INTERCITY SEMINAR ALGEBRAIC GEOMETRY, autumn 2007

Lecture 1: General Introduction

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Theme. $D^b(X) := D^b(\mathbf{Coh}X)$ as an invariant of a noetherian scheme X, where $D^b(\mathbf{Coh}X)$ denotes the derived category of bounded complexes in $\mathbf{Coh}X$ = the abelian category of coherent $\mathcal{O}_X$-modules.

1. Think of it as a variation of: $K_0(X) = K(\mathbf{Coh}X)$, $K^0(X)$, $\mathrm{CH}(X)$, $\mathbf{Coh}(X)$ with appropriate structures.
2. Note that $\mathbf{Coh}(X)$, $D^b(X)$ are categories...
3. Of course, this seminar is a good opportunity to learn about derived categories.

References: See Moonen's program. **Main reference**: Huybrechts' book [6].

1 Qcoh(X) and Coh(X)

For $(X, \mathcal{O}_X)$ a ringed space, and $\mathcal{F}$ an $\mathcal{O}_X$-module:

$\mathcal{F}$ is quasi-coherent $\overset{\text{def}}{\iff}$ $\forall x \in X\ \exists U \ni x\ \exists (\mathcal{O}_X|_U)^{(I)} \to (\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U \to 0$ (see [3]).

For X a scheme: $\mathcal{F}$ is quasi-coherent $\iff$ $\forall$open affine $U = \mathrm{Spec}\,(A), \exists A-$module M s.t. $\mathcal{F}|_U \cong \widetilde{M}\quad (M = \mathcal{F}(U))$. $(f \in A : \mathcal{F}(D(f)) = (\mathcal{F}U)_f)$.

For $(X, \mathcal{O}_X)$ a ringed space, again:

$$\mathcal{F} \text{ is coherent} \quad \overset{\text{def}}{\iff} \begin{cases} 1. & \mathcal{F} \text{ is finitely generated} : \forall x \exists U, n, (\mathcal{O}_X|_U)^n \twoheadrightarrow \mathcal{F}|_U; \\ 2. & \forall U \subset X, \forall n, \forall \mathcal{O}_U^n \xrightarrow{\phi} \mathcal{F}|_U : \mathrm{Ker}\,\phi \text{ is finitely generated.} \end{cases}$$

For X a locally noetherian scheme:

[0] Typesetting by Wen-Wei Li.

$$\begin{aligned}\mathcal{F} \text{ is coherent} &\iff (\mathcal{F} \text{ is quasi-coherent and of finite type})\\ &\iff (\forall \text{ open affine } U = \operatorname{Spec} A : \mathcal{F}|_U = \widetilde{\mathcal{F}U}\\ &\qquad \text{and } \mathcal{F}U \text{ is a finitely generated } A\text{-module})\end{aligned}$$

For $X = \operatorname{Spec} A$: $\mathbf{Qcoh}(X) \rightleftharpoons A-\text{mod}$, $\mathcal{F} \mapsto \mathcal{F}(X)$, $\widetilde{M} \leftarrow M$ is an equivalence; if A is noetherian: $\mathbf{Coh}(X) \rightleftharpoons$ f.g. $A-\text{mod}$ is an equivalence.

$\mathbf{Qcoh}(X)$ and $\mathbf{Coh}(X)$ are full subcategories of $\mathcal{O}_X$-mod; they are abelian categories, closed under $- \otimes_{\mathcal{O}_X} -$.

For $f : X \to Y$ in $(\mathbf{Sch})$, $\mathcal{F} \in \mathbf{Qcoh}(Y)$: $f^*\mathcal{F} \in \mathbf{Qcoh}(X)$; if X is noetherian, or f quasi-compact and separated, $\mathcal{F} \in \mathbf{Qcoh}(X) \Rightarrow f_*\mathcal{F} \in \mathbf{Qcoh}(Y)$ ([5], Prop II 5.8). If Y is locally noetherian and f proper: $f_* : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ ([4]).

Theorem 1.1 (Gabriel)**.** *Let k be a field, X, Y k-schemes of finite type, such that $\mathbf{Coh}(X)$ and $\mathbf{Coh}(Y)$ are equivalent. Then $X \cong Y$.*

Proof. We reconstruct X from $\mathbf{Coh}(X)$.

Simple objects of $\mathbf{Coh}(X)$: $i_{x,*}\kappa(x)$, where $x \in X$ closed, $i_x : \{x\} \to X$ the inclusion and $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ (indeed, all other nonzero $\mathcal{F} \in \mathbf{Coh}(X)$ have nontrivial quotients).

So we know X^o as a set.

Topology on X^o: $U \subset X$ is open $\iff$

$$\exists \mathcal{F} \in \mathbf{Coh}(X) : \overbrace{\forall x \in X^o : x \in U \iff \operatorname{Hom}(\mathcal{F}, i_{x,*}\kappa(x)) = 0}^{\operatorname{Supp}\mathcal{F} = X \setminus U}$$

So we have X (add generic points for all irreducible closed subsets of X^o).

Now $\mathcal{O}_X$. We have: $\mathcal{O}_X(X) = \operatorname{End}(\mathrm{id}_{\mathbf{Coh}(X)})$, because $\operatorname{End}_{\mathbf{Coh}X}(\mathcal{O}_X) = \mathcal{O}_X(X)$. ($\forall$ ringed space $(X, \mathcal{O}_X) : \mathcal{O}_X(X) = \operatorname{End}(\mathrm{id}_{\mathbf{Qcoh}(X)})$)

Now comes Gabriel's input: for $U \subset X$ open, $\mathbf{Coh}(X) \to \mathbf{Coh}(U)$ is the *quotient* Q by the full subcategory $\mathbf{Coh}_Z(X)$ of $\mathcal{F}$ with $\operatorname{Supp}(\mathcal{F}) \subset X \setminus U =: Z$. (thick, or Serre subcategory).

$$\begin{aligned}\operatorname{Ob}(Q) &:= \operatorname{Ob}(\mathbf{Coh}(X))\\ \operatorname{Hom}_Q(\mathcal{F}, \mathcal{G}) &:= \{\mathcal{F} \overset{s}{\leftarrow} \mathcal{H} \overset{g}{\rightarrow} \mathcal{G} \text{ in } \mathbf{Coh}(X) \text{ s.t. } \operatorname{Ker}(s), \operatorname{Coker}(s) \in \mathbf{Coh}_Z(X)\}/\sim\end{aligned}$$

where

$$(\mathcal{F} \overset{s}{\leftarrow} \mathcal{H} \overset{g}{\rightarrow} \mathcal{G}) \sim (\mathcal{F} \overset{s'}{\leftarrow} \mathcal{H} \overset{g'}{\rightarrow} \mathcal{G}) \iff \exists \quad \begin{array}{ccccc} & & \mathcal{H} & & \\ & \overset{s}{\swarrow} & \uparrow & \overset{f}{\searrow} & \\ \mathcal{F} & \overset{s''}{\longleftarrow} & \mathcal{H}'' & \overset{f''}{\longrightarrow} & \mathcal{G} \\ & \underset{s'}{\nwarrow} & \downarrow & \underset{f'}{\nearrow} & \\ & & \mathcal{H}' & & \end{array}$$

For details, see e.g. [8] Example 10.3.2, or [2] Exercise II 5.9.

Note: Fortunately $\mathbf{Coh}(X)$ is a *small* category.

The universal property of $q : \mathbf{Coh}(X) \to \mathbf{Coh}(U)$ that we apply to $\mathbf{Coh}(X) \to \mathbf{Coh}(U)$: as $\mathbf{Coh}(X) \to \mathbf{Coh}(U)$ satisfies that if $\mathrm{Ker}\,(f), \mathrm{Coker}\,(f) \in \mathbf{Coh}_Z(X)$, then the image of f is an isomorphism, then

$$\begin{array}{ccc} \mathbf{Coh}(X) & \xrightarrow{\;q\;} & Q \\ & \searrow & \swarrow \exists! \\ & \mathbf{Coh}(U) & \end{array}$$

Surjectivity of $Q \to \mathbf{Coh}(U)$ follows from:

$$\forall \mathcal{F} \in \mathbf{Coh}(U)\ \exists \mathcal{F}' \in \mathbf{Coh}(X) \text{ s.t. } \mathcal{F}'|_U \cong \mathcal{F} \quad ([5], \text{Exer. II. 5.15})$$

So: for $U \subset X$ open: $\mathcal{O}_X(U) = \mathrm{End}(\mathrm{id}_{\mathbf{Coh}(X)/\mathbf{Coh}_Z(X)})$. □

Exercise 1.2. (Finitely generated $\mathbb{Z}$-modules)/(finite $\mathbb{Z}$-modules) $\xrightarrow{\text{equivalence}}$ (finite-dimensional $\mathbb{Q}$-vector spaces). How for $\mathbb{Z}$-modules?

2 $D^b(X)$, derived categories.

References: [2], [8]

Let $\mathcal{A}$ be an abelian category, i.e.:

- Additive:
  - $\forall X, Y :$ $\mathrm{Hom}(X, Y)$ is a $\mathbb{Z}$-module, composition is bilinear.
  - $\exists 0 \in \mathrm{Ob}(\mathcal{A})$, final and initial.
  - $\forall X, Y : \exists X \oplus Y,\ X \oplus Y = X \times Y$
- $\exists$ kernel and cokernels, and

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \downarrow & & \uparrow \\ \mathrm{Coker}\,\mathrm{Ker}\,(f) & \xrightarrow{\;\sim\;} & \mathrm{Ker}\,\mathrm{Coker}\,(f) \end{array}$$

Kom($\mathcal{A}$) := Category of complexes in (over) $\mathcal{A}$:
Objects: $((K^n)_{n\in\mathbb{Z}},(d^n)_{n\in\mathbb{Z}}), K^n \in \mathrm{Ob}\mathcal{A}, d^n : K^n \to K^{n+1}, d^n \circ d^{n-1} = 0$.
Morphisms: $(f^n)_{n\in\mathbb{Z}} : f^n : K^n \to L^n$ such that all diagrams

$$\begin{array}{ccc} K^n & \xrightarrow{d^n} & K^{n+1} \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle f^{n+1}} \\ L^n & \xrightarrow{d^n} & L^{n+1} \end{array}$$

are commutative.

$\mathrm{Kom}^+(\mathcal{A})$: Full subcategory of (K,d) s.t. $K^n = 0$ for $n \ll 0$.
$\mathrm{Kom}^-(\mathcal{A})$: Full subcategory of (K,d) s.t. $K^n = 0$ for $n \gg 0$.
$\mathrm{Kom}^b(\mathcal{A})$: Full subcategory of (K,d) s.t. $K^n = 0$ for $|n| \gg 0$.

($\mathrm{Kom}^*(\mathcal{A})$ is abelian)

Homology: $H^n : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A} : K^\bullet \mapsto \mathrm{Ker}\, d^n/\mathrm{Im} d^{n-1}$ (indeed functorial). So: $K^\bullet$ exact (acylcic) $\iff H^n(K^\bullet) = 0$ for all n.

Definition 2.1. $f : K^\bullet \to L^\bullet$ is a quasi-isomorphism $\iff \forall n \in \mathbb{Z} : H^n(f)$ is an isomorphism.

Example 2.2. If $A \rightarrowtail I^0 \to I^1 \to \cdots$ is exact, then $A[0] \to I^\bullet$ is a quasi-isomorphism ($A[0]$ is the complex with A in degree 0, 0 elsewhere).

Definition-Theorem 2.3 ([2] III 2.1)**.** There exists $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ such that

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ {\scriptstyle F}\downarrow & \swarrow{\scriptstyle \exists! G} & \\ D & & \end{array}$$

where $D(\mathcal{A})$ is just a category (not additive, ...) such that f qis $\Rightarrow Q(f)$ isom.

The proof is a "formal" localisation procedure for arbitrary categories

$$Q : B \to B[S^{-1}], S \text{ arbitrary class of morphisms in } B$$

Morphisms in $B[S^{-1}]$: $f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \cdots \circ f_n \circ s_n^{-1}$, modulo equivalence relation.

Similarly: $\mathrm{Kom}^*(\mathcal{A}) \xrightarrow{Q} D^*(\mathcal{A}), * \in \{+,-,b\}$.

Fact 2.4. $D^*(\mathcal{A}) \to D(\mathcal{A})$ *are full subcategories, essential image : those* K *with* $H^n(K) = 0$ *for* $n \ll 0, n \gg 0, \ldots$.

Let $\mathrm{Kom}_0(\mathcal{A})$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ of (K,d) with $d = 0$. Then we have

$$\begin{array}{ccc} h : \mathrm{Kom}(\mathcal{A}) & \longrightarrow & \mathrm{Kom}_0(\mathcal{A}) \\ & \searrow \quad \nearrow & \\ & D(\mathcal{A}) & \end{array}$$

equivalence $\iff \mathcal{A}$ is semi-simple i.e. all short exact sequences split

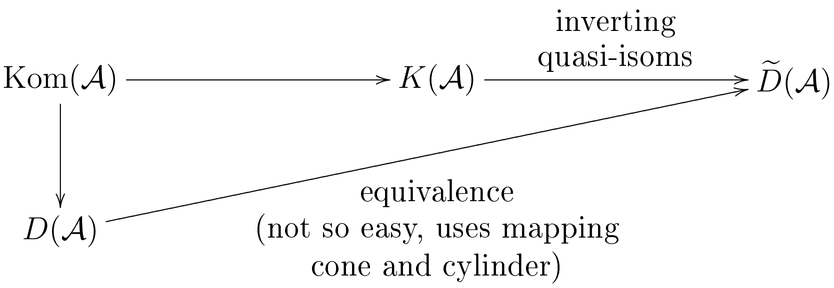
To find out more about $D(\mathcal{A})$, one uses the *homotopy category* $K(\mathcal{A})$.

$$\mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}\mathcal{A}),\ \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}(K, L)/\text{homotopy}.$$

(Recall: $f \sim g \iff \exists h^n : K^n \to L^{n-1}$ s.t. $f - g = h \circ d + d \circ h$; the f with $f \sim 0$ form an ideal in $\mathrm{Kom}(\mathcal{A})$. This terminology comes from topology, via a suitable functor; triangulated spaces, or singular homology.)

Similarly: $K^*(\mathcal{A})$ (additive categories). Note: $f \sim g \Rightarrow H^n(f) = H^n(g)$, so quasi-isomorphism in $K(\mathcal{A})$ makes sense.

Now:



The quasi-isoms in $K(\mathcal{A})$ form a *localising class*:

$S \subset \mathrm{Hom}_B$ is localising $\begin{cases} (a)\ \forall X \in B : \ \mathrm{id}_X \in S, \text{ closed under composition.} \\ (b)\ \begin{array}{ccc} W & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} Z & \overset{g}{\dashrightarrow} & W \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} \\ X & \xrightarrow{f} & Y \end{array}. \\ (c) \text{ if } f, g : X \to Y, \text{ then} \\ \qquad (\exists s \in S : sf = sg) \iff (\exists t \in S : ft = gt). \end{cases}$

Left $S \subset \mathrm{Hom}_B$ be localising, X, Y in $\mathrm{Ob}(B)$. Then

$$\mathrm{Hom}_{B[S^{-1}]}(X, Y) = \{X \xleftarrow{s \in S} X' \xrightarrow{f} Y\}/\sim \ \text{ as before}$$

$D^*(\mathcal{A})$ is additive. $\mathcal{A} \to D^*(\mathcal{A}), X \mapsto X[0]$, is fully faithful, and induces an equivalence of $\mathcal{A}$ with the full subcategory of $K^\bullet$ in $D^\bullet(\mathcal{A})$ with $H^i(K^\bullet) = 0\ \forall i \neq 0$.

Translation functor: for $n \in \mathbb{Z} : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A}), K^\bullet \mapsto K[n]^\bullet : K[n]^i = K^{n+i}, f[n]^i = f^{n+i}, d[n] = (-1)^n d$.

Definition 2.5. For $i \in \mathbb{Z}$, $X, Y \in \mathcal{A}$:

$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{D^*(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D^*(\mathcal{A})}(X[k], Y[i+k])$ for all $k \in \mathbb{Z}$.

Note: $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \times \mathrm{Ext}^j_{\mathcal{A}}(Y,Z) \to \mathrm{Ext}^{i+j}_{\mathcal{A}}(X, Z[i])$ is now just *composition.*

Theorem 2.6. $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0$ *if* $i < 0$.

For $i > 0$: every element of $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ is of the form:

$$\begin{array}{ccccccc} & [0 \longrightarrow & K^{-i} = Y & \longrightarrow \cdots \longrightarrow & K^0 & \longrightarrow 0] & \\ & \swarrow s & & & & \searrow f & \\ X[0] & & & & & & Y[i] \end{array}$$

with $0 \to Y \to K^{-i+1} \to \cdots \to K^0 \to X \to 0$ exact.

Proposition 2.7 (See Keller, *Derived categories and tilting* in [7] § 2.5)**.** *Assume* $\forall X, Y$ *in* $\mathcal{A}$*:* $\mathrm{Ext}^2_{\mathcal{A}}(X,Y) = 0$*. Then* $\forall K^\bullet$ *in* $\mathrm{Kom}(\mathcal{A})$*:* $\exists$ *quasi-isomorphisms:* $K^\bullet \leftarrow S^\bullet \to H^\bullet(K^\bullet)$*.*

$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \{(f_n, \epsilon_n)_{n\in\mathbb{Z}} | f : H^n(K^\bullet) \to H^n(L^\bullet), \epsilon \in \mathrm{Ext}^1_{\mathcal{A}}(H^n(K^\bullet), H^{n-1}(L^\bullet))\}$.

3 Distinguished triangles in $D(\mathcal{A})$

(Generalise short exact sequences of complexes, and give rise to long exact homology sequences)

Definition 3.1. A triangle in $\mathrm{Kom}(\mathcal{A}), K(\mathcal{A}), D(\mathcal{A})$ is a diagram

$$K^\bullet \xrightarrow{u} L^\bullet \xrightarrow{v} M^\bullet \xrightarrow{w} K[1]^\bullet$$

A morphisms of triangles is a commutative diagram:

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{u} & L^\bullet & \xrightarrow{v} & M^\bullet & \xrightarrow{w} & K[1]^\bullet \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ K_1^\bullet & \longrightarrow & L_1^\bullet & \longrightarrow & M_1^\bullet & \longrightarrow & K_1[1]^\bullet \end{array}$$

This makes a category, and we have the notion of isomorphism.

A triangle is *distinguished* if it is isomorphic to some diagram:

$$K^\bullet \xrightarrow{\bar{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} K[1]^\bullet \quad \text{(to be explained)}.$$

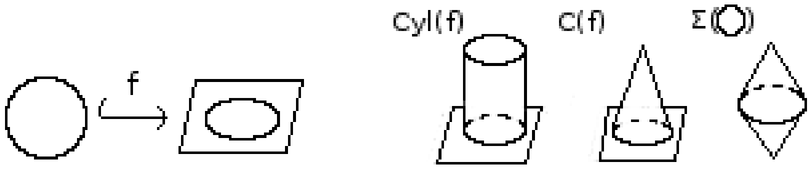
$$\boxed{\text{rows exact, comm.}}\quad \begin{array}{ccccccccc}
 & & 0 & \longrightarrow & L^\bullet & \xrightarrow{\bar\pi} & C(f) & \xrightarrow{\delta} & K[1]^\bullet \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle\alpha} & & \| & & \\
0 & \longrightarrow & K^\bullet & \xrightarrow{\bar f} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & C(f) & & \\
 & & \| & & \downarrow{\scriptstyle\beta} & & & & \\
 & & K^\bullet & \xrightarrow{f} & L^\bullet & & & &
\end{array}$$

where $f \in \mathrm{Kom}(\mathcal{A})$ is arbitrary.

Let $f : X \to Y$ in Top. Then we have:

$$\begin{array}{ccccc}
 & Y & \longrightarrow & C(f) & \longrightarrow \Sigma(X) \\
 & \downarrow & & \| & \\
 & \mathrm{Cyl}(f) & \longrightarrow & C(f) & \\
 & \downarrow & & & \\
X \xrightarrow{f} & Y & & &
\end{array}$$

$$\begin{array}{rcl}
\mathrm{Cyl}(f) & = & (X \times [0,1] \coprod Y)/\langle (x,1) = f(x) \rangle \\
C(f) & = & \mathrm{Cyl}(f)/\langle (x,0) = (x',0) \rangle \\
\Sigma(X) & = & X \times [0,1]/\langle (x,0) = (x',0), (x,1) = (x',1) \rangle
\end{array}$$



- Mayer-Vietoris for $\Sigma(X) = \underbrace{\frac{X \times [0,1/2]}{\langle (x,0) = (x',0) \rangle}}_{U} \cup \underbrace{\frac{X \times [1/2,1]}{\langle (x,1) = (x',1) \rangle}}_{V}$,

  $U \cap V = X$.

- Suspension: $H^i(X) \xrightarrow{\sim} H^{i+1}(\Sigma X) \quad \forall i > 0$.

Theorem 3.2. *Let $K^\bullet \xrightarrow{u} L^\bullet \xrightarrow{v} M^\bullet \xrightarrow{w} K[1]^\bullet$ be a distinguished triangle in $D(\mathcal{A})$. Then the sequence $\cdots \to H^i(K^\bullet) \to H^i(L^\bullet) \to H^i(M^\bullet) \to H^{i+1}(K^\bullet) \to \cdots$ is exact.*

We view $D^*(\mathcal{A})$ as a triangulated category (the notion has been axiomatised). In particular, the "invariant" $D^b(\mathbf{Coh}X)$ is considered as such.

Remark 3.3. $K^*(\mathcal{A})$ is also triangulated.

4 Derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $F : \mathcal{A} \to \mathcal{B}$ left exact. Let $R \subset \mathrm{Ob}(\mathcal{A})$ be *adapted to* F:

- stable under finite direct sums.
- $\forall K^\bullet$ in $\mathrm{Kom}^+(R)$ exact, $FK^\bullet$ in $\mathrm{Kom}(B)$ exact.
- $\forall X \in \mathrm{Ob}\mathcal{A}\ \exists I \in \mathrm{Ob}R$ s.t. $X \rightarrowtail I$.

Let $S_R :=$ quasi-isomorphisms in $K^+(R)$.

Then S_R is localising, and $K^+(R)[S_R^{-1}] \to D^+(\mathcal{A})$ is an equivalence. (Similar for left exact, D^-.)

Let Φ be an inverse.

Then one defines: $RF : D^+(\mathcal{A}) \xrightarrow{\Phi} K^+(R)[S_R^{-1}] \xrightarrow{F} D^+(B)$.

$\exists$ characterisation of RF be a universal property.

Remark 4.1. RF is *exact* in the sense that it preserves the distinguished triangles.

Proposition 4.2 (3.5 in Huybrechts' book)**.** *Let X be a noetherian scheme. Then $D^b(X) \to D^b_{\mathbf{Coh}}(\mathbf{Qcoh}X)$ is an equivalence, where $D^b_{\mathbf{Coh}}(\mathbf{Qcoh}X)$ is the full subcategory of $D^b(\mathbf{Qcoh}X)$ of complexes $K^\bullet$ with $H^i(K^\bullet)$ in $\mathbf{Coh}(X)$ for all $i \in \mathbb{Z}$.*

$\mathbf{Qcoh}(X)$ *has enough injectives.*

5 Serre functors

Theorem 5.1 (Serre duality, 3.12 in [6])**.** *let X/k be smooth and projective. Then $\exists$ functorial isomorphisms, $\forall \mathcal{E}^\bullet, \mathcal{F}^\bullet$ in $D^b(X)$:*

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n])^\vee.$$

Remark 5.2. These Hom's are finite dimensional k-vector spaces; $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes \omega_X[n]$ is the Serre functor.

Exercise 5.3. Derive ordinary Serre duality from this, and vice versa.

So: $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes \omega_X[n]$ is unique up to equivalence.

Theorem 5.4 (Bondal, Orlov)**.** *Let X, Y be smooth projectives such that $\exists$ exact equivalence $D^b(X) \to D^b(T)$ and such that $\omega_X^{\pm}$ is ample. Then $X \simeq Y$.*

Theorem 5.5 (Balmer)**.** *"A reduced noetherian scheme X can be reconstructed from $(D^{\mathrm{perf}}(X), \otimes)$".*

6 Grothendieck groups, Chow rings

Reference: [1] Ch. 15 (page 294), [6], p.124.

Let X be a nonsingular projective variety $/k$ a field (smooth projective k-scheme).

$K_0(X)$: Grothendieck group of coherent $\mathcal{O}_X$-modules: $\mathcal{F} \in \mathbf{Coh}(X) \rightsquigarrow [\mathcal{F}] \in K_0(X)$, $\forall \mathcal{F}' \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{F}'' : [\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$; $\mathbb{Z}$-module.

$K^0(X)$: Similar for locally free $\mathcal{O}_X$-modules, $\otimes$ makes it into a commutative $\mathbb{Z}$-algebra.

$K^0(X) \to K_0(X)$ is an isomorphism of $\mathbb{Z}$-modules because $\forall \mathcal{F} \in \mathbf{Coh}(X)$ $\exists$ finite resolution by locally free $\mathcal{O}_X$-modules of finite rank.

For $\mathcal{F}^\bullet$ in $D^b(X)$: $[\mathcal{F}^\bullet] := \sum_i (-1)^i [\mathcal{F}^i] = \sum_i (-1)^i [h^i(\mathcal{F}^\bullet)] \in K_0(X)$, additive for distinguished triangles.

Finally: $\mathbb{Q} \otimes K^0(X) \xrightarrow{\text{ch}} \mathbb{Q} \otimes \text{CH}(X)$ is an isomorphism of $\mathbb{Q}$-algebras.

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