INTERCITY SEMINAR ALGEBRAIC GEOMETRY, autumn 2007

Exercises of lecture 1: General Introduction

Exercise 1: Localization

Let $\mathbb{Z}-\mathbf{Mod}_0$ be the full subcategory of $\mathbb{Z}-\mathbf{Mod}$ consisting of

$$\{M \in \mathbb{Z} - \mathbf{Mod} | M \otimes_{\mathbb{Z}} \mathbb{Q} = (0)\}$$

or equivalently, the full subcategory consisting of inductive limits of finite $\mathbb{Z}$-modules (or the $\mathbb{Z}$-modules in which every element has finite order). Set

$$S = \{s : M \to N | \mathrm{Ker}(s), \mathrm{Coker}(s) \in \mathbb{Z} - \mathbf{Mod_0}\}$$

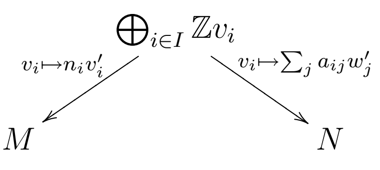
I claim that the localization with respect to S is equivalent to the category of $\mathbb{Q}$-vector spaces. Recall that $\mathbb{Q}$ is flat over $\mathbb{Z}$, hence $S = \{s : M \to N | s \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphism$\}$. From this, one can deduce that S satisfies all the good properties (my reference is the chapter by Grivel in [2]), hence there exists a localization $Q : \mathbb{Z}-\mathbf{Mod} \to \mathcal{C}$; $\mathcal{C}$ is additive and Q is an additive functor.

Consider the additive functor $- \otimes_{\mathbb{Z}} \mathbb{Q} : \mathbb{Z}-\mathbf{Mod} \to \mathbb{Q}-\mathbf{Vec}$. By the universal property of localization, we get a canonical additive functor $L : \mathcal{C} \to \mathbb{Q}-\mathbf{Vec}$. I claim that L is essentially surjective and fully faithful. This will impy that L is the desired equivalence.

Essential surjectivity: Let V be a $\mathbb{Q}$-vector space, then $V \otimes_{\mathbb{Z}} \mathbb{Q} = V$.

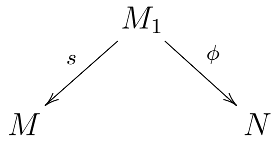
Fullness: Let $\phi : M \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$ be any $\mathbb{Q}$-linear map. We may choose basis $\{v_i | i \in I\}$ for $M \otimes_Z \mathbb{Q}$ and $\{w_j | j \in J\}$ for $N \otimes_Z \mathbb{Q}$, such that $v_i \in M \otimes 1$ and $w_j \in N \otimes 1$. Choose v'_i [resp. w'_j] in M [resp N] such that $v'_i \otimes 1 = v_i$ [resp. $w'_j \otimes 1 = w_j$]. For each i, choose $n_i \in \mathbb{Z}$ such that $\phi(v_i) = \frac{1}{n_i} \sum_j a_{ij} w_j$, where $a_{ij} \in \mathbb{Z}$.

Consider the diagram:

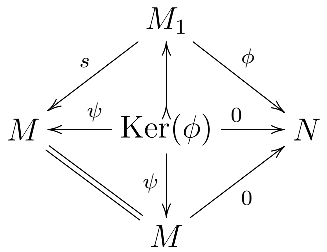


Then $-\otimes_{\mathbb{Z}} \mathbb{Q}$ gives the map $v_i \mapsto \frac{1}{n_i} v_i \mapsto \frac{1}{n_i} \sum_j a_{ij} w_j = \phi(v_i)$ from $M \otimes_{\mathbb{Z}} \mathbb{Q}$ to $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Faithfulness: Suppose that the diagram



satisfies $(\phi \otimes_{\mathbb{Z}} \mathbb{Q}) \circ (s \otimes_{\mathbb{Z}} \mathbb{Q})^{-1} = 0$. Again, by flatness of $\mathbb{Q}$ over $\mathbb{Z}$, this implies that $\mathrm{Ker}(\phi) \otimes_{\mathbb{Z}} \mathbb{Q} = M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider the diagram:



The morphism $\mathrm{Ker}(\phi) \to M_1$ lies in S, thus so is $\psi : \mathrm{Ker}(\phi) \to M$. This shows that (s, ϕ) is equivalent to 0.

Exercise 2: Serre duality

Preliminaries on sheaves

We will make use of the following results:

Proposition 2.1 ([4] 3.3)**.** *Let X be a noetherian scheme, then any quasi-coherent sheaf F admits a resolution by quasi-coherent sheaves which are injective as $\mathscr{O}_X$-modules.*

More precisely, we have:

Proposition 2.2 ([4] 3.4, 3.5)**.** *Let X be a noetherian scheme, then*

- $D^*(\mathbf{Qcoh}(X)) \to D^*_{\mathbf{Qcoh}(X)}(\mathscr{O}_X - \mathbf{Mod})$ *is an equivalence which preserves distinguished triangles, where $* = b, +$, and $D^*_{\mathbf{Qcoh}(X)}(\mathscr{O}_X - \mathbf{Mod})$ is the full triangulated subcategory of $D^*(\mathscr{O}_X - \mathbf{Mod})$ of objects with quasi-coherent homologies.*

- $D^b(X) \to D^b_{\mathbf{Coh}}(\mathbf{Qcoh}(X))$ *is an equivalence which preserves distinguished triangles, where $D^b_{\mathbf{Coh}}(\mathbf{Qcoh}(X))$ is the full triangulated subcategory of $D^b(\mathbf{Qcoh}(X))$ of objects with coherent homologies.*

On the other hand, we also have locally free resolutions of coherent sheaves:

Proposition 2.3. *Let X be a regular quasi-projective scheme over a field. For any coherent sheaf F on X, there exists a finite resolution $0 \leftarrow F \leftarrow E^\bullet$ by locally free $\mathscr{O}_X$-modules of finite rank.*

Proof. Given a coherent sheaf F on X. For $n \gg 0$, $F \otimes \mathscr{O}_X(n)$ will be generated by global sections, hence there exists m and an epimorphism

$$0 \leftarrow F \otimes \mathscr{O}_X(n) \leftarrow \mathscr{O}_X^{\oplus m}$$

Twisting back gives $0 \leftarrow F \leftarrow \mathscr{O}_X(-n)^{\oplus m}$. The finiteness of such resolutions follows from the regularity of X and Serre's theorem on global homological dimension. □

Remark 2.4. From this, one can deduce that for any bounded complex $E^\bullet$ of coherent sheaves, there exists a bounded complex $F^\bullet$ of locally free $\mathscr{O}_X$-modules and a quasi-isomorphism $F^\bullet \twoheadrightarrow E^\bullet$ in $\mathrm{Kom}(\mathbf{Coh}(X))$.

Remark 2.5. Suppose X is projective. One can actually resolve a coherent sheaf E by *negative* enough locally free $\mathscr{O}_X$-modules of finite rank. This means: for any coherent sheaf M, there exists a finite resolution by locally free $\mathscr{O}_X$-modules of finite rank:

$$0 \leftarrow E \leftarrow F_0 \leftarrow \cdots F_n \leftarrow 0$$

such that $\mathrm{Ext}^p(F_q, M) = 0$ for all $p > 1$ and $0 \leq q \leq n$.

We can assume that $n \geq 2 \dim X$. For $q < n$ the *negativeness* follows from the proof and the cohomological criterion of ampleness. As to $q = n$, note that $\mathrm{Ext}^p(F_n, M) = \mathrm{Ext}^{p+n}(E, M)$ if $F_1, \ldots, F_{n-1}$ are *negative enough* for M. However, we have a Leray's spectral sequence

$$E_2^{i,j} = H^i(X, \mathcal{E}xt^j(E, M)) \Rightarrow \mathrm{Ext}^{i+j}(E, M)$$

For $i + j = p + n > 2 \dim X$, either $i > n$ or $j > n$, which implies that either $H^i(X, -) = 0$ or $\mathcal{E}xt^j(E, -) = 0$, hence $E^{p+n}(E, M) = 0$.

Proposition 2.6. *With the conditions of 2.5,* $\mathrm{Hom}_{D^b(X)}(-, -)$ *will be finite-dimensional k-vector spaces.*

Generalities on RHom and $\overset{\mathrm{L}}{\otimes}$

These materials can also be found in [4] Chapter 3. In particular, I will follow his sign convention for tensor products of complexes, which is different from that in [2].

Assume from nom on that X is a smooth projective variety over a field. Assume for simplicity that our complexes are complexes over $\mathbf{Coh}(X)$. Note that $D^-(X)$ admits finite locally free (hence flat) resolutions. There is almost no injective objects in $D^b(X)$, however one can embed $D^b(X)$ in $D^b_{\mathbf{Coh}}(\mathbf{Qcoh}(X))$; this justifies all the upcoming discussions on $-\overset{\mathrm{L}}{\otimes}-$ and $\mathrm{RHom}(-,-)$ for the bounded derived category of coherent sheaves.

Definition 2.7 (Hom complex)**.** Let $A^\bullet, B^\bullet$ be two complexes, define a complex

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) := \prod_{p\in\mathbb{Z}} \mathrm{Hom}(A^p, B^{p+n})$$

with differentials $d^n : \mathrm{Hom}^n(A, B) \to \mathrm{Hom}^{n+1}(A, B)$ defined by

$$d^n\phi = d_B \circ \phi - (-1)^n \phi \circ d_A$$

Here are some properties of $\mathrm{Hom}^\bullet$ and its derived functor(s) $\mathrm{RHom}(-,-)$:

- $\mathrm{Hom}^\bullet(-,-)$ preserves mapping cones in each variable ([2] 11.3). When the second variable is fixed i.e. $\mathrm{Hom}^\bullet(-, B)$, one must employ a dual version of mapping cone.

- If either $A^\bullet$ or $B^\bullet$ is acyclic, $B^\bullet$ consists of injective objects, then $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is acyclic ([2] 10.5).

- If our category has enough injectives as well as projectives, then the derived functor $\mathrm{RHom}(-,-)$ can be computed by taking injective resolution in the second variable, or by taking projective in the first variable; the resulting functors are isomorphic ([2] 10.8).

- We have a functorial isomorphism for all $n \in \mathbb{Z}$:

$$\mathrm{Ext}^n(A^\bullet, B^\bullet) := \mathrm{Hom}_{D(X)}(A^\bullet, B^\bullet[n]) = H^n\mathrm{RHom}(A^\bullet, B^\bullet)$$

Definition 2.8 (tensor product of complexes)**.** Let $A^\bullet, B^\bullet \in \mathrm{Kom}(\mathscr{O}_X - \mathbf{Mod})$, define a complex

$$(A^\bullet \otimes B^\bullet)^n := \bigoplus_{i+j=n} A^i \otimes B^j$$

with differentials $d^n = (d^{i,j})_{i+j=n} : (A \otimes B)^n \to (A \otimes B)^{n+1}$, $d^{i,j} : A^i \otimes B^j \to (A^{i+1} \otimes B^j) \oplus (A^i \otimes B^{j+1})$ being defined by

$$d^{i,j}(a \otimes b) = d_{A^i} a \otimes b + -(1)^i a \otimes d_{B^j} b$$

Remark 2.9. There is an isomorphism $\tau : A \otimes B \xrightarrow{\sim} B \otimes A$ by defining

$$\tau : a \otimes b \mapsto (-1)^{ij} b \otimes a \ \text{ on } A^i \otimes B^j$$

For more generalities on $- \otimes -$, see [2] 11.3.

Proposition 2.10. *Let $A^\bullet, B^\bullet, C^\bullet$ be three complexes*

- *If $\phi : B^\bullet \to C^\bullet$ is a chain map, then the canonical map*

  $$\phi_* : \mathrm{Hom}^\bullet(A^\bullet, B^\bullet) \to \mathrm{Hom}^\bullet(A^\bullet, C^\bullet)$$

  is a chain map.

- *The canonical map*

  $$\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) \to \mathrm{Hom}^\bullet(A^\bullet \otimes C^\bullet, B^\bullet \otimes C^\bullet)$$

  given by $\phi \mapsto \phi \otimes \mathrm{id}_C$ is a chain map.

Proof. The first assertion is obvious, and the second one is a straightforward verification: Consider $(f^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}^n(A, B)$, where $f^p \in \mathrm{Hom}(A^p, B^{p+n})$. Its image is $(g^p)_{p\in\mathbb{Z}}$, where

$$g^p : \bigoplus_{i+j=p} A^i \otimes C^j \to \bigoplus_{i+j=p} B^{i+n} \otimes C^j, \ g^p = \oplus_{i+j=p}(f^i \otimes \mathrm{id}_{C^j})$$

Consider $d(g^\bullet)$, its restriction on $A^i \otimes C^j$ $(i + j = p)$ is

$$\begin{aligned}
&d_{B\otimes C} \circ (f^i \otimes \mathrm{id}_{C^j}) - (-1)^n g^p \circ (d_{A^i} \otimes \mathrm{id}_{C^j} + (-1)^i \mathrm{id}_{A^i} \otimes d_{C^j}) \\
=&d_{B^{i+n}} f^i \otimes \mathrm{id}_{C^j} + (-1)^{i+n} f^i \otimes d_{C^j} - (-1)^n (f^{i+1} d_{A^i} \otimes \mathrm{id}_{C^j} + (-1)^i f^i \otimes d_{C^j}) \\
=&d_{B^{i+n}} f^i \otimes \mathrm{id}_{C^j} - (-1)^n f^{i+1} d_{A^i} \otimes \mathrm{id}_{C^j}
\end{aligned}$$

On the other hand, the restriction of $d(f^\bullet)$ on A^p is $d_{B^{p+n}} f^p - (-1)^n f^{p+1} d_{A^p}$, for all $p \in \mathbb{Z}$. Hence the restriction on $A^i \otimes C^j$ of its image under $- \otimes C$ is

$$(d_{B^{i+n}} f^i - (-1)^n f^{i+1} d_{A^i}) \otimes \mathrm{id}_{C^j} = d_{B^{i+n}} f^i \otimes \mathrm{id}_{C^j} - (-1)^n f^{i+1} d_{A^i} \otimes \mathrm{id}_{C^j}$$

□

Here are some properties of $-\otimes-$ and its derived functor $-\overset{\mathrm{L}}{\otimes}-$:

- $-\otimes-$ preserves mapping cones in each variable ([2] 11.3).
- If either $A^\bullet$ or $B^\bullet$ is acyclic, and one of them consists of flat objects, then $A^\bullet \otimes B^\bullet$ is acyclic ([2] 11.5).
- If our category has enough flat objects, then $-\overset{\mathrm{L}}{\otimes}-$ can be computed by taking flat resolutions in either variable ([2] 11.6).

From now on, I will omit the bullet sign of a complex if there is no confusion.

Definition 2.11 (The dual of a complex)**.** For any complex E, we can form the dual complex

$$E^\vee := \mathrm{Hom}^\bullet(E, \mathscr{O}_X)$$

Its components are given by $E^{\vee -n} = \mathrm{Hom}^{-n}(E, \mathscr{O}_X) = \mathrm{Hom}(E^n, \mathscr{O}_X)$. By applying the rules above, the differential is given by

$$\lambda \in E^{\vee -n} \mapsto (-1)^{n+1}\lambda \circ d_{E^{n-1}} \in E^{\vee -n+1}.$$

Also note that, if $E \to F$ is a chain map, so is the induced map $F^\vee \to E^\vee$.

Definition-Proposition 2.12 (Trace, and its dual version)**.** For any $F \in \mathrm{Kom}^b(\mathbf{Coh}(X))$, we can define chain maps

$$\begin{aligned}
\alpha :& F^\vee \otimes F \to \mathscr{O}_X \\
& \lambda \otimes f \mapsto \begin{cases} \lambda(f) & \text{, if } \deg\lambda + \deg f = 0 \\ 0 & \text{, otherwise.} \end{cases} \\
\beta' :& \mathscr{O}_X \to \mathrm{Hom}^\bullet(F, F) \\
& 1 \mapsto \mathrm{id}_F \\
\gamma :& F \otimes F^\vee \to \mathrm{Hom}(F, F) \\
& f \otimes \lambda \mapsto \lambda(-)f \in \mathrm{Hom}^{i-j}(F, F) \text{ if } f \in F^i, \lambda \in F^{\vee(-j)}
\end{aligned}$$

Moreover, if F consists of locally free $\mathscr{O}_X$-modules of finite rank, then γ is an isomorphism. In this case, define $\beta := \gamma^{-1} \circ \beta' : \mathscr{O}_X \to F \otimes F^\vee$.

Proof. For α: In view of the condition $\deg\lambda + \deg f = 0$, it suffices to check that $\alpha(d(\lambda \otimes f)) = 0$ when $\lambda \in F^{\vee(-i-1)}$ and $f \in F^i$. We have $d(\lambda \otimes f) = d\lambda \otimes f + (-1)^{i+1}\lambda \otimes df = (-1)^i(\lambda \circ d) \otimes f + (-1)^{i+1}\lambda \otimes df$, which is mapped to $(-1)^i\lambda(df) + (-1)^{i+1}\lambda(df) = 0$.

For β': It is clearly a chain map, since $d(\mathrm{id}_F) = d_F - d_F = 0$.

For γ: For any $f \in F^i, \lambda \in F^{\vee -j}$, we have

$$d(f \otimes \lambda) = df \otimes \lambda + (-1)^i f \otimes d\lambda = df \otimes \lambda + (-1)^{i+j+1} f \otimes (\lambda \circ d)$$

which corresponds to $\lambda(-)df - (-1)^{i+j}\lambda(d(-))f \in \mathrm{Hom}^{i-j+1}(F, F)$. On the other hand:

$$d(\lambda(-)f) = \lambda(-)df - (-1)^{i+j}\lambda(d(-))f$$

which is just the same thing. Hence γ is a chain map. If F consists of locally free $\mathscr{O}_X$-modules of finite rank, then γ is an isomorphism by basic module theory. □

Remark 2.13. In [4] p.77, he defines $\mathrm{tr} : \mathrm{Hom}^\bullet(F, F) \to \mathscr{O}_X$ as $\mathrm{tr}(\phi) = \sum_i (-1)^i \mathrm{tr}\, \phi|_{F^i}$ if $\phi \in \mathrm{Hom}^0(F, F) = \bigoplus_i \mathrm{Hom}(F^i, F^i)$, and 0 elsewhere. In our formalism, this is just the composition

$$\mathrm{Hom}^\bullet(F, F) \xrightarrow{\gamma^{-1}} F \otimes F^\vee \xrightarrow{\tau} F^\vee \otimes F \xrightarrow{\alpha} \mathscr{O}_X.$$

Definition-Proposition 2.14. Consider $E \mapsto E^\curlyvee := \mathrm{RHom}(E, \mathscr{O}_X)$, it is a functor from $D^\pm(X)$ to $D^\mp(X)$, and its restriction on $D^b(X)$ maps into $D^b(X)$ itself.

Since there is almost no projective objects in $\mathbf{Coh}(X)$, the standard way to calculate $\mathcal{E}^\curlyvee$ is taking an injective resolution $0 \to \mathscr{O}_X \to I$ then take $\mathrm{Hom}^\bullet(E, I)$. Alternatively, recall that $\mathrm{Hom}^\bullet(-,-)$ preserves mapping cones. Note that the mapping cone of this injective resolution is

$$C = [0 \to \mathscr{O}_X \to I^0 \to I^1 \to \cdots]$$

Suppose that the terms of E are locally free of finite rank and are *negative enough* so that $\mathrm{Ext}^i(E^j, \mathscr{O}_X) = 0$ $(i > 0, j \in \mathbb{Z})$, then $\mathrm{Hom}^\bullet(E, C)$ is acyclic by a spectral sequence argument, thus $\mathrm{Hom}^\bullet(E, I) \to \mathrm{Hom}^\bullet(E, \mathscr{O}_X) = E^\vee$ is a quasi-isomorphism.

In conclusion: If E is a bounded complex $E^\bullet$ of *negative enough* locally free $\mathscr{O}_X$-modules of finite rank, then $E^\curlyvee = E^\vee$ in $D^b(X)$. Note that every bounded complex is quasi-isomorphic to such a complex.

Now we are ready to enounce the main result of this subsection:

Proposition 2.15 (cf. [3]). *With the definitions above, we have a functorial isomorphism*

$$\mathrm{RHom}(E, G \overset{\mathrm{L}}{\otimes} F^\curlyvee) \xrightarrow{\sim} \mathrm{RHom}(E \overset{\mathrm{L}}{\otimes} F, G)$$

where $E, F, G \in D^b(X)$.

Proof. To prove this, assume that E and F consist of locally free $\mathscr{O}_X$-modules of finite rank, $F^\bullet$ consists of *negative enough* locally frees, and replace G by a resolution by quasi-coherent sheaves which are injective as $\mathscr{O}_X$-modules.

Using the maps $\alpha : F^\vee \otimes F \to \mathscr{O}_X$ and $\beta : \mathscr{O}_X \to F \otimes F^\vee$ and the previous propositions, we get two maps from composition:

$$\mathrm{Hom}^\bullet(E, G \otimes F^\vee) \longrightarrow \mathrm{Hom}^\bullet(E \otimes F, G \otimes F^\vee \otimes F) \xrightarrow{\mathrm{id}\otimes\alpha_*} \mathrm{Hom}^\bullet(E \otimes F, G)$$

$$\mathrm{Hom}^\bullet(E, G \otimes F^\vee) \xleftarrow{\mathrm{id}\otimes\beta^*} \mathrm{Hom}^\bullet(E \otimes F \otimes F^\vee, G \otimes F^\vee) \longleftarrow \mathrm{Hom}^\bullet(E \otimes F, G)$$

They are chain maps – Indeed, push forward by $\mathrm{id} \otimes \alpha$ [resp. pull back by $\mathrm{id} \otimes \beta$] is evidently a chain map, and we have seen that the canonical map $\mathrm{Hom}^\bullet(-,-) \to \mathrm{Hom}^\bullet(- \otimes F, - \otimes F)$ [resp. $F^\vee$] is also a chain map.

By writing down their explicit expressions, one sees that the two maps are mutually inverse and functorial in E, F, G. As F consists of locally free $\mathscr{O}_X$-modules, so is $F^\vee$, thus $G \otimes F^\vee$ and $G \otimes F^\vee \otimes F$ consist of injective quasi-coherent sheaves [1]. On the other hand, all the tensor products in sight are tensor products with flat objects. Therefore, we can pass from $\mathrm{Hom}^\bullet$ to RHom, from $\otimes$ to $\overset{\mathrm{L}}{\otimes}$, and they yield mutually inverse functorial morphisms on the level of $D^b(X)$. □

Remark 2.16. The adjunction relation also yields a trace map $t : E^\curlyvee \overset{\mathrm{L}}{\otimes} E \to \mathscr{O}_X$ [resp. its dual version $\mathscr{O}_X \to E \overset{\mathrm{L}}{\otimes} E^\curlyvee$] in $D^b(X)$. They can be constructed directly from their complex version, and they can be calculated by using $E^\vee \otimes E \to \mathscr{O}_X$ [resp. $\mathscr{O}_X \to E \otimes E^\vee$] by taking a locally free resolution $E^\bullet$ whose terms are *negative enough*.

The pairing

Now, put $\omega_X := \Omega_X^n$, and view it as a complex concentrated at degree 0. There is a natural isomorphism $H^n(X, \omega_X) \simeq k$. Composition induces a functorial pairing for E, F in $D^b(X)$:

$$\mathrm{Hom}_{D^b(X)}(E, F) \times \mathrm{Hom}_{D^b(X)}(F, E \overset{\mathrm{L}}{\otimes} \omega_X[n]) \to \mathrm{Hom}_{D^b(X)}(E, E \overset{\mathrm{L}}{\otimes} \omega_X[n])$$

Recall that $H^0\mathrm{RHom} = \mathrm{Hom}_{D^b}$, and use the adjunction relation, we can

[1] Use the adjointness, or recall that injectivity of $\mathscr{O}_X$-modules is a local property (Hint: locally injective ⇒ locally flabby ⇒ flabby, now use the Leray spectral sequence for Ext).

go further:

$$\begin{aligned}\mathrm{Hom}_{D^b(X)}(E, E \overset{\mathrm{L}}{\otimes} \omega_X[n]) &= \mathrm{Hom}_{D^b(X)}(\mathscr{O}_X, E \overset{\mathrm{L}}{\otimes} \omega_X[n] \overset{\mathrm{L}}{\otimes} E^\vee) \\ &\xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(\mathscr{O}_X, E \overset{\mathrm{L}}{\otimes} E^\vee \overset{\mathrm{L}}{\otimes} \omega_X[n]) \\ &\to \mathrm{Hom}_{D^b(X)}(\mathscr{O}_X, \omega_X[n]) \xrightarrow{t\otimes 1} k\end{aligned}$$

This gives a functorial k-bilinear pairing

$$\mathrm{Hom}_{D^b(X)}(E, F) \times \mathrm{Hom}_{D^b(X)}(F, E \overset{\mathrm{L}}{\otimes} \omega_X[n]) \to k$$

Relations with Yoneda pairing

In order to do the exercise, we must check that the pairing constructed above coincides with that in Serre duality. Let $E^\bullet, F^\bullet, G^\bullet$ be objects in $D^b(X)$. In order have a concrete idea of the map $\mathrm{Hom}_{D^b}(E^\bullet, F^\bullet) \times \mathrm{Hom}_{D^b}(F^\bullet, G^\bullet) \to \mathrm{Hom}_{D^b}(E^\bullet, G^\bullet)$, we work in $D^b_{\mathbf{Coh}}(\mathbf{Qcoh}(X))$ to get resolutions by injective $\mathscr{O}_X$-modules $F^\bullet \to J^\bullet$, $G^\bullet \to K^\bullet$. Form the Hom complexes $\mathrm{Hom}^\bullet(E^\bullet, J^\bullet)$, $\mathrm{Hom}^\bullet(J^\bullet, K^\bullet)$, then the composition in $D^b(X)$ is just the composition between Hom complexes:

$$\mathrm{Hom}^i(E^\bullet, J^\bullet) \times \mathrm{Hom}^j(J^\bullet, K^\bullet) \to \mathrm{Hom}^{i+j}(E^\bullet, K^\bullet)$$

Taking homology at $i = j = 0$ will give the composition law for $\mathrm{Hom}_{D^b(X)}$; in general, taking homology at i, j gives a pairing $\mathrm{Ext}^i(E^\bullet, F^\bullet) \times \mathrm{Ext}^j(F^\bullet, G^\bullet) \to \mathrm{Ext}^{i+j}(E^\bullet, G^\bullet)$, where $\mathrm{Ext}^i(-,-) := \mathrm{Hom}_{D^b(X)}(-, -[i])$.

When $E^\bullet, F^\bullet, G^\bullet$ are complexes concentrated at degree 0, this is exactly the same construction as the Yoneda pairing who appears in the Grothendieck duality (my reference is [1] IV-1 [2]):

$$\mathrm{Ext}^i(E^\bullet, F^\bullet) \times \mathrm{Ext}^j(F^\bullet, G^\bullet) \to \mathrm{Ext}^{i+j}(E^\bullet, G^\bullet)$$

Equivalence with Serre duality

Suppose that E, F are coherent sheaves, considered as complexes concentrated at degree 0 and i, respectively; according to the preceding remark on Yoneda pairing, what we get is exactly the same pairing in Grothendieck duality (where E is usually taken to be $\mathscr{O}_X$):

$$\mathrm{Ext}^i(E, F) \times \mathrm{Ext}^{n-i}(F, E \otimes \omega_X) \to k$$

[2] In [1], they also take an injective resolution of $E^\bullet$, but this seems to be unnecessary.

Now let's set out to do the exercise.

($\Rightarrow$): Suppose that the pairing on level of $D^b(X)$ gives an isomorphism

$$\mathrm{Hom}_{D^b(X)}(E,F) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(F, E \overset{\mathrm{L}}{\otimes} \omega_X[n])^\vee$$

For any $0 \le i \le n$ and any coherent sheaf F on X, let F be the complex with only one term F concentrated at degree i, and let E be the complex with only $\mathscr{O}_X$ concentrated at degree 0. It follows that

$$\mathrm{Ext}^i(\mathscr{O}_X, F) \xrightarrow{\sim} \mathrm{Ext}^{n-i}(F, \omega_X)^\vee$$

that is,

$$H^i(X,F) \xrightarrow{\sim} \mathrm{Ext}^{n-i}(F,\omega_X)^\vee$$

which is the usual form of Serre duality on smooth projective varieties.

($\Leftarrow$): Conversely, suppose that Serre duality holds. Since $\mathrm{Ext}^i(E,F) = \mathrm{Ext}^i(\mathscr{O}_X, F \otimes E^\vee)$ if E is a locally free sheaf of finite rank, it follows that

$$\mathrm{Ext}^i(E,F) \times \mathrm{Ext}^{n-i}(F, E \otimes \omega_X) \to k$$

is a perfect pairing for any coherent sheaf F and any locally free E of finite rank. A routine diagram chasing shows that this is the same homomorphism defined using trace map $E \otimes E^\vee \to \mathscr{O}_X$ (express the isomorphisms in terms of push forward via $E \otimes E^\vee \to \mathscr{O}_X$ and pull back via $E \otimes E^\vee \to \mathscr{O}_X$).

To show that

$$\mathrm{Hom}_{D^b(X)}(E,F) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(F, (E \overset{\mathrm{L}}{\otimes} \omega_X)[n])^\vee$$

for any E, F in $D^b(X)$. We can assume that E is a complex of locally free $\mathscr{O}_X$-modules of finite rank.

For any F in $D^b(X)$, let $p(F)$ be the minimal length among bounded complexes quasi-isomorphic to F. Define $q(E)$ similarly, but the minimum is taken among bounded complexes with locally free terms. Proceed by double induction on $(p(F), q(E))$.

When $p(F), q(E) \le 1$, this is Serre duality. In general, suppose that $p(F) > 1$. Take a complex $F^\bullet$ of length $p = p(F)$ quasi-isomorphic to F, and consider the truncation

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & F^{p-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
\| & & \| & & \uparrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & F^{p-1} & \longrightarrow & F^p & \longrightarrow & 0 \\
{\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^p & \longrightarrow & 0
\end{array}$$

This gives a short exact sequence, hence a distinguished triangle $F' \to F \to F'' \to F'[1]$ in $D^b(X)$ with $p(F'), p(F'') < p(F)$. Apply the derived Hom functor $R\mathrm{Hom}$, and then the long exact sequence + induction hypothesis + 5-lemma yield the desired isomorphism [3].

The induction step to reduce $q(E)$ is similar. This completes the proof.

References

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[2] A. Borel et al. *Algebraic D-modules.* Perspectives in Mathematics. Academic Press, 1986.

[3] S. Gelfand and Y. Manin. *Methods of homological algebra.* Springer monographs in mathematics. Springer, second edition, 2003.

[4] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Oxford mathematical monographs. Oxford University Press, 2006.

[3] This amounts to say that the one-term complexes form a *classe génératrice* of $D^b(X)$ in the sense of [2] §12.