Lecture notes Algebraic Geometry
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Examples of fibred products

1. Let $f\colon X \to S$ be a morphism of schemes, and let $j\colon U \hookrightarrow S$ be an open immersion. Then the diagram

$$\begin{array}{ccc} f^{-1}U & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ U & \longrightarrow & S \end{array}$$

   is Cartesian, i.e. $X \times_S U \cong S$.

2. Let S be a scheme, and let $\mathbf{P}^n_{\mathbf{Z}}$ be n-dimensional projective space over $\mathbf{Z}$. There are unique morphisms $S \to \operatorname{Spec} \mathbf{Z}$ and $\mathbf{P}^n_{\mathbf{Z}} \to \operatorname{Spec} \mathbf{Z}$, and we define $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times_{\operatorname{Spec} \mathbf{Z}} S$, so that we have a Cartesian diagram

$$\begin{array}{ccc} \mathbf{P}^n_S & \longrightarrow & \mathbf{P}^n_{\mathbf{Z}} \\ \big\downarrow & \square & \big\downarrow \\ S & \longrightarrow & \operatorname{Spec} \mathbf{Z}. \end{array}$$

3. Let $X \to S$ be a morphism of schemes, let $s \in S$, and let $\kappa(s)$ be the residue class field of S at s. Then the *fibre of f at s* is defined by the Cartesian diagram

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \big\downarrow & \square & \big\downarrow \\ \operatorname{Spec} \kappa(s) & \longrightarrow & S. \end{array}$$

   As a topological space, X_s equals $f^{-1}\{s\}$ with the induced topology from X.

Intermezzo on integral dependence

A general reference for today's stuff is J.-P. Serre's book "Algèbre locale, multiplicités" (Lecture Notes in Mathematics 11).

Definition. Let $\phi\colon A \to B$ be a morphism of rings. We view B as an A-algebra and the elements of A as elements of B via ϕ. An element $b \in B$ is said to be *integral over* A if there exist $n \geq 1$ and $a_0, \dots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$$

in B. Such a relation is called an *integral dependence relation.*

Proposition. *Let $\phi\colon A \to B$ be as above.*

1. *The following are equivalent for $b \in B$:*
   (i) *b is integral over A;*
   (ii) *$A[b] \subset B$ is a finitely generated A-module (in other words, $A[b]$ is a finite A-algebra);*
   (iii) *there is a finite sub-A-algebra $C \subset B$ with $b \in C$;*
   (iv) *there is a finitely generated sub-A-module $M \subset B$ such that $bM \subset M$ and $1 \in M$;*
   (v) *there is a finitely generated sub-A-module $M \subset B$ such that $bM \subset M$ and M contains an element m that is not a zero-divisor of B (i.e. multiplication by m is an injective function $B \to B$; note that 0 is a zero-divisor unless B is the zero ring).*
2. *Define the integral closure of A in B by*

$$B' = \{b \in B \mid b \text{ is integral over } A\}.$$

   Then B' is a sub-A-algebra of B.
3. *Suppose that ϕ is injective and that A, B are integral domains such that A is integrally closed in its field of fractions K. Then b is integral over A if and only if the minimal polynomial of b over K has coefficients in A.*
4. *Let A be an integral domain with field of fractions K, let $K \hookrightarrow L$ be a field extension, and let $\tilde{A}$ be the integral closure of A in L. Let $S \subset A$ be a multiplicative system not containing 0. Then*

$$\widetilde{S^{-1}A} = S^{-1}\tilde{A} \quad \text{(as subsets of } L\text{)}.$$

Proof.

1. The implications (i)⇒(ii)⇒(iii)⇒(iv)⇒(v) are easy. For (v)⇒(i), take a surjection $A^n \to M$ of A-modules, and let a be an $n \times n$-matrix such that the diagram

$$\begin{array}{ccc} A^n & \longrightarrow & M \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A^n & \longrightarrow & M \end{array}$$

   commutes, where the vertical arrows are given by multiplication by the matrix a and by the element b, respectively. Let f be the characteristic polynomial of a over A. By the Cayley-Hamilton theorem, $f(a) = 0$ as a map from A^n to A^n. But then multiplication by $f(b)$ is also the zero map $M \to M$. Because m is not a zero-divisor, $f(b)m = 0$ implies that $f(b) = 0$.
2. Using the fact that b is integral over A if and only if $A[b]$ is a finitely generated A-module, it is easy to see that if x, y are integral over A, then so are $x \pm y$ and xy. This proves that B' is a sub-A-algebra of B.
3. It is clear that an element $b \in B$ is integral over A if the minimal polynomial of $b \in B$ over $K = \operatorname{Frac} A$ has coefficients in A. Conversely, suppose b is integral over A. Let L be the field of fractions of B, and let L' be a normal closure of L over K. Then the minimal polynomial f of b over K splits into linear factors over L', and all of its roots are integral over A since they satisfy the same integral dependence relation as b does. The coefficients of f are in K, and they are polynomial expressions in the roots of f, so they are integral over A. Because A is by assumption integrally closed in K, it follows that f has coefficients in A.
4. This is just a computation with polynomials and fractions.

Normalisation of integral schemes

This section is adapted from Exercise 3.8 in Hartshorne's book.

Let X be an integral scheme. Then X is irreducible, so it has a generic point η. The local ring $K = \mathcal{O}_{X,\eta}$ of X at η is a field, called the *function field* of X. Let $K \to L$ be a field extension. We construct the *normalisation*

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array}$$

of X in L as follows. For each open affine subset $U = \operatorname{Spec} A$ of X, we let $\tilde{A}$ be the integral closure of A in L, and we put $\tilde{U} = \operatorname{Spec} \tilde{A}$. If $V = \operatorname{Spec} B$ is another affine open subset of X, we have a commutative diagram

$$\begin{array}{ccccccc} \tilde{U} & \supset & f^{-1}(U \cap V) & \longrightarrow & g^{-1}(U \cap V) & \subset & \tilde{V} \\ & \searrow^{f} & & \searrow \quad \swarrow & & \swarrow_{g} & \\ & U & \supset & U \cap V & \subset & V & \end{array}$$

where f and g are induced by the inclusions $A \subset \tilde{A}$ and $B \subset \tilde{B}$. The map $f^{-1}(U \cap V) \to g^{-1}(U \cap V)$ still needs to be constructed; if we do this in a 'sufficiently canonical' way, then the glueing condition (also called cocycle condition) will automatically be satisfied. Now $U \cap V$ can be covered by open subsets that are simultaneously of the form $\operatorname{Spec}(A_a) \subset U$ and $\operatorname{Spec}(B_b) \subset V$. By the proposition from the previous section, we have canonical isomorphisms

$$(\tilde{A})_a \cong \widetilde{A_a} \cong \widetilde{B_b} \cong (\tilde{B})_b,$$

giving us the required map.

As an application, we note that all non-singular projective curves over a field k can be constructed by normalizing $\mathbf{P}^1_k$ (which has function field $k(t)$, with t transcendental over k) in a finite field extension of $k(t)$. We now show that this extension can always be written as $k(t, x)$ with x separable algebraic over $k(t)$.

Let C be an absolutely irreducible and absolutely reduced curve over k (i.e. C is irreducible and reduced, and remains so when we extend scalars to an algebraic closure $\bar{k}$ of k), and let K be the function field of C. Then K is a finitely generated extension field of k (i.e. K is the field of fractions of a sub-k-algebra of K which is of finite type over k) of transcendence degree 1 over k. Because C is absolutely irreducible and absolutely reduced, the curve $C_{\bar{k}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ obtained by base extension is integral, and its function field is

$$K' = K \otimes_k \bar{k}.$$

Now K' is a finitely generated field extension of $\bar{k}$ of transcendence degree 1, so K' is separably generated over $\bar{k}$ (Hartshorne, Theorem I.4.8A) and hence $\Omega_{K'/\bar{k}}$ is a 1-dimensional K'-vector space (Hartshorne, Theorem II.8.6A). Since

$$\Omega_{K'/\bar{k}} = K' \otimes_K \Omega_{K/k},$$

this implies that $\dim_K \Omega_{K/k} = 1$. Choose $t \in K$ such that $dt \neq 0$. Then the exact sequence

$$K \otimes_{k(t)} \Omega_{k(t)/k} \to \Omega_{K/k} \to \Omega_{K/k(t)} \to 0$$

of K-vector spaces (Hartshorne, Prop. II.8.3A) gives $\Omega_{K/k(t)} = 0$, and this implies that K is separable over $k(t)$ (Hartshorne, Prop. II.8.6A). Since this extension is finitely generated, applying the primitive element theorem shows that $K = k(t, x)$ for some $x \in K$.

Definition. An integral domain is said to be *normal* if it is integrally closed in its field of fractions.

Proposition. *Let A be a Noetherian normal integral domain, $K = \operatorname{Frac} A$, let $K \to L$ be a finite extension, and let B be the integral closure of A in L. Then B is a finitely generated A-module.*

Proof. We give the proof under the assumption that $K \to L$ is separable. Write $L = K[x]/(f)$ using the primitive element theorem, where

$$f = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in K[x]$$

is a monic irreducible polynomial. After multiplying x by a suitable element of A if necessary, we may assume that the f_i are in A. Let B' be the subring $A[x]/(f)$ of L, so that

$$B' = A \cdot 1 \oplus A \cdot \bar{x} \oplus \cdots \oplus A \cdot \bar{x}^{d-1}.$$

We have $B' \subset B$ since $\bar{x}$ is integral over A. To set an upper bound for B, we use a duality: We have the *trace form*

$$\begin{aligned} L \times L &\to K \\ (y, z) &\mapsto \operatorname{tr}(yz), \end{aligned}$$

where $\operatorname{tr} y$ is the trace of the K-linear 'multiplication by y' map from L to itself. The trace form is a symmetric K-bilinear form, giving a *perfect pairing* $L \times L \to K$ (i.e. it gives an identifcation of L with its K-linear dual). To see this, use the fact that the trace is unchanged under extension of the base, and extend scalars to an algebraic closure $\bar{K}$ of K:

$$\begin{array}{ccc} L & \xrightarrow{\operatorname{tr}} & K \\ \downarrow & & \downarrow \\ L \otimes_K \bar{K} & \xrightarrow{\operatorname{tr} \otimes \operatorname{id}} & \bar{K} \end{array}$$

We have an isomorphism of $\bar{K}$-algebras

$$L \otimes_K \bar{K} \cong \bar{K}[x]/(f) \cong \bar{K}^{\deg f} = \bar{K} \times \bar{K} \times \cdots \times \bar{K}$$

by the Chinese Remainder Theorem, and the matrix of the trace form $\bar{K}^{\deg f} \times \bar{K}^{\deg f} \to \bar{K}$ with regard to the standard basis of $\bar{K}^{\deg f}$ is the unit matrix. This means that the trace form is a perfect pairing.

Now we put

$$B^\vee := \{y \in L \mid \operatorname{tr}(yz) \in A \text{ for all } z \in B\}.$$

For $y \in B$, $\operatorname{tr} y$ is integral over A, hence it is in A; from this we see that $B \subset B^\vee$. Furthermore, we define

$$(B')^\vee = \{y \in L \mid \operatorname{tr}(yz) \in A \text{ for all } z \in B'\};$$

then $B' \subset B$ implies $B^\vee \subset (B')^\vee$, so we have inclusions

$$B' \subset B \subset B^\vee \subset (B')^\vee \cong A^d,$$

where the last isomorphism follows from $B' \cong A^d$. As A is Noetherian, B is finitely generated as an A-module (A^n is a finitely generated module over a Noetherian ring, hence it is a Noetherian module, which means that every submodule is finitely generated).

Proposition. *Let A be an integral domain that unique factorisation. Then A is normal.*

Proof. Nobody wanted to see the proof, or wanted to admit that they wanted to see it.

Definition. An integral scheme X is *normal* if and only if A is normal for every open affine subset $U = \operatorname{Spec} A$ of X.

Corollary. *Let k be a field. Then $\mathbf{A}_k^n$ and consequently $\mathbf{P}_k^n$ are normal.*

Proof. The ring $k[x_1, \ldots, x_n]$ is a unique factorisation domain, so its spectrum $\mathbf{A}_k^n$ is normal. Since $\mathbf{P}_k^n$ can be covered by open subsets isomorphic to $\mathbf{A}_k^n$, it is normal as well.

Some steps towards dimension theory

The following lemma does not have as much to do with normal integral domains as its name suggests.

Noether's normalisation lemma. *Let k be a field, and let A be a finitely generated k-algebra which is a domain. Then there exist $d \geq 0$ and $a_1, \ldots, a_d \in A$ such that the following hold:*

1. *$a_1, \ldots, a_d$ are algebraically independent over k, i.e. the map*

$$\begin{aligned} k[x_1, \ldots, x_d] &\to A \\ x_i &\mapsto a_i \end{aligned}$$

   is injective.
2. *A is integral (hence finite) over its subring $k[x_1, \ldots, x_d]$.*

Proof. Choose a finite number of generators $a_1, \ldots, a_n$ of A as a k-algebra, and order them such that $a_1, \ldots, a_d$ are algebraically independent over k and $a_{d+1}, \ldots, a_n$ are algebraic over $k(a_1, \ldots, a_d)$. If $d = n$, then we are done, so assume $d < n$. Choose a polynomial $P \in k[x_1, \ldots, x_n]$ of degree $m > 0$ as a polynomial in x_n, such that $P(a_1, \ldots, a_n) = 0$. Write

$$P = P_m x_n^m + \cdots + P_1 x_n + P_0$$

with $P_0, P_1, \ldots, P_m \in k[x_1, \ldots, x_{n-1}]$. We may assume that $P_m(a_1, \ldots, a_{n-1}) \neq 0$. We claim that there exist $e_i \geq 0\,(1 \leq i \leq n-1)$ such that $P(x_1 + x_n^{e_1}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$ is monic as a polynomial in x_n, up to multiplication by a constant in k. Note that replacing $a_1, \ldots, a_n$ by

$$\begin{aligned} a'_1 &= a_1 + a_n^{e_1} \\ &\vdots \\ a'_{n-1} &= a_{n-1} + a_n^{e_{n-1}} \\ a'_n &= a_n \end{aligned}$$

gives a system of generators of A satisfying the same conditions as the a_i. To prove the claim, write

$$P = \sum_{i \in S} p_i x_1^{i_1} \cdots x_n^{i_n} \quad (p_i \in k)$$

where i runs over a finite set S of n-tuples of non-negative integers. Then the highest power of x_n occurring in

$$(x_1 + x_n^{e_1})^{i_1} \cdots (x_{n-1} + x_n^{e_{n-1}})^{i_{n-1}} x_n^{i_n}$$

equals $e_1 i_1 + \cdots + e_{n-1} i_{n-1} + i_n$. Take the e_i such that these numbers are all distinct for the $i \in S$. (For example, take $b > \max_{i \in S, 1 \leq i \leq n} i_j$, and let $e_j := b^j$.) Then after expanding all the products in

$$P(x_1 + x_n^{e_1}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = \sum_{i \in S} p_i (x_1 + x_n^{e_1})^{i_1} \cdots (x_{n-1} + x_n^{e_{n-1}})^{i_{n-1}} x_n^{i_n},$$

there is no cancellation of the monomials $p_i x_n^{e_1 i_1 + \cdots + e_{n-1} i_{n-1} + i_n}$, which proves the claim. After replacing the a_i by the a'_i, we may therefore assume that the extension $k[a_1, \ldots, a_{n-1}] \subset k[a_1, \ldots, a_n]$ is integral. Now continue until $n = d$.

Proposition. *Let $\phi\colon A \to B$ be integral, and let $f := \operatorname{Spec} \phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Then f is a closed morphism.*

Proof. Let $I \subset B$ be an ideal; we assume without loss of generality that I is radical. We must show that the set

$$f(V(I)) = \{\phi^{-1}\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} B, \mathfrak{q} \supset I\} \subset \operatorname{Spec} A$$

is closed. In other words, we must show that if $\mathfrak{p} \in \operatorname{Spec} A$ contains the ideal

$$\bigcap_{\mathfrak{q} \supset I} \phi^{-1}\mathfrak{q} = \phi^{-1}\sqrt{I} = \phi^{-1}I,$$

then $\mathfrak{p}$ is itself of the form $\phi^{-1}\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \supset I$. We have a commutative diagram

$$\begin{array}{ccc} \phi^{-1}I & & I \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ \bar{A} & \xrightarrow{\bar{\phi}} & \bar{B} \end{array}$$

where $\bar{\phi}$ is injective. After replacing A, B, ϕ by $\bar{A}, \bar{B}, \bar{\phi}$ we may assume that $I = 0$ and $\phi^{-1}I = 0$. Now we need to prove that $\mathfrak{p}$ is in the image of f for every $\mathfrak{p} \in \operatorname{Spec} A$ which contains the ideal

$$\bigcap_{\mathfrak{q} \in \operatorname{Spec} B} \phi^{-1}\mathfrak{q} = \phi^{-1}\{0\} = (0),$$

where we used that B is reduced (we divided out by a radical ideal) and that ϕ is injective. This means that we must show that f is surjective. If $A = 0$, there is nothing to prove. Otherwise, let $\mathfrak{p} \in \operatorname{Spec} A$. Then there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & A_{\mathfrak{p}} \otimes_A B \end{array}$$

or, in terms of the spectra of these rings,

$$\begin{array}{ccc} \operatorname{Spec}(A_{\mathfrak{p}} \otimes_A B) & \longrightarrow & \operatorname{Spec} A_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \xrightarrow{f} & \operatorname{Spec} A. \end{array}$$

From this we see that it is sufficient to demonstrate the existence of a point $\mathfrak{q} \in \operatorname{Spec}(A_{\mathfrak{p}} \otimes_A B)$ that maps to $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Spec} A_{\mathfrak{p}}$. Namely, the image of such $\mathfrak{q}$ in $\operatorname{Spec} A$ equals $\mathfrak{p}$ via the top and right arrows, so by commutativity of the diagram the left arrow maps $\mathfrak{q}$ to an inverse of $\mathfrak{p}$ under f. Thus we replace A by $A_{\mathfrak{p}}$ and B by $A_{\mathfrak{p}} \otimes_A B$. Then A is local, and we must show that the maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$ of A is in the image of f. As ϕ is injective, $B \neq 0$. Take a maximal ideal $\mathfrak{m}'$ of B. Then we have an integral morphism

$$A/\phi^{-1}\mathfrak{m}' \hookrightarrow B/\mathfrak{m}'.$$

The next lemma shows that $A/\phi^{-1}\mathfrak{m}'$ is a field, thereby proving that $\phi^{-1}\mathfrak{m}' = \mathfrak{m}$.

Lemma. *Let $A \hookrightarrow B$ be a homomorphism of integral domains such that B is integral over A. Then A is a field if and only if B is a field.*

Proof. Suppose A is a field, and let $x \in B - \{0\}$. Then multiplication by x is injective on the finite-dimensional A-vector space $A[x] \subset B$, hence it is also surjective. This implies that x is invertible. Conversely, suppose B is a field, and let $x \in A - \{0\}$. Then $x^{-1} \in B$ is integral over A. Let

$$(x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \cdots + a_0 = 0$$

be an integral dependence relation for x^{-1}. Then the identity

$$1 + a_{n-1}x + \cdots + a_0x^n = 0$$

in A shows that x is invertible in A.