Algebraic Geometry: Exercises May 23

1. Let k be a field, $d \geq 0$ an integer and let V be a k-vector space of dimension d. Fix a k-linear endomorphism F of V and consider V as a module over the polynomial ring $k[x]$ via $xv = Fv$ for all $v \in V$. Let $X = \operatorname{Spec} k[x]$; the coherent $\mathcal{O}_X$-module associated to the $k[x]$-module V we will denote by $\mathcal{V}$. Let P_F be the characteristic polynomial of F.

   a) By using the Cayley-Hamilton theorem, show that the support of $\mathcal{V}$ is contained in the closed subset $V(P_F)$. Conclude that there is a decomposition of coherent $\mathcal{O}_X$-modules

   $$\mathcal{V} = \bigoplus_{s \in V(P_F)} \mathcal{V}_s,$$

   where each $\mathcal{V}_s$ is a skyscraper sheaf with support in s.

   b) Suppose $s \in V(P_F)$ and let φ be the monic irreducible factor of P_F to which s corresponds. Show that $V_s := \Gamma(X, \mathcal{V}_s) \subset V$ is the subspace consisting of those $v \in V$ for which $\varphi^r(F)v = 0$ for some $r \geq 0$.

   c) Suppose P_F has all its zeroes in k. Let D be the endomorphism of V which acts on V_s as $x(s)$ (here $x(s)$ is the image of x in the residue field of X at s). Show that $N := F - D$ is nilpotent and conclude that $F = D + N$ is the Jordan decomposition of F.

2. Let k be a field. Define $X = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$.

   a) Consider the ideal sheaf $\mathcal{I}$ of the diagonal in X. If we restrict $\mathcal{I}$ to the diagonal, we obtain a line bundle. Determine its degree. [*Hint*: show that this line bundle trivialises on the two standard open affines U_0 and U_1 of $\mathbf{P}^1_k$ and determine the factor that relates these two trivialisations on $U_0 \cap U_1$.]

   b) Show that the Picard group $\operatorname{Pic} X$ is the free abelian group generated by the classes of a horizontal and vertical line in X.

   c) Describe the intersection pairing on $\operatorname{Pic} X$. Check that your answer is consistent with part (a).