EXERCISES *Introduction to Dynamical Systems '17-'18* **: Series I**

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Exercise 1. Consider the planar system

$$\ddot{x} - x + x^2 = 0 \text{ or } \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & x - x^2. \end{array} \right. \tag{1}$$

This system defines a flow $\phi(t; x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ parameterized by time t: $\phi(t; x_0, y_0) \in \mathbb{R}^2$ is the value of the solution of (1) with initial condition (x_0, y_0) at time t; orbits are denoted by $\Gamma(x_0, y_0)$ or by $\Gamma^{\pm}(x_0, y_0)$.

a) System (1) is integrable with Hamiltonian H, i.e. solutions of (1) are given as level sets of a function $H(x, y)$. Determine $H(x, y)$.
Hint: Apply the standard procedure: multiply $\ddot{x} - x + x^2 = 0$ by $\dot{x}(= y)$ and integrate over time.

b) Give a sketch of the phase portrait of (1), or equivalently, of the orbits of the flow generated by (1).

ci) Define $\mathcal{S}_p \subset \mathbb{R}^2$ as the set of all initial conditions such that the orbit $\Gamma(x_0, y_0)$ is periodic (in time, with minimal period $T > 0$). Determine $\mathcal{S}_p$ and give a sketch.

cii) Define $\mathcal{S}_+ \subset \mathbb{R}^2$ as the set of all initial conditions such that the limit for $t \to \infty$ of the orbit $\Gamma^+(x_0, y_0)$ exists. Determine $\mathcal{S}_+$ and give a sketch. Do the same for $\mathcal{S}_- \subset \mathbb{R}^2$ – the set of all initial conditions such that the limit $t \to -\infty$ of $\Gamma^-(x_0, y_0)$ exists.

ciii) Define $\mathcal{S}_\infty \subset \mathbb{R}^2$ as the set of all initial conditions such that both limits $t \to \infty$ and $t \to -\infty$ exist for the (full) orbit $\Gamma(x_0, y_0)$. Determine $\mathcal{S}_\infty$ and give a sketch.

d) The solutions $\gamma(t; (x_0, y_0))$ of (1) with $(x_0, y_0) \in \mathcal{S}_\infty$ can be determined explicitly. To see this, introduce $\tilde{\gamma}(t) = \alpha(\cosh \beta t)^{-2}$, with $\alpha, \beta \in \mathbb{R}$ parameters that can a priori be chosen freely (note that $\lim_{t \to \pm\infty} \tilde{\gamma}(t) = 0$). Substitute $\tilde{\gamma}(t)$ in (1) and determine α and β. For these (special) values of α and β, a one-parameter family of solutions of (1) is determined by $\tilde{\gamma}(t - \tau)$, $\tau \in \mathbb{R}$. Express $\gamma(t; (x_0, y_0))$ in terms $\tilde{\gamma}(t - \tau)$ (or more explicitly: express (x_0, y_0) in terms of τ (or vice versa)).

e) The solutions $\gamma(t; (x_0, y_0))$ of (1) with $(x_0, y_0) \in \mathcal{S}_- \backslash \mathcal{S}_\infty$, i.e. solutions that have a well-defined limit as $t \to -\infty$ but not as $t \to \infty$, can also be determined along these lines. Show that these solutions blow up in a finite time T_*. Relate T_* to (x_0, y_0).
Hint: Replace $\cosh \beta t$ in $\tilde{\gamma}(t)$ of (d) by $\sinh \beta t$.

f) It follows from (e) that the flow $\phi(t, x, y)$ is not defined for all $t \in \mathbb{R}$ for general (x_0, y_0) (!). Discuss whether this is also the case (or not?) for the system $\ddot{x} - x + x^3 = 0$. And for $\ddot{x} - x - x^3 = 0$?
Hint: The equivalents of the special solutions constructed in (d) and (e) can be found by considering $\tilde{\gamma}(t) = \alpha(\cosh \beta \text{ or } \sinh \beta t)^{-n}$, with $\alpha, \beta \in \mathbb{R}$, $n > 0$.

Exercise 2. Consider the non-autonomous equation,

$$\dot{x} = t^2 + [\sin(x + t)]x, \text{ with } x(0) = x_0, \tag{2}$$

and its autonomous equivalent,

$$\left\{ \begin{array}{rcl} \dot{x} & = & y^2 + [\sin(x + y)]x, \\ \dot{y} & = & 1, \end{array} \right. \quad \text{with } (x(0), y(0)) = (x_0, 0). \tag{3}$$

Note that it is clear from the theory of Chapter 3 in the book that equation (2)/system (3) must have a uniquely defined solution on a certain time interval.

a) Explain why we cannot conclude from Theorems 4.3 and 4.5 (in the book) that equation (2)/system (3) defines a complete flow.

b) Use (2) to prove that $|x(t)| \leq |x(0)| + \frac{1}{3}t^3 + \int_0^t |x(s)| ds$.

c) Introduce the functions $\alpha(t), z(t) \geq 0$ by $|x(t)| = z(t) - \alpha(t)$ and substitute this into the estimate in (b). Construct an explicit function $\alpha(t)$ in such a way that $z(t)$ satisfies the estimate $z(t) \leq K + \int_0^t z(s) ds$ for some $K > 0$.

d) Apply Grönwall's Lemma (Lemma 3.13 in the book) to the estimate on $z(t)$ in (c) and conclude from that that $|x(t)| \leq Ke^t$ for all $t \geq 0$.

e) Prove that equation (2)/system (3) defines a complete flow.