Front Interactions in a Three-Component System*

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Abstract. The three-component reaction-diffusion system introduced in [C. P. Schenk et al., Phys. Rev. Lett., 78 (1997), pp. 3781–3784] has become a paradigm model in pattern formation. It exhibits a rich variety of dynamics of fronts, pulses, and spots. The front and pulse interactions range in type from weak, in which the localized structures interact only through their exponentially small tails, to strong interactions, in which they annihilate or collide and in which all components are far from equilibrium in the domains between the localized structures. Intermediate to these two extremes sits the semistrong interaction regime, in which the activator component of the front is near equilibrium in the intervals between adjacent fronts but both inhibitor components are far from equilibrium there, and hence their concentration profiles drive the front evolution. In this paper, we focus on dynamically evolving $N$-front solutions in the semistrong regime. The primary result is use of a renormalization group method to rigorously derive the system of $N$ coupled ODEs that governs the positions of the fronts. The operators associated with the linearization about the $N$-front solutions have small eigenvalues, and the $N$-front solutions may be decomposed into a component in the space spanned by the associated eigenfunctions and a component projected onto the complement of this space. This decomposition is carried out iteratively at a sequence of times. The former projections yield the ODEs for the front positions, while the latter projections are associated with remainders that we show stay small in a suitable norm during each iteration of the renormalization group method. Our results also help extend the application of the renormalization group method from the weak interaction regime for which it was initially developed to the semistrong interaction regime. The second set of results that we present is a detailed analysis of this system of ODEs, providing a classification of the possible front interactions in the cases of $N = 1, 2, 3, 4$, as well as how front solutions interact with the stationary pulse solutions studied earlier in [A. Doelman, P. van Heijster, and T. J. Kaper, J. Dynam. Differential Equations, 21 (2009), pp. 73–115; P. van Heijster, A. Doelman, and T. J. Kaper, Phys. D, 237 (2008), pp. 3335–3368]. Moreover, we present some results on the general case of $N$-front interactions.

Key words. three-component reaction-diffusion systems, front interactions, renormalization group

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1. Introduction. Patterns are ubiquitous in science and engineering. They form when key physical quantities—for example, the concentrations of chemical species—exhibit nontrivial spatial and/or temporal dependence. Stripes, hexagons, spots, fronts, pulses, spirals, targets, sand ripples, and roll cells are all examples of patterns that may be observed.

Patterns may be classified as being near-equilibrium or far-from-equilibrium. In the former, the amplitudes of the key physical quantities are close to their equilibrium values everywhere in the domain. Such patterns arise, for example, when stable homogeneous (or equilibrium) states are destabilized by diffusion, as in the classical Turing bifurcation. By contrast, in far-from-equilibrium patterns, the key physical quantities exhibit large excursions away from equilibrium. Often, such patterns have a localized character; i.e., they are close to equilibrium on large parts of the domain and far from equilibrium on relatively small or narrow subdomains. Examples include fronts, which connect two different equilibria, pulses that may be the concatenations of two fronts, spots, and other more complicated spatially localized structures.

In the last decade, the three-component reaction-diffusion equation introduced in [21] has become a paradigm model for investigating the rich variety of front, pulse, and spot dynamics. As shown numerically and experimentally in [2, 10, 15, 16, 17, 18, 21, 25, 26], these localized structures can undergo repulsion, annihilation, attraction, breathing, collision, scattering, self-replication, and spontaneous generation. This three-component model consists of a well-studied bistable equation for the activator component and linear equations for the two inhibitor components, with bidirectional linear coupling. Hence, it may be interpreted as a FitzHugh–Nagumo-type equation augmented with a second inhibitor component. It has become a paradigm problem because, among other reasons, it is simultaneously complex enough to support the rich dynamics of these localized structures and simple enough to permit extensive analysis, as has been shown recently in one space dimension in [6, 24].

A scaled version of this paradigm model in one dimension is

\[
\begin{aligned}
U_t &= U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma), \\
\tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\
\theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W,
\end{aligned}
\]

where \(0 < \varepsilon \ll 1, D > 1, \tau, \theta > 0, \alpha, \beta, \gamma \in \mathbb{R}, \mathcal{O}(1)\) with respect to \(\varepsilon\), and \((\xi, t) \in \mathbb{R} \times \mathbb{R}^+\). Here, \(U\) represents the activator concentration, and \(V\) and \(W\) represent the concentrations of the inhibitors (for \(\alpha, \beta > 0\)). This partial differential equation (PDE) has homogeneous steady states \(\mathcal{O}(\varepsilon)\) close to \((U, V, W) = (\pm 1, \pm 1, \pm 1)\) and to \((0, 0, 0)\), with the former being stable and the latter unstable. Fronts are solutions that are close to the stable homogeneous steady state near \((-1, -1, -1)\) on a certain interval and then jump to the other stable homogeneous state near \((1, 1, 1)\). Backs are the opposites of fronts, and they are related to fronts via the symmetry \((U, V, W, \gamma) \rightarrow (-U, -V, -W, -\gamma)\) of (1.1), so that one may simply refer to both as fronts. Finally, pulses, which are the concatenation of a front and a back, are biasymptotic to either the homogeneous state near \((-1, -1, -1)\) or to that near \((1, 1, 1)\). By symmetry, any result about the former pulse solution also holds for the latter type, and vice versa. Hence, one may focus on the former type, without loss of generality. The fronts, backs, and pulses we study approach the steady states exponentially fast as \(\xi \to \pm \infty\).
The third component $W$ was introduced in [21] to stabilize traveling spot solutions in two dimensions. In [6, 24], the relation between the three-component model and its two-component limit has been investigated in detail (in one spatial dimension). In [6, 24], we have shown that the third component significantly increases the richness of the dynamics generated by the model. For instance, stationary 2-pulse (4-front) solutions cannot exist in the two-component limit [6]. Here, we will also establish that uniformly traveling 3-front solutions can exist only in the three-component model; see Lemma 4.11.

The existence and stability of traveling 1-pulse solutions and standing 1-pulse and 2-pulse solutions was proved in [6, 24]. We used and extended classical methods from geometric singular perturbation theory and from Evans function theory. Moreover, we note that it was critical for the application of these methods that the localized structures were either constant in time or fixed in a comoving frame.

This paper may be viewed as the next natural step in the analysis of the three-component model (1.1). We study dynamically evolving solutions consisting of $N$ fronts. It is not clear how to use the classical techniques to rigorously establish the existence of these solutions or their stability, since there is not a single global comoving frame in which all $N$ fronts are constant. Indeed, any two adjacent fronts may move in opposite directions and/or with different speeds; see Figures 1 and 2.

Our objectives in this paper are to derive and to analyze the system of $N$ coupled ordinary differential equations (ODEs) that governs the velocities of the fronts in an $N$-front solution in the parameter regime $\tau, \theta = O(1)$—see Remark 1.1. The derivation is readily carried out formally using matched asymptotic expansions. However, a rigorous justification of the validity of these ODEs—i.e., of the validity of reducing the three PDEs in (1.1) to a system of $N$ ODEs for the front velocities—requires significant new analysis. This justification is the primary result of this paper. It will be achieved by modifying the renormalization group (RG) method to consider the stability in a bounded variation (BV)-type norm. The second main result is an analysis of the reduced ODEs. In particular, we classify the different possible front dynamics for these $N$-front solutions, as well as how interacting fronts may pair up into (interacting) pulses.

As a preparatory result, we will show that 1-front solutions travel with velocity $\dot{\Gamma}(t) = \frac{3}{2} \sqrt{2\tau \gamma}$, where $\xi = \Gamma(t)$ denotes the position of the fronts at time $t$. Moreover, we show that
they are stable (see Lemma 2.1). The first substantial case involves 2-front solutions. We will show that the front velocities are given by
\[ \hat{\Gamma}_1 = \frac{3}{2} \sqrt{2} \varepsilon \left( \gamma - \alpha \varepsilon (\Gamma_1 - \Gamma_2) - \beta \varepsilon \xi (\Gamma_1 - \Gamma_2) \right), \quad \hat{\Gamma}_2 = -\hat{\Gamma}_1, \]
to leading order. Analysis of these ODEs reveals that a 2-front solution converges asymptotically to a standing 1-pulse solution if and only if this 1-pulse solution is stable and there are no unstable 1-pulse solutions between it and the initial fronts. Otherwise, the fronts may converge asymptotically to \( \pm \infty \) or annihilate.

The dynamics exhibited by 3-front and 4-front solutions is more varied. We show, among other things, that 3-front solutions and 4-front solutions for which one (or more) of the outer fronts travels to \( \pm \infty \) can be stable. Also, the 4-front solutions can converge asymptotically to a ground state, a stable 1-pulse solution, or a stable 2-pulse solution.

For general \( N \geq 1 \), we will show that the velocities of the fronts are given to leading order by (2.1). Analyzing these ODEs in the generic case when \( \gamma \neq 0 \), we show that uniformly traveling solutions are possible when the number of fronts is odd but not when the number of fronts is even. Similarly, in the generic case, we find that stationary \( N \)-front solutions can exist when \( N \) is even, but not when \( N \) is odd. See Lemma 4.4.

In proving the existence and stability of the dynamically evolving \( N \)-front solutions, we focus exclusively on the case \( N = 2 \), in order to keep the analysis of the RG method as transparent as possible. Nevertheless, the ideas and arguments in the proof also suffice to rigorously justify the ODE reduction for \( N \)-front solutions for general \( N \). There are \( N \) eigenvalues near zero, and the spectral splitting holds uniformly for the \( N \)-front solutions, as follows from the
analysis in [24]. See also [14] for a detailed study of the stability of \(N\)-pulses using the RG method.

The validity of the ODE system (2.1) will be established using an RG method. Indeed, the method will simultaneously give the existence and stability of the \(N\)-front solutions, as long as no two adjacent fronts get too close. One begins with the manifold of approximate \(N\)-front solutions obtained from a formal derivation. Initial data \(\Phi_N(\xi, t = 0) = (U_N(\xi, 0), V_N(\xi, 0), W_N(\xi, 0))\) for the PDE (1.1) that lies close to a point on this manifold may be decomposed into the sum of an approximating “skeleton” \(N\)-front solution on the manifold and a remainder which lies in the directions transverse to the manifold and whose norm is of the size of the distance to the manifold. Based on leading order matched asymptotic expansions, one expects that \(\Phi_N(\xi, t)\) will remain close to the skeleton solution as it evolves on that manifold, i.e., that the remainder remains small. However, proving that this is the case requires a stability analysis about the time-dependent solution on the manifold. With the RG method, we show that there exists a sequence of times \(\{t^*_i\}_{i=0}^\infty\), with \(t^*_0 = 0\), at which one may freeze the skeleton solution on the manifold and linearize about this frozen solution in order to approximate the linearization about \(\Phi_N(\xi, t)\) on the interval \([t^*_i, t^*_{i+1}]\). Then, at the end of each time interval, one renormalizes the skeleton solution by taking an appropriate point on the manifold, and repeats the above procedure. Projection of the solutions onto the eigenspace associated with the \(N\) small \(O(\varepsilon)\) eigenvalues of the linearized operator leads to the ODEs for the positions of the fronts, and projection onto the complementary eigenspace leads to the bounds on the resolvent and semigroup, and hence also to the bounds on the remainder.

There are several competing factors, akin to normal hyperbolicity, which determine whether or not the RG approach succeeds. On the one hand, the lengths of the intervals, \(t^*_{i+1} - t^*_i\), must be sufficiently long so that the contraction estimates obtained from the semigroup estimates are sufficient. On the other hand, the lengths of these intervals must be sufficiently short so that the secular errors which accumulate in making the frozen linearization approximation do not become too big.

Front and pulse interactions have been studied using RG methods in [5, 14, 9, 19]. The underlying strategy in applying the method here is similar to that used in these other studies. The main challenge we face in applying the RG method to the three-component model (1.1) is that we cannot use variants of an \(H^1\)-norm, such as those used in [5]. These norms are singular when comparing functions with small differences in their asymptotic states at spatial infinity. To overcome this, we define the \(\chi\)-norm (see (3.1)), which can be seen as a variant of the usual BV norm.

We observe that the interactions between the fronts and pulses that we study is classified as semistrong; see [4, 5, 12, 14, 22]. Semistrong interaction of two adjacent fronts means that the interaction is driven essentially by the component(s) that are not near equilibrium in the intervals between the fronts. In the case of (1.1), the front interactions are driven by \(V\) and \(W\); see Figure 1. Hence, the semistrong interaction of fronts and pulses in (1.1) stems from the separation of length scales in the PDEs, i.e., from their singularly perturbed nature. The interaction in the semistrong regimes is stronger, and hence the observed front interactions are richer, than that in the weak interaction regime [7, 8, 19, 20]. In the weak regime, the pulses are assumed to be “sufficiently far apart” that the pulses can be considered as “particles” to leading order. Semistrong interacting localized structures change shape, and
the interaction may even cause “bifurcations.” On the other hand, semistrong interactions are weaker than strong interactions, which occur, for example, when fronts collide or when a pulse self-replicates. For the three-component model (1.1), numerical simulations suggest that when two fronts enter the strong interaction regime, where \( \Gamma_{i+1}(\xi) - \Gamma_{i}(\xi) \ll \epsilon^{-1} \) for some \( i \), the fronts collide and disappear; see Figures 7 and 10 (in section 4). It is a challenge to analyze strong interactions and to apply the RG method to strongly interacting fronts.

This paper is organized as follows. In section 2, we present the formal derivation of the ODE (2.1). The renormalization group method that rigorously justifies the derivation of this ODE is presented in section 3. Then, a detailed analysis of the ODEs for the cases \( N = 1, 2, 3, 4 \) is presented in section 4. Moreover, we present some general results for \( N \) odd or even.

**Remark 1.1.** The fact that the parameters \( \tau \) and \( \theta \) are \( O(1) \) is a key assumption in this paper. For these values of \( \tau \) and \( \theta \), the terms involving \( c \) are to leading order absent in the slow fields [6, 24]. This is crucial, since \( c = c(t) \) is not even well defined in the slow fields. It is a fundamental challenge to adapt the methods used in this paper for problems where the speeds of the fronts do have a leading order influence in the slow fields. Here, this occurs if \( \tau, \theta \) are \( O(\epsilon^{-2}) \) large; see also [6, 24]. In this parameter regime, traveling 1-pulse solutions and breathing 1-pulse solutions exist and bifurcate from stationary 1-pulse solutions. The proof of section 3 breaks down in this regime, since the essential spectrum converges asymptotically to the origin in the limit \( \epsilon \to 0 \).

2. Formal derivation of \( N \)-front dynamics. In this section, we formally derive an \( N \)-component ODE describing the dynamics of the \( N \) different fronts of an \( N \)-front solution. A priori, the fronts of an \( N \)-front solution all travel with different speeds. Therefore, it is not possible to introduce one comoving frame which travels along with every front. We formally overcome this problem by introducing \( N \) comoving frames such that every frame travels along with one of the fronts. This way we obtain \( N \) different independent “fast” ODEs. To leading order, we then solve each of these ODEs by singular perturbation techniques and obtain \( N \) jump conditions (2.4). Since the speeds of the fronts have no leading order influence on any of the intermediate slow fields, we can formally “glue” the \( N \) different fast solutions together in the slow fields. Formally, we then obtain an \( N \)-component ODE (2.1) describing the evolution of the \( N \) fronts. The key underlying assumption in this construction is that the speeds of the various fronts appear at higher order in the slow fields; see Remark 1.1. The perturbation analysis can be summarized as follows.

Assume that all parameters of (1.1) are \( O(1) \) with respect to \( \epsilon \), and let \( \epsilon \) be small enough. Moreover, assume that the speeds of the fronts of an \( N \)-front solution \( \Phi_{N}(\xi, t) \) to (1.1) are all \( O(\epsilon) \). Then, to leading order, the \( i \)th front \( \Gamma_{i} \) of this \( N \)-front solution formally evolves as

\[
\dot{\Gamma}_{i}(t) = (-1)^{i+1} \frac{3}{2} \sqrt{2\epsilon} \left[ \gamma + \alpha \left( -e^{\epsilon(\Gamma_{1}\cdots \Gamma_{i})} + \cdots + (-1)^{i-1}e^{\epsilon(\Gamma_{1}\cdots \Gamma_{i})} \right) \right. \\
\left. + (-1)^{i}e^{\epsilon(\Gamma_{i-1}\cdots \Gamma_{i})} + \cdots + (-1)^{N-1}e^{\epsilon(\Gamma_{1}\cdots \Gamma_{N})} \right) \\
+ \beta \left( -e^{\frac{\epsilon}{\theta}(\Gamma_{1}\cdots \Gamma_{i})} + \cdots + (-1)^{i-1}e^{\frac{\epsilon}{\theta}(\Gamma_{1}\cdots \Gamma_{i})} + (-1)^{i}e^{\frac{\epsilon}{\theta}(\Gamma_{i-1}\cdots \Gamma_{i})} \right) \\
+ \cdots + (-1)^{N-1}e^{\frac{\epsilon}{\theta}(\Gamma_{1}\cdots \Gamma_{N})} \right] \quad \text{for} \quad i = 1, \ldots, N.
\]

\[ (2.1) \]
Here, \( \Gamma_i \) is the \( \xi \)-coordinate of the \( i \)th time the \( U \) component crosses zero, and \( \dot{\Gamma} \) is the time-derivative of \( \Gamma \).

Note that \( \Gamma_i < \Gamma_j \) if \( i < j \), and therefore all the exponentials in (2.1) have a negative exponent. Moreover, since we use the fast scaling, the distance between two fronts is of order \( O(\varepsilon^{-1}) \). Thus, the interactions between the fronts are not exponentially small, as in the case of weak interaction. Also observe that the influence of the \( i \)th front on the \( j \)th front is independent of the number of fronts in between.

This formal result is derived as follows. Since the \( i \)th front of an \( N \)-front solution is located at \( \Gamma_i \) and moves with speed \( \varepsilon c_i \), we have that

\[
(2.2) \quad \Gamma_i(t) = \Gamma_i(0) + \varepsilon \int_0^t c_i(s) ds \quad \Rightarrow \quad \dot{\Gamma}_i(t) = \varepsilon c_i(t).
\]

Since the various speeds \( \varepsilon c_i \) of the fronts have no leading order influence on the slow equations, the PDE (1.1) to leading order reduces to the following ODE system:

\[
\begin{aligned}
\frac{d}{d\xi} p &= u, \\
\frac{d}{d\xi} q &= \varepsilon(v - u) + O(\varepsilon^3), \\
\frac{d}{d\xi} r &= \frac{\varepsilon}{2}(w - u) + O(\varepsilon^3).
\end{aligned}
\]

In the \( N \) fast fields, the regions around the fronts, the solution is governed by the first two ODEs with different speeds \( \varepsilon c_i \) and with different fixed \( v \) and \( w \) components; that is, \( (v(\Gamma_i), w(\Gamma_i)) = (v_i, w_i) \). In the fast fields, the \( U \) component to leading order jumps from a locally invariant manifold \( \mathcal{M}_{\varepsilon}^- \) to the other \( \mathcal{M}_{\varepsilon}^+ \), where \( \mathcal{M}_{\varepsilon}^\pm = \{ u = \pm 1 - \frac{1}{2} \varepsilon (\alpha v + \beta w + \gamma) + O(\varepsilon^2), p = O(\varepsilon^2) \} \). Therefore, the solution has to lie in the intersection of their unstable and stable manifold; i.e., it has to lie in \( W^u(\mathcal{M}_{\varepsilon}^\pm) \cap W^s(\mathcal{M}_{\varepsilon}^\pm) \). The distance between those two manifolds, which has to be zero to leading order, is measured by a Melnikov integral [6, 23]. This integral yields \( N \) conditions

\[
\varepsilon(\alpha v_i + \beta w_i + \gamma) \int_{-\infty}^{\infty} p_0(\xi) d\xi + (-1)^i c_i \int_{-\infty}^{\infty} p_0(\xi)^2 d\xi = O(\varepsilon\sqrt{\varepsilon}), \quad \text{for } i = 1, \ldots, N,
\]

with \( p_0(\xi) \) the derivative of the leading order integrable flow; that is, \( p_0(\xi) = p \)-solution of the \((u, p)\)-system of (2.3) with \( \varepsilon = 0 \). In particular, \( p_0 = \frac{1}{2} \sqrt{2} \) sech\( ^2 \left( \frac{1}{2} \sqrt{2} \xi \right) \). Integrating gives \( N \) jump conditions,

\[
(2.4) \quad \alpha v_i + \beta w_i + \gamma = (-1)^{i+1} \frac{1}{3} \sqrt{2} c_i \quad \text{for } i = 1, \ldots, N.
\]

In the \( N + 1 \) slow fields, the regions in between the fronts, the solution is governed by the last four ODEs of (2.3) with \( u \) fixed at either +1 or −1. To leading order, these ODEs can be solved explicitly,

\[
(2.5) \quad v(\xi) = A_j e^{\varepsilon \xi} + B_j e^{-\varepsilon \xi} + (-1)^j, \quad w(\xi) = C_j e^{\varepsilon \xi} + D_j e^{-\varepsilon \xi} + (-1)^j,
\]

\[ j = 1, \ldots, N + 1. \]
Note that \( v(\xi) \) and \( w(\xi) \) do not change in leading order when \( |\xi| \ll \varepsilon^{-1} \) during the passage along a slow manifold. Therefore, we assume that \( \Delta \Gamma_i = \Gamma_{i+1} - \Gamma_i = O(\varepsilon^{-1}) \); see Remark 3.3.

To determine the constants \( A_i, B_i, C_i, D_i \), and thus \((v_i, w_i)\), as functions of the front locations \( \Gamma_i \), we implement the asymptotic boundary conditions and match the slow solutions (2.5) and their derivatives over the fast regions. That is, \( A_{N+1} = B_1 = C_{N+1} = D_{N+1} = 0 \), and

\[
\begin{align*}
A_i e^{\varepsilon \Gamma_i} + B_i e^{-\varepsilon \Gamma_i} &= A_{i+1} e^{\varepsilon \Gamma_i} + B_{i+1} e^{-\varepsilon \Gamma_i} + 2(-1)^{i-1}, \\
A_i e^{\varepsilon \Gamma_i} - B_i e^{-\varepsilon \Gamma_i} &= A_{i+1} e^{\varepsilon \Gamma_i} - B_{i+1} e^{-\varepsilon \Gamma_i}, \\
C_i e^{\frac{\varepsilon}{2} \Gamma_i} + D_i e^{-\frac{\varepsilon}{2} \Gamma_i} &= C_{i+1} e^{\frac{\varepsilon}{2} \Gamma_i} + D_{i+1} e^{-\frac{\varepsilon}{2} \Gamma_i} + 2(-1)^{i-1}, \\
C_i e^{\frac{\varepsilon}{2} \Gamma_i} - D_i e^{-\frac{\varepsilon}{2} \Gamma_i} &= C_{i+1} e^{\frac{\varepsilon}{2} \Gamma_i} - D_{i+1} e^{-\frac{\varepsilon}{2} \Gamma_i},
\end{align*}
\]

for \( i = 1, \ldots, N \). Since we have as many unknowns as equations, we can determine the remaining unknowns \( \{A_i, B_i, C_i, D_i\}_{i=1}^{N+1} \). Rewriting (2.6) gives

\[
\begin{align*}
A_i &= A_{i+1} + (-1)^{i-1} e^{-\varepsilon \Gamma_i}, & B_{i+1} &= B_i - (-1)^{i-1} e^{\varepsilon \Gamma_i}, \\
C_i &= C_{i+1} + (-1)^{i-1} e^{-\frac{\varepsilon}{2} \Gamma_i}, & D_{i+1} &= D_i - (-1)^{i-1} e^{\frac{\varepsilon}{2} \Gamma_i},
\end{align*}
\]

for \( i = 1, \ldots, N \). Therefore,

\[
\begin{align*}
A_i &= \sum_{j=i}^{N} (-1)^{j-1} e^{-\varepsilon \Gamma_j}, & B_i &= \sum_{j=1}^{i-1} (-1)^j e^{\varepsilon \Gamma_j}, \\
C_i &= \sum_{j=i}^{N} (-1)^{j-1} e^{-\frac{\varepsilon}{2} \Gamma_j}, & D_i &= \sum_{j=1}^{i-1} (-1)^j e^{\frac{\varepsilon}{2} \Gamma_j},
\end{align*}
\]

for \( i = 1, \ldots, N + 1 \), and where an empty summation is defined to be zero. Since \((v_i, w_i) = (v(\Gamma_i), w(\Gamma_i))\), we obtain that

\[
\begin{align*}
v_i &= A_i e^{\varepsilon \Gamma_i} + B_i e^{-\varepsilon \Gamma_i} + (-1)^i \\
&= \left( \sum_{j=i}^{N} (-1)^{j-1} e^{-\varepsilon \Gamma_j} \right) e^{\varepsilon \Gamma_i} + \left( \sum_{j=1}^{i-1} (-1)^j e^{\varepsilon \Gamma_j} \right) e^{-\varepsilon \Gamma_i} + (-1)^i \\
&= \sum_{j=1}^{i-1} (-1)^j e^{\varepsilon (\Gamma_i - \Gamma_j)} + \sum_{j=i+1}^{N} (-1)^{j-1} e^{\varepsilon (\Gamma_i - \Gamma_j)}, \quad \text{for } i = 1, \ldots, N.
\end{align*}
\]

\[
w_i = \sum_{j=1}^{i-1} (-1)^j e^{\frac{\varepsilon}{2} (\Gamma_i - \Gamma_j)} + \sum_{j=i+1}^{N} (-1)^{j-1} e^{\frac{\varepsilon}{2} (\Gamma_i - \Gamma_j)},
\]

for \( i = 1, \ldots, N \). Combining (2.9) with the jump conditions (2.4) and using (2.2), we obtain (2.1).

**Lemma 2.1.** Assume that all conditions of the formal construction are met. Moreover, assume that \( N = 1 \); that is, we look at 1-front solutions. Then these 1-front solutions travel with speed

\[
\dot{\Gamma}(t) = \varepsilon c = \frac{3}{2} \sqrt{2 \varepsilon \gamma}.
\]
Moreover, these 1-front solutions are stable.

Proof. Equation (2.10) is a direct consequence of (2.1) with \( N = 1 \). However, since we only have to introduce one comoving frame, this result can be made rigorous by the method used in [6]. Likewise, the stability of these 1-front solutions directly follows from the pulse-stability analysis in [24]: the 1-front solutions can have only one small eigenvalue, the translational eigenvalue at \( \lambda = 0 \). ■

3. An RG method. We reformulate the results of the previous section in a rigorous manner (see Theorem 3.2 in section 3.2) and use the RG method developed in [5, 19] to rigorously prove this theorem. In order to focus on the essence of the method and avoid technical details, we consider only the case \( N = 2 \) of (2.1) in full detail. The proof of the general case runs along the same lines modulo certain technicalities, such as the uniform spectral compatibility; see section 3.4. In order to formulate the theorem, we first need to introduce a suitable norm.

3.1. The \( \chi \)-norm. We define the \( \chi \)-norm by

\[
\| (U, V, W) \|_\chi := \| U \|_\chi + \| V \|_\chi + \| W \|_\chi,
\]

with \( \| \cdot \|_\chi := \| \chi \cdot \|_{L^1} + \| \partial_\xi \cdot \|_{L^1} \),

where \( \chi(\xi) \) is a positive function with mass 1, that is, \( \bar{\chi} = \int_{-\infty}^{\infty} \chi(\xi) d\xi = 1 \), and is exponentially decaying with an \( O(1) \) parameter with respect to \( \varepsilon \) (for example, \( \chi(\xi) = \frac{1}{\varepsilon} e^{-|\xi|} \)). It is straightforward to check that the \( \chi \)-norm is indeed a norm; in essence, it is a weighted \( W^{1,1} \)-norm. We also define the normed space \( X \):

\[
X := \{ (U, V, W) \mid \| (U, V, W) \|_\chi < \infty \}.
\]

The reason for using this particular norm, instead of a more usual one such as the scaled variant of the \( H^1 \)-norm used in [5], is that this \( \chi \)-norm is well behaved with respect to differences in asymptotic behavior at spatial infinity. The need for this is explained as follows. An \( N \)-front solution to (1.1) only converges asymptotically to leading order to \( (-1, -1, -1) \) at \( \xi = -\infty \) [6]. However, the skeleton solution which we use to approximate an \( N \)-front solution (see (3.6)) converges asymptotically exactly to \( (-1, -1, -1) \) at \( \infty \). Therefore, although the error is only of \( O(\varepsilon) \) size at spatial infinity, the \( H^1 \)-norm of this error is unbounded. Since the tails of the \( N \)-front solution and the skeleton solution are exponentially flat, the seminorm \( \| \partial_\xi \cdot \|_{L^1} \) (all constants have norm zero) does not yield an unbounded error. To make this seminorm into a norm, we add the component \( \| \chi \cdot \|_{L^1} \), which, by the third assumption on \( \chi \), also does not penalize errors at infinity. The first two properties that we impose on the weight \( \chi \), positivity and mass one, are to make sure that the \( \chi \)-norm uniformly dominates the \( L^\infty \)-norm.

Lemma 3.1. Let \( u, v \) be integrable functions such that \( \| u \|_\chi, \| v \|_\chi < \infty \). Then, the \( \chi \)-norm has the following three properties:

\[
\begin{align*}
\| u \|_{L^\infty} & \leq \| u \|_\chi, \\
\| G \ast u \|_\chi & \leq 2 \| G \|_{L^1} \| u \|_\chi, \\
\| uv \|_\chi & \leq 2 \| u \|_\chi \| v \|_\chi,
\end{align*}
\]

where \( G \) in (3.4) is an \( L^1 \)-function (in this paper typically a Green’s function), and \( \ast \) the usual convolution.
Proof. The first property, (3.3), is established via the following inequalities:

$$u(x) - u(y) = \int_{y}^{x} u_\xi d\xi \implies |u(x)| \leq \int_{-\infty}^{\infty} |u_\xi| d\xi + |u(y)|.$$  

Multiplying by $\chi(y)$, integrating over all $y$ in $(-\infty, \infty)$, and recalling that $\chi$ is positive and has mass one, we find that

$$|u(x)| \leq \|u_\xi\|_{L^1} + \int_{-\infty}^{\infty} |\chi(y)u(y)| dy \implies \|u\|_{L^\infty} \leq \|u\|_{\chi}.$$  

The proofs of the second and third properties, (3.4) and (3.5), heavily rely on the first property. To prove the second property (3.4), we use Hölder’s inequality, the fact that

$$(G * u)_\xi = G * u_\xi \quad \text{[11]},$$

the inequality $\|G * u\|_{L^p} \leq \|G\|_{L^1} \|u\|_{L^p}$ for $1 \leq p \leq \infty$, and finally the above result (3.3),

$$\|G * u\|_{\chi} = \|\chi(G * u)\|_{L^1} + \|(G * u)_\xi\|_{L^1} \leq \|\chi\|_{L^1} \|G * u\|_{L^\infty} + \|G * u_\xi\|_{L^1} \leq \|G\|_{L^1} \|u\|_{L^\infty} + \|G\|_{L^1} \|u_\xi\|_{L^1} \leq 2\|G\|_{L^1} \|u\|_{\chi}.$$  

To prove the third property (3.5) observe that

$$\|uv\|_{\chi} \leq \|u\chi v\|_{L^1} + \|uv_\xi\|_{L^1} + \|v\chi u\|_{L^1} + \|vu_\xi\|_{L^1} \leq \|u\|_{L^\infty} \|v\|_{\chi} + \|v\|_{L^\infty} \|u\|_{\chi} \leq 2\|u\|_{\chi} \|v\|_{\chi}.$$  

3.2. The main result. In order to give an accurate formulation of the main result of this paper, i.e., that the dynamics of an $N$-front solution of (1.1) is indeed determined by the formally derived (2.1), we first need to introduce some more notation. We define the stationary skeleton $N$-front solution $\Phi_\Gamma(\xi)$ by

$$\Phi_\Gamma(\xi) = \begin{pmatrix} \Phi_1(\xi) \\ \Phi_2(\xi) \\ \Phi_3(\xi) \end{pmatrix} = \begin{pmatrix} U_0(\xi; \Gamma) \\ G_V * U_0(\xi; \Gamma) \\ G_W * U_0(\xi; \Gamma) \end{pmatrix},$$

in which $U_0(\xi, \Gamma)$ is the leading order approximation of the $U$ component of a stationary $N$-front solution of (1.1),

$$U_0(\xi, \Gamma) = -1 + \sum_{i=1}^{N} (-1)^{i-1} \tanh \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_i) \right).$$

Here, $\Gamma_i$ determines the location of the $i$th front; more precisely, $U_0(\xi)$ has its $i$th sign change at $\xi = \Gamma_i$. By definition, we have that $\Gamma_i < \Gamma_{i+1}$, and since the interaction of the fronts is semi-strong, we may assume that $\Delta \Gamma_i = \Gamma_{i+1} - \Gamma_i = \mathcal{O}(\varepsilon^{-1})$; see Remark 3.3. The functions $G_V(\xi)$...
and \( G_W(\xi) \) are the Green’s functions associated with the stationary \( V \)- and \( W \)-equations of (1.1) with \( U(\xi, t) = U_0(\xi) \). For example, \( G_V \ast U_0 \) is the (exact!) solution of

\[
0 = \frac{1}{\varepsilon^2} V_{\xi\xi} + U_0 - V.
\]

Straightforward computations yield that

\[
G_V = -\frac{1}{2} \varepsilon e^{-\varepsilon |\xi|} \quad \text{and} \quad G_W = -\frac{1}{2} \varepsilon \frac{1}{D} e^{-\frac{1}{D} |\xi|},
\]

which are both \( L^1 \)-functions with norm 1.

The graph of the functions \( \Phi(t) \) forms an \( N \)-dimensional manifold \( \mathcal{M}_{N,0} \). Note that \( \mathcal{M}_{N,0} \) has a boundary \( \partial \mathcal{M}_{N,0} \) consisting of \( N - 1 \) codimension 1 hyperplanes at which \( \Gamma_i = \Gamma_{i+1} \) \((i = 1, \ldots, N-1)\). The evolution within \( \mathcal{M}_{N,0} \) is (to leading order) determined by (2.1). The dynamical skeleton \( N \)-front solution \( \Phi(t) \) is defined to be an \( N \)-front solution (3.6) whose fronts \( \Gamma_i(t) \) evolve according to the ODE (2.1).

This ODE has been obtained under the assumptions that \( \Gamma_i < \Gamma_{i+1} \) and \( \Delta \Gamma_i = O(\varepsilon^{-1}) \) (Remark 3.3). However, these properties are not necessarily conserved by the flow generated by (2.1): two components \( \Gamma_i(t) \) and \( \Gamma_{i+1}(t) \) of a solution of (2.1) may in principle cross and thus change order. (See section 4, in which the dynamics generated by (2.1) is studied.) In other words, the evolution of (2.1) may drive a solution toward the boundary \( \partial \mathcal{M}_{N,0} \). Our methods—and in fact all methods considered in the literature—break down in the strong interaction regime, i.e., for solutions of the PDE (1.1) that have two fronts \( \Gamma_i(t) \) and \( \Gamma_{i+1}(t) \) that become too close. In fact, we will see in the simulations presented in section 4 that these fronts will in general annihilate each other in the PDE, while their approximating counterparts will survive the collision and move through each other in the ODE simulations—something that is impossible in the PDE. Therefore, we define \( t_m = t_m(\Gamma(0)) \) of a solution \( \Gamma(t) \) of (2.1) as the maximal time for which \( \min_i \Delta \Gamma_i(t) > \varepsilon^{-1/2} \) for all time \( 0 \leq t < t_m \). Thus, \( \Gamma(t_m) = O(\varepsilon^{-1/2}) \) close to \( \partial \mathcal{M}_{N,0} \), and \( t_m = O(\varepsilon^{-2}) \) since \( \Delta \Gamma_i(0) = O(\varepsilon^{-1}) \) by definition and \( \dot{\Gamma}_i(t) = O(\varepsilon) \) for all \( i \). Note that our methods in principle allow us to extend our results into regions in which \( \Delta \Gamma_i(t) = O(\varepsilon^{-2}) \) for any \( \varepsilon \in (0, 1) \); see Remark 3.3. In that sense the choice for the critical distance, \( \sigma = \frac{1}{2} \), is somewhat arbitrary. However, it does provide us with a unique definition of \( t_m(\Gamma(0)) \), and none of the other possible choices for \( \sigma \) appear to give more insight than the present one. Note also that the fronts do not necessarily collide. In fact, \( \Gamma(t) \) remains bounded away from \( \partial \mathcal{M}_{N,0} \) for many choices of \( \Gamma(0) \). In other words, \( t_m(\Gamma(0)) = \infty \) for large sets of initial conditions; see section 4.

We can now formulate our main result.

**Theorem 3.2.** Let \( \varepsilon > 0 \) be sufficiently small, and assume that all parameters of (1.1) are \( O(1) \) with respect to \( \varepsilon \). Let \( \Phi_N(\xi, t) = (U_N(\xi, t), V_N(\xi, t), W_N(\xi, t)) \) be a solution of (1.1) which is \( O(\varepsilon) \) close to the \( N \)-front manifold \( \mathcal{M}_{N,0} \) at \( t = 0 \); i.e., there is a \( \Gamma(0) \) such that \( \Delta \Gamma_i(0) = O(\varepsilon^{-1}) \) and

\[
\|\Phi_N(\cdot, 0) - \Phi(0)\|_\infty < \tilde{C} \varepsilon,
\]

for some \( \tilde{C} > 0 \). Then, \( \Phi_N(\xi, t) \) remains \( O(\varepsilon) \) close to \( \mathcal{M}_{N,0} \) for \( 0 \leq t < t_m \), and its evolution is governed by (2.1), the leading order dynamics of the fronts of \( \Phi(0)(\xi) \). In particular,
\(\Phi_N(\xi, t)\) can be decomposed into

\[
\Phi_N(\xi, t) = \Phi_T(t) + Z(\xi, t)
\]

with

\[
\|Z(\cdot, t)\|_X \leq C\varepsilon \quad \text{for all } 0 \leq t < t_m.
\]

This theorem establishes the validity of the \(N\)-front dynamics formally obtained in section 2.

By using an improved skeleton solution \(\Phi_T(\xi)\) and the same RG procedure as in this section, it is possible to improve on this result. For instance, we can prove the existence of an attracting manifold \(\mathcal{M}_{N,1}\) with the property that a solution \(\Phi_N(\xi, t)\) with initial conditions \(\Phi_N(\xi, 0)\) starting only \(O(1)\) close to \(\mathcal{M}_{N,1}\) will eventually be \(O(\varepsilon^2)\) close to it. This manifold \(\mathcal{M}_{N,1}\) is an \(O(\varepsilon)\) correction to the manifold \(\mathcal{M}_{N,0}\). To determine the improved skeleton solution \(\Phi_T(\xi)\) we need the results of this section (with the normal skeleton solution \(\Phi_T(\xi)\) (3.6)). Therefore, this section can be seen as a first step in an iteration procedure to obtain an attracting \(N\)-dimensional set \(\mathcal{M}_{N,\varepsilon}\) with boundary \(\partial \mathcal{M}_{N,\varepsilon}\) in the solution space associated with (1.1). Away from \(\partial \mathcal{M}_{N,\varepsilon}\) the dynamics on \(\mathcal{M}_{N,\varepsilon}\) is to leading order governed by (2.1). Note that this analysis is somewhat subtle, for instance since the speed of the fronts influences the corrections to the shape of the front solutions in the higher order approximations (and vice versa). Nevertheless, this iteration procedure can be performed by embedding the geometrical approach of [4, 5] into the higher order RG method analysis—see [5], where the speed of the interacting pulses determines the amplitude of the pulses to leading order. We refrain from going into the details here. It should be observed that an iterated refinement of the theorem does not yet necessarily establish whether or not \(\mathcal{M}_{N,\varepsilon}\) is actually a manifold. See also [1, 27].

We emphasize that the dynamics of the skeleton solution \(\Phi_T(\xi)\) is only to leading order determined by (2.1). Because of accumulation of error, the predicted front position could diverge by an \(O(1)\) for nonstationary solutions \(\Phi_N(\xi, t)\) after \(O(\varepsilon^{-1})\) time. However, at all points on the manifold the front dynamics is given to leading order by (2.1), particularly for the configuration of steady states and traveling waves.

The remainder of this section is devoted to the proof of Theorem 3.2, using the RG method, as developed in [5, 19]. As was already stated, we consider only the case \(N = 2\) in full detail. For clarity, we note that (2.1) reduces to

\[
\begin{align*}
\dot{\Gamma}_1 &= -\frac{3}{2}\sqrt{2\varepsilon} \left( \gamma - \alpha \varepsilon^{\varepsilon_{1}}(\Gamma_1 - \Gamma_2) - \beta \varepsilon \frac{\partial}{\partial \Gamma_1}(\Gamma_1 - \Gamma_2) \right), \\
\dot{\Gamma}_2 &= -\frac{3}{2}\sqrt{2\varepsilon} \left( \gamma - \alpha \varepsilon^{\varepsilon_{2}}(\Gamma_1 - \Gamma_2) - \beta \varepsilon \frac{\partial}{\partial \Gamma_2}(\Gamma_1 - \Gamma_2) \right)
\end{align*}
\]

in this case.

Substituting the decomposition (3.10) into the PDE (1.1), we find

\[
Z_t + \frac{\partial \Phi_T}{\partial \Gamma} \dot{\Gamma} = R(\Phi_T) + L_\Gamma Z + N(Z).
\]

The residual \(R(\Phi_T)\) is defined as the error made by the skeleton solution (3.6) and is determined by plugging (3.6) into the right-hand side of (1.1). Since, by construction, \(\Phi_2(\xi)\)
\((\Phi_3(\xi))\) solves the second (third) component of the right-hand side of (1.1) exactly for given \(\Phi_1(\xi) = U_0(\xi, \Gamma)\) (3.8), we obtain that the second (third) component of the residual is zero. However, the first component \(R_1 \neq 0:\)

\[
R(\Phi) = \begin{pmatrix}
R_1 \\
R_2 \\
R_3
\end{pmatrix} = \begin{pmatrix}
(U_0)_{\xi\xi} + U_0 - (U_0)^3 - \varepsilon (\alpha (G_V * U_0) + \beta (G_W * U_0) + \gamma) \\
0 \\
0
\end{pmatrix}.
\]

(3.14)

The linear operator reads

\[
L_\Gamma = \begin{pmatrix}
\frac{\partial^2}{\partial \xi^2} + 1 - 3\Phi_1^2 & -\varepsilon\alpha & -\varepsilon\beta \\
\frac{1}{\tau} & \frac{1}{\tau} \left( \frac{\Phi_1^2}{\varepsilon^2} - 1 \right) & 0 \\
\frac{1}{\theta} & 0 & \frac{1}{\theta} \left( \frac{D^2}{\varepsilon^2} \Phi_1^2 - 1 \right)
\end{pmatrix}.
\]

(3.15)

Finally, the nonlinear term is given by

\[
N(Z) = \begin{pmatrix}
-3\Phi_1 Z_1^2 - Z_3^3 \\
0 \\
0
\end{pmatrix}^t.
\]

(3.16)

The proof of the theorem now consists of several steps, which are all essential to the RG method used \([5, 14, 9, 19]\).

1. \textbf{First, we bound the nonlinear growth term} \(N(Z)\) (3.16) \textbf{and the residual} \(R(\Phi_{\Gamma(t)})\) (3.14) \textbf{that occur in the PDE (3.13)}. See section 3.3.

2. In section 3.4, \textbf{we analyze the linear operator} \(L_\Gamma\) \textbf{in Lemmas 3.7–3.10}. \textbf{We determine that} \(L_\Gamma\) \textbf{has two small eigenvalues and that the rest of its spectrum is well into the left half complex plane}. Moreover, \textbf{we obtain a bound on the \(\chi\)-norm of functions which do not have a contribution in the direction of the eigenvectors \(\Psi_\pm\) belonging to the small eigenvalues associated with} \(L_\Gamma\).

3. \textbf{Next, we start the RG method}. \textbf{We freeze a basepoint} \(\Gamma^0 := (\Gamma_1^0, \Gamma_2^0)\); that is, \textbf{we fix the front location, and we rewrite (3.13) once more}; see section 3.5. \textbf{Then, we project onto the eigenspace of the small eigenvalues} \(\lambda_\pm\), \textbf{to obtain the motion of the two fronts} \(\Gamma_1\) \textbf{and} \(\Gamma_2\) (3.45); \textbf{section 3.6}. \textbf{In section 3.7, we project onto the eigenspace} \(X_{\Gamma_0}\) \textbf{the space perpendicular to the eigenspace of the small eigenvalues} \(\lambda_\pm\). \textbf{The analysis of these projected equations gives a bound on the size of the remainder} \(Z(\xi, t)\) \textbf{on some time interval} \([0, t^*]\); \textbf{see Lemma 3.13}.

4. \textbf{At time} \(t = t^*\), \textbf{we renormalize by choosing a new basepoint} \(\Gamma^1 := (\Gamma_1^1, \Gamma_2^1)\), \textbf{and we show that the} \(\chi\)-\textbf{norm of the remainder} \(Z(\xi, t)\) \textbf{has the same asymptotic magnitude as before renormalization}; \textbf{see Lemma 3.17. Moreover, we show that the new basepoint} \(\Gamma^1\) \textbf{is near the location of the fronts from the previous step at time} \(t^*\); \textbf{\(\Gamma(t^*)\)}. \textbf{A repetition of step 3 and the above observation then bounds the remainder} \(Z(\xi, t)\) \textbf{for all time} (3.11). \textbf{See section 3.8}.

5. \textbf{With this estimate on the remainder} \(Z(\xi, t)\), \textbf{we further investigate the evolution of the two fronts} \(\Gamma_1\) \textbf{and} \(\Gamma_2\) (3.45); \textbf{section 3.9}. \textbf{This validates (3.12) and completes the proof}.
Remark 3.3. The results established in this paper are valid under the assumption that the fronts $\Gamma_{i+1}$ and $\Gamma_i$ do not interact strongly. To leading order, this translates into $\Delta \Gamma_i = O(\varepsilon^{-1})$, the assumption that has been imposed throughout this paper. However, the interaction between two neighboring fronts remains semistrong as long as $\Delta \Gamma_i = O(\varepsilon^{-\sigma})$ for some $\sigma > 0$. All results remain valid under this somewhat weaker assumption. The proofs for the more general results may become slightly more technical, though, since it may be necessary to incorporate (straightforward) higher order calculations. Therefore, we refrain from going into the details here.

Since the notation $O(\varepsilon^{-\sigma})$, $\sigma > 0$, plays a crucial role in this paper, we recall its definition. A quantity $Q(\varepsilon)$ is of $O(\varepsilon^{-\sigma})$ for some $\sigma > 0$ if there exists a $C > 0$, independent of $\varepsilon$, and an $\varepsilon_0 > 0$ such that $\varepsilon^\sigma|Q(\varepsilon)| > C$ for all $0 < \varepsilon < \varepsilon_0$.

3.3. Nonlinearity and residual. In this section, we bound the norms of the nonlinear term $N$ and the residual $R$, but before we do so we compute bounds on $\Phi_1$, the first component of $\Phi_T$ (3.6), in several norms.

Lemma 3.4. $\|\Phi_1\|_X = O(1)$.

Proof. To compute the $L^1$-norm of $(\Phi_1)_\xi$, observe that

$$(\Phi_1)_\xi = \frac{1}{2} \sqrt{2} \left( \text{sech}^2 \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_1) \right) - \text{sech}^2 \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_2) \right) \right).$$

Since $|\Gamma_1 - \Gamma_2| \geq C \varepsilon^{-\sigma}$ for some $\sigma > 0$, we obtain that

$$\|((\Phi_1)_\xi)_{L^1}\| = \left\| \frac{1}{2} \sqrt{2} \text{sech}^2 \left( \frac{1}{2} \sqrt{2}\xi \right) \right\| + \text{exp. small} = O(1).$$

By the assumptions on $\chi(\xi)$, we observe that also $\|\chi \Phi_1\|_{L^1} = O(1)$. $\blacksquare$

Now, we establish the following bound on the $\chi$-norm of $N(Z)$.

Lemma 3.5. $\|N(Z)\|_X \leq C \left\{ \|Z_1\|_X^2 + \|Z_1\|_X \right\}$, where $C$ is an $O(1)$ constant.

Proof. This follows immediately from (3.16), (3.5), and Lemma 3.4. $\blacksquare$

Next, we bound the residual $R$.

Lemma 3.6. $\|R\|_{L^\infty} = O(\varepsilon)$ and $\|R\|_X = O(\varepsilon)$.

Proof. We need only to prove the second bound on $R$, since the first bound then follows from (3.3). Moreover, since $R_2 = 0$ and $R_3 = 0$, we need only consider the $\chi$-norm of $R_1$. A short calculation shows that

$$(U_0)_\xi \xi + U_0 - (U_0)^3 = \frac{3}{2} \left( e^{\sqrt{2}(\Gamma_1 - \Gamma_2)} - 1 \right) \text{sech}^2 \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_1) \right) \text{sech}^2 \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_2) \right),$$

which is exponentially small. Therefore, the leading order behavior of $R_1$ is given by

$$(3.17) \quad R_1(\Phi_T) = -\varepsilon (\alpha (G_V * U_0) + \beta (G_W * U_0) + \gamma).$$

Now, by (3.4), the fact that $\|G_V\|_{L^1} = 1$, $\|G_W\|_{L^1} = 1$, and Lemma 3.4, we obtain the following bound for the leading order terms in $R_1$:

$$\|R_1\|_X = \varepsilon \|\alpha (G_V * U_0) + \beta (G_W * U_0) + \gamma\|_X$$

$$\leq \varepsilon \left( 2\|\alpha\|_{L^1} \|U_0\|_X + 2\|\beta\|_{L^1} \|U_0\|_X + |\gamma| \right)$$

$$\leq \varepsilon (2(|\alpha| + |\beta|)\|U_0\|_X + |\gamma|) = O(\varepsilon). \quad \blacksquare$$
We start by computing its spectrum \( \sigma(L_\Gamma) \) (3.15), with \( \Gamma \) fixed.

Moreover, the small adjoint eigenvectors \( \lambda \) are of leading order the same as the small eigenvectors.

Also observe that in [24],

\[
O(1) \text{ rate, which implies that } |R_i|_{L^\infty} = O(1).
\]

Since \( R_2 \) (\( R_3 \)), the second (third) diagonal entry of \( L_\Gamma \), it will decay slowly with an exponential rate of \( O(\varepsilon) \). Therefore, both \( \|R_2\|_{L^1} \) (\( \|R_3\|_{L^1} \)) and \( \|R_2\|_{L^2}^2 \) (\( \|R_3\|_{L^2}^2 \)) are \( O(\varepsilon^{-1}) \), while \( \|R_2\|_{L^\infty} = O(1) \) (\( \|R_3\|_{L^\infty} = O(1) \)).

Note that \( \psi_1(\xi) \) and \( \psi_2(\xi) \) are strongly localized functions around \( \xi = \Gamma_1 \) and \( \xi = \Gamma_2 \), respectively. Moreover, \( \|\psi_1\|_{L^1} = \|\psi_2\|_{L^1} = 2 \) (see Lemma 3.4), and \( \|\psi_1\|_{L^2}^2 = \|\psi_2\|_{L^2}^2 = \frac{2}{3}\sqrt{2} \).

Also observe that in [24], \( \mu = \infty \), which corresponds to the translation invariant eigenvalue \( \lambda^+ = 0 \).

**Lemma 3.8.** The adjoint operator \( L_\Gamma^* \) of \( L_\Gamma \) has the same two small eigenvalues \( \lambda_\pm \) as \( L_\Gamma \).

Moreover, the small adjoint eigenvectors \( \Psi_\pm \) associated with these small eigenvalues \( \lambda_\pm \) are to leading order the same as the small eigenvectors \( \Psi_\pm \) (3.19). Although the correction terms

]\( (3.19) \)

\[
\Psi_\pm = \begin{pmatrix} \psi_1 \pm \psi_2 \\ 0 \\ \varepsilon \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \end{pmatrix},
\]

with

\[
\psi_i(\xi; \Gamma_i) = \frac{\partial U_0}{\partial \Gamma_i} = \frac{1}{2} \sqrt{2} \sech^2 \left( \frac{1}{2} \sqrt{2}(\xi - \Gamma_i) \right), \quad i = 1, 2,
\]

and where a straightforward computation yields that \( \|R_i\|_{L^\infty} = O(1) \) for \( i = 1, 2, 3 \), \( \|R_1\|_{L^1} = O(1) \), and \( \|R_j\|_{L^1} = O(\varepsilon^{-1}) \), \( \|R_j\|_{L^2}^2 = O(\varepsilon^{-1}) \) for \( j = 2, 3 \).

**Proof.** The operator \( L_\Gamma \) is, to leading order, the same as the linear operator associated with a stationary 2-front solution, as studied in section 4 of [24]. Hence, its spectrum and its eigenfunctions are to leading order the same. Therefore, only the statements about the error terms \( R_i(\xi) \) do not follow immediately from [24]. Nevertheless, these estimates follow directly from the structure of the linear operator \( L_\Gamma \) and its eigenfunctions \( \Psi_\pm \).

Clearly, all the \( R_i \)'s must be bounded and integrable (since \( \Psi_\pm \) is an eigenfunction).

The structure of \( R_1 \) is determined by \( L_1 \), the operator at the (1,1) entry of \( L_\Gamma \). Thus \( R_1 \) must decay exponentially with a \( O(1) \) rate, which implies that \( \|R_1\|_{L^{\infty,1,2}} = O(1) \).

Since \( R_2 \) (\( R_3 \)), the second (third) diagonal entry of \( L_\Gamma \), it will decay slowly with an exponential rate of \( O(\varepsilon) \). Therefore, both \( \|R_2\|_{L^1} \) (\( \|R_3\|_{L^1} \)) and \( \|R_2\|_{L^2} \) (\( \|R_3\|_{L^2} \)) are \( O(\varepsilon^{-1}) \), while \( \|R_2\|_{L^\infty} = O(1) \) (\( \|R_3\|_{L^\infty} = O(1) \)).

Note that \( \psi_1(\xi) \) and \( \psi_2(\xi) \) are strongly localized functions around \( \xi = \Gamma_1 \) and \( \xi = \Gamma_2 \), respectively. Moreover, \( \|\psi_1\|_{L^1} = \|\psi_2\|_{L^1} = 2 \) (see Lemma 3.4), and \( \|\psi_1\|_{L^2}^2 = \|\psi_2\|_{L^2}^2 = \frac{2}{3}\sqrt{2} \).

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**Lemma 3.8.** The adjoint operator \( L_\Gamma^* \) of \( L_\Gamma \) has the same two small eigenvalues \( \lambda_\pm \) as \( L_\Gamma \).

Moreover, the small adjoint eigenvectors \( \Psi_\pm \) associated with these small eigenvalues \( \lambda_\pm \) are to leading order the same as the small eigenvectors \( \Psi_\pm \) (3.19). Although the correction terms
to the small adjoint eigenvectors, \( R_\Gamma \), may differ from (3.19), their norms are of the same order.

Proof. The adjoint operator of \( L_\Gamma \) is given by \( L_\Gamma^\dagger = L_\Gamma^\dagger \), so that \( L_\Gamma^\dagger \) has the same spectrum as \( L_\Gamma \). The associated eigenfunctions can be computed by the variation of constants formula, combined with the observation that the eigenvalues \( \lambda_\pm \) are small.

With these small (adjoint) eigenvectors at hand we split the normed space \( X \) (3.2) into the eigenspace \( X_\Gamma^C \) and its spectral complement \( X_\Gamma \), where the eigenspace \( X_\Gamma^C \) is spanned by the two small eigenvectors \( \Psi_\pm \) (3.19). To project on these two spaces, we introduce the spectral projection \( \pi_\Gamma \), which, in terms of the small (adjoint) eigenfunctions \( \Psi_\pm, \Psi_\pm^\dagger \), is given by

\[
\pi_\Gamma \Phi = \frac{(\Phi, \Psi_\pm^\dagger)}{(\Psi_\pm, \Psi_\pm^\dagger)} \Psi_\pm + \frac{(\Phi, \Psi_\pm^\dagger)}{(\Psi_\pm, \Psi_\pm^\dagger)} \Psi_\pm^\dagger,
\]

where \((\cdot, \cdot)\) denotes the standard \( L^2 \)-inner product. Note that we have, by Lemmas 3.7 and 3.8,

\[
(\Psi_\pm, \Psi_\pm^\dagger) = \begin{pmatrix} (\psi_1 \mp \psi_2) & \varepsilon (R_1 \mp \psi_2) & (\psi_1 \mp \psi_2) & \varepsilon (R_1) \\ 0 & 0 & 0 & 0 \end{pmatrix} = (\psi_1 \mp \psi_2) + \varepsilon (R_1 \mp \psi_2) + \varepsilon (R_1) + \mathcal{O}(\varepsilon).
\]

The complementary projection is defined by \( \tilde{\pi}_\Gamma = I - \pi_\Gamma \). The spaces \( X_\Gamma \) and \( X_\Gamma^C \) are thus determined by

\[
X_\Gamma = \{ \Phi \in X \mid \pi_\Gamma \Phi = 0 \} \quad \text{and} \quad X_\Gamma^C = \{ \Phi \in X \mid \tilde{\pi}_\Gamma \Phi = 0 \}.
\]

Since \( L_\Gamma \) is an analytic operator we can generate its semigroup by the Laplace transform of the resolvent. We define the contour \( \mathcal{C} \) as

\[
\mathcal{C}(t) := \{ t - i \mid t \in (-\infty, -\nu) \} \cup \{ -\nu + t \frac{\nu}{\nu} i \mid t \in (-\nu, \nu) \} \cup \{ -t + i \mid t \in (\nu, \infty) \},
\]

with \( \nu := \frac{1}{2} \min\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \} \). The contour \( \mathcal{C} \) splits the complex plane into two pieces, one containing the small eigenvalues \( \lambda_\pm \), while the other piece contains the rest of the spectrum of \( L_\Gamma \) and is bounded away from the origin in an \( \mathcal{O}(1) \) fashion. Moreover, the spectrum \( \sigma(L_\Gamma) \) is an \( \mathcal{O}(1) \) distance away from contour \( \mathcal{C} \) (Lemma 3.7 and [24]); see Figure 3. Thus, we generate the semigroup \( S \) associated with \( L_\Gamma \) restricted to the space \( X_\Gamma \) (3.22) by the contour integral

\[
S(t)F = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\lambda t} (\lambda - L_\Gamma)^{-1} F d\lambda,
\]

where we assume that \( F \in X_\Gamma \).
Figure 3. The contour $C$ splits the complex plane into two pieces, one containing the small eigenvalues $\lambda_\pm$, while the other piece contains the rest of the spectrum of $L_G$. Moreover, the spectrum $\sigma(L_G)$ is an $O(1)$ distance away from contour $C$. See [24] for more details on the structure of $\sigma_{ess}$.

**Lemma 3.9.** Assume that $F \in X_G$; then $\Phi = S(t)F$ satisfies

$$
\|\Phi\|_X \leq C e^{-\nu t} \|F\|_X.
$$

To prove this semigroup estimate, we first need to prove an intermediate lemma on the resolvent, as follows.

**Lemma 3.10.** There exists a constant $C > 0$ such that for all $\lambda$ an $O(1)$ distance from $\sigma(L_G)$, and for all $F \in X_G$ (3.22), the solutions $G$ to the inhomogeneous problem $(L_G - \lambda)G = F$ satisfy $\|G\|_X \leq C \|F\|_X$.

**Proof.** First, we observe by (3.4) and (3.9) that the solution $\tilde{g}_i$ to the inhomogeneous problem $(L_i - \lambda)\tilde{g}_i = \tilde{f}_i$, where $L_i$ is the operator in the $i$th element of the diagonal of (3.15), obeys

$$
\|\tilde{g}_i\|_X \leq C \|\tilde{f}_i\|_X,
$$

as long as $\lambda$ is at an $O(1)$ distance from the spectrum $\sigma(L_i)$ of $L_i$. Note that this is automatically satisfied for $\lambda$ which are $O(1)$ distance away from $\sigma(L_G)$.

Next, we write $G = (g_1, g_2, g_3)^t$ and $F = (f_1, f_2, f_3)^t$. Then,

$$
g_2 = (L_2 - \lambda)^{-1} \left( f_2 - \frac{g_1}{\tau} \right), \quad g_3 = (L_3 - \lambda)^{-1} \left( f_3 - \frac{g_1}{\eta} \right).
$$

By the above result (3.26), we know

$$
\|g_i\|_X \leq C (\|f_i\|_X + \|g_1\|_X), \quad i = 2, 3.
$$

Next, we define

$$
h(\xi) := f_1 + \varepsilon \alpha (L_2 - \lambda)^{-1} f_2 + \varepsilon \beta (L_3 - \lambda)^{-1} f_3,
$$
so that $g_1$ is implicitly determined by

$$g_1 = (L_1 - \lambda)^{-1} \left( h - \frac{\varepsilon \alpha}{\tau} (L_2 - \lambda)^{-1} g_1 - \frac{\varepsilon \beta}{\theta} (L_3 - \lambda)^{-1} g_1 \right).$$

Hence, solving for $g_1$,

$$g_1 = \left( I + \frac{\varepsilon \alpha}{\tau} (L_1 - \lambda)^{-1} (L_2 - \lambda)^{-1} + \frac{\varepsilon \beta}{\theta} (L_1 - \lambda)^{-1} (L_3 - \lambda)^{-1} \right)^{-1} (L_1 - \lambda)^{-1} h.$$ 

From (3.26) we obtain

$$\|(L_1 - \lambda)^{-1} (L_i - \lambda)^{-1}\|_{\chi \rightarrow \chi} = O(1), \quad i = 2, 3.$$ 

From the Neumann expansion of the inverse we have

$$\left\| \left( I + \frac{\varepsilon \alpha}{\tau} (L_1 - \lambda)^{-1} (L_2 - \lambda)^{-1} + \frac{\varepsilon \beta}{\theta} (L_1 - \lambda)^{-1} (L_3 - \lambda)^{-1} \right)^{-1} \right\|_{\chi \rightarrow \chi} = O(1).$$

Thus, we find, again by (3.26),

$$(3.28) \quad \|g_1\|_\chi \leq C \|(L_1 - \lambda)^{-1} h\|_\chi \leq C \left( \|f_1\|_\chi + \varepsilon \|f_2\|_\chi + \varepsilon \|f_3\|_\chi \right).$$

The proof of the lemma follows from the combination of (3.27) and (3.28).

Proof of Lemma 3.9. The contour $\mathcal{C}$ divides the complex plane into two pieces, and the spectrum $\sigma(L_\Gamma) \setminus \lambda_\pm$ is completely contained in one of these pieces. Moreover, the spectrum is an $O(1)$ distance away from the contour $\mathcal{C}$. Since by assumption $F \in X_\Gamma$, the result follows from Lemma 3.10. ■

Remark 3.11. Because of the specific $\chi$-norm we use, it is possible that we get extra point spectrum at the tip of the essential spectrum (compared to [24]). However, the essential spectrum is in the left half plane and an $O(1)$ distance away from the imaginary axis (3.18). Therefore, this “new” point spectrum does not generate instabilities, and we can neglect it.

3.5. Initializing the RG method. We use the RG method developed in [19] and adapted to singularly perturbed problems in [5, 14, 9]. We assume that the initial condition $\Phi_2(\xi, 0) = (U_2(\xi, 0), V_2(\xi, 0), W_2(\xi, 0))$ is close to the skeleton solution $\Phi_{T^*}(\xi)$,

$$(3.29) \quad \|Z_0^*\|_\chi := \|\Phi_2(\cdot, 0) - \Phi_{T^*}\|_\chi < \delta,$$

for some $T^*$ (see (3.6)) and some $\delta > 0$. Then, the following lemma holds.

Lemma 3.12. Let $0 < \varepsilon \leq \delta \ll 1$ be sufficiently small, and let $\Phi_2(\xi, t)$ and $\Phi_{T^*}(\xi)$ satisfy (3.29). Then the following hold:

(i) There exists a unique smooth operator $\mathcal{H} : X \rightarrow \mathbb{R}^2$ such that the function $\Phi_{T^*}(\xi)$ with $\Gamma^0 := G^* + \mathcal{H}(Z_0^*)$ satisfies $Z_0^*(\xi) := \Phi_2(\xi, 0) - \Phi_{T^*}(\xi) \in X_{\Gamma^0}$. Moreover, $\|Z_0^*\|_\chi = O(\delta)$.

(ii) If $Z_0^*(\xi) \in X_{\Gamma}$, for a $\Phi_{T^*}(\xi)$ of the form (3.6), then there exists a $C > 0$ such that

$$(3.30) \quad |\Gamma^0 - \Gamma^*| \leq C \|Z_0^*\|_\chi |\Gamma^* - \tilde{\Gamma}| \leq C \delta |\Gamma^* - \tilde{\Gamma}|.$$

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The first part of the lemma states that if the initial condition \( \Phi_2 \) is close to a function \( \Phi_{\Gamma^*} \) of the form (3.6), then there exists a basepoint \( \Gamma^0 = (\Gamma^0_1, \Gamma^0_2) \) such that \( \Phi_2 - \Phi_{\Gamma^0} \) is also small, and it is perpendicular to the space spanned by the small eigenvalues associated with \( L_{\Gamma^0} \). Moreover, the mapping \( (\Phi_2, \Gamma^*) \rightarrow \Gamma^0 \) given by \( \Gamma^0 := \Gamma^* + \mathcal{H}(Z^*_{\Gamma^0}) \) is smooth. The second part of the lemma concerns situations in which one wants to shift from one basepoint to another: if the initial perturbation is already perpendicular to the small eigenvalue space associated with an \( L_{\Gamma^0} \), then the distance \( |\Gamma^0 - \Gamma^*| \) between the new basepoint \( \Gamma^0 \) and \( \Gamma^* \) is small compared to the distance \( |\Gamma^* - \Gamma| \) between the old basepoint \( \Gamma \) and \( \Gamma^* \); see Figure 4.

**Proof.** Consider a \( \Gamma^0 \) such that \( \Phi_2(\xi, 0) = \Phi_{\Gamma^*}(\xi) + Z^0_*(\xi) = \Phi_{\Gamma^0}(\xi) + Z^0_0(\xi) \). The condition \( Z^0_0(\xi) \in X_{\Gamma^0} \) is equivalent to

\[
\pi_{\Gamma^0}(Z^0_0) = \pi_{\Gamma^0}(Z^0_0 + \Phi_{\Gamma^*} - \Phi_{\Gamma^0}) = 0.
\]

By (3.19) and Lemma 3.8, we obtain that

\[
\begin{align*}
\Lambda_1(\Gamma^0, Z^0_0) &:= (|Z^0_0 + \Phi_{\Gamma^*} - \Phi_{\Gamma^0}|_1, \psi_1(\Gamma^0_1)) = \mathcal{O}(\varepsilon), \\
\Lambda_2(\Gamma^0, Z^0_0) &:= (|Z^0_0 + \Phi_{\Gamma^*} - \Phi_{\Gamma^0}|_1, \psi_2(\Gamma^0_2)) = \mathcal{O}(\varepsilon),
\end{align*}
\]

where \( \psi_1 \) and \( \psi_2 \) are defined by (3.20) with \( \Gamma_1 = \Gamma^0_1 \) and \( \Gamma_2 = \Gamma^0_2 \), respectively. Note that, since the adjoint eigenvectors are zero to leading order in the second and third components (3.19), we do not need to consider the second and third components of (3.31).

Observe that \( \Lambda_i(\Gamma^*, 0) = 0 \) for \( i = 1, 2 \). The gradient of the map \( \Lambda = (\Lambda_1, \Lambda_2) \) with respect to \( \Gamma^0 \) at \( (\Gamma^*, 0) \) is given by

\[
\nabla_{\Gamma^0} \Lambda|_{(\Gamma^0=\Gamma^*, Z^0_0=0)} = \begin{pmatrix} ||\psi_1||^2_{L^2} & 0 \\ 0 & -||\psi_2||^2_{L^2} \end{pmatrix} + \mathcal{O}(\varepsilon),
\]

\[
\text{Figure 4. Schematic plot of the geometry of a curve of the 2-front solutions with initial condition } \Phi_2 \text{ in the } (\Gamma, \Phi)-\text{plane. It gives a geometrical interpretation of the spaces } X_{\Gamma^*} \text{ and } X_{\Gamma^0} \text{ and the perturbations } Z^0_0 \text{ and } Z^0_* \text{ analyzed in Lemma 3.12. Moreover, it also illustrates the manifold } M_{2,0}. \text{ The first part of the lemma states that } Z^0_0 \in X_{\Gamma^0} \text{ is small as long as } Z^0_* \text{ is small. According to the second part, if } Z^* \text{ belongs to a } X_{\Gamma}, \text{ then the distance between } \Gamma^0 \text{ and } \Gamma^* \text{ is of the asymptotic magnitude of the distance between } \Gamma^* \text{ and } \bar{\Gamma} \text{ times the asymptotic magnitude of } Z^*_0.\]

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where we have used that \( \frac{\partial}{\partial y}[\Phi_{10}]_1 = \mp \psi_i \) for \( i = 1, 2 \) to leading order; see (3.20). Thus, the map \( \Gamma \) is uniformly invertible near \((\Gamma^*, 0)\). Hence, for \( \delta \) sufficiently small, the implicit function theorem guarantees the existence of a unique smooth function \( H(Z_0^0) \) such that \( \Phi_{10} \) with \( \Gamma^0 := \Gamma^* + H(Z_0^0) \) satisfies (3.32), i.e., \( Z_0^0 \in X_{\Gamma^0} \), as introduced in the lemma.

To prove the second part of (i), we observe that the implicit function theorem also guarantees that \( H(Z_0^0) \) is uniformly \( O(1) \) Lipschitz and that \( H(0) = 0 \). This yields
\[
\|Z_0^0\|_\chi = \|Z_0^0 + \Phi_{10} + \Phi_{10} - \Phi_{10} - \Phi_{10}\|_\chi \leq \|Z_0^0\|_\chi + \|\Phi_{10} - \Phi_{10}\|_\chi \leq \delta + C|\Gamma^* - \Gamma^0| \leq \delta + C|\Gamma^* - \Gamma^0| \leq C\delta.
\]

For part (ii), we observe that if \( Z_0^0 \in X_\Gamma \), then \( ([Z_0^0], \psi_i(\tilde{\Gamma}_i)) = O(\varepsilon) \). Since \( \delta \geq \varepsilon \), substitution of this into (3.32) yields to leading order
\[
\left| \left[ \Phi_{10} - \Phi_{10} - \Phi_{10}, \psi_i(\tilde{\Gamma}_i) \right] = \left| \left[ Z_0^0, \psi_i(\Gamma_0^0) \right] \right| \leq M \left| \left[ Z_0^0, \psi_i(\Gamma_0^0) \right] \right| \leq M \left| \left[ Z_0^0, \psi_i(\Gamma_0^0) \right] \right|, \quad i = 1, 2.
\]

Next, we use the mean value theorem and (3.20) to obtain
\[
\left| \left[ \psi_i(\Gamma_i^{\text{mer}}, \psi_i(\Gamma_0^0)) \right] = \left| \psi_i(\Gamma_i^{\text{mer}}, \psi_i(\Gamma_0^0)) \right| \right| |\Gamma_0^0 - \Gamma_i^*|, \quad i = 1, 2,
\]
where \( \Gamma_i^{\text{mer}} \in (\Gamma_i^*, \Gamma_i^*) \). From part (i) we know that \( |\Gamma_i^0 - \Gamma_i^*| = O(\delta) \), so that to leading order in \( \delta \)
\[
|\psi_i(\Gamma_i^{\text{mer}}, \psi_i(\Gamma_0^0))| = |\psi_i|_2 = O(1), \quad i = 1, 2.
\]
Combining (3.35) with (3.36), we find that the left-hand side of (3.34) is proportional to \( |\Gamma_0^0 - \Gamma^*| \). To bound the right-hand side of (3.34), we use property (3.3),
\[
\left| \left[ Z_0^0, \psi_i(\Gamma_i^0) \right] \right| \leq \|Z_0^0\|_\chi \|\psi_i(\Gamma_i^0)\|_{L^1} \leq \|Z_0^0\|_\chi \|\psi_i(\Gamma_i^0)\|_{L^1}, \quad i = 1, 2.
\]
In order to control this \( L^1 \)-norm, we distinguish between two cases. First, assume that \( |\tilde{\Gamma} - \Gamma^0| > 4 \); then
\[
\|\psi_i(\tilde{\Gamma}_i) - \psi_i(\Gamma_0^0)\|_{L^1} \leq 2 \int_{-\infty}^{\infty} |\psi_i|_{d\xi} = 4 \leq |\tilde{\Gamma} - \Gamma^0|, \quad i = 1, 2.
\]
If \( |\tilde{\Gamma} - \Gamma^0| < 4 \), then we once more use the mean value theorem,
\[
\|\psi_i(\tilde{\Gamma}_i) - \psi_i(\Gamma_0^0)\|_{L^1} \leq |\tilde{\Gamma} - \Gamma^0| \int_{-\infty}^{\infty} |\psi_i(\Gamma_i^{\text{mer}}(\xi))|_{d\xi} = C|\tilde{\Gamma} - \Gamma^0|, \quad i = 1, 2.
\]
Thus, the right-hand side of (3.34) is bounded by \( C \|Z_0^0\|_\chi |\tilde{\Gamma} - \Gamma^0| \). Using the triangle inequality on \( |\tilde{\Gamma} - \Gamma^0| \), we obtain the desired result. □
Before we can initialize the first iteration step of the RG method, we need an a priori bound on its time step. Let \( t^*_t \) be the upper bound on the time step such that the remainder \( Z \) stays smaller than \( \sqrt{\epsilon} \), that is,

\[
(3.37) \quad t^*_t = \inf \{ t \mid \| Z(\cdot, t) \|_\chi > \sqrt{\epsilon} \}.
\]

This time step bound is well defined, and positive, by continuity of the remainder and by the assumption that the remainder is \( \mathcal{O}(\epsilon) \) small at \( t = 0 \); see Theorem 3.2 and Lemma 3.12(i) with \( \delta = \epsilon \). So, by construction, the remainder \( Z(\xi, t) \) stays \( \mathcal{O}(\sqrt{\epsilon}) \) small for all \( 0 \leq t \leq t^*_t \). A very rough estimate shows that \( t^*_t \) is at least \( \mathcal{O}(\epsilon^{-1/2}) \). The second time step bound, \( t^*_u \), is

\[
(3.38) \quad t^*_u := \frac{1}{4\nu} \log \epsilon,
\]

where we recall the definition of \( \nu \) from the line under (3.23). This second bound arises naturally from the forthcoming analysis; see Lemma 3.13. The actual time step bound, \( t^* \), is now defined as the minimum of the above two time step bounds,

\[
(3.39) \quad t^* := \min \{ t^*_u, t^*_t \}.
\]

We will show that \( t^*_u < t^*_t \), so that \( t^* = t^*_u \). See Lemma 3.13.

With this definition of the time step bound, we begin the first iteration of the RG method. We freeze the basepoint \( \Gamma = (\Gamma_1, \Gamma_2) = (\Gamma^0_1, \Gamma^0_2) = \Gamma^0 \) with \( \Gamma^0_1 < \Gamma^0_2 \) and \( |\Gamma^0_2 - \Gamma^0_1| = \mathcal{O}(\epsilon^{-1}) \), and we decompose the actual solution \( \Phi_2 \) into

\[
(3.40) \quad \Phi_2(\xi, t) = \Phi_{\Gamma(t)}(\xi) + Z^0(\xi, t) \quad \text{such that } Z^0(\xi, t) \in X_{\Gamma^0} \text{ for all } t \leq t^*,
\]

which can be done by Lemma 3.12(i). This decomposition transforms the nonlinear PDE (3.13) into

\[
(3.41) \quad \begin{aligned}
Z^0_t + \frac{\partial \Phi_{\Gamma}}{\partial \Gamma} \dot{\Gamma} &= R + L_{\Gamma^0} Z^0 + \Delta LZ^0 + N(Z^0), \\
Z^0(\xi, 0) &= Z^0_0,
\end{aligned}
\]

where \( \Delta L := L_{\Gamma} - L_{\Gamma^0} \), the secular term which measures the growth of the remainder \( Z^0 \) while \( \Gamma \) slides away from \( \Gamma^0 \), and \( Z^0_0 = Z^0_0 - \Phi_{\Gamma^0} - \Phi_{\Gamma^0} \) (see Lemma 3.12).

### 3.6. Projecting onto the small eigenspace \( X_{\Gamma^0} \)

In the next section, we project (3.41) onto the space \( X_{\Gamma^0} \) to derive estimates on the remainder \( Z^0(\xi, t) \). Here, we project onto the eigenspace \( X_{\Gamma^0} \), the space spanned by the small eigenvectors of the operator \( L_{\Gamma^0} \), to derive a rough version of the equation of the motion of the two fronts \( \Gamma_1 \) and \( \Gamma_2 \). Since \( Z^0(\xi, t) \in X_{\Gamma^0} \) for all \( t \leq t^* \) (3.40) and since the projection \( \pi_{\Gamma^0} \) commutes with the operator \( L_{\Gamma^0} \), we have that \( \pi_{\Gamma^0} L_{\Gamma^0} Z^0 = L_{\Gamma^0} \pi_{\Gamma^0} Z^0 = 0 \). We obtain the projected equation

\[
\pi_{\Gamma^0} \left( \frac{\partial \Phi_{\Gamma}}{\partial \Gamma} \right) = \pi_{\Gamma^0} \left( R + \Delta LZ^0 + N(Z^0) \right).
\]
By definition of \( \pi_{\Gamma_0} (3.21) \), this is equivalent to

\begin{equation}
\left( \frac{\partial \Phi_\Gamma}{\partial \Gamma} \hat{\Gamma}, \Psi_+^i \right) = \left( R + \Delta LZ^0 + N(Z^0), \Psi_+^i \right).
\end{equation}

Observe by (3.6), (3.7), and (3.20) that

\begin{equation}
\left( \frac{\partial \Phi_\Gamma}{\partial \Gamma} \hat{\Gamma} = \left( \begin{array}{c}
\frac{\partial \Phi_1}{\partial \Gamma} \\
\frac{\partial \Phi_2}{\partial \Gamma} \\
\frac{\partial \Phi_3}{\partial \Gamma}
\end{array} \right) \right)
\left( \begin{array}{c}
\hat{\Gamma}_1 \\
\hat{\Gamma}_2
\end{array} \right)
= \left( \begin{array}{c}
-\psi_1 \hat{\Gamma}_1 + \psi_2 \hat{\Gamma}_2 \\
-(G_V \ast \psi_1) \hat{\Gamma}_1 + (G_V \ast \psi_2) \hat{\Gamma}_2 \\
-(G_W \ast \psi_1) \hat{\Gamma}_1 + (G_W \ast \psi_2) \hat{\Gamma}_2
\end{array} \right).
\end{equation}

On the other hand,

\begin{equation}
(\psi_1, \psi_2) = \text{exp. small}, \quad \left( \psi_i, R_j^\dagger \right) = O(1),
\end{equation}

\begin{equation}
\left( G_V \ast \psi_i, R_j^\dagger \right) \leq \|G_V \ast \psi_i\|_{L^\infty} \|R_j^\dagger\|_{L^1} = O(\varepsilon) O(\varepsilon^{-1}) = O(1),
\end{equation}

\begin{equation}
\left( G_W \ast \psi_i, R_j^\dagger \right) \leq \|G_W \ast \psi_i\|_{L^\infty} \|R_j^\dagger\|_{L^1} = O(\varepsilon) O(\varepsilon^{-1}) = O(1),
\end{equation}

for \( i = 1, 2 \), and \( j = 2, 3 \), and where \( R_j^\dagger \) and \( \bar{R}_j^\dagger \) are defined in the proof of Lemma 3.8. Therefore, (3.42) reduces to leading order to

\begin{equation}
\left( \begin{array}{c}
-\|\psi_1\|^2_{L^2} + O(\varepsilon) & -\|\psi_2\|^2_{L^2} + O(\varepsilon) \\
-\|\psi_1\|^2_{L^2} + O(\varepsilon) & \|\psi_2\|^2_{L^2} + O(\varepsilon)
\end{array} \right)
\left( \begin{array}{c}
\hat{\Gamma}_1 \\
\hat{\Gamma}_2
\end{array} \right)
= \left( \begin{array}{c}
R + \Delta LZ^0 + N(Z^0), \Psi_+^i \\
R + \Delta LZ^0 + N(Z^0), \Psi_+^i
\end{array} \right).
\end{equation}

Note that the second and third components of \( R + \Delta LZ + N(Z) \) are identically zero. Therefore, inverting the matrix of the left-hand side, we obtain

\begin{equation}
\hat{\Gamma}_1 = \frac{-1}{\|\psi_1\|^2_{L^2}} \left[ R_1 + |\Delta LZ^0|_1 + [N(Z^0)]_1, \psi_1 \right] (1 + O(\varepsilon)),
\end{equation}

\begin{equation}
\hat{\Gamma}_2 = \frac{1}{\|\psi_2\|^2_{L^2}} \left[ R_1 + |\Delta LZ^0|_1 + [N(Z^0)]_1, \psi_2 \right] (1 + O(\varepsilon)).
\end{equation}

In section 3.9, we will further investigate these two ODEs, and we will validate (3.12). However, to do so, we first need a better bound on the remainder \( Z^0(\xi, t) \). This bound is obtained by projecting (3.41) onto \( X_{\Gamma_0} \).

**3.7. Projecting onto \( X_{\Gamma_0} \).** Projection of (3.41) onto \( X_{\Gamma_0} \) yields

\begin{equation}
Z_0^0 = \tilde{R} + L_{\Gamma_0} Z^0 + \tilde{\pi}_{\Gamma_0} \left( \Delta LZ^0 + N(Z^0) \right),
\end{equation}

\begin{equation}
Z^0(\xi, 0) = Z_0^0,
\end{equation}

with \( \tilde{R} = \tilde{\pi}_{\Gamma_0} \left( R - \frac{\partial \Phi_\Gamma}{\partial \Gamma} \hat{\Gamma} \right) \). The variation of constants formula applied to (3.46) yields

\begin{equation}
Z^0(\xi, t) = S(t) Z_0^0 + \int_0^t S(t - s) \left( \tilde{R} + \tilde{\pi}_{\Gamma_0} \left( \Delta LZ^0 + N(Z^0) \right) \right) ds,
\end{equation}
where $S$ is the semigroup generated by $L_{1^0}$; see (3.24). The main result of this section reads as follows.

**Lemma 3.13.** There exists a constant $C > 0$, independent of $\varepsilon$, such that the remainder $Z^0(\xi, t)$ stays $O(\varepsilon)$ small during one time step $t^* = t_1^* = \frac{1}{\varepsilon} \log |\varepsilon|$. More precisely,

\begin{equation}
\|Z^0(\cdot, t)\|_\chi \leq C (e^{-\nu t} \|Z_0^0\|_\chi + \varepsilon) \leq \varepsilon C \quad \text{for } t \in [0, t^*].
\end{equation}

Before we can prove this lemma, we need some intermediate results, Lemmas 3.14–3.16.

As a preliminary step we define two useful quantities:

\begin{equation}
T_0(t) := \sup_{0 \leq s \leq t} e^{\nu s} \|Z^0(\cdot, s)\|_\chi,
\end{equation}

\begin{equation}
T_1(t) := \sup_{0 \leq s \leq t} |\Gamma(s) - \Gamma^0|.
\end{equation}

The first quantity measures the growth of the remainder $Z^0$ in a weighted $\chi$-norm, and the latter measures the maximal distance a 2-front solution $\Phi$ has travelled from its basepoint $\Gamma^0$. Observe that, by the assumptions on the time step $t^*$, $T_0(t) = O(\varepsilon^{1/4})$. The fact that we have an a priori upper bound on $T_0$ is one of the reasons for imposing the special bounds (3.37) and (3.38). To bound $T_1(t)$ in terms of $T_0(t)$, we need estimates on the nonlinear term $N(Z^0)$ and the secular term $\Delta LZ^0$.

**Lemma 3.14.** There exists a constant $C > 0$ such that $\|N(Z^0)\|_\chi \leq C \|Z^0\|_\chi^2$ and $\|\Delta LZ^0\|_\chi \leq C |\Gamma - \Gamma^0| \|Z^0\|_\chi$ for $t \leq t^*$.

**Proof.** The nonlinear term $N(Z^0)$ has already been analyzed in Lemma 3.5. However, we now have an extra assumption on the magnitude of the remainder (3.37)–(3.39). Therefore, the bound on $N(Z^0)$ can be sharpened:

\begin{equation}
\|N(Z^0)\|_\chi \leq C \left( \|Z^0\|_\chi^2 + \|Z^0\|_\chi^3 \right) \leq C \|Z^0\|_\chi^2 \quad \text{for } t \leq t^*.
\end{equation}

The bound on the secular term $\Delta LZ^0$ follows from

\begin{equation}
\|\Delta LZ^0\|_\chi = \|3 (\Phi_1^2(\Gamma) - \Phi_2^2(\Gamma^0)) \|Z^0\|_\chi
\end{equation}

\begin{equation}
= \|3 (\Phi_1(\Gamma) - \Phi_2(\Gamma^0)) \|Z^0\|_\chi + \|3 (\Phi_1(\Gamma) - \Phi_2(\Gamma^0)) \|Z^0\|_\chi
\end{equation}

\begin{equation}
\leq C \left\{ \|\Phi_1(\Gamma) - \Phi_2(\Gamma^0)\|_{L^\infty} \|Z^0\|_\chi + \|\Phi_1(\Gamma) - \Phi_2(\Gamma^0)\|_{L^1} \right\}
\end{equation}

\begin{equation}
\leq C |\Gamma - \Gamma^0| \|Z^0\|_\chi + \|\Phi_1(\Gamma) - \Phi_2(\Gamma^0)\|_{L^1} \|Z^0\|_{L^\infty}
\end{equation}

where we used (3.3) and the Lipschitz continuity of $\Phi_1^2$ and $\Phi_2^2$.

Using (3.45), we can estimate $T_1(t)$ as follows.

**Lemma 3.15.** There exists a constant $C > 0$, independent of $\varepsilon$, such that $T_1(t) \leq C (\varepsilon t + T_0(t)^2)$ for $t \in [0, t^*]$.

Note that this implies that $T_1(t)$ is at most $O(\sqrt{\varepsilon})$.  

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Lemma 3.6. By assumption, it is now obvious that the \( \chi \)-norms of the components \( \tilde{\Gamma} \) are \( C \)-bounded. By Lemmas 3.6, 3.14, and 3.15, all three of the above norms are \( C \)-bounded. To prove the last bound, we need to show that the asymptotic magnitude of \( \tilde{\Gamma} \) can be estimated by

\[
\| S(t-s) \tilde{\Gamma}^0 (\Delta LZ^0) \|_{\chi} \leq Ce^{-\nu(t-s)} T_1 \| Z^0 \|_{\chi},
\]

(3.53)

where, besides (3.51) and (3.52), we used the facts that \( \psi_1 \in L^1 \) and \( \psi_2 \in L^1 \), (3.3), and Lemma 3.6. By assumption, \( T_0(t) \) is at most \( O(\varepsilon^{1/4}) \); this completes the proof.

Proof. The first two inequalities of (3.53) immediately follow from Lemma 3.14 combined with the observation that projections are bounded in the \( \chi \)-norm. To prove the last bound, we need to show that the asymptotic magnitude of \( \tilde{\Gamma} \) is \( O(\varepsilon) \). To do so, we observe that (3.45) combined with (3.3) gives

\[
\tilde{\Gamma}_i \leq C \left( \| R_1 \|_{\chi} + \| \Delta LZ^0 \|_{\chi} + \| N(Z^0) \|_{\chi} \right) , \quad i = 1, 2.
\]

By Lemmas 3.6, 3.14, and 3.15, all three of the above norms are \( O(\varepsilon) \) for \( t \in [0, t^*] \). Therefore, \( \tilde{\Gamma}_1 \leq \varepsilon C \) and \( \tilde{\Gamma}_2 \leq \varepsilon C \). Now, the second and third component of \( R - \frac{\partial}{\partial t} \tilde{\Gamma} \) can be estimated using the above observation together with (3.43) and (3.44),

\[
\left\| \left( \frac{\partial \Phi_2}{\partial \Gamma_1} \tilde{\Gamma}_1 + \frac{\partial \Phi_2}{\partial \Gamma_2} \tilde{\Gamma}_2 \right) \right\|_{\chi} \leq \varepsilon C \| G_V * (\psi_1 + \psi_2) \|_{\chi} \leq \varepsilon C,
\]

\[
\left\| \left( \frac{\partial \Phi_3}{\partial \Gamma_1} \tilde{\Gamma}_1 + \frac{\partial \Phi_3}{\partial \Gamma_2} \tilde{\Gamma}_2 \right) \right\|_{\chi} \leq \varepsilon C.
\]

It is now obvious that the \( \chi \)-norms of the components \( \tilde{R}_2 \) and \( \tilde{R}_3 \) are \( O(\varepsilon) \). The first component of \( \tilde{R} \) will also be \( O(\varepsilon) \). To see this, observe that, up to exponentially small terms,

\[
\pi_{\Gamma_0} \left( \frac{\partial \Phi_1}{\partial \Gamma_1} \tilde{\Gamma}_1 - \frac{\partial \Phi_1}{\partial \Gamma_2} \tilde{\Gamma}_2 \right) = \tilde{\Gamma}_1 \left[ \frac{(\psi_1^0, \psi_1^0)}{\| \psi_1^0 \|_{L_2}} \psi_1^0 - \psi_1^0 \right] + \tilde{\Gamma}_2 \left[ \psi_1^0 - \frac{(\psi_2^0, \psi_2^0)}{\| \psi_2^0 \|_{L_2}^2} \psi_2^0 \right],
\]

(3.54)
where \( \psi^O_i = \psi_i(\xi; \Gamma^O_i) \) while \( \psi^I_i = \psi_i(\xi; \Gamma_i(t)) \). Since the functions \( \psi_i(\xi) \) are Lipschitz continuous and in \( L^1 \), we can bound the \( \chi \)-norm of (3.54) by

\[
\left\| \tilde{\pi}_{T^0} \left( -\frac{\partial \Phi_1}{\partial \Gamma_1} \dot{\Gamma}_1 - \frac{\partial \Phi_1}{\partial \Gamma_2} \dot{\Gamma}_2 \right) \right\|_{\chi} \leq C \varepsilon \sqrt{\varepsilon},
\]

where we again used that \( \dot{\Gamma}_1 \leq \varepsilon C, \dot{\Gamma}_2 \leq \varepsilon C \), and Lemma 3.15. The \( \chi \)-norm of \( \tilde{\pi}_{T^0} R_1 \) is not larger than the \( \chi \)-norm of \( R_1 \), and from Lemma 3.6 we recall that \( \| R_1 \|_{\chi} = O(\varepsilon) \). Therefore, \( \| \tilde{R} \|_{\chi} = O(\varepsilon) \) for \( t \in [0, t^*] \).}

With the above three lemmas, we can now prove Lemma 3.13.

**Proof of Lemma 3.13.** Taking the \( \chi \)-norm of \( Z^O(\xi, t') \) (3.47) at \( t = t' \in [0, t^*] \), we find

\[
\left\| Z^O(\cdot, t') \right\|_{\chi} \leq C \left\{ e^{-\nu t'} \left\| Z^O_0 \right\|_{\chi} + \int_0^t e^{-\nu(t'-s)} \left( \varepsilon + T_1(s) \right) \left\| Z^O(\cdot, s) \right\|_{\chi} + \left\| Z^O(\cdot, s) \right\|_{\chi}^2 ds \right\}.
\]

Multiplying the above inequality by \( e^{\nu t'} \) and taking the supremum over \( t' \in (0, t) \), we find

\[
T_0(t) \leq C \left\{ T_0(0) + \varepsilon \int_0^t e^{\nu s} ds + T_1(t)T_0(t) \int_0^t ds + (T_0(t))^2 \int_0^t e^{-\nu s} ds \right\}
\]

\[
= \Rightarrow T_0(t) \leq C \left\{ T_0(0) + \varepsilon e^{\nu t} + T_1(t)T_0(t) + (T_0(t))^2 \right\}.
\]

Next, we eliminate \( T_1(t) \) from the above inequality by using Lemma 3.15,

\[
T_0(t) \left( 1 - \varepsilon Ct^2 \right) \leq C \left\{ T_0(0) + \varepsilon e^{\nu t} + t(T_0(t))^3 + (T_0(t))^2 \right\}.
\]

Since the time step \( t^* \ll \min\{T_0(t)^{-1}, \varepsilon^{-1/2}\} \), we can incorporate the cubic term into the quadratic term, and we can conservatively underestimate the left-hand side by \( T_0(t)/2 \). Thus, we obtain a simple quadratic inequality,

\[
T_0(t) \leq C \left\{ T_0(0) + \varepsilon e^{\nu t} + (T_0(t))^2 \right\}.
\]

Plainly, if (3.56) is satisfied, then so is (3.55). To study the inequality (3.56), we look at the related quadratic equation in \( T_0(t) \),

\[
(T_0(t))^2 - \frac{1}{C} T_0(t) + T_0(0) + \varepsilon e^{\nu t} = 0.
\]

Since \( T_0(0) + \varepsilon e^{\nu t} \leq O(\varepsilon^{3/4}) \ll 1 \), both roots of the quadratic, \( r_1 \) and \( r_2 \), are real and positive. To leading order, they have the form

\[
r_1 = 2C \left( T_0(0) + \varepsilon e^{\nu t} \right) \quad \text{and} \quad r_2 = \frac{1}{2C}.
\]

Thus, the values of \( T_0(t) \) satisfying (3.56) lie in the domain \( 0 < T_0(t) < r_1 \) and \( T_0(t) > r_2 \). Moreover, since \( T_0(0) < r_1 \) and \( T_0(t) \) is continuous, we know that

\[
T_0(t) \leq r_1 = 2C \left( T_0(0) + \varepsilon e^{\nu t} \right)
\]

for all \( t \leq t^* \). Using the definition of \( T_0(t) \) (3.49), we have completed the proof. 

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3.8. Iteration. In sections 3.5–3.7, we performed one step of the RG procedure. We found that in the time interval \([0, t^*]\) the remainder \(Z^0(\xi, t)\) with respect to the decomposition (3.40) stays \(O(\varepsilon)\) small. The next step of the RG procedure is to choose at time \(t = t^*\) a different basepoint, \(\Gamma^1 := (\Gamma_1^0, \Gamma_1^1)\), and to decompose the 2-front solution \(\Phi_2\) with respect to this new basepoint, as follows:

\[
\Phi_2(\xi, t) = \Phi_{\Gamma(0)}(\xi) + Z^1(\xi, t) \quad \text{such that} \quad Z^1(\xi, t) \in X_{\Gamma^1} \text{ for all } t^* \leq t \leq 2t^*;
\]

see Figure 4 (with, in the notation of this section, \(\tilde{\Gamma} \to \Gamma^0, \Gamma^* \to \Gamma(t^*), \Gamma^0 \to \Gamma^1, Z \to Z^0_0, Z^*_0 \to Z^0(t^*), \text{ and } Z^0_0 \to Z^1_0\)). The idea of the RG method is now to restart the procedure of sections 3.5–3.7 with the same PDE (3.41), but with the new basepoint \(\Gamma^1\) and the new initial condition \(Z^1_0\), and to show that the remainder \(Z^1(\xi, t)\) stays \(O(\varepsilon)\) small in the interval \([t^*, 2t^*]\). Of course, one first has to prove that this new basepoint \(\Gamma^1\) is not too far away from the location of the front at the end of the first time step \(\Gamma(t^*)\)—more precisely, that the renormalization has no leading order influence on the dynamics of the fronts, and that the new initial condition \(Z^1_0(\xi) := Z^1(\xi, t^*)\) is of order \(O(\varepsilon)\). This first step will be proved in Lemma 3.17. Then, to show that \(Z^1(\xi, t)\) stays \(O(\varepsilon)\) on the time interval \([t^*, 2t^*]\), we note that the analysis on \([0, t^*]\) presented in sections 3.5–3.7 depended only on the asymptotic quantities are of the same size for this second time interval, the analysis of those sections may be repeated to directly yield that \(Z^1(\xi, t)\) stays \(O(\varepsilon)\) small on \([t^*, 2t^*]\).

**Lemma 3.17.** Let \(Z^1_0(\xi)\) denote the initial condition of (3.41) on the second interval \([t^*, 2t^*]\), (3.58). Then,

\[
\|\Gamma^1 - \Gamma(t^*)\| = O(\varepsilon^2|\log \varepsilon|) \quad \text{and} \quad \|Z_0^1\|_X = O(\varepsilon).
\]

**Proof.** Since \(Z^0(\xi, t) \in X_{\Gamma^0}\) for all \(t \leq t^*\) by (3.40), (3.30) in Lemma 3.12 yields

\[
\|\Gamma^1 - \Gamma(t^*)\| \leq C\|\Gamma(t^*) - \Gamma^0\||Z^0(\cdot, t^*)\|_X.
\]

We use the definition (3.49) of \(T_1\) and Lemma 3.15, together with Lemma 3.13, to further estimate the right-hand side of the above inequality:

\[
\|\Gamma^1 - \Gamma(t^*)\| \leq C\|\Gamma(t^*) - \Gamma^0\||Z^0(\cdot, t^*)\|_X \leq C\varepsilon|\log \varepsilon| + T_1^2 \|Z^0(\cdot, t^*)\|_X \leq C\varepsilon|\log \varepsilon| + (\|Z_0^0\|_X + \varepsilon^{3/4})^2 \|Z^0(\cdot, t^*)\|_X \leq C\varepsilon|\log \varepsilon| + (\varepsilon + \varepsilon^{3/4})^2 \varepsilon \leq C\varepsilon^2|\log \varepsilon|.
\]

By continuity of \(\Phi_2(\xi, t)\) in \(t = t^*\) and Lipschitz continuity of \(\Phi_{\Gamma(0)}(\xi)\), we can now estimate \(\|Z^1_0\|_X\):

\[
\|Z^1_0\|_X \leq \|Z^1_0 - Z^0(\cdot, t^*)\|_X + \|Z^0(\cdot, t^*)\|_X \leq \|\Phi_2(\xi, t^*) - \Phi_{\Gamma(0)}(\xi) - \Phi_2(\xi, t^*) + \Phi_{\Gamma^1}(\xi)\|_X + \|Z^0(\cdot, t^*)\|_X \leq C\|\Gamma^1 - \Gamma(t^*)\| + \|Z^0(\cdot, t^*)\|_X \leq C\varepsilon^2|\log \varepsilon| + \varepsilon \leq C\varepsilon.
\]
3.9. Completion of the proof of Theorem 3.2. In this section, we finish the proof of Theorem 3.2. In the previous section, we established that the remainder $Z(\xi, t)$ also stays $O(\varepsilon)$ small in the second time interval. Repeating the same arguments, we can show that by our choosing a new basepoint $\Gamma^2$ at $2t^*$, the remainder $Z^2(\xi, t)$ also stays small in the interval $[2t^*, 3t^*]$. By this iterative procedure, and since $t^* \gg 1$, we can now conclude that the remainder $Z(\xi, t)$ stays $O(\varepsilon)$ small until $t^m$, that is, up to the moment we approach the boundary $\partial M_{20}$. Here, the analysis of the previous sections breaks down since the fronts approach the strong interaction regime, so that the inner product $(\psi_1, \psi_2)$ is no longer exponentially small; i.e., the fast fronts are no longer strongly localized. For instance, the derivation of (3.45) heavily relies on this fact. This proves (3.11).

To prove (3.12), we further analyze the expressions for $\dot{\Gamma}_1$ and $\dot{\Gamma}_2$ (3.45). Since we know that the $\chi$-norm of the remainder $Z(\xi, t)$ is $O(\varepsilon)$ small for all time up to $t^m$, we conclude from (3.52) and (3.51) that
\[
([\Delta LZ]_1, \psi_i) = O(\varepsilon^2 \log \varepsilon), \quad ([N(Z)]_1, \psi_i) = O(\varepsilon^2), \quad i = 1, 2.
\]
Now, the inner product involving $R_1$ is $O(\varepsilon)$. Therefore, the above terms are of higher order, and we can neglect them in (3.45). After neglecting exponentially small terms, the residual $R_1$ is given by (see (3.17))
\[
R_1(\Phi_1) = -\varepsilon (\alpha (G_V * U_0) + \beta (G_W * U_0) + \gamma).
\]
The projections of $G_V * U_0$ and $G_W * U_0$ can be explicitly computed using (3.7), (3.9), and (3.20)—see also the Melnikov integrals of [6]. To leading order, we obtain
\[
(R_1, \psi_i) = -2\varepsilon \left( \gamma - \alpha e^{\varepsilon (\Gamma_1 - \Gamma_2)} - \beta e^{\varepsilon (\Gamma_1 - \Gamma_2)} \right), \quad i = 1, 2.
\]
Finally, since $\|\psi_i\|_{L^2}^2 = \frac{2}{3} \sqrt{2}$, the evolution of $\Gamma_1$ and $\Gamma_2$ is to leading order indeed given by (3.12). This completes the proof of Theorem 3.2. 

4. The dynamics of $N$-front solutions. In this section, we analyze the dynamics of the fronts of $N$-front solutions. The system of ODEs describing the evolution of these fronts was formally derived in section 2, and this derivation was made rigorous in the previous section; see Theorem 3.2. In section 4.2, we classify the dynamics of all possible 2-front solutions, showing that the two fronts move toward a stable 1-pulse solution if and only if the system parameters are such that the 1-pulse solution is stable (for these parameters) and such that there is no unstable 1-pulse solution between the initial data and the attractor. In sections 4.3 and 4.4, we prove similar results for 3-front and 4-front solutions, respectively. For example, we determine the stability of front-solutions for which one or more of the fronts travel to infinity. To prove all the statements of these three sections, it is useful first to prove a statement for general $N$-front solutions. That is, $N$-front solutions for $N$ odd are not stationary, while $N$-front solutions for $N$ even do not travel with a uniform $O(\varepsilon)$ speed. This will be proved in the first section.

Note that Theorem 3.2 states that the derivation is valid up to time $t_m$, which can be $+\infty$. If $t_m < \infty$, the fronts enter the strong interaction regime after $t_m$. Here, the system of ODEs
no longer describes the dynamics of the fronts. For example, two components of the system of ODEs can cross, while this is not possible in the PDE case. The following conjecture is motivated by the numerical simulations; see Figures 7 and 10 below.

Conjecture 4.1. A pair of colliding fronts of the PDE (1.1) disappears for \( O(1) \) parameters with respect to \( \varepsilon \).

This yields that after collision between two of the fronts in the ODE description, these two fronts should be removed from the system. Therefore, the \( N \)-dimensional system of ODEs reduces to an \( N - 2 \)-dimensional system.

Remark 4.2. The trivial dynamics of a 1-front solution is completely captured by Lemma 2.1.

Remark 4.3. The derived systems of ODEs for the \( N \)-front dynamics (2.1) is a gradient flow. That is,

\[
\dot{\Gamma}(t) = -\nabla G(\Gamma(t)),
\]

where

\[
G = \frac{3}{2} \sqrt{2\varepsilon} \left( \gamma \left( \sum_{i=1}^{N} (-1)^{i+1} \Gamma_i \right) + \frac{\alpha}{\varepsilon} \left( \sum_{i,j=1, i<j}^{N} (-1)^{i+j} \varepsilon \delta(\Gamma_i - \Gamma_j) \right) \right.
\]

\[
+ \frac{\beta D}{\varepsilon} \left( \sum_{i,j=1, i<j}^{N} (-1)^{i+j} \varepsilon \delta(\Gamma_i - \Gamma_j) \right). \]

A direct consequence is that there do not exist solutions which are periodic in time (besides the stationary solutions). Therefore, it is immediately clear that the derived systems of ODEs cannot be valid for large \( \tau \) and \( \theta \), i.e., \( \tau, \theta = O(\varepsilon^{-2}) \), since in the full PDE stationary 1-pulse solutions (2-front solutions) may undergo a Hopf bifurcation for \( \tau, \theta = O(\varepsilon^{-2}) \) and thus create periodic motion; see Remark 1.1 and [6].

4.1. \( N \)-front solutions with \( N \) even and \( N \) odd. In this section, we investigate the differences between odd and even \( N \)-front solutions. By studying the total movement of the \( N \) fronts, we can prove the following lemma.

Lemma 4.4. Let \( 0 < \varepsilon \ll 1 \) be sufficiently small, and assume that all other parameters in (1.1) are \( O(1) \) with respect to \( \varepsilon \). Moreover, assume that \( \gamma \neq 0 \). Then, for \( N \) odd, there exist no stationary \( N \)-front solutions to (1.1), and for \( N \) even there exist no uniformly traveling \( N \)-front solutions with \( O(\varepsilon) \) velocity.

Proof. We begin with the case of \( N \) odd. The speed of the \( i \)th front is given by (2.1). Summing these front velocities over all \( N \) and noting the pairwise cancellations of terms for adjacent fronts, we find that for \( N \) odd,

\[
\sum_{i=1}^{N} \dot{\Gamma}_i(t) = \frac{3}{2} \sqrt{2\varepsilon} \gamma.
\]

Thus, there must be a net movement of the fronts in the direction given by the sign of \( \gamma \). The assumption that \( \gamma \neq 0 \) now proves the first part of the lemma.
For $N$ even, the sum of $N$ components is to leading order zero; see again (2.1). This yields that there can be no net movement of the $N$ fronts. Therefore, it is not possible to have a uniformly traveling $N$-front solution with an $O(\varepsilon)$ speed.

**Corollary 4.5.** For $N$ odd, at least one of the fronts of an $N$-front solution has to travel to $\pm\infty$, where the sign is determined by the sign of $\gamma$. If $N$ is even and one of the fronts of an $N$-front solution travels to $\pm\infty$, then at least one of the other fronts has to travel in the opposite direction, i.e., to $\mp\infty$.

It is of interest to observe that the dynamics of an $N$-front solution for which one of the fronts is far away from all of the others can be completely understood from the dynamics of an $(N - 1)$-front solution and those of a single front, independently. In this case, to leading order, the system of ODEs for the $N$-front solution breaks up into the ODE (2.10) for a 1-front solution and a system of ODEs (2.1) for an $(N - 1)$-front solution. Since for $N$ odd at least one of the fronts always travels to $\pm\infty$, this yields that, when looking at a fixed interval, an $N$-front solution with $N$ odd will eventually behave like an $M$-front solution, with $M < N$ even.

**Remark 4.6.** In the nongeneric case where $\gamma = 0$ there do exist stationary odd $N$-front solutions.

### 4.2. The 2-front solutions

The dynamics of the fronts of a 2-front solution can be deduced in a straightforward fashion from (3.12). Observe that the fronts travel with opposite velocities. Therefore, we can rewrite the system of ODEs for the front dynamics as one ODE for the dynamics of the distance, $\Delta \Gamma := \Gamma_2 - \Gamma_1$,

\[
\Delta \Gamma = 3\sqrt{2\varepsilon} \left( \alpha e^{\varepsilon \Delta \Gamma} + \beta e^{-\varepsilon \Delta \Gamma} - \gamma \right).
\]

The fixed points $\Delta \Gamma_i^*$, $i \in \mathbb{N}$, of this ODE coincide with the solutions of the existence condition for standing 1-pulse solutions as derived in [6]. In particular, there are either zero, one, or two solutions, depending on the signs of $\alpha$ and $\beta$ and on the size of $\alpha + \beta$ relative to $\gamma$; see Lemma 2.1 in [6]. The stability of these particular standing 1-pulse solutions is determined by the sign of the small eigenvalue $\lambda_1 := -3\sqrt{2\varepsilon}^2 \left( \alpha e^{-\varepsilon \Delta \Gamma^*_i} + \beta e^{-\varepsilon \Delta \Gamma^*_i} \right)$; see [24]. On a case-by-case basis, we draw the following conclusions about solutions $\Delta \Gamma(t)$: if (4.2) has no roots, then $\Delta \Gamma(t)$ tends to $\infty$ for all initial data $\Delta \Gamma(t)$ if and only if $\gamma < 0$; otherwise it tends to 0. If (4.2) has one root $\Delta \Gamma^*_i$, then if it is stable, $\Delta \Gamma(t)$ tends to $\Delta \Gamma^*_i$, whereas if it is unstable, $\Delta \Gamma(t)$ tends to 0 or to $\infty$, depending on the sign of $\Delta \Gamma(0) - \Delta \Gamma^*_i$. Finally, if (4.2) has two roots, one stable $\Delta \Gamma^*_1$ and the other unstable $\Delta \Gamma^*_2$, then we distinguish two cases: first, if $\Delta \Gamma^*_1 > \Delta \Gamma^*_2$, then initial conditions larger than the unstable root, that is, $\Delta \Gamma(0) > \Delta \Gamma^*_2$, tend to $\Delta \Gamma^*_1$, while smaller initial conditions tend to 0. Second, if $\Delta \Gamma^*_1 < \Delta \Gamma^*_2$, then initial conditions smaller than the unstable root, that is, $\Delta \Gamma(0) < \Delta \Gamma^*_2$, tend to $\Delta \Gamma^*_1$, while larger initial conditions tend to $\infty$. This yields the following lemma.

**Lemma 4.7.** The fronts of a 2-front solution $\Delta \Gamma(t)$ converge asymptotically to a standing 1-pulse solution with width $\Delta \Gamma = \Delta \Gamma^*_1$ if and only if this 1-pulse solution is stable and there is no unstable standing 1-pulse solution determined by $\Delta \Gamma^*_2$ with $\Delta \Gamma(0) < \Delta \Gamma^*_2 < \Delta \Gamma^*_1$ or $\Delta \Gamma(0) > \Delta \Gamma^*_2 > \Delta \Gamma^*_1$. 

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Figure 5. In these frames the parameters are \((\alpha, \beta, \gamma, D, \tau, 0, \varepsilon) = (6, -5, -2.5, 1, 1, 0.01)\). In the upper left frame, we plot \(\dot{\Delta \Gamma} (4.2)\) as a function of \(\Delta \Gamma\). Observe that \(\dot{\Delta \Gamma}\) has two zeroes, \(\Delta \Gamma^1_*\) and \(\Delta \Gamma^2_*\). The first zero is \(\Delta \Gamma^1_* \approx 109\), and the associated eigenvalue is \(\lambda_1(\Delta \Gamma^1_*) < 0\). The second zero is \(\Delta \Gamma^2_* \approx 440\), and the associated eigenvalue is \(\lambda_1(\Delta \Gamma^2_*) > 0\). Therefore, the 1-pulse solution with width \(\Delta \Gamma^1_*\) is stable, while the 1-pulse solution with width \(\Delta \Gamma^2_*\) is unstable. In the upper right frame, we plot the evolution of the pulse distance \(\Delta \Gamma\) according to the ODE (4.2) for two different initial conditions, one just below the unstable stationary width \(\Delta \Gamma^2_*\), and one just above this value. In the lower two frames, we plot the evolution of the \(U\) component of the PDE (1.1) for (approximately) the same two initial conditions. These plots are obtained from numerical simulations of (1.1). We observe that the dynamics of the two fronts \(\Gamma_1\) and \(\Gamma_2\) agrees to within the error of the asymptotic approximation with the dynamics of the derived ODE (4.2). More specifically, in the lower left frame the distance between the two fronts, \(\Delta \Gamma\), approaches 115, which is to leading order the same as \(\Delta \Gamma^1_* \approx 109\) (since \(115 - 1.09/\varepsilon + \mathcal{O}(1)\) for \(\varepsilon = 0.01\)). In the lower right frame the two fronts diverge, as described by Lemma 4.7.

Proof. Equation (4.2) has at most two fixed points \(\Delta \Gamma^1,^2_*\) for a given parameter combination [6]. The stability of these fixed points is determined by

\[
\lambda_1 = \frac{\partial \Delta \Gamma}{\partial \Delta \Gamma}.
\]

Since (4.2) is a one-dimensional, autonomous ODE, this proves the lemma.

See also Figure 5, where we plotted \(\dot{\Delta \Gamma}\) as a function of \(\Delta \Gamma\), as well as the solutions of the ODE (4.2) and the PDE (1.1) for two different initial conditions.

Due to the symmetry of the PDE (1.1), we immediately obtain a result on “2-back solutions,” that is, solutions that converge asymptotically to \((+1,+1,+1) + \mathcal{O}(\varepsilon)\) at \(-\infty\). These 2-back solutions turn out to be relevant for understanding the dynamics of 3- and 4-front solutions; see the next two sections.
Lemma 4.8. The ODE

\[ \dot{\Delta \hat{\Gamma}} = 3\sqrt{2\varepsilon} \left( \alpha e^{-\varepsilon \Delta \hat{\Gamma}} + \beta e^{-\varepsilon \Delta \hat{\Gamma}} + \gamma \right) \]

describes the evolution of the distance between the fronts of a 2-back solution \( \Delta \hat{\Gamma}(t) \). The fronts approach a standing 1-pulse solution (which converges asymptotically to \((+1,+1,+1) + O(\varepsilon)\)) with width \( \Delta \hat{\Gamma} = \Delta \hat{\Gamma}_{1}^* \) if and only if this 1-pulse solution is stable and there is no unstable standing 1-pulse solution (which converges asymptotically to \((+1,+1,+1) + O(\varepsilon)\)) determined by \( \Delta \hat{\Gamma}_{2}^* \) with \( \Delta \hat{\Gamma}(0) < \Delta \hat{\Gamma}_{2}^* < \Delta \hat{\Gamma}_{1}^* \) or \( \Delta \hat{\Gamma}(0) > \Delta \hat{\Gamma}_{2}^* > \Delta \hat{\Gamma}_{1}^* \).

Finally, it is worth noting that for \( \Delta \Gamma \gg \varepsilon^{-1} \), (4.2) reduces, to leading order, to \( \Delta \Gamma = -3\sqrt{2}\varepsilon\gamma \), and the dynamics is, just as in the case of 1-front solutions, completely determined by the sign of \( \gamma \). More specifically, the two fronts have a weak tail-tail interaction \( \Delta \Gamma \gg 1/\varepsilon \), and they can be interpreted as two single 1-front solutions; see Lemma 2.1.

4.3. The 3-front solutions. The dynamics of the fronts in 3-front solutions is quite rich. We deduce conditions under which 3-front solutions for which one of the outer fronts travels to \( \pm \infty \) are stable; see Corollaries 4.9 and 4.10. Moreover, we prove the existence of uniformly traveling 3-front solutions; see Lemma 4.11. The presence of the second inhibitor component \( W \) is necessary for the validity of this lemma; i.e., Lemma 4.11 does not hold for a two-component version of (1.1).

The system of ODEs describing the leading order behavior of the dynamics of these three fronts, up to collision, reads

\[
\begin{align*}
\dot{\Gamma}_{1} &= \frac{3}{2} \sqrt{2\varepsilon} \left( \gamma + \alpha \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) + \beta \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) \right), \\
\dot{\Gamma}_{2} &= -\frac{3}{2} \sqrt{2\varepsilon} \left( \gamma + \alpha \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) + \beta \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) \right), \\
\dot{\Gamma}_{3} &= \frac{3}{2} \sqrt{2\varepsilon} \left( \gamma + \alpha \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) + \beta \left( -e^{\varepsilon (\Gamma_{1}-\Gamma_{2})} + e^{\varepsilon (\Gamma_{1}-\Gamma_{3})} \right) \right).
\end{align*}
\]

By Lemma 4.4 and Corollary 4.5, we know that there are no stationary 3-front solutions and that at least one front has to travel to \( \pm \infty \).

To get additional insight in the dynamics of a 3-front solution, we first assume that (at least) the third front travels to \( +\infty \). Another case, in which \( \Gamma_{3} \) does not go to \( +\infty \) and in which the leftmost pulse travels to \( -\infty \), will be discussed later on. We introduce four new coordinates,

\[ B_{i} = e^{-\varepsilon \Gamma_{i}}, \quad \text{for} \quad i = 1, 2, 3, \quad \text{and} \quad t' = \frac{3}{2} \sqrt{2\varepsilon^{2}} t. \]

The rescaling of time absorbs the terms in front of the parentheses of (4.5) into the time-variable. The transformations of \( \Gamma_{i} \) are such that the fronts traveling to \( +\infty \) now travel to 0, while fronts which previously traveled to \( -\infty \) now travel to \( +\infty \). In the new coordinate system, the assumption \( \Gamma_{1} < \Gamma_{2} < \Gamma_{3} \) reads \( B_{3} < B_{2} < B_{1} \), and the system of ODEs (4.5) transforms into
Therefore, this eigenvalue can be seen as a translation invariant eigenvalue; i.e., only the stable if and only if
\[ \gamma > 0 \] implies a net movement to \(-\infty\); see (4.1). The sign of the second eigenvalue \(\lambda_2\) is the same as the sign of the small eigenvalue of a standing 1-pulse solution; see (4.3) and [24].

The eigenvectors belonging to the eigenvalues read
\[
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ B_x \\ 0 \end{pmatrix}, & e_2 &= \begin{pmatrix} 1 \\ -B_x \\ 0 \end{pmatrix}, & e_3 &= \begin{pmatrix} K_1 \\ K_2 \\ 1 \end{pmatrix},
\end{align*}
\]
where \(K_1\) and \(K_2\) are some computable constants. We observe that the eigenspace belonging to the neutral eigenvalue \(\lambda_1\) is, as expected, exactly the line of fixed points belonging to \(B_f\). Therefore, this eigenvalue can be seen as a translation invariant eigenvalue; i.e., only the distance between \(\Gamma_1\) and \(\Gamma_2\) (or the ratio of \(B_1\) and \(B_2\)) is important and not the actual location. Therefore, \(\lambda_1\) generates no instabilities, and we have shown that a fixed point \(B_f\) is stable if and only if \(\gamma > 0\) and \(\alpha DB_x^D + \beta B_x > 0\), where \(B_x\) solves (4.8). Transforming back to the original coordinates, we have proved the following corollary.

**Corollary 4.9.** Let \(B_x\) solve (4.8). Then, the 3-front solution for which the third front \(\Gamma_3\) travels to \(+\infty\) and the other two fronts \(\Gamma_1\) and \(\Gamma_2\) converge asymptotically to a 1-pulse solution with width \(-\frac{D}{\varepsilon} \log B_x\) is attracting if and only if \(\gamma > 0\) and \(\alpha DB_x^D + \beta B_x > 0\).
Figure 6. In these six frames, we plot six different types of behavior of the fronts of a 3-front solution with \( \text{sgn}(\alpha) = \text{sgn}(\beta) \). The values of \((\alpha, \beta, \gamma)\) vary from frame to frame, and \((D, \varepsilon)\) are fixed at \((5, 0.01)\). In the first and third frames, the 3-front solution evolves into a stable 1-front solution combined with a front traveling to \( \pm \infty \). In the other four cases, two of the fronts collide, and the system of ODEs (4.5) no longer describes the dynamics of the fronts of a 3-front solution to the PDE (1.1) after the collision. Compare this figure also with Figure 7, which shows the front locations in the corresponding PDE simulations.

See also frame III of Figures 6 and 7 for a plot of the system of ODEs in the original coordinates and a contour plot of a numerical simulation of the PDE (1.1), with system parameters satisfying the above corollary.

A similar analysis can be performed for the dynamics of fronts traveling to \(-\infty\). However, we now have to use a slightly different coordinate transformation,

\[
C_i = e^{\frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{D}}, \quad i = 1, 2, 3, \quad \text{and} \quad t' = \frac{3}{2} \sqrt{\frac{\varepsilon^2}{2D}},
\]

In these new coordinates, the system of ODEs has the line(s) of fixed points \((C_1, C_2, C_3) = (0, C_*, C_3, C_3)\), where \(C_*\) solves

\[
\alpha C_*^D + \beta C_* = -\gamma,
\]

which is the existence condition for a standing 1-pulse solution with \((U, V, W)(\pm \infty) \approx (1, 1, 1) + \mathcal{O}(\varepsilon)\), as well as the condition to have fixed points of (4.4); see also [6]. A linear stability analysis comparable to that of previous paragraph yields the following corollary.

**Corollary 4.10.** Let \(C_*\) solve (4.10). Then, the 3-front solution for which the first front \(\Gamma_1\) travels to \(-\infty\) and the other two fronts \(\Gamma_2\) and \(\Gamma_3\) converge asymptotically to a 1-pulse solution with width \(-\frac{D}{2} \log C_*\) is attracting if and only if \(\gamma < 0\) and \(\alpha D C_*^D + \beta C_* > 0\).

From Lemma 4.4, we know that uniformly traveling 3-front solutions may exist. It turns out from the next lemma that a necessary condition for this type of solution to exist is that the parameters \(\alpha\) and \(\beta\) have different signs. Thus, the third component of the PDE (1.1) is strictly necessary for the system to support traveling 3-front solutions; the two-component limit of the PDE, i.e., the system of PDEs without the third component and with \(\beta = 0\) (see [6, 24]), does not support uniformly traveling 3-front solutions.
Figure 7. In these six frames, we show the contour plots of six simulations of the PDE (1.1) for the same parameter values and the same initial conditions as in Figure 6. Thus, the values of \((\alpha, \beta, \gamma)\) vary from frame to frame, while the other parameters \((D, \varepsilon, \tau, \theta)\) are kept fixed at \((5, 0.01, 1, 1)\). The black regions indicate that the value of \(U = -1\), while white indicates \(U = 1\). In all the six frames, one of the outer fronts travels to \(\pm \infty\), depending on the sign of \(\gamma\); see Corollary 4.5. Frames 1 and 3, the two other fronts form a stable 1-pulse solution. In frames 2, 4, 5, and 6, the other two fronts collide and disappear; see Conjecture 4.1. The dynamics of the remaining fronts after these collisions is described by (2.10). Note that in all six cases the actual spatial domain of the simulation was \([-1000, 1000]\). So, we have zoomed in on the spatial domain where the interesting dynamics takes place.

Lemma 4.11. If the signs of \(\alpha\) and \(\beta\) are the same, then there exist no 3-front solutions traveling with uniform \(O(\varepsilon)\) speed to \(\pm \infty\). However, when the assumption is dropped, there exist parameter combinations for which (1.1) supports uniformly traveling 3-front solutions.

Proof. By (4.1) each of the fronts of a uniformly traveling 3-front solution has to travel with speed \(\frac{1}{2}\sqrt{2\varepsilon}\gamma\). Plugging this into (4.5), we find

\[
\begin{align*}
-2\gamma &= 3 \left( \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_1-\Gamma_3) \right) + \beta \left( -\frac{e^\varepsilon}{\tau}(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_1-\Gamma_3) \right) \right), \\
-4\gamma &= 3 \left( \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_2-\Gamma_3) \right) + \beta \left( -\frac{e^\varepsilon}{\tau}(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_2-\Gamma_3) \right) \right), \\
-2\gamma &= 3 \left( \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_3) + e^\varepsilon(\Gamma_2-\Gamma_3) \right) + \beta \left( -\frac{e^\varepsilon}{\tau}(\Gamma_1-\Gamma_3) + e^\varepsilon(\Gamma_2-\Gamma_3) \right) \right).
\end{align*}
\]

The third equation is a linear combination of the first two equations. Therefore, we can neglect the third equation and solve the system of the first two equations. Moreover, since only the distances between the fronts are important, there are only two unknowns, \(\Gamma_1-\Gamma_2\) and \(\Gamma_2-\Gamma_3\).
So, a priori, system (4.11) is solvable. Rewriting the first two equations, we find the equality

\[ \frac{1}{2} \alpha \left( e^{e(\Gamma_1 - \Gamma_2)} + e^{e(\Gamma_2 - \Gamma_3)} \right) + \frac{1}{2} \beta \left( e^{\frac{\Gamma_1}{\gamma}(\Gamma_1 - \Gamma_2)} + e^{\frac{\Gamma_2}{\gamma}(\Gamma_2 - \Gamma_3)} \right) = \alpha e^{e(\Gamma_1 - \Gamma_3)} + \beta e^{\frac{\Gamma_1}{\gamma}(\Gamma_1 - \Gamma_3)}. \]

(4.12)

By construction \( \Gamma_1 < \Gamma_2 < \Gamma_3 \), and therefore the following two inequalities hold:

\[ e^{e(\Gamma_1 - \Gamma_3)} < e^{e(\Gamma_2 - \Gamma_3)}, \quad e^{e(\Gamma_1 - \Gamma_3)} < e^{e(\Gamma_1 - \Gamma_2)}. \]

(4.13)

This yields that equality (4.12) cannot be fulfilled if \( \text{sgn}(\alpha) = \text{sgn}(\beta) \). Therefore, there cannot be uniformly traveling 3-front solutions if \( \alpha \) and \( \beta \) have the same sign.

To show that there are parameter combinations for which (4.12) holds if \( \text{sgn}(\alpha) \neq \text{sgn}(\beta) \), we prescribe the parameters \( \alpha \neq 0, D, \varepsilon \) and the front positions \( \Gamma_1, \Gamma_2, \Gamma_3 \), and choose \( \beta \) as the solution of (4.12), i.e.,

\[ \beta = -\frac{1}{2} \frac{e^{e(\Gamma_1 - \Gamma_2)} + e^{e(\Gamma_2 - \Gamma_3)}}{e^{\frac{\Gamma_1}{\gamma}(\Gamma_1 - \Gamma_2)} + e^{\frac{\Gamma_2}{\gamma}(\Gamma_2 - \Gamma_3)}} - e^{e(\Gamma_1 - \Gamma_3)}. \]

By (4.13) the numerator as well as the denominator are positive, and therefore \( \beta \) is well defined and nonzero (and \( \text{sgn}(\alpha) \neq \text{sgn}(\beta) \)). The first equality of (4.11) now determines a value of the last free parameter \( \gamma \) for which (4.11) is solved and a uniformly traveling 3-front solution thus exists. Note that this construction works for all given initial parameter combinations \( \alpha, D, \varepsilon, \Gamma_1, \Gamma_2, \Gamma_3 \).

See Figure 8 for a uniformly traveling 3-front solution. In the same figure a uniformly traveling 5-front solution is shown; the existence of this type of solution can be proved by similar (but more involved) arguments. This gives rise to the following conjecture.

**Conjecture 4.12.** For every odd \( N \) there exist uniformly traveling \( N \)-front solutions. For every even \( N \) there exist stationary \( N \)-front solutions.

We refer to [6, 24] for the proof of this conjecture for \( N = 2, 4 \).
After collision there are therefore three scenarios possible: the two remaining fronts merge, they form a stable solution simultaneously travel very slowly to $\pm \infty$. In frames IV and V, the solutions converge asymptotically to a stable 1-pulse solution homoclinic to $(U,V,W) = (1,1,1) + O(\varepsilon)$, while the other two fronts travel to $\pm \infty$. In frame VI, we see a stable 2-pulse solution; see also section 5 of [6] and section 6 of [24].

4.4. The 4-front solutions. As $N$ increases, the dynamics of the $N$ fronts naturally becomes more and more complex. A 4-front solution may, for example, evolve toward one of three types of stationary patterns, the ground state, a 1-pulse solution, or a 2-pulse solution; see Figures 9 and 10. The system of ODEs describing the evolution of the four fronts, up to collision, is obtained from (2.1) with $N = 4$,

\begin{align}
\dot{\Gamma}_1 &= \frac{3}{2} \sqrt{2} \varepsilon \left( \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1 - \Gamma_2)} + e^{\varepsilon(\Gamma_1 - \Gamma_3)} - e^{\varepsilon(\Gamma_1 - \Gamma_4)} \right) \\
&\quad + \beta \left( -e^{\varepsilon(\Gamma_1 - \Gamma_2)} + e^{\varepsilon(\Gamma_1 - \Gamma_3)} - e^{\varepsilon(\Gamma_1 - \Gamma_4)} \right) \right), \\
\dot{\Gamma}_2 &= -\frac{3}{2} \sqrt{2} \varepsilon \left( \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1 - \Gamma_2)} + e^{\varepsilon(\Gamma_2 - \Gamma_3)} - e^{\varepsilon(\Gamma_1 - \Gamma_4)} \right) \\
&\quad + \beta \left( -e^{\varepsilon(\Gamma_1 - \Gamma_2)} + e^{\varepsilon(\Gamma_2 - \Gamma_3)} - e^{\varepsilon(\Gamma_2 - \Gamma_4)} \right) \right), \\
\dot{\Gamma}_3 &= \frac{3}{2} \sqrt{2} \varepsilon \left( \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1 - \Gamma_3)} + e^{\varepsilon(\Gamma_2 - \Gamma_3)} - e^{\varepsilon(\Gamma_3 - \Gamma_4)} \right) \\
&\quad + \beta \left( -e^{\varepsilon(\Gamma_1 - \Gamma_3)} + e^{\varepsilon(\Gamma_2 - \Gamma_3)} - e^{\varepsilon(\Gamma_3 - \Gamma_4)} \right) \right), \\
\dot{\Gamma}_4 &= -\frac{3}{2} \sqrt{2} \varepsilon \left( \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1 - \Gamma_4)} + e^{\varepsilon(\Gamma_2 - \Gamma_4)} - e^{\varepsilon(\Gamma_3 - \Gamma_4)} \right) \\
&\quad + \beta \left( -e^{\varepsilon(\Gamma_1 - \Gamma_4)} + e^{\varepsilon(\Gamma_2 - \Gamma_4)} - e^{\varepsilon(\Gamma_3 - \Gamma_4)} \right) \right). 
\end{align}
Figure 10. In these frames, we plot the possible behaviors of a 4-front solution as seen in numerical simulations of the PDE (1.1). The black color indicates that $U = -1$, while white indicates $U = 1$. The first three frames, the upper row, correspond to the first frame of Figure 9. Two of the outer fronts collide and disappear. The two remaining fronts now behave in three possible ways: they collide and disappear (Frame 1), they form a stable 1-pulse solution (Frame 2), or they diverge (Frame 3); see also Lemma 4.7 and Figure 5. The next three frames, the middle row, correspond to the second frame of Figure 9. Here, the inner two fronts collide and disappear, and the remaining two fronts disappear (Frame 4), stabilize (Frame 5), or diverge (Frame 6); see again Lemma 4.7 and Figure 5. The last three frames, the lower row, correspond to III, IV, and VI, respectively, of Figure 9. None of the fronts collide, and we obtain two slowly diverging 1-pulse solutions (Frame 7), the outer fronts diverging while the inner fronts form a stable 1-pulse solution (Frame 8), or a stable 2-pulse solution (Frame 9). Note that for all the nine simulations the actual spatial domain was $\xi \in [-1000, 1000]$, $\tau = \theta = 1$, and $\varepsilon = 0.01$.

Of course, (4.14) has quite some structure, for instance, $\sum_{i=1}^{4} \dot{\Gamma}_i = 0$. In fact, the system has a two-dimensional invariant manifold

\begin{equation}
\mathcal{M}_0 := \{(\Gamma_1(t), \Gamma_2(t), \Gamma_3(t), \Gamma_4(t)) \mid \Gamma_4(t) = -\Gamma_1(t), \, \Gamma_3(t) = -\Gamma_2(t)\}.
\end{equation}
The manifold $\mathcal{M}_0$ can be interpreted as representing the dynamics of symmetric 2-pulse solutions within the larger family of 4-front interactions. Hence, if $\mathcal{M}_0$ is attracting, then the fronts will organize into two pairs of pulses; i.e., the front dynamics evolves into pulse dynamics.

Moreover, the fixed points of $\mathcal{M}_0$ can be determined by solving $\dot{\Gamma}_1(t) = 0$ and $\dot{\Gamma}_2(t) = 0$. These equations yield

$$
\begin{align*}
0 &= \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1-\Gamma_2)} + e^{\varepsilon(\Gamma_1+\Gamma_2)} - e^{2\varepsilon \Gamma_1} \right) \\
&\quad + \beta \left( -e^{\frac{\varepsilon}{2}(\Gamma_1-\Gamma_2)} + e^{\frac{\varepsilon}{2}(\Gamma_1+\Gamma_2)} - e^{2\frac{\varepsilon}{2} \Gamma_1} \right), \\
0 &= \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1-\Gamma_2)} - e^{\varepsilon(\Gamma_1+\Gamma_2)} + e^{2\varepsilon \Gamma_2} \right) \\
&\quad + \beta \left( -e^{\frac{\varepsilon}{2}(\Gamma_1-\Gamma_2)} - e^{\frac{\varepsilon}{2}(\Gamma_1+\Gamma_2)} + e^{2\frac{\varepsilon}{2} \Gamma_2} \right),
\end{align*}
$$

which coincides with the existence condition of stationary 2-pulse solutions constructed in Theorem 5.1 of [6]. The analysis of [24] establishes the (in)stability of the fixed points of $\mathcal{M}_0$, i.e., of the symmetric stationary 2-pulse solutions.

**Lemma 4.13.** If $\alpha > 0$ and $\beta > 0$, then the manifold $\mathcal{M}_0$ (4.15) is normally attracting (linearly stable), while it is normally repelling if $\alpha < 0$ and $\beta < 0$.

**Proof.** We linearize about points on $\mathcal{M}_0$. After a suitable rescaling of the time, we obtain that the linear evolution of the perturbation $v$ is given by the matrix equation

$$
\begin{pmatrix}
\dot{v}_1 \\
\dot{v}_2 \\
\dot{v}_3 \\
\dot{v}_4
\end{pmatrix} =
\begin{pmatrix}
A & C & -D & E \\
C & B & F & -D \\
-D & F & B & C \\
E & -D & C & A
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix},
$$

where

$$
\begin{align*}
A &= -C + D - E, \quad B = -C + D - F, \quad C = \alpha A_1 A_2^{-1} + \beta A_1^\frac{1}{2} A_2^{-\frac{1}{2}}, \\
D &= \alpha A_1 A_2 + \beta A_1^\frac{1}{2} A_2^\frac{1}{2}, \quad E = \alpha A_1^2 + \beta A_2^2, \quad F = \alpha A_2^2 + \beta A_2^2, \quad \text{and} \quad A_i = e^{\varepsilon \Gamma_i}, i = 1, 2.
\end{align*}
$$

Note that this matrix is singular since adding all the rows yields $(0, 0, 0, 0)$, and that it is symmetric across both diagonals. The eigenvalues and eigenvectors of the matrix read

$$
\begin{align*}
\lambda_1 &= 0, \quad e_1 = [1 \quad 1 \quad 1 \quad 1]^T, \\
\lambda_2 &= -2( C - D ), \quad e_2 = [1 \quad -1 \quad -1 \quad 1]^T, \\
\lambda_3 &= K_3, \quad e_3 = [1 \quad L_3 \quad -L_3 \quad -1]^T, \\
\lambda_4 &= K_4, \quad e_4 = [1 \quad L_4 \quad -L_4 \quad -1]^T,
\end{align*}
$$

where $K_i$ and $L_i$, $i = 3, 4$, are known constants. The eigenvalue-eigenvector pair $\lambda_1$ and $e_1$ correspond to uniform translations of the 4-front solution. Small perturbations in the direction of $e_1$ cause the positions of the four fronts to shift by the same constant small amount. Hence, the relative distances between pulses stay the same, and such perturbations do not destabilize the 4-front solution, since linear stability is only up to translates. Then, in order to study perturbations in the directions of the other three eigenvectors, we may mod out the translation.
invariance and assume, without loss of generality, that the components of these perturbations sum to zero. The third and fourth eigenvalues \( \lambda_3 \) and \( \lambda_4 \) have eigenvectors \( e_{3,4} \) along the direction of \( M_0 \). Therefore, these eigenvalues are not important for the stability result. Thus, the second eigenvalue \( \lambda_2 \), whose eigenvector \( e_2 \) is perpendicular to \( M_0 \), yields the stability result. It is given by

\[
\lambda_2 = -2(C - D) = 2\alpha A_1 (A_2 - A_2^{-1}) + 2\beta A_1^B (A_2^B - A_2^{-B}).
\]

Note that the eigenvalue \( \lambda_2 \) explicitly depends on time via \( A_1 \) and \( A_2 \). By construction \( \Gamma_1 < \Gamma_2 < 0 \), hence \( 0 < A_1 < A_2 < 1 \). This yields that \( \lambda_2(t) < 0 \) for all time \( t \) if \( \alpha > 0 \) and \( \beta > 0 \), while \( \lambda_2(t) > 0 \) for all time \( t \) if \( \alpha < 0 \) and \( \beta < 0 \). Thus, \( M_0 \) is normally attracting if \( \alpha, \beta > 0 \) and normally repelling if \( \alpha, \beta < 0 \) [3].

Recall from Corollary 4.5 that if one front escapes to \(+\infty\), there is always another front traveling to \(-\infty\). For a stability analysis of the fixed points at infinity, we therefore have to combine the coordinate transformations (4.6) and (4.9) of the last section. We introduce the new coordinates

\[(4.17) \quad C_i = e^\frac{t}{\beta} \Gamma_i \quad \text{for} \quad i = 1, 2, \quad B_i = e^{-\frac{t}{\beta} \Gamma_i} \quad \text{for} \quad i = 3, 4, \quad \text{and} \quad t' = \frac{3}{2} \sqrt{2} e^2 D t.\]

In these new coordinates, the system of ODEs (4.14) reads

\[
\begin{align*}
\dot{C}_1 &= C_1 \left( \gamma + \alpha \left( -\frac{C_i^D}{C_2^D} + C_i^P B_3^D - C_i^P B_4^D \right) + \beta \left( -\frac{C_1}{C_2} + C_1 B_3 - C_1 B_4 \right) \right), \\
\dot{C}_2 &= -C_2 \left( \gamma + \alpha \left( -\frac{C_i^D}{C_2^D} + C_i^P B_3^D - C_i^P B_4^D \right) + \beta \left( -\frac{C_1}{C_2} + C_2 B_3 - C_2 B_4 \right) \right), \\
\dot{B}_3 &= -B_3 \left( \gamma + \alpha \left( -C_i^D B_3^D + C_i^P B_3^D - B_3^D \right) + \beta \left( -C_1 B_3 + C_2 B_3 - \frac{B_3}{B_3} \right) \right), \\
\dot{B}_4 &= B_4 \left( \gamma + \alpha \left( -C_i^D B_4^D + C_i^P B_4^D - B_4^D \right) + \beta \left( -C_1 B_4 + C_2 B_4 - \frac{B_4}{B_3} \right) \right). 
\end{align*}
\]

This system has the line(s) of fixed points \((C_1, C_2, B_3, B_4) = (0, C_2, K_s/C_2, 0)\), where \( K_s \) solves

\[(4.19) \quad \alpha K_s^D + \beta K_s = -\gamma; \]

see (4.10). To determine the linear stability of one of these fixed points \( K_f = K_f(C_2; K_s) \), we linearize around this point. This yields the matrix

\[
\begin{pmatrix}
\gamma & 0 & \beta & 0 \\
0 & -(\alpha D K_s^D + \beta K_s) & 0 & \beta K_s^2 C_2^2 \\
\beta K_s^2 \frac{C_2}{c_2^2} & -(\alpha D K_s^D + \beta K_s) & \gamma & 0 \\
0 & 0 & \beta & \gamma
\end{pmatrix}.
\]

The four eigenvalues of this matrix read \( \lambda_1 = 0 \), \( \lambda_2 = -2(\alpha D K_s^D + \beta K_s) \), \( \lambda_3 = \gamma \), and \( \lambda_4 = \gamma \). The third and fourth eigenvalues are stable if \( \gamma < 0 \). The eigenvector belonging to
the neutral eigenvalue $\lambda_1$ points in the direction of the line of fixed points generated by $K_f$, so it yields no instabilities. The second eigenvalue is stable as long as $\alpha DK_1^D + \beta K_* > 0$, which is the same condition as for 3-front solutions (Corollary 4.10) and 1-pulse solutions (to $(U,V,W)(\pm \infty) = (-1, -1, -1) + O(\varepsilon)$; see [6, 24]). This proves the following corollary.

**Corollary 4.14.** A 4-front solution for which the outer two fronts $\Gamma_1$ and $\Gamma_4$ travel to $\pm \infty$, respectively, and for which the other two fronts $\Gamma_2$ and $\Gamma_3$ converge asymptotically to a 1-pulse solution with width $-\frac{D}{2} \log K_*$ is stable if and only if $\gamma < 0$ and $\alpha DK_1^D + \beta K_* > 0$, where $K_*$ solves (4.19).

See also Figure 9 frames IV and V, and Figure 10 frame 8.

There is another fixed point at infinity, namely the fixed point where two left fronts, $\Gamma_1$ and $\Gamma_2$, travel to $-\infty$, while the other two fronts, $\Gamma_3$ and $\Gamma_4$, travel to $+\infty$. To determine the stability of these fixed points we again need a transformation. First, we define $K_1$ ($K_4$) as the distance between $\Gamma_1$ and $\Gamma_2$ ($\Gamma_3$ and $\Gamma_4$). This way, we get a system of ODEs with the variables $\Gamma_1$, $\Gamma_4$, $K_1$, and $K_4$. Next, we use a transformation similar to (4.17),

$$C_1 = e^{\frac{D}{2} \Gamma_1}, \quad B_4 = e^{-\frac{D}{2} \Gamma_4}, \quad L_1 = e^{\frac{D}{2} K_1}, \quad L_4 = e^{\frac{D}{2} K_4},$$

and $t' = \frac{3}{2} \sqrt{\frac{\varepsilon^2}{D}} t$, to obtain a system of ODEs with the variables $C_1$, $B_4$, $L_1$, and $L_4$. Analyzing the fixed points of this system yields the following corollary.

**Corollary 4.15.** A 4-front solution for which the two left fronts $\Gamma_1$ and $\Gamma_2$ travel to $-\infty$ with a fixed width $-\frac{D}{2} \log L_1^*$ and for which the two right fronts $\Gamma_3$ and $\Gamma_4$ travel to $+\infty$ with a fixed width $-\frac{D}{2} \log L_4^*$ is stable if and only if $L_1^* = L_4^*$ and $\alpha D (L_1^*)^{-D} + \beta (L_4^*)^{-1} > 0$, where $L_1^*$ solves (4.19).

See also Figure 9 frame III, and Figure 10 frame 7.

**Remark 4.16.** System (4.14) actually possesses a 1-parameter family of invariant manifolds by translation invariance of the underlying PDE:

$$\mathcal{M}_K := \{ (\Gamma_1(t), \Gamma_2(t), \Gamma_3(t), \Gamma_4(t)) \mid \Gamma_4(t) = -\Gamma_1(t) + K, \Gamma_3(t) = -\Gamma_2(t) + K \},$$

where $K \gg 2\Gamma_2$. Each of these manifolds is normally attracting if $\alpha > 0$ and $\beta > 0$, while they are normally repelling if $\alpha < 0$ and $\beta < 0$.

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