EXERCISES *Introduction to Dynamical Systems '18-'19* **: Series III**

Date: 12-11-'18.

Exercise 1.

A. Exercise 12, page 156 book ($\longleftrightarrow$ exercise 12, page 162 first edition book).

B. Exercise 13, page 156 book ($\longleftrightarrow$ exercise 13, page 163 first edition book).

C. Let $\phi(t;a)$ define a smooth n-dimensional flow in $\mathbb{R}^n$ and let A be a subset of $\mathbb{R}^n$. The omega limit set of the set A is defined as the union of all $\omega(a)$ over all $a \in A$, i.e. $\omega(A) = \cup_{a\in A}\omega(a)$. Since for a given a_0, $\omega(a_0)$ also is a subset of $\mathbb{R}^n$, one can thus define $\omega(\omega(a_0))$. Is it true that $\omega(\omega(a_0)) = \omega(a_0)$? If you agree, give a proof; if not, give a counter example.

Exercise 2. Consider the *complex* differential equation,

$$\dot{z}_0 = z_0 - (1+i)z_0|z_0|^2 \quad \text{with} \quad z_0 : \mathbb{R} \to \mathbb{C}. \tag{1}$$

a-i) Transform (1) into a real two-dimensional system by introducing $(x_0(t), y_0(t)) : \mathbb{R} \to \mathbb{R}^2$ through $z_0 = x_0 + iy_0$ and deriving an equation for $(\dot{x}_0, \dot{y}_0)$.

a-ii) Determine the periodic solution $\phi_r(t) = (x_r(t), y_r(t))$ of the real system found in (a-i): introduce polar coordinates $(r_0(t), \theta_0(t))$, sketch the phase portrait and determine an explicit expression for $\phi_r(t)$.

a-iii) Show that $\phi_r(t)$ is asymptotically stable (<u>*without*</u> using Floquet theory and/or the subsequent exercises).

b-i) Determine the (real) linearized system about $\phi_r(t)$, i.e. introduce $(\xi_0(t), \eta_0(t))$ by $x_0 = x_r + \xi_0$, $y_0 = y_r + \eta_0$ and linearize.

b-ii) Show explicitly that $\dot{\phi}_r$ solves the (ξ_0, η_0)-system.

c) Determine the (two) Floquet exponents of the (ξ_0, η_0)-system.

The Floquet exponents can also be determined by directly working with complex system (1).

d-i) Determine the solution $\phi_c(t) : \mathbb{R} \to \mathbb{C}$ of (1) that corresponds to $\phi_r(t) \in \mathbb{R}^2$ as found in (a-ii).

d-ii) Linearize about ϕ_c by setting $z_0 = \phi_c + \zeta_0$.

d-iii) Introduce $w_0(t)$ by $\zeta_0 = e^{-it}w_0$. Show that w_0 is a solution of,

$$\dot{w}_0 = -(1+i)w_0 - (1+i)w_0^* \quad \text{with} \quad w_0^* = \text{ the complex geconjugate of } w_0, \tag{2}$$

a linear (complex) equation <u>*with constant coefficients*</u>.

d-iv) Use (2) to determine the stability of $\phi_c(t)$.

d-v) <u>*Explain in detail*</u> why the eigenvalues associated to (2) are equal to the Floquet exponents found in c).

Now consider the following extension of (1),

$$\left\{ \begin{array}{rcl} \dot{z}_0 & = & z_0 - (1+i)\left[z_0|z_0|^2 + 2z_1^2 z_0^* + 4z_0|z_1|^2\right], \\ \dot{z}_1 & = & \alpha z_1 - (1+i)\left[3z_1|z_1|^2 + z_0^2 z_1^* + 2z_1|z_0|^2\right], \end{array} \right. \tag{3}$$

with $(z_0(t), z_1(t)) : \mathbb{R} \to \mathbb{C}^2$ and $\alpha \in \mathbb{R}$ a parameter.

e-i) Show that $\phi_{c,2}(t) = (\phi_c(t), 0) \in \mathbb{C}^2$, with $\phi_c(t)$ as (d-i), is a periodic solution of (3).

e-ii) Determine the Floquet exponents associated to the linearization of (3) about $\phi_{c,2}(t)$. Determine α_0 such that $\phi_{c,2}(t)$ is stable for $\alpha < \alpha_0$.