EXERCISES *Introduction to Dynamical Systems '17-'18* **: Series III**

Date: 13-11-'17.

Exercise 1. Consider the n-dimensional linear equation with T-periodic coefficients ($T > 0$),

$$\dot{x} = A(t)x,\ x \in \mathbb{R}^n \text{ and } A(t+T) = A(t). \tag{1}$$

It follows from Flocquet's Theorem (Theorem 2.13 in the book) that any fundamental matrix solution $\Phi(t)$ of (1) can be written as $\Phi(t) = P(t)e^{Bt}$ with $P(t)$ a T-periodic $n \times n$ matrix and B an $n \times n$ matrix with constant coefficients.

(a) Consider the case $n = 1$ with $A(t) = f(t)$, a T-periodic function.

(i) Determine B.
Hint: Introduce and use $\bar{f} = \frac{1}{T}\int_0^T f(s)\,ds$, the average of $f(t)$.
Give necessary and sufficient conditions on $f(t)$ such that

(ii) every solution of (1) remains bounded, both for $t \to \infty$ as well as for $t \to -\infty$;

(iii) every solution of (1) is T-periodic.

(b) Now consider the case $n = 2$ with $A(t) = g(t)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $g(t)$ a T-periodic function and $a, b, c, d \in \mathbb{R}$.

(i) Determine B.
Give necessary and sufficient conditions on $A(t)$ such that

(ii) every solution of (1) remains bounded, both for $t \to \infty$ as well as for $t \to -\infty$;

(iii) every solution of (1) is T-periodic.

(c) Based on (b) one might conjecture that solutions of (1) with a matrix $A(t)$ for which not only $\text{Tr}A(t) \equiv 0$ but also all coefficients have average 0 will certainly remain bounded for all t. Consider the example,

$$A(t) = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

(i.e. again $n = 2$). Is the conjecture correct?
Hint: Show that in this case (1) is equivalent to the complex equation $\dot{z} = e^{it}\bar{z}$ and solve this equation.

Exercise 2. Consider the 2-dimensional system,

$$\begin{cases} \dot{x} &= 1 + y - x^2 - y^2, \\ \dot{y} &= 1 - x - x^2 - y^2. \end{cases} \tag{2}$$

a) Determine the critical points of (2). What can you say about their stability?
Note. We will come back to the issue of stability in (f), do not (yet) use (f) to answer (a).

b) Show that the flow generated by (2) is symmetric with respect to the line $\{x + y = 0\}$.

c) Find an explicit periodic solution $\phi(t)$ of (2).
Hint: Transform (2) into polar coordinates form.

d) Derive the linearization of (2) about the periodic solution $\phi(t)$.

e) Determine both Flocquet exponents of the linearized (time-periodic) system obtained in (d). Can you draw a conclusion about the stability of $\phi(t)$?

f) Sketch the phase portrait associated to (2). What can you now say about the stability of the critical points?
Hint: Introduce new variables $u = x - y$ and $v = x + y$ that exploit the symmetry found in (c), write (2) as system in u and v, and find an integral for this system.
Extra hint: Introduce $w = v^2$ and solve the equation for $\frac{dw}{du}$ to obtain the first integral.

g) Show that $\phi(t)$ is stable (but not asymptotically stable).
Hint: What can you say about the Poincaré map?

h) Assume we perturb system (2) with a small term $\varepsilon\vec{h}(x, y) = \varepsilon(h_1(x, y), h_2(x, y))$, i.e. consider,

$$\begin{cases} \dot{x} &= 1 + y - x^2 - y^2 + \varepsilon h_1(x, y), \\ \dot{y} &= 1 - x - x^2 - y^2 + \varepsilon h_2(x, y), \end{cases} \tag{3}$$

with $0 < \varepsilon \ll 1$. Give an *explicit* example of a perturbation $\varepsilon\vec{h}(x, y)$ that preserves the periodic orbit $\phi(t)$ and that makes it asymptotically stable. Establish the (asymptotic) stability by computing the (perturbed) Floquet exponents.
Remark: To do this last part (h), it is not necessary to first have done (f) and (g).