m\mathbb{Z}$. 
The book explains, in about 400 pages, that one can compute, for $\ell$ prime, $\rho_\ell$ in time polynomial in $\ell$, and then $\tau(p)$ in time polynomial in $\log p$. More generally: for $M_k(\text{SL}_2(\mathbb{Z}))$. 

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Bas Edixhoven (Universiteit Leiden)  Number theory, computer algebra, geometry  2011/05/06  20 / 26
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For $\sigma$ in $\text{Aut}(\mathbb{C})$ and $z$ in $\text{Roots}(f)$:

$$0 = \sigma(0) = \sigma(f(z)) = \sigma(z^n + \cdots + a_1z + a_0)$$

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$\text{Gal}(f)$ is the group of permutations of $\text{Roots}(f)$ given by elements of $\text{Aut}(\mathbb{C})$. 
Example: roots of unity

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For $n = 5$:

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Conclusion: in terms of the labelling $\text{Gal}(f)$ is given by elements of $\text{GL}_1(\mathbb{Z}/n\mathbb{Z})$. 
A 2-dimensional Galois representation mod $n$ is a polynomial $f = x^{n^2} + \cdots + a_1 x + a_0$ in $\mathbb{Q}[x]$ of degree $n^2$, with a bijection $\mathbb{Z}/n\mathbb{Z}^2 \to \text{Roots}(f)$, such that each element of $\text{Gal}(f)$ acts as multiplication by an element of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$. These objects play the most important role in Andrew Wiles’s proof of Fermat’s Last Theorem (1993-1994). 40 years ago the Langlands program started, relating Galois representations and automorphic forms. Question: can one efficiently compute the Galois representations whose existence is guaranteed by the Langlands program? It looks as if the answer will be ‘yes’.
Two-dimensional Galois representations

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An example by Johan Bosman

The polynomial:

\[ f = x^{24} - 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} \\
+ 9223x^{18} + 121141x^{17} + 1837654x^{16} - 800032x^{15} \\
+ 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\
+ 279370098x^{11} + 1464085056x^{10} + 1129229689x^9 \\
+ 3299556862x^8 + 14586202192x^7 + 29414918270x^6 \\
+ 45332850431x^5 - 6437110763x^4 - 111429920358x^3 \\
- 12449542097x^2 + 93960798341x - 31890957224 \]

has Galois group \( \text{PGL}_2(\mathbb{Z}/23\mathbb{Z}) \), and (reduced) discriminant \( 23^{43} \); it comes from étale cohomology of degree 21 of a variety of complex dimension 21.
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This is avoided by numerical approximations with a precision that suffices to derive exact results from them.

Bounds for the required precision—in other words, bounds for the height of the rational numbers that describe the Galois representation to be computed—are obtained from Arakelov theory...
Thank you for your attention!

Questions?

With: Jean-Marc Couveignes (Toulouse), Robin de Jong, Franz Merkl (München), Johan Bosman, Peter Bruin, Ila Varma.