EXERCISES *Introduction to Dynamical Systems '18-'19* **: Series IV**

Date: 10-12-'18.

Exercise 1. Consider the 2-dimensional system,

$$\left\{ \begin{array}{rcl} \dot{x} & = & 1 + y - x^2 - y^2, \\ \dot{y} & = & 1 - x - x^2 - y^2. \end{array} \right. \tag{1}$$

a) Determine the critical points of (1) and their local character; show that the flow generated by (1) is symmetric with respect to the line $\{x + y = 0\}$.

b) Show that system (1) is integrable by constructing an integral $K(x, y)$.
Hint: Introduce new variables $u = x - y$ and $v = x + y$ that exploit the symmetry found in (a), write (1) as system in u and v, and determine an integral $\tilde{K}(u, v)$ for this system by introducing $w = v^2$ and solving the equation for $\frac{dw}{du}$.

c) Sketch the phase portrait associated to (1) and conclude that system (1) has a homoclinic solution.

Now consider a more general version of (1),

$$\left\{ \begin{array}{rcl} \dot{x} & = & 1 + y - x^2 - y^2 + h(x, y), \\ \dot{y} & = & 1 - x - x^2 - y^2 + h(x, y), \end{array} \right. \quad \text{with } h : \mathbb{R}^2 \to \mathbb{R}, h(0, 0) = 0, \text{ `sufficiently smooth'}. \tag{2}$$

d) Take $h(x, y) = \varepsilon(x + y)$ with $0 < \varepsilon \ll 1$: show that the homoclinic orbit of system (1) does not survive the perturbation of (2).
Hint: Determine $\dot{K}$ or $\dot{\tilde{K}}$.

e) Take $h(x, y) = \alpha(x - y)^3$, $\alpha \in \mathbb{R}$: show that system (2) is integrable by deriving an integral $K_\alpha(x, y)$ (or $\tilde{K}_\alpha(u, v)$) such that $K_0(x, y) = K(x, y)$, with $K(x, y)$ as in (b).

f) Take $h(x, y)$ as in (e) with $\alpha = \varepsilon$ and $0 < \varepsilon \ll 1$: show that system (1) has a homoclinic orbit and give a sketch of the phase portrait.

g) Take $h(x, y)$ as in (e) with $\alpha = A \gg 1$: show that system (1) does not have a homoclinic orbit and give a sketch of the phase portrait.

Exercise 2. Consider for $\beta \in \mathbb{R}$ the 2-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} & = & \beta xy - x^3 + y^2, \\ \dot{y} & = & -y + x^2 + xy. \end{array} \right. \tag{3}$$

Determine the center manifold $W^c((0, 0))$ up to and including terms of order three. Determine the (approximate) flow on $W^c((0, 0))$ near $(0, 0)$. Determine the stability of $(0, 0)$ for all $\beta \in \mathbb{R}$.

Exercise 3. Consider for $\gamma \in \mathbb{R}$ the 2-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x^3, \\ \dot{y} & = & -y + x^2 + \gamma x^4. \end{array} \right. \tag{4}$$

a) Take $\gamma = -2$. Determine the stable manifold $W^s((0, 0))$ and the center manifold(s) $W^c((0, 0))$ *explicitly* by solving the appropriate equations. Is $W^c((0, 0))$ uniquely determined? Is it analytic? If so, give an expression of $W^c((0, 0))$ in terms of a power series.

b) Sketch, for γ still equal to -2, the phase portrait, including the manifolds $W^s((0, 0))$ and $W^c((0, 0))$.

c) Consider the general case $\gamma \in \mathbb{R}$. What can you say about $W^c((0, 0))$? Is it unique? Is it analytic? Can you give an explicit expression, or a power series expansion?