Why there exist no Division Algebras over $\mathbb{R}$ of uneven dimension greater than 1

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Wednesday, 14th of May 2008

**Definition.** Identify the $n$-sphere $S^n$ with all points in $\mathbb{R}^{n+1}$ of euclidean norm 1. The $n$-sphere is said to be parallelisable if there exist $n$ continuous maps $\phi_i : S^n \rightarrow S^n$ such that for every $a \in S^n$, $a, \phi_1(a), \phi_2(a), \ldots, \phi_n(a)$ is linearly independent.

The concept of parallelisability is relevant to the existence of division algebras by way of the following implication:

**Proposition.** Suppose that for $n \geq 0$, there exists an $n$-dimensional division algebra $A$ over $\mathbb{R}$. Then the $(n-1)$-sphere is parallelisable.

In the uneven-dimensional case we can then show that parallelisability of the $n$-sphere leads to a contradiction using the Brouwer degree of a map from $S^n$ to $S^n$:

**Claim.** Let $n \geq 0$. Let $f, g \in \text{Mor}(S^n, S^n)$. The Brouwer degree satisfies the following properties:

1. $\deg(g \circ f) = \deg(f) \deg(g)$.
2. $\deg(\text{Const}) = 0$.
3. $\deg(\text{Id}_{S^n}) = 1$.
4. If $f \sim g$, then $\deg(f) = \deg(g)$.
5. Let $0 \leq i \leq n$, then $\text{Ref}_i$ is the map that sends a point $(v_0, v_1, \ldots, v_i, \ldots, v_n)$ to $(v_0, v_1, \ldots, -v_i, \ldots, v_n)$. We have $\deg(\text{Ref}_i) = -1$.

For the purpose of this talk, these properties will only be assumed, not proven, but it will be shown how this leads to a contradiction, and if there is time, the construction of the Brouwer degree will shortly be discussed.