EXERCISES *Introduction to Dynamical Systems '17-'18*: **Series II**

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Exercise 1. Consider the second order equation,

$$\ddot{x} + \alpha(t)\dot{x} + \beta(t)x = 0, \tag{1}$$

or, equivalently as two-dimensional system,

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -\beta(t) & -\alpha(t) \end{pmatrix} \vec{x} \text{ with } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \tag{2}$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are continuous differentiable functions of t.

a-i) Consider $\alpha(t) \equiv 0$ and $\beta(t)$ such that $\lim_{t\to\infty} \beta(t) = 1$. Thus, (1) approaches $\ddot{x} + x = 0$ and we may expect the critical point $(0,0)$ of (2) to be stable. To study this we employ the method of 'variation of constants' and introduce $X(t)$ and $Y(t)$ by setting $x(t) = X(t)\cos t + Y(t)\sin t$. Show that the evolution of $(X(t), Y(t))$ is determined by

$$\left\{ \begin{array}{lcl} \dot{X} & = & (\beta(t) - 1)\left[X \cos t \sin t + Y \sin^2 t\right], \\ \dot{Y} & = & -(\beta(t) - 1)\left[X \cos^2 t + Y \cos t \sin t\right], \end{array} \right. \tag{3}$$

Hint. $\dot{x} = \dot{X}\cos t + \dot{Y}\sin t - X\sin t + Y\cos t = -X\sin t + Y\cos t$ by assuming that X, Y satisfy $\dot{X}\cos t + \dot{Y}\sin t = 0$; the second equation for $\dot{X}, \dot{Y}$ follows by plugging $\ddot{x}$ into (1).

a-ii) Introduce $R(t) = X^2(t) + Y^2(t)$ and show that $R(t) = \|\vec{x}(t)\|^2$ with $\vec{x}(t)$ as in (2). Use (3) to show that,

$$\dot{R} \leq 4|\beta(t) - 1|R. \tag{4}$$

a-iii) Use (4) to prove – directly by the definition of stability – that if $\int_0^\infty |\beta(t) - 1|\, ds < \infty$ the critical point $(0,0)$ of (2) indeed is stable.
Comment. This stability condition is sufficient, however, it is not necessary. On the other hand, Fatou's conjecture that $(0,0)$ is stable in (2) for any (smooth) $\beta(t)$ with $\lim_{t\to\infty} \beta(t) = 1$ has also be shown to be incorrect.

b) Now consider $\alpha(t) \equiv \bar{\alpha} > 0$ and $\beta(t)$ such that $\lim_{t\to\infty} \beta(t) = \bar{\beta} > 0$ with $\bar{\alpha}, \bar{\beta}$ such that $\bar{\beta} - \frac{1}{4}\bar{\alpha}^2 > 0$. Show that $(0,0)$ is *asymptotically* stable stable as solution of (2).
Hint. Introduce $z(t)$ by $x(t) = z(t)e^{-\frac{1}{2}\bar{\alpha}t}$, ω by $\omega^2 = \bar{\beta} - \frac{1}{4}\bar{\alpha}^2$ and τ by $\tau = \omega t$. Derive the differential equation for z *as function of* τ and follow the procedure of (a-i,ii,iii).

c) Consider $\alpha(t), \beta(t)$ such that $\lim_{t\to\infty} \alpha(t) = \bar{\alpha} \geq 0$ and $\lim_{t\to\infty} \beta(t) = \bar{\beta} \geq 0$ with (again) $\bar{\beta} - \frac{1}{4}\bar{\alpha}^2 > 0$. *Assuming that* $\lim_{t\to\infty} \dot{\alpha}(t) = 0$, show that $(0,0)$ is *asymptotically* stable if $\bar{\alpha} > 0$ and obtain a condition similar to the one in (a-iii) that establishes the stability – but not necessarily asymptotical stability – of $(0,0)$ if $\bar{\alpha} = 0$.
Hint. Introduce $z(t)$ by the Liouville transform $x(t) = z(t)e^{-\frac{1}{2}\int_0^t \alpha(s)\, ds}$ and follow (a) and (b).

d) Consider as in (c) $\alpha(t), \beta(t)$ with $\lim_{t\to\infty} \alpha(t) = \bar{\alpha} \geq 0, \lim_{t\to\infty} \beta(t) = \bar{\beta} \geq 0$ and $\bar{\beta} - \frac{1}{4}\bar{\alpha}^2 > 0$, but now for functions $\alpha(t)$ for which $\dot{\alpha}(t)$ does not converge as $t \to \infty$. What goes wrong? Formulate a condition on $\dot{\alpha}(t)$ that guarantees asymptotic stability of $(0,0)$ (and provide a proof). Can you 'optimize' the condition, i.e. make it as weak and simple as possible (within the present approach)?
Note. A function $f(t)$ that converges as $t \to \infty$ does not necessarily have a converging derivative $\dot{f}(t)$ – consider for instance $f_n(t) = \frac{1}{t}\sin t^n$ for $n = 1, 2, 3$.

Exercise 2. Consider for $p, q \in \mathbb{R}$ the nonlinear planar system,

$$(p - q\dot{x}^2)\ddot{x} = f(x), \tag{5}$$

with $f(0) = 0$ and $f : \mathbb{R} \to \mathbb{R}$ analytic as function of x in a neighborhood of 0 – i.e. $f(x)$ has a converging Taylor series expansion around $x = 0$. As in exercise 1, (5) can be interpreted as a two-dimensional system in $\vec{x} = (x_1, x_2) = (x, \dot{x})$. For $(p,q) \neq (0,0)$, $(0,0)$ is a critical point of the two-dimensional flow associated to (5).

a) Linearize around $(0,0)$ in the $\vec{x}$-system associated to (5) and conclude that $(0,0)$ is unstable if $pf'(0) > 0$.

b) Show that $V(x_1, x_2) = \frac{1}{2}px_2^2 - \frac{1}{4}qx_2^4 - F(x_1)$, with $F(x) = \int_0^x f(\xi)\, d\xi$, is a *constant of motion* for (5), i.e. show that $\frac{d}{dt}V(x_1(t), x_2(t)) = \frac{d}{dt}V(x(t), \dot{x}(t)) \equiv 0$ for solutions of (5).

c) Assume that $p \neq 0$ and $f'(0) \neq 0$ and show that $(0,0)$ is stable if $pf'(0) < 0$.
Hint. Use $\pm V(x_1, x_2)$ as a Lyapunov function (for (x_1, x_2) sufficiently close to $(0,0)$).

d) Assume that $p \neq 0$ that $f(x)$ is such that $f'(0) = 0$ and $f''(0) \neq 0$. Show that $(0,0)$ is unstable.
Hint. By (b), orbits of the flow associated to (5) lie on level sets $V(x_1, x_2) = h$ of V. Determine the local structure of the level sets of V near $(0,0)$ and the direction of the flow on these level sets. (Give a sketch!) Argue that $(0,0)$ cannot be stable by the definition of (in)stability.

e) Assume that $p = 0$ and show that $(0,0)$ is stable if $qf'(0) > 0$ and unstable if $qf'(0) < 0$.

f) Determine the stability of $(0,0)$ for general $p, q \in \mathbb{R}$ (with $(p,q) \neq (0,0)$) and general $f(x)$ (analytic near 0 with $f(0) = 0$).