EXERCISES *Introduction to Dynamical Systems '17-'18* **: Series IV**

Date: 11-12-'17.

Exercise 1. Consider the integrable problem,

$$\ddot{x} + x - x^3 + C = 0, \quad \text{with } C \in \mathbb{R}. \tag{1}$$

a) Determine the set $\mathcal{I}_{\text{hom}}$ such that system (1) has a homoclinic orbit if $C \in \mathcal{I}_{\text{hom}}$. For which $C = C_{\text{het}}$ does (1) have heteroclinic orbits? Give sketches of the phase portraits of (1) for C such that $C_{\text{het}} > C \in \mathcal{I}_{\text{hom}}$, $C = C_{\text{het}}$, $C_{\text{het}} < C \in \mathcal{I}_{\text{hom}}$, and $C_{\text{het}} \neq C \notin \mathcal{I}_{\text{hom}}$.

b) Now consider a more general version of (1),

$$\ddot{x} + x + Ax^2 + Bx^3 + C = 0, \quad \text{with } A, B, C \in \mathbb{R}. \tag{2}$$

For which A, B, C does (2) have heteroclinic orbits?

Exercise 2.
Let $V : \mathbb{R}^n \to \mathbb{R}$ be given as a smooth (at least C^2) map; consider the associated *gradient flow*,

$$\dot{x} = -\nabla V(x), \quad \text{or } \dot{x}_i = -\frac{\partial V}{\partial x_i}(x), \ i = 1, ..., n. \tag{3}$$

Prove that system (3) cannot have a homoclinic solution.
Hint: Consider (and use) $\dot{V}$.

Exercise 3.

Consider the two-dimensional system,

$$\left\{ \begin{array}{rcrcrcr} \dot{x} & = & xy & - & x^3 & - & xy^2, \\ \dot{y} & = & -\ \ y & + & ax^2 & + & bx^2y, \end{array} \right. \quad \text{with parameters } a, b \in \mathbb{R}. \tag{4}$$

Determine for all $a, b \in \mathbb{R}$ the stability of the critical point $(0,0)$ of (4).

Exercise 4.

Consider the system,

$$\left\{ \begin{array}{rcrcr} \dot{x} & = & & - & x^3, \\ \dot{y} & = & -\ \ y & + & x^2. \end{array} \right. \tag{5}$$

The vector field, i.e. the righthand side of system (5), is – of course – analytic in x and y.

a) Show that the center manifold $W^c((0,0))$ associated to the critical point $(0,0)$ of (5) is **not** analytic. *Hint:* Let $W^c((0,0))$ be given by $y = h(x)$ and assume that $h(x)$ is analytic, i.e. assume that $h(x)$ can locally be represented by a (full) power series, $h(x) = \sum_{n=1}^{\infty} c_n x^n$; determine all coefficients c_n.

b) Is $W^c((0,0))$ determined uniquely? Explain!

Exercise 5.
Consider the 3-dimensional system,

$$\left\{ \begin{array}{rcl} \dot{x} & = & -y + xz - x^4, \\ \dot{y} & = & x + yz + xyz, \\ \dot{z} & = & -z - x^2 - y^2 + z^2 + \sin x^3. \end{array} \right. \tag{6}$$

Determine the stability of the critical point $(0,0,0)$.
Hint: Determine the flow on the center manifold and use polar coordinates.