EXERCISES *Introduction to Dynamical Systems '18-'19* **: Series I**

Date: 17-09-'18.

Exercise 1.

Ia) Consider $g : [0, \infty) \to \mathbb{R}$ given by $g(t) = \frac{\cos t^2}{t+1}$: show that $\lim_{t\to\infty} g(t)$ exists, while $\lim_{t\to\infty} \dot{g}(t) (= \frac{dg}{dt}(t))$ does not.

Ib) Consider the autonomous ODE $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, with initial condition $x(0) = x_0$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ (at least) continuously differentiable. Let $\phi(t; x_0)$ be a solution such that $\lim_{t\to\infty} \phi(t; x_0) = a$ for a certain $a \in \mathbb{R}^n$. Prove that a must be a critical point of the system.
Warning: Be aware of functions that behave like $g(t)$ in (Ia).

Ic) Explain why $g(t)$ of (Ia) cannot be a solution of a system as described in (Ib) (with $n = 1$).

Id) Can $g(t)$ be a solution of a non-autonomous (smooth) system $\dot{x} = f(x, t)$? Explain! Is the situation different from that of (Ib) and (Ic)? Why?

IIa) Consider the flow $\phi(t; x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Prove 'continuous dependence on initial conditions' for flows, i.e. let the initial condition $x_0 \in \mathbb{R}^n$ be given and show that for all $T > 0$ and $\delta > 0$ there is an $\varepsilon > 0$ such that $\|\phi(T; x_0) - \phi(T; \tilde{x}_0)\| < \delta$ for all $\tilde{x}_0$ with $\|x_0 - \tilde{x}_0\| < \varepsilon$.
Note: This flow is not necessarily associated to an ODE.

IIb) Can you improve this result to a version that is uniform in t, i.e. to: for all $T > 0$ and $\delta > 0$ there is an $\varepsilon > 0$ such that $\|\phi(t; x_0) - \phi(t; \tilde{x}_0)\| < \delta$ for all $\tilde{x}_0$ with $\|x_0 - \tilde{x}_0\| < \varepsilon$ and all $t \in [0, T]$?

IIc) Now establish the equivalent of (Ib) for flows: assume that $\phi(t; x_0) \to a \in \mathbb{R}^n$ as $t \to \infty$ (and x_0 fixed), show that $\phi(t; a) \equiv a$ i.e. that $x = a$ is a fixed point of the flow ϕ.
Hint: Assume that $\phi(t; a) \not\equiv a$ so that $\|\phi(T; a) - a\| = \sigma > 0$ for some σ and T and apply (IIa).

Exercise 2.

Consider for $n \in \mathbb{N}$, $n \geq 1$, the planar – thus 2-dimensional – system,

$$\ddot{x} + x^n = 0 \ \text{ or } \ \begin{cases} \dot{x} &= y, \\ \dot{y} &= -x^n. \end{cases} \tag{1}$$

This system defines a flow $\phi(t; x, y) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, parameterized by time t (a priori $\in \mathbb{R}$); solutions of (1) with initial condition (x_0, y_0) are denoted by $(x(t; x_0, y_0), y(t; x_0, y_0))$, orbits of the flow are denoted by $\Gamma(x_0, y_0)$.

a) Explain why neither Theorem 4.6 nor Theorem 4.8 (in section 4.3 of the book) can be applied to system (1) on the unbounded domain $\mathbb{R}^2$ to conclude that solutions of (1) exist for all $t \in \mathbb{R}$, i.e. to conclude that (1) generates a complete flow.
Note: Theorems 4.6 and 4.8 in the revised edition of the book correspond to Theorems 4.3 and 4.5 in the original version of the book.

b) System (1) is integrable with Hamiltonian H, i.e. orbits of (1) are given as level sets of a function $H(x, y)$. Determine $H(x, y)$.
Hint: Apply the standard procedure: multiply $\ddot{x} + x^n = 0$ by $\dot{x} (= y)$ and integrate over time.

c) Consider n odd: show that (1) generates a complete flow, i.e. show that all solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) exist for all $t \in \mathbb{R}$.
Hint: Give a sketch of the phase portrait of (1) – or equivalently, of the orbits of the flow generated by (1) – and conclude that all orbits $\Gamma(x_0, y_0)$ – with $(x_0, y_0) \neq (0, 0)$ – are closed. What does this mean?

d) Consider n even: define $\mathcal{S}_+ \subset \mathbb{R}^2$ as the set of all initial conditions such that the limit of $t \to \infty$ for solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) with $(x_0, y_0) \in \mathcal{S}_+$ exists (note that $\mathcal{S}_+$ coincides with the orbits $\Gamma(x_0, y_0)$ with $(x_0, y_0) \in \mathcal{S}_+$). Determine $\mathcal{S}_+$ explicitly. Do the same for $\mathcal{S}_- \subset \mathbb{R}^2$ – the set of all initial conditions such that the limit $t \to -\infty$ of solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) exists.

e) (n even) Give a sketch of the phase portrait of (1), including $\mathcal{S}_+$ and $\mathcal{S}_-$.

f) (n even) The solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) with $(x_0, y_0) \in \mathcal{S}_-$ can be determined explicitly. Show that these solutions blow up in finite (positive) time.

g) (n even) Consider the solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) with $(x_0, y_0) \notin \mathcal{S}_+$. Show that for all $K > 0$ there is a T such $x(t; x_0, y_0) < -K$ and $y(t; x_0, y_0) < -K$ for all $t \geq T$.

h) (n even) Show that – except for the trivial, critical point, solution $(x(t; 0, 0), y(t; 0, 0)) \equiv (0, 0)$ – there are no solutions $(x(t; x_0, y_0), y(t; x_0, y_0))$ of (1) that exist for all $t \in \mathbb{R}$.