EXERCISES *Introduction to Dynamical Systems '18-'19* **: Series II**

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Exercise 1. Consider the n-dimensional system,

$$\dot{X} = AX + F(X), \tag{1}$$

with A an $n \times n$ constant coefficients matrix and $F(X)$ a nonlinear expression that satisfies $\|F(X)\| \leq C\|X\|^2$ as $\|X\| \to 0$, for some $C > 0$. Recall that we cannot draw any conclusions on the stability of the critical point $\bar{X} = 0$ of (1) if A has eigenvalues λ with $\mathrm{Re}(\lambda) = 0$. Take $n = 2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and write X as (x, y). Show that the critical point $\bar{X} = (\bar{x}, \bar{y}) = (0, 0)$ of (1) is,

a) unstable for $F(X) \equiv 0$.

a) stable for $F(X) = F(x, y) = \begin{pmatrix} 0 \\ -x^3 \end{pmatrix}$.
Hint: Write (1) as equation in $\ddot{x}$ and determine a *first integral* $V(x, y)$ of this equation, i.e. a function $V(x, y)$ such that $\frac{d}{dt}V(x, y) = 0$ for solutions of the equation. Use this $V(x, y)$ as Lyapunov function.

c) asymptotically stable for $F(X) = F(x, y) = \begin{pmatrix} -x^3 \\ -x^3 - y^3 \end{pmatrix}$.
Hint: Use the Lyapunov function constructed in b).

Exercise 2. Consider the linear system,

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + B(t)x, \quad x \in \mathbb{R}^n, n \geq 1, \\ x(0) & = & x_0, \end{array} \right. \tag{2}$$

with A a $n \times n$ matrix (with constant coefficients) and $B(t)$ a $n \times n$ matrix with coefficients that depend continuously on time t; define $\phi(t; x_0)$ as the solution of (2). Moreover, it is given that there is a (positive) continuous function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|B(t)x\| < C(t)\|x\|$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.
(*Note*: For given $B(t)$, functions $C(t)$ like defined here always exist; the infimum over all possible $C(t)$'s defines the norm $\|B(t)\|$ of the matrix $B(t)$.)

a) Explain why $\tilde{\phi}(t; x_0) = e^{At + \int_0^t B(s)ds} x_0$ is, in general, not a solution of (2).

Assume first that,

(Ai) all eigenvalues λ_j of A satisfy $\mathrm{Re}\lambda_j < 0$;
(Bi) $\lim_{t\to\infty} C(t) = 0$.

b) Show that it follows from assumptions *(Ai)* and *(Bi)* that $\lim_{t\to\infty} \phi(t; x_0) \to 0$ for all $x_0 \in \mathbb{R}^n$ and that the critical point $x^* = 0$ of (2) is asymptotically stable (use the definition!).
Hint. Apply the arguments of the proof of Theorem 4.17 (on page 117 in the book) with $g(y)$ replaced by $B(t)x$. [For the first edition of the book: page 121-122 for the proof of Theorem 4.6.]

Now assume that,

(Aii) all eigenvalues λ_j of A satisfy $\mathrm{Re}\lambda_j \leq 0$;
(Aiii) the eigenvalues λ_j of A with $\mathrm{Re}(\lambda_j) = 0$ do not coincide (i.e. these eigenvalues have – algebraic and geometric – multiplicity 1);
(Bii) $C(t)$ is integrable: $\int_0^\infty C(t)\, dt = D < \infty$.

c) Show that it follows from assumptions *(Aii)*, *(Aiii)*, and *(Bii)* that the critical point $x^* = 0$ of (2) is stable. Use the definition.
Note: You may use that *(Aii)* and *(Aiii)* together imply that there is a $K > 0$ such that $|e^{At}x_0| < K|x_0|$ for all $x_0 \in \mathbb{R}^n$ and $t \geq 0$.

d) Additional assumption *(Aiii)* is not that strong and assumption *(Bii)* is only slightly stronger than *(Bi)*: can you establish the result of c) under (slightly) weaker conditions, i.e. without *(Aiii)* and/or with *(Bi)* instead of *(Bii)*? Explain!