Spatially periodic multi-pulse patterns in a generalized Klausmeier–Gray–Scott model

Lotte Sewalt† ‡ and Arjen Doelman†

Abstract. Semi-arid ecosystems form the stage for a plethora of vegetation patterns; a feature that has been captured in terms of mathematical models since the beginning of this millennium. To study these patterns, we use a reaction-advection-diffusion model that describes the interaction of vegetation and water supply on gentle slopes. As water diffuses much faster than vegetation, this model operates on multiple timescales. While many types of patterns are observed in the field, our focus is on two-dimensional stripe patterns. These patterns are typically observed on sloped terrains. In the present idealized setting they correspond to solutions of the model that are spatially periodic in one direction – the $x$-direction – and are extended trivially in the perpendicular $y$-direction. The existence of long wavelength patterns in our model is established analytically using methods from geometric singular perturbation theory, in which a correct parameter scaling is crucial. Subsequently, an Evans function approach yields statements about their stability. Previous work has shown numerically that in order for stripe patterns to be stable, a sufficient slope is needed. The instability of the two-dimensional patterns in parameter regimes corresponding to gentle slopes is confirmed analytically in this article, and we show that the ecological resilience of stripe patterns increases with increasing slope. A full stability proof for steep slopes is, however, beyond the scope of this asymptotic analysis. Since the main destabilization mechanism for the constructed two-dimensional stripes is via perturbations in the transverse direction, we provide a detailed overview of stability results of one-dimensional, spatially periodic patterns and show that they can be stable.

Key words. spatial ecology, reaction-advection-diffusion equations, singular perturbations, Evans function

AMS subject classifications. 34C25, 34C37, 35B25, 35B35, 35B36, 35C07, 92D40

1. Introduction. Human activity and climate change have stressed life on the Earth’s surface. Especially in the drylands the effects are tangible, as persistent soil degradation has led to barren areas, unsuitable for agriculture. To combat this process of desertification is set as one of seventeen global goals on the United Nations’ sustainable development agenda [47]. The need for a better insight in arid ecosystems is thus widely acknowledged. In the absence of grazing, vegetation growth is mainly limited by scarcity of water and nutrient. Therefore, homogeneously vegetated areas may turn into bare soil as a result of decreasing precipitation. Several intermediate stages, where the terrain is partly vegetated and partly barren, are observed, all with strikingly regular patterns [41]. In the case of a flat terrain, the transition from homogeneous vegetation to bare soil goes via hole, labyrinth, and spot patterns, [52, 54]. On a sloped terrain, however, the labyrinth patterns self-organize spatially as stripe patterns, parallel to the terrain’s contours, [73]. Moreover, patterns on flat terrain have been reported to be stationary, while the stripe patterns slowly move uphill, [9, 35]. Although the evolution from fertile to barren soil is not instantaneous, it is catastrophic in the sense that it is nearly irreversible; an increase of water availability does not automatically cultivate barren terrain to be fertile again. Therefore, vegetation patterns not only announce an early warning signal for

†Leiden University, Mathematical Institute, Niels Bohrweg 1, 2333CA Leiden, The Netherlands.
‡Supported by the Netherlands Organization of Scientific Research (NWO) through the NDNS+ cluster.
patterns in a gKGS model
desertification, but conversely provide a starting point for the development of a homogeneously vegetated state.

The mathematical models developed to study the mechanisms responsible for this pattern formation are mostly of reaction-diffusion type. Some models are two-component systems regarding the interaction of plant density with water density, [53, 67, 74, e.g], while others include competition for surface water, see [30, 51], or [25, 42, e.g.]. The effect of grazing of the region in question is also expected to have a significant impact, and is taken into account in [25, 43, e.g.]. Typically, the patterns originate from a Turing bifurcation in which a homogeneously vegetated state becomes unstable with respect to spatially heterogeneous perturbation [62]. The catastrophe is then explained by the fact that there is a multi-stable parameter regime, in which various vegetation patterns and the bare soil state are attracting [52, 54, 57, 64]. As water availability decreases, the vegetation pattern that originates from the Turing bifurcation destabilizes. Typically the system will ‘jump’ – i.e. evolve at a relatively fast time scale – to another vegetation pattern, that destabilizes again as the water availability increases further. After a number of such jumps, the system collapses into the bare soil state [55, 58, 61]. This is subtle and largely unexplained process, both the rate at which the water availability decreases and the magnitude of noise in the system have a decisive but so far not understood impact [7, 55, 58, 60, 61]. Moreover, the process is not simply reverted as water increases, because the bare soil state typically remains stable. The water availability should be increased far beyond the value at which the collapse takes to re-enter again into a vegetated state. This is also referred to as hysteresis.

The focus of this article is on the striped vegetation patterns, which are observed on sloped terrains, [2, 9, 10, 73]. This slope induces a downwards flow of precipitation in the form of surface water, which is then modeled by an advection term, and was introduced by C.A. Klausmeier in 1999, [35] – see [56] and the references therein for an extended analysis of this model. Klausmeier’s model has two components, for water $U$, and vegetation $V$, and is subjected to a rescaling in [64], after which it is of the form

$$
\begin{align*}
U_t &= A(1 - U) - UV^2 + CU_x, \\
V_t &= D\Delta_{x,y}V - BV + UV^2,
\end{align*}
$$

(1.1)

where $A$ controls water input, $B$ is the natural death rate of vegetation, $C$ is the rate at which water flows downhill and $D$ is a diffusion coefficient. All coefficients are positive. The diffusion $\Delta$ models vegetation spread on a two-dimensional, infinite domain, $(x, y) \in \mathbb{R}^2$. Naturally, $t \in \mathbb{R}^+$ is the temporal coordinate.

The observation that vegetation patterns also occur in the absence of a slope, motivated an extension of model (1.1). By introducing (nonlinear) diffusion of water – a priori in the form of an additional term $\Delta_{x,y}U^\nu$ in (1.1) (for some $\nu > 1$) [25, 64] – the spread is also modeled by a term that has no preferred direction. Extensive comparisons in [64] show that nonlinear diffusion does not induce significant qualitative differences to the dynamics exhibited by the model – especially since the patterns considered here have uniformly positive $U$-values [44, 64]. This motivates that in this article, we focus specifically on linear diffusion of water, i.e. we
Lotte Sewalt and Arjen Doelman

set $\nu = 1$ and study

\begin{align}
U_t &= \Delta_{x,y} U + A(1 - U) - UV^2 + CU_x, \\
V_t &= \delta^2 \Delta_{x,y} V - BV + UV^2. 
\end{align} (1.2)

Because it is natural to assume that water diffuses faster than vegetation, $0 < \delta \ll 1$ is a small parameter. The homogeneous background state $(U, V) = (1, 0)$ is interpreted as a desert state; a constant water availability, yet with no vegetation. The main parameter, $A$, relates to precipitation, $B$ models the vegetation’s death rate and $C$ models advection of water induced by a sloped terrain. Furthermore we assume $A, B > 0$, and $C \geq 0$. Under the assumption $A > 4B^2$, there are two more homogeneous steady states, $(U_{\pm}, V_{\pm})$ which are vegetated. We refer to (1.2) as the \emph{generalized Klausmeier–Gray–Scott (gKGS) model}, because for $C = 0$, the model reduces to the Gray–Scott model, \cite{28}. The model in \cite{28} describes autocatalytic reactions, but as their setting is in a continuously stirred tank reactor, no diffusion is taken into account \cite{28}. The Gray–Scott ODE system, extended with diffusion, is what is currently referred to as the Gray–Scott model, and was introduced in \cite{5, 50}.

It is widely known that both the Gray–Scott and the Klausmeier model exhibit a plethora of spatial patterns, \cite{35, 49}, and the gKGS-model captures patterns on both sloped (Klausmeier) and flat (Gray–Scott) terrains. To describe this phenomena mathematically, the existence of solutions with certain properties (wave number, amplitudes) is usually proved. However, stability analysis of these patterns is a necessary next step, as one does not expect to observe patterns which are unstable. To bridge the gap between existence of regular patterns in a mathematical setting and observable patterns, we consider the so-called \emph{Busse balloon}, introduced in \cite{4} and generalized to reaction-diffusion equations in \cite{19, 45}. This balloon is a region in (wave number, parameter)-space in which patterns are stable against perturbations. In the context of vegetation patterns, the natural choice for the parameter is precipitation, $A$. Multistability of patterns with different wavelengths explains the hysteresis mentioned earlier. The onset of patterns, an extremum of the Busse balloon, occurs through a Turing bifurcation, while at the other end of the balloon, only patterns with very small wave numbers, are stable. The latter coincides with the homoclinic limit of periodic patterns confirmed by, for example \cite{37, 38, 48, 63}.

Research in the setting of system (1.2) has been fruitful at the interface of ecology and mathematics. Several Busse balloons for the gKGS system were derived using numerical continuation in \cite{64}. In the same article, analytic control over the beginning of pattern formation is gained through the analysis of the complex Ginzburg-Landau equation associated with the pattern’s amplitude. Although system (1.2) is written in a slightly different way in \cite{58, 61}, the instability mechanism for stripe patterns is unfolded further in these two articles. Using numerical continuation and simulations, \cite{61} reports an extensive study of the destabilization of stripe patterns under the influence of a slowly decreasing parameter $A$. For flat terrains, the existence of nontrivial two-dimensional patterns of stripe and rhomb type and to some extent also their stability, is established in \cite{58}. An analytical result is formulated in theorem 3 of that article, which states that stripe patterns are unstable in the Gray–Scott model, i.e. on flat terrain, described by the $C = 0$ case of (1.2). For nonzero slope $C$, however, the stability is derived only using numerical continuation. These numerics suggest that as the slope of the
terrain increases, the stability regime of stripe patterns spreads from the Turing bifurcation down. Eventually, for a slope large enough, even patterns with the smallest wave numbers are stable. Observations in nature also support this theory, because stripe patterns occur on sloped terrain [2,9,10]. Yet, analytical results on existence and stability of stripe patterns with small wave numbers have not been reported so far. To justify both numerics and to test the ability of the gKGS model to describe natural observations, this analysis is precisely the focus of the current paper. Our methods are based on those used for multiscale pulses in the Gray–Scott model, which as a long history of analytical studies that are discussed in section 1.1. Due to the advective term in (1.2), this truly is a nontrivial task; the symmetries that are used extensively in the proofs in for example [13,18], break as soon as \( C \neq 0 \). Hence, the proof of existence of pulse patterns with small wave numbers in (1.2) – theorem 3.1 of this paper – requires mathematical methods that go beyond the established framework of geometric singular perturbation theory.

1.1. Existence and stability results of the Gray–Scott model. In this article, we exploit the singularly perturbed nature of (1.2), (recall \( 0 < \delta \ll 1 \)) and use asymptotic analysis to prove existence and stability of stripe patterns. Our approach mainly follows the techniques developed for generalized Gray–Scott type systems in [13–15,18,45]. It involves geometric singular perturbation methods as well as an Evans function formulation, using the slow/fast structure of the problem and the evaluation of the solutions of a nonlocal eigenvalue problem.

In order to clearly compare the results, we summarize briefly the relevant results for the Gray–Scott model, (1.2) with \( C = 0 \). In one spatial dimension, the existence of homoclinic \( N \)-pulses with \( N \geq 1 \) is covered in theorem 4.1 of [18] (for a certain special scaling of the parameters) – see also Figure 1. These pulse solutions correspond to a single strip or \( N \) stripes of vegetation in an elsewhere bare terrain. An \( N \)-pulse is constructed as a homoclinic solution to the desert background state \((U, V) = (1,0)\) that makes \( N \) fast excursions in a small spatial regime in which \( V \) is large. Already in the early publications on existence of pulses in the Gray–Scott model, the asymptotic scaling of the parameters and variables proves to be a crucial step in the analysis; a feature we also encounter for \( C \neq 0 \) throughout this article. In terms of the small parameter \( \delta \), the \( V \)-component of the pulse solutions constructed in [18] have asymptotically large amplitudes in a small spatial interval, while being exponentially small outside the pulse region. The pulse solutions of [45] have less restrictions on relative magnitude of the \( U \) and \( V \)-component, and in [37,38], an even broader parameter spectrum is analyzed in which pulse solutions exist. The rescaled, singularly perturbed system gives rise to a geometric singular perturbation analysis, where solutions of the limiting slow and fast system are concatenated according to classical Fenichel theory [21,22,33]. Periodic extensions of the \( N \)-pulses are characterized by slow components with long length scaled with a negative power of \( \delta \), in which the \( V \)-coordinate is exponentially small. That is, the slow/fast structure of these periodic solutions remains to be clearly distinguished. As stated in theorem 4.2 of [18], there exists a 1-parameter family of stationary periodic solutions, and the proof relies highly on the reversible symmetry of (1.2) with \( C = 0 \). In [58], this result has been extended to a much wider region in parameter space than considered in [18]. With a trivial extension in \( y \)-direction, the existence proof remains valid for two dimensions, but of course the extra spatial freedom gives way to more complex patterns. The existence of spot and multispot
patterns on bounded two-dimensional domains have been proved in [69, 70].

![Figure 1: Simulations of homoclinic 1-pulses in system (1.2) in one spatial dimension x and δ = 0.01 with C = 0 (Gray-Scott) in the top panels and C = 1 (gKGS) in the bottom panels. Left, a plots of U at t = 50 are depicted, in the middle V at t = 50 is depicted, and on the right we see a surface plot of V against both x and t. As parameter values we have used the sets A = 4, B = 1.8, C = 0 in the Gray-Scott case, and A = 1.4, B = 0.6, C = 1 in the gKGS case. Note that the V-pulse is stationary for C = 0, and travels with a constant speed to the right for C = 1.](image)

The stability of homoclinic and periodic pulse patterns in the one-dimensional Gray–Scott is further analyzed in [13], where again scaling is essential, especially the relative magnitude of parameter B. From the stability problem, a nonlocal eigenvalue problem (NLEP) is formulated, and the eigenfunctions are constructed using matched asymptotic expansions. The NLEP is subsequently solved using hypergeometric functions, which reveals that for B small enough with respect to A, the 1-pulse is stable. There is a Hopf bifurcation through which it loses/gains stability in a specific B-regime, and the 1-pulse is unstable if B becomes larger than that. In the more general setting, the terminology of a Busse balloon for patterns in the Gray–Scott model was used as early as in [45]. Of course, as the authors are considering only one spatial dimension, the stability results do not necessarily hold for two-dimensional solutions, albeit with a trivial extension in the y-direction, as perturbations in the transverse direction are not taken into account. For spot and stripe patterns in two dimensions, the instability mechanisms of the Gray–Scott model are analyzed in [39] and in more detail in [6].

Another method to calculate the eigenvalues in a setting like this, is using the Evans
function, see [1]. The embedding of this a priori formal stability analysis using the NLEP into the Evans function approach is established in [15], and studied in full detail in a more general context in [14].

As stated earlier, the one-dimensional patterns are trivially (constantly), extended in the transverse \( y \)-direction form 2D stripe patterns. To show existence of these patterns, no extra analysis is needed compared to the 1D existence proof. Stability, however, now needs to be tested against perturbations in both the \( x \)- and \( y \)-direction. This analysis is done in [58], and in the parameter regimes considered in that article, the transverse perturbations cause instability.

1.2. Outline of this article. In this article, we exploit the singularly perturbed nature of the system (recall \( \delta \) is a small parameter), and analytically prove the existence of traveling stripe patterns, with a singular character, on a sloped terrain. That is, we establish the existence of single pulses with an asymptotically large amplitude, as well as periodic extensions of these pulses. For the one-dimensional \( N \)-pulse, our approach goes along the lines of [18]. However, the advection term breaks the reversible symmetry so that the solutions now travel with a constant, a priori unknown, wavespeed, \( S \) which we introduce by setting \( \chi = x - St \). Hence, we use the following system to construct traveling patterns,

\[(1.3) \begin{align*}
0 &= U_{\chi\chi} + A(1 - U) - UV^2 + (C + S)U_{\chi}, \\
0 &= \delta^2 V_{\chi\chi} - BV + UV^2 + SV_{\chi},
\end{align*}\]

where we make specific choices for the magnitude of the parameters and coordinates with respect to \( \delta \), in section 2.1. Note that it thus follows that a pattern travels uphill if the sign of its speed \( S \) has the same sign as the coefficient \( C \) of the slope term in (1.3). We will find that this is indeed the case. This is natural from the ecological point of view since vegetation patterns are expected to move towards down flowing water, moreover it has been confirmed observationally [9] and by the simulations in [35,58]. The simulation of (1.3) with \( C > 0 \) in Figure 1 clearly exhibits a stable uphill traveling homoclinic vegetation pulse. Note that it is asymmetric – contrary to its stationary counterpart (with \( C = 0 \)) – and indeed travels with constant speed.

Stripe patterns are typically oriented perpendicular to the slope of the terrain [2,9,10, 35,58]. Since the generalized Klausmeier-Gray-Scott model (1.2) describes the formation of vegetation patterns on an unbounded plane with constant inclination, this makes the existence analysis of stripe patterns trivial, once the one-dimensional problem has been resolved. We choose the \( y \)-coordinate of the stripe pattern perpendicular to the \( x \)-direction (which is defined as the direction in which the plane is sloped, see (1.2)). Hence, the spatial derivatives with respect to \( y \) vanish in the existence problem for stripes, which is thus identical to (1.3).

Compared to the localized homoclinic patterns, the loss of symmetry has a more fundamental effect on the existence analysis of spatially periodic solutions: the introduction of the advective slope term requires a novel approach. In section 3, we employ a contraction argument to show that, for each given slope \( C \), there is an interval of speeds \( S \) for which traveling periodic pulse solutions exist.

In section 4, we analyze the Evans function to evaluate the eigenvalues corresponding to the stability of homoclinic and periodic pulse patterns. We require the constructed pulse
patterns to have a sufficiently long wave length so that solutions to the stability problem are exponentially small in between the pulses. This implies that every family of eigenvalues one would expect in the stability analysis of spatially periodic patterns may be asymptotically approximated by a single discrete eigenvalue. See also section 4.1.3 and, for a rigorous validation, [8, 23, 24, 63]. Using the slow/fast splitting of the solutions, a leading order approximation of the eigenvalue problem is then formulated. This is directly equivalent to the nonlocal eigenvalue problem (NLEP) considered in [13] and can be solved by the methods of that paper (using hypergeometric functions). Similar to the results for the Gray–Scott model, the stability of patterns depends on the asymptotic magnitude of $B$, but the slope also affects the stability. The results are presented in theorems 4.1–4.3. One of the findings is that perturbations with (asymptotically) large transverse wave numbers eventually destabilize any pattern. This confirms in some sense the numerical results of [58], which state that stripe patterns can only be stable on sufficiently steep slopes; a parameter regime that is not covered by the scaling restrictions of our asymptotic analysis. Moreover, one may argue that not all perturbations are representative to model ecological resilience of a pattern. Therefore, we also present corollary 4.5, which analyzes the stability with respect to a more restrictive and specifically chosen function space. Since it is the transverse perturbations that form the main destabilization mechanism of stripe patterns, theorems 4.2 and 4.3 restrict to one-dimensional patterns for which we derive saddle-node and Hopf bifurcations in several parameter regimes. Lastly, a detailed characterization of destabilization through a decrease of $A$ or $C$ is discussed in section 5. One of the conclusions that can be drawn is that indeed, a larger value of both precipitation and slope is preferable for stripe patterns, confirming the natural observations.

2. Existence of homoclinic traveling multi-pulse patterns. In this section, we construct $N$-pulse solutions and periodic solutions of (1.3) that consist of slow and fast components. Specifically, these solutions have large $\chi$-intervals for which the $V$-solution is close to zero. The building block for these solutions is the 1-pulse, i.e. a solution homoclinic to $(1,0)$, with a localized pulse where $V$ is $O(1)$ or larger compared to $\delta$ and $U$ small within the localized $\chi$-interval. In this section, we will prove the following theorem.

**Theorem 2.1 (Existence of 1-pulse patterns).** Let $A = \delta^2\gamma a$, $B = \delta^3 b$, $C = \delta^4 c$, and $S = \delta^{1-\frac{2}{3} \beta + 2\gamma} s$, where $\gamma, \beta$ satisfy the following assumptions, also illustrated by figure 2

- **A1** $\beta < \gamma$,
- **A2** $\beta > \frac{2}{3}\gamma (\gamma - 1)$,
- **A3** $\beta \geq 2(\gamma - 1)$,

and $a, b, c, s \in O(1)$. Then, there exists a $\delta_0$, such that for all $\delta < \delta_0$ and $a, b, c$ given, there exists a uniquely determined speed $s$ such that the four-dimensional dynamical system (1.3) has a homoclinic solution to the critical point $(U, U_\chi, V, V_\chi) = (1, 0, 0, 0)$. This solution corresponds to a traveling wave solution $\gamma_{\text{hom}}(\chi)$ of (1.2), with speed $s$. Its spatial profile is in $x$ biasymptotic to $(U, V) = (1, 0)$, and trivially extends into the $y$-direction. The orbit $\gamma_{\text{hom}}$ consists of two slow components and a single fast excursion. The magnitude of the $U$ and $V$ components during the fast excursion are $\delta^{1-\frac{2}{3} \beta - \gamma}$ and $\delta^{1-\frac{2}{3} \beta + \gamma}$, respectively. Moreover, the traveling speed of this wave is to leading order in $\delta$ given by

$$s = \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}}.$$
During the fast excursion, the $U$-component is constant to leading order, $U = \delta^{1+\frac{1}{2}\beta-\gamma} u_0$, with

$$u_0 = \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}.$$  

In the remainder of this section, we prove this theorem using geometric singular perturbation techniques.

**Figure 2:** Scaling regime in the $(\gamma, \beta)$-plane, that satisfies assumptions $\textbf{A1–A3}$ in theorem 2.1.

**2.1. Rescaling.** The model proposed by Klausmeier, (1.1), is in dimensionalized form, meaning that it does not take into account the relative magnitudes of parameters and coordinates. However, in the case of Gray–Scott models, patterns are observed in regimes where not all quantities are $\mathcal{O}(1)$. In fact, scaling is essential for the analysis and plays an important role in the results. Hence, scaling of all parameters and coordinates is also appropriate in system (1.2), and here too, it is crucial for the analysis.

In the system (1.3), which already incorporates traveling solutions with a trivial $y$-extension, we introduce a new traveling coordinate,

$$\xi = \delta^{\frac{1}{2}\beta-1} \chi,$$
which we will refer to as the fast variable. The original $\chi$ will be referred to as the super slow variable. Furthermore, we introduce the following rescaled parameters and coordinates,

\begin{equation}
\begin{aligned}
S &= \delta^{-1-\frac{3}{2}\beta+2\gamma}s, \\
U &= \delta^{\frac{3}{2}\beta-\gamma}u, \\
V &= \delta^{-1-\frac{1}{2}\beta+\gamma}v
\end{aligned}
\end{equation}

where $a, b, c, s, u, v$ are $O(1)$. This specific scaling is motivated by the scalings for the Gray-Scott model of $[11,12]$ that generalize those of earlier work on the Gray–Scott model $[13,18]$. Most important is assumption $A1$, without which our rescaled system would not have distinct slow and fast behavior. This assumption makes it possible to perform geometric singular perturbation analysis in the first place. The fact that $A$ is scaled quadratically compared to $C$, is the most significant choice because this is the only way that neither the effects of $a$ nor those of $c$ (the most important parameters) are negligible. The factor $\delta^{-1-\frac{3}{2}\beta+2\gamma}$ of $s$ is chosen such that the speed has a measurable effect in the two-dimensional Hamiltonian system associated with the fast dynamics. Furthermore, we require the $U$-equation to exhibit slow behavior with small amplitudes in the fast interval as opposed to the fast dynamics with large amplitudes governed by the $V$-equation, this results in assumptions $A2$ and $A3$, respectively. Lastly, we balance $BV$ and $UV^2$, so that the two-dimensional Hamiltonian system associated with the fast $V$-equation has a homoclinic orbit for $\delta \to 0$ and we take into account the forthcoming matching of slow and fast behavior, which leads to the specific choices of the scalings of $U$ and $V$. The details of the limit behavior in the slow and fast variable, are laid out in sections 2.3 and 2.4, respectively. In the end, all quantities are scaled with only two scaling parameters, $\beta$ and $\gamma$, combined with three assumptions $A1$–$A3$. Note that the calibration of the scalings is largely consistent with that of the Gray–Scott case, where only the scaling parameter $\beta$ was left undetermined. Upon introducing an extra parameter, the slope $C$, one may expect the degree of freedom in scaling to increase by one which is indeed the case here. In figure 2, the allowed regime of $\beta, \gamma$ according to the assumptions is graphically outlined.

Rescaling system (1.3) according to (2.4) gives rise to,

\begin{equation}
\begin{aligned}
u_{\xi\xi} &= \delta^{2(\gamma-\beta)}\left[uv^2 - \delta^{2+\beta}a(\delta^{-\frac{3}{2}\beta-1} - u) - \delta^{1-\gamma+\frac{3}{2}\beta}cu_{\xi} - \delta^2 su_{\xi}\right], \\
v_{\xi\xi} &= bv - uv^2 - \delta^{2(\gamma-\beta)}sv_{\xi}.
\end{aligned}
\end{equation}

Note that $\gamma > \beta$ ($A1$), so that the right hand side of $u$-equation is indeed asymptotically small for $0 < \delta \ll 1$. The homogeneous equilibrium state $(U, V) = (1, 0)$ is now represented by $(u, v) = (\delta^{-1-\frac{3}{2}\beta+\gamma}, 0)$. We label this desert state $P$. We write (2.5) as a four-dimensional system of first order differential equations, by introducing the variables $p, q$.

\begin{equation}
\begin{aligned}
u_{\xi} &= \delta^{\gamma-\beta}p, \\
p_{\xi} &= \delta^{\gamma-\beta}\left[uv^2 - \delta^{2+\beta}a(\delta^{-\frac{3}{2}\beta-1} - u) - \delta^{1+\frac{3}{2}\beta}cp - \delta^{2+\gamma-\beta}sp\right], \\
v_{\xi} &= q, \\
q_{\xi} &= bv - uv^2 - \delta^{2(\gamma-\beta)}sq.
\end{aligned}
\end{equation}
Assumptions A1–A3 guarantee that the right hand side of the equation for $p_\xi$ is always asymptotically small. We now introduce a second slow variable, see A1, to write system (2.6) in the slow form,

$$\zeta = \delta^{\gamma - \beta} \xi.$$  

Note that $\zeta$ is, however, faster or of the same speed as $\chi$ because of A3,

$$\chi = \delta^{1 - \frac{1}{2} \beta} \xi \lesssim \zeta = \delta^{\gamma - \beta} \xi \ll \xi.$$  

From now on, we work with $\zeta, \xi$ exclusively and refer to those as the slow and fast variable, respectively. In terms of the slow variable $\zeta$, (2.6) becomes

$$u_\zeta = p,$$

$$p_\zeta = uv^2 - \delta^{2 + \beta} a(\delta^{\gamma - \frac{3}{2} \beta - 1} - u) - \delta^{1 + \frac{1}{2} \beta} cp - \delta^{2 + \gamma - \beta} sp,$$

$$\delta^{\gamma - \beta} v_\zeta = q,$$

$$\delta^{\gamma - \beta} v_\zeta = bv - uv^2 - \delta^{2(\gamma - \beta)} sq.$$  

Using the limiting behavior for $\delta \to 0$ of the two equivalent systems (2.6) and (2.7), we will establish the existence of solutions for $\delta \neq 0$.

### 2.2. Alternative scalings of the generalized Klausmeier-Gray-Scott model.

With our choice to present the generalized Klausmeier-Gray-Scott in the form of (1.2), we follow [64] and emphasize the similarities between this model and the original Gray-Scott reaction-diffusion model as it was introduced in [49]. There are, however, other possible choices. In [58,61], the model is called the extended Klausmeier model and is scaled to emphasize its relation with the original Klausmeier model,

$$\hat{U}_i = \Delta_{x,y} \hat{U} + \hat{A} - \hat{U} - \hat{U} \hat{V}^2 + \hat{C} \hat{U}_x,$$

$$\hat{V}_i = \delta^2 \Delta_{x,y} \hat{V} - \hat{B} \hat{V} + \hat{A} \hat{V}^2.$$  

Moreover, in [37–39] an alternative scaling for the Gray-Scott model has been developed. In the generalized Klausmeier-Gray-Scott setting, the scaling of [37–39] would take the form,

$$\tilde{B} \hat{U}_i = \Delta_{x,y} \tilde{U} + (1 - \tilde{U}) - \tilde{U} \tilde{V}^2 + \tilde{C} \hat{U}_x,$$

$$\tilde{V}_i = \tilde{\delta}^2 \Delta_{x,y} \tilde{V} - \tilde{V} + \tilde{A} \tilde{U} \tilde{V}^2.$$  

(in which $\tilde{B}$ is typically denoted by $\tau$). Of course, the three models (1.2), (2.8) and (2.9) are completely equivalent. More precise, (1.2) can be brought into the forms of (2.8) or (2.9) by the transformations

$$U = \sqrt{A} \hat{U}, \quad V = \sqrt{A} \hat{V}, \quad (x, y) = (\frac{x}{\sqrt{A}}, \frac{y}{\sqrt{A}}), \quad t = \frac{t}{A},$$

$$U = \hat{U}, \quad V = \sqrt{A} \hat{V}, \quad (x, y) = (\frac{x}{\sqrt{A}}, \frac{y}{\sqrt{A}}), \quad t = \frac{t}{A},$$

so that

$$\tilde{\delta} = \frac{\delta}{\sqrt{A}}, \quad \tilde{A} = \frac{A}{\sqrt{A}}, \quad \tilde{B} = \frac{B}{A}, \quad \tilde{C} = \frac{C}{\sqrt{A}},$$

$$\hat{\delta} = \frac{\delta}{\sqrt{A}}, \quad \hat{A} = \frac{A}{\sqrt{A}}, \quad \hat{B} = \frac{B}{A}, \quad \hat{C} = \frac{C}{\sqrt{A}}.$$
Nevertheless, there are also differences between the models. The extended Klausmeier representation (2.8) of [58, 61] is closer to the ecological background: for instance, the parameter $\hat{A}$ is directly related to rainfall [35], while this relation is less obvious for $A$ in (1.2) – see [64] for a discussion. In fact, the parameter values of (2.8) in [58, 61] are chosen for their ecological relevance (and follow those of [35]) – see however also the discussion in section 5. Although one has to be very careful with interpreting the relation between mathematical results that hold for a certain scaling in a ‘sufficiently small’ asymptotic parameter (as is the case here) and simulations done with explicitly chosen numerical parameter values, the (far-from-equilibrium) simulations of [58, 61] on terrains with slopes that are not too steep may be captured by parameter regions in the (1.2)-scaling for which our analysis holds. We refer to section 5 for a more extensive discussion of the ecological interpretation of our analysis.

Especially in the Gray-Scott setting, i.e. if $\tilde{C} = C = 0$, there are various advantages to the formulation of (2.9): in Gray-Scott, patterns are typically stationary which means that the existence problem has one parameter ($\tilde{B} = \tau$) less in the form (2.9). As in the present analysis, assumptions on the relative magnitudes of the parameters such as in the statement of theorem 2.1 also are necessary in the scaled system (2.9). One could argue that these take a somewhat more natural form in that scaling, for instance, hypotheses $A_1$ and $A_2$ are in (2.9) equivalent to

$$\sqrt{\delta} \ll \tilde{A} \ll 1,$$

In fact, in the terminology of [37], hypotheses $A_1$ and $A_2$ imply that it is assumed that (2.9) is in the intermediate regime between the low feed case $\tilde{A} = \mathcal{O}(\sqrt{\delta})$ and the pulse splitting regime $\tilde{A} = \mathcal{O}(1)$. Naturally, these regimes also appear in the analysis of the Gray-Scott model in the scalings of this paper (that are based on those of [11, 12]): the low feed and splitting regimes of (2.9) correspond to the curves $C_{SN}$ and $C_{SPLIT}$ defined in [12] (and to the lines $\beta = \gamma$ and $\beta = \frac{2}{3}(\gamma - 1)$ in Figure 2).

In the end, the main reason to choose between the scalings (1.2), (2.8) and (2.9) seems to be the background from which one is working: (2.8) provides the most direct interpretation in ecological terms, (2.9) is the most natural starting point for a matched asymptotics approach along the lines of [32], while (1.2) with (2.4) is tailored to the geometric singular perturbation approach originally developed in [13, 18] that we follow here.

### 2.3. Slow limit behavior.

For $\delta \to 0$, the last two equations of system (2.7) become,

\begin{equation}
\begin{aligned}
0 &= q, \\
0 &= bv - uq^2,
\end{aligned}
\end{equation}

which define two two-dimensional critical manifolds in $\mathbb{R}^4$, $\{v = 0, q = 0\}$ and $\{v = b/u, q = 0\}$. Since $P$ is on the former, and as the latter is not normally hyperbolic for all $v, q$, we focus on the dynamics on

\begin{equation}
\mathcal{M} = \{u \geq 0, v = 0, \ q = 0\},
\end{equation}

where we restrict to $u \geq 0$ because this is the only ecologically relevant regime for $u$. Fenichel theory [3, 21, 22], guarantees the persistence of $\mathcal{M}$ for $0 < \delta \ll 1$ as a slow manifold. In the
present case, (2.11) exactly defines the invariant slow manifold associated to the full problem. Fenichel theory and its implications are discussed in more detail in section 2.5.

The slow dynamics on $\mathcal{M}$ is described by

\begin{equation}
\begin{aligned}
    u_\zeta &= p, \\
    p_\zeta &= -\delta^{2+\beta} a(\delta^{-\frac{3}{2}\beta-1} - u) - \delta^{1+\frac{1}{2}\beta} cp,
\end{aligned}
\end{equation}

from which we can derive that $P$ is a saddle point on $\mathcal{M}$, with eigenvalues

\[ \Lambda_{\pm} = \frac{1}{2} \delta^{1+\frac{1}{2}\beta} \left( -c \pm \sqrt{c^2 + 4a} \right), \]

and eigenvectors through $P$ in the $(u, \hat{p})$-plane,

\begin{equation}
\begin{aligned}
    \ell^u : \hat{p} &= \frac{1}{2} \left( c - \sqrt{c^2 + 4a} \right) (1 - \delta^{1+\frac{1}{2}\beta-\gamma} u), \\
    \ell^s : \hat{p} &= \frac{1}{2} \left( c + \sqrt{c^2 + 4a} \right) (1 - \delta^{1+\frac{1}{2}\beta-\gamma} u),
\end{aligned}
\end{equation}

with a new variable $\hat{p} = \delta^{\beta-\gamma} p$. When assumption $A2$ is satisfied, the eigenvectors do not depend on $u$ in the leading order. The point $P$ then has an asymptotically large $u$-value, and hence in the $(\hat{p}, u)$-plane $\ell^{s,u}$ are vertical at leading order. This is depicted in the middle panel of figure 3. However, as $\beta = \frac{2}{3}(\gamma - 1)$, the fixed point comes into frame and $U = u$. In this case the eigenvectors do depend on $u$, see the right panel in figure 3. This case is also explained in more detail in remarks 1 and 2. Contrary to the Gray–Scott case, where $c = 0$, the behavior on $\mathcal{M}$ is in both panels of figure 3 not symmetric about the $u$-axis.

![Figure 3: Left: The phase plane corresponding to (2.15), with in purple the homoclinic orbit to $(v, q) \in \mathcal{M}$. Center: Dynamics on $\mathcal{M}$ dictated by (2.12) in the $(\hat{p}, u)$ plane when $A2$ is satisfied. The equilibrium $P$ has an asymptotically large $u$-value and does not appear in this frame. To leading order, all dynamics, as well as $\ell^{s,u}$ are vertical. Right: Dynamics on $\mathcal{M}$ in the $(\hat{p}, U)$ plane, when $\beta = \frac{2}{3}(\gamma - 1)$ and $u = U$ (see remarks 1 and 2). The equilibrium $P$ is $O(1)$ and the eigenvectors vary with $U$.](image-url)
2.4. Fast limit behavior. The limiting fast problem, system (2.6) with \( \delta \to 0 \), implies that \( u \) and \( p \) are constant, to leading order, i.e.

\[
(2.14) \quad u(\xi) \equiv u_0, \quad p(\xi) \equiv p_0.
\]

Hence, the fast dynamics is determined by

\[
(2.15) \quad \begin{align*}
v_\xi &= q, \\
q_\xi &= bv - u_0 v^2,
\end{align*}
\]

where \( u_0 \) occurs as a parameter. The fast limit problem (2.15) is Hamiltonian, with conserved quantity

\[
(2.16) \quad H(v, q) = \frac{1}{2} q^2 - \frac{1}{2} bv^2 + \frac{1}{3} u_0 v^3.
\]

Note that there exists a symmetry in (2.15),

\[
(2.17) \quad q \to -q, \quad \xi \to -\xi.
\]

The equilibrium of (2.15) of our focus, \((v, q) = (0, 0)\), has \( H = 0 \) and is a saddle. There is a homoclinic connection to \((0, 0)\), and we find an explicit solution by integrating \( H(v, q) = 0 \) with respect to \( \xi \),

\[
(2.18) \quad \begin{align*}
v_0(\xi) &= \frac{3b}{2u_0} \text{sech}^2 \left( \frac{1}{2} \sqrt{b} \xi \right), \\
q_0(\xi) &= -\frac{3b \sqrt{b}}{2u_0} \text{sech}^2 \left( \frac{1}{2} \sqrt{b} \xi \right) \tanh \left( \frac{1}{2} \sqrt{b} \xi \right).
\end{align*}
\]

The phase plane of (2.15) is depicted in the left panel of figure 3. Since the homoclinic orbit exists for any \( u_0 \) and \( p_0 \), the equilibrium \((v, q) = (0, 0)\) represents the entire critical manifold \( M \) (2.11). The critical manifold has three-dimensional stable and unstable manifolds, \( W^{s,u}(M) \). Because the homoclinic orbit exists for all \( u_0, p_0 \), we find \( W^u(M) = W^s(M) \) in the limit \( \delta \downarrow 0 \). This is represented in three dimensions in the left hand panel of figure 4.

2.5. Persistence of solutions in the perturbed problem. With the results of sections 2.3 and 2.4, we construct a singular homoclinic orbit to \( P \) with both slow and fast parts obtained by the \( \delta = 0 \) limits. For \( \delta \downarrow 0 \), the eigenvectors \( \ell^{s,u} \) tend to and eventually collide on the \( u \)-axis. In other words, the stable and unstable manifolds of \( P \), restricted to \( M \) are in fact the \( u \)-axis in this limit. To construct a singular slow/fast homoclinic solution to \( P \), we may match any point on the \( u \)-axis to the fast homoclinic excursion, because for \( \xi \to \pm \infty \), the eigenvectors \( \ell^{s,u} \) will connect it to \( P \). As a result, we have a one parameter family of singular homoclinic connections to \( P \), parametrized by the value of \( u \) at the concatenation point of the slow and fast parts, say \( u_0 \).

For \( \delta \neq 0 \), the stable and unstable manifolds of \( M \) persist as \( W^{s,u}_\delta(M) \) according to Fenichel theory [21, 22], but they no longer coincide. Instead, the intersection of \( W^{s,u}_\delta(M) \)
Figure 4: Schematic three-dimensional representation of the intersection of $W^u(M)$ and $W^s(M)$ of (2.6) for $\delta = 0$ (left) and $0 < \delta \ll 1$ in four dimensions. In reality, $M$, here the one-dimensional vertical axis, is two-dimensional. Left: The stable and unstable manifolds of $M$ are three-dimensional, and coincide. For every point on $M$ there exists a fast homoclinic connection. Right: The stable (blue) and unstable (red) manifolds persist, but no longer coincide. Their intersections correspond with fast $N$-pulses, homoclinic to $M$. Here, only the first two intersections are drawn: the 1-pulse, in white, and the 2-pulse in purple.

perturbs, like depicted in figure 4. A fast connection to $M$ is thus no longer guaranteed for every pair $(u, \hat{p})$.

A solution homoclinic to $P$, with a slow/fast structure must leave $P$ via the perturbed $\ell^u$ on $M$, connect to a fast orbit homoclinic to $M$, and returning to $M$ exactly such that it enters $P$ along the perturbed stable manifold, $\ell^s$. These eigenvectors $\ell^{s,u}$ have also perturbed by $\delta$, and no longer coincide on the $u$-axis. Schematically, we represent a homoclinic slow/fast solution in four dimensions in a three-dimensional illustration in the left hand panel of Figure 5.

Although the fast connection to $M$ generally does not persist, a perturbed homoclinic orbit is still approximated by the $\delta = 0$ solution in the fast field, (2.18). To prove the statement of theorem 2.1, we show that there still exists a homoclinic solution to $P$ for $0 < \delta \ll 1$. We label this solution $\gamma_{\text{hom}}(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi))$. Without a loss of generality, we choose $\gamma_{\text{hom}}(0) = (u_0, p_0, v_{\text{max}}, 0)$. That is, we select the $\xi = 0$ point exactly there where $q = 0$; it is
at the maximum value of the $V$-pulse.

We first show that there exists a one-dimensional manifold for which $W^s_{\delta}(\mathcal{M})$ and $W^u_{\delta}(\mathcal{M})$ intersect transversally in the hyperplane $\{q = 0\}$. Then, following the approach developed in [14, 18], we construct take-off and touch-down curves that dictate how to connect the intersection $W^s_{\delta}(\mathcal{M}) \cap W^u_{\delta}(\mathcal{M}) \cap \{q = 0\}$ to the slow stable and unstable manifolds of $P$ restricted to $\mathcal{M}$, which are the eigenvectors $\ell^s,u$.

**Lemma 2.2.** Assume the conditions on $a, b, c$ from theorem 2.1 are satisfied. Then there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the stable and unstable manifold of $\mathcal{M}$ in system (2.6) intersect transversally in $\{q = 0\}$; the $(u_0, p_0)$ coordinates of this one-dimensional intersection are at leading order given by

$$p_0 = \delta^{\gamma - \beta} su_0.$$  

(2.19)

In other words, the homoclinic connection of (2.15) to $\mathcal{M}$ persists for $\delta > 0$ in (2.6) if (2.19) is satisfied.

For $0 < \delta \ll 1$, the critical manifold $\mathcal{M}$, as well as the stable and unstable manifolds $W^{s,u}(\mathcal{M})$ persist as a slow manifold $\mathcal{M}_\delta$ and $W^{s,u}_\delta(\mathcal{M})$ in an $O(\delta^{\gamma - \beta})$-neighborhood. This follows directly from Fenichel’s first and second theorem, see [21, 33]. In fact, we may choose $\mathcal{M}_\delta = \mathcal{M}$, because $v = q = 0$ is still invariant under the flow for $0 < \delta \ll 1$. This is
not the case for $W^s,u_\delta(M)$; the stable and unstable manifolds that used to coincide (see the left hand side of figure 4), no longer do so. In general, the two three-dimensional manifolds intersect in two-dimensional surfaces, and in these intersections lie the only trajectories that are biasymptotic (homoclinic) to $M$, see an illustration of such a perturbation and a persisting connection in the right-hand panel of figure 4.

To detect these intersections, and hence proving Lemma 2.2, we apply a Melnikov method, much like [14,18]. We will work with the fast variable $\xi$ and define the fast interval $I_f$, as

$$I_f = \left[ -\frac{1}{\delta \mu}, \frac{1}{\delta \mu} \right], \quad \text{with} \quad 0 < \mu < \gamma - \beta.$$  

If $\gamma(\xi)$ is a solution of (2.6), then, for $\xi \in I_f$, $u$ and $p$ are constant and equal to $u_0, p_0$ at leading order. We measure the distance between $W^s(M)$ and $W^u(M)$ in the hyperplane \{$q = 0\}$, using the Hamiltonian (2.16) and the fact that on $M$, $H \equiv 0$ even though $\delta \neq 0$, see (2.16). We now determine the first order corrections of $u$ and $p$ in the fast variable $\xi$. That is, we write

$$u(\xi) = u_0 + \delta^{\gamma-\beta} u_1(\xi) + \text{h.o.t.},$$

$$p(\xi) = p_0 + \delta^{\gamma-\beta} p_1(\xi) + \text{h.o.t.},$$

and determine $u_1(\xi), p_1(\xi)$. We assume $u(0) = u_0$, and set $u_j(0) = 0$ for all $j \geq 1$. Using a standard asymptotic analysis and the boundary conditions that $U$ and $V$ must remain bounded on the real line, we find that $u_1(\xi) \equiv 0$. From the leading order analysis of section 2.3, we derive that $p = O(\delta^{\gamma-\beta})$, so $p_0 = 0$. As a matter of fact, $p_1 = \hat{p}$, the variable introduced in section 2.3. We compute $\hat{p}$ from the first order terms of system (2.6),

$$\hat{p} = \int_0^\xi \hat{p}_\xi d\xi + \hat{p}(0) = \int_0^\xi u_0 v_0^2(\xi) d\xi + \hat{p}(0),$$

$$= \frac{3b\sqrt{b}}{2u_0} \tanh \left( \frac{1}{2} \sqrt{b} \xi \right) \left( 2 + \text{sech}^2 \left( \frac{1}{2} \sqrt{b} \xi \right) \right) + \hat{p}(0),$$

to leading order in $\delta$. Now we can measure the change in the Hamiltonian $H$ in $I_f$ over the fast homoclinic orbit. For $\delta \neq 0$, it is given by

$$H_\xi = \delta^{2(\gamma-\beta)} \left( \frac{1}{3} v^3 \hat{p} - s q^2 \right),$$

which implies

$$\Delta H = \delta^{2(\gamma-\beta)} \int_{I_f} \left( \frac{1}{3} v^3 \hat{p} - s q^2 \right) d\xi,$$

$$= \delta^{2(\gamma-\beta)} \frac{6b^2 \sqrt{b}}{5u_0^2} \left( \frac{2\hat{p}(0)}{u_0} - s \right),$$

where $\Delta H$ means the change in $H$ over $I_f$. Hence, for given $p_0 = \delta^{\gamma-\beta} \hat{p}(0)$, $\Delta H$ changes sign once and in a transversal way at

$$\hat{p}(0) = \frac{1}{2} s u_0,$$
which implies that $W^s_{\delta}(\mathcal{M})$ and $W^u_{\delta}(\mathcal{M})$ indeed intersect transversally. This proves Lemma 2.2.

Note that equation (2.24) is actually the first moment we explicitly focus on the construction of a 1-pulse, by measuring the change in $H$ over exactly one fast excursion. In Corollary 2.4, we explain how this step changes to establish the existence of $N$-pulses with $N > 1$.

**Lemma 2.3 (Take-off and touch-down curves).** Let $\Gamma(\xi; u_0)$ be the one-parameter family of solutions of (2.6) with $\Gamma(0; u_0) \in W^s_{\delta}(\mathcal{M}) \cap W^u_{\delta}(\mathcal{M}) \cap \{q = 0\}$. In other words, let $\Gamma(0; u_0) = (u_0, 1/2 \delta^{\gamma-\beta} \delta u_0, v(0), 0)$ at leading order, see (2.25). Moreover, let $\Gamma^{o,d}(\xi; u^{o,d}, p^{o,d})$ be trajectories strictly on $\mathcal{M}$ with initial condition at $(u^{o,d}, p^{o,d})$. Then, there exist two curves $T_o(u)$ and $T_d(u)$ on $\mathcal{M}$, such that there are $\Gamma^{o,d}$ satisfying

$$
||\Gamma(\xi; u_0) - \Gamma^o(\xi; u_0, T_o(u_0))|| < k_1 e^{k_3 \delta^{\gamma-\beta}}, \text{ for } -\xi \geq O(\delta^{\gamma-\beta}),
$$

$$
||\Gamma(\xi; u_0) - \Gamma^d(\xi; u_0, T_d(u_0))|| < k_2 e^{k_3 \delta^{\gamma-\beta}}, \text{ for } \xi \geq O(\delta^{\gamma-\beta}),
$$

for some $O(1)$ constants $k_{1,2,3} > 0$. The curves $T_o(u)$ and $T_d(u)$ are called take-off and touch-down curve, respectively, and are given by

$$
T_o(u) = \frac{1}{2} \delta^{\gamma-\beta} \left( su - \frac{6b\sqrt{b}}{u} \right),
$$

$$
T_d(u) = \frac{1}{2} \delta^{\gamma-\beta} \left( su + \frac{6b\sqrt{b}}{u} \right),
$$

to leading order. Thus, this lemma explicitly describes trajectories on $\mathcal{M}$, for which the distance to a solution homoclinic to $\mathcal{M}$ gets exponentially small as $\xi$ gets large in either the negative or the positive half-line.

**Proof.** The existence of the take-off and touch-down curves is established by Fenichel theory, where the points in $T_o$ and $T_d$ are referred to as base points of the Fenichel fibers, see [21,22]. The geometry of $W^s(\mathcal{M}) \cap W^u(\mathcal{M})$ implies that $T_o$ and $T_d$ can indeed be characterized as graphs over $u$. Their quantification is obtained by explicitly determining the change of $p$ from the $\xi = 0$ point in the intersection $W^s_{\delta}(\mathcal{M}) \cap W^u_{\delta}(\mathcal{M}) \cap \{q = 0\}$, over half a fast pulse, to an exponentially small neighborhood of $\mathcal{M}$ for both negative and positive $\xi$,

$$
\int_{-\frac{1}{2\pi}}^{0} p_\xi d\xi = \int_{0}^{\frac{1}{2\pi}} p_\xi d\xi,
$$

for $\xi < 0$ and $\xi > 0$, respectively. The boundaries of the integrals are the boundaries of $I_f$, see (2.20). The fast behavior is symmetric to leading order, see (2.17), so both integrals are half of the integral over the full fast interval, and equal to

$$
\frac{1}{2} \int_{I_f} p_\xi d\xi = \frac{1}{2} \delta^{\gamma-\beta} \int_{I_f} u_0 v_0(\xi)^2 d\xi + \text{h.o.t.},
$$

$$
= \frac{1}{2} \delta^{\gamma-\beta} \frac{6b\sqrt{b}}{u_0} + \text{h.o.t.}
$$
So, combining this rate of change of $p$ with $p(0)$ from (2.22), we arrive at (2.27). The estimates of (2.26) are derived from Fenichel theory, and the constant $k_3$ is related to the largest eigenvalue of the fast field. ■

Using lemmas 2.2 and 2.3, we now prove theorem 2.1.

**Proof.** (of theorem 2.1). From all the solutions $\gamma(\xi)$ that are biasymptotic to $M$, the homoclinic 1-pulse $\gamma_{\text{hom}}$ is by construction exactly that one that connects to $\ell^u$ for $\xi < 0$ and to $\ell^s$ for $\xi > 0$. Thus, we need

$$T_o(u_0) = \ell^u(u_0) \quad \text{and} \quad T_d(u_0) = \ell^s(u_0).$$

This happens when

$$u_0 = \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}, \quad \text{and} \quad s = \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}},$$

to leading order in $\delta$, see the left panel of figure 5. ■

**Corollary 2.4 (Existence of $N$-pulse patterns).** Let $A, B, C$ satisfy the assumptions A1–A3. Then there exists a $\delta_0$, such that for all $\delta < \delta_0$, and for each $N$ of $O(1)$, there exists a solution $\gamma_{N,\text{hom}}$ of (1.3) homoclinic to $(U, U_X, V, V_Y) = (1, 0, 0, 0)$ for a uniquely defined speed $s_N$. The orbit $\gamma_{N,\text{hom}}$ corresponds to a traveling wave solution of (1.2). Its spatial profile that is in $x$ biasymptotic to $(U, V) = (1, 0)$, and trivially extends into the $y$-direction. It consists of two slow components and $N$ fast excursions. The traveling speed of this wave is given by

$$s_N = \frac{c\sqrt{c^2 + 4a}}{6Nb\sqrt{b}}.$$ 

During the fast excursions, the $U$-component is constant to leading order, $U = \delta^{1 + \frac{3}{2}\beta - \eta} u_0$, with

$$u_{0,N} = \frac{6Nb\sqrt{b}}{\sqrt{c^2 + 4a}}.$$

For $N = 2$ the homoclinic orbits to $P$ are schematically drawn in the right hand panels of figures 4 and 5. The key principle to realize is that manifolds $W^s(M)$ and $W^u(M)$ may intersect more than once in $\{q = 0\}$ . In [14], it is shown, using the symmetry of the fast system, that indeed, there may be many intersections $W^s(M) \cap W^u(M) \cap \{q = 0\}$, all of dimension 1. Following $\xi$ in forward and backward time along $W^s$ and $W^u$, starting from $M$, the first intersection corresponds to the 1-pulse of theorem 2.1 and has $v = O(1)$. The second intersection happens for $0 < v \ll 1$, and corresponds to the 2-pulse, the third intersection has again $v = O(1)$ and corresponds to the 3-pulse, et cetera.

**Proof.** The condition (2.22) must still hold, but since the trajectory makes $N$ circuits in the fast regime, we must measure the change of $p$ a total of $N$ times. Symmetry (2.17) makes that we can construct take-off and touch-down curves for every $N$, as follows

$$T_o^N(u) = \frac{1}{2} \delta^{\gamma - \beta} \left( su - \frac{6Nb\sqrt{b}}{u} \right), \quad T_d^N(u) = \frac{1}{2} \delta^{\gamma - \beta} \left( su + \frac{6Nb\sqrt{b}}{u} \right).$$
Again, an intersection of $T^N_0, T^N_d$ with $\ell^{s,u}$ constructs the homoclinic $N$-pulse, and we arrive at the conditions for $u^0, N$ and $s_N$ as stated in the theorem.

As opposed to the homoclinic 1-pulse, we will not dive into the details of the existence of periodic extensions of these $N$-pulses nor their stability. The main reason for this is the fact they can be argued to be unstable by a simple argument, explained in section 4.1.

Remark 1. Even in the case that $\beta = \frac{2}{3}(\gamma - 1)$, when $A_2$ is violated, homoclinic pulse solutions continue to exist. In that case, the expressions for $\ell^{s,u}$ from (2.13) depend on $u$ in the leading order and the behavior on $M$ is as described by the right panel of figure 3. The fast limit results remain unchanged but the fine tuning for the values (2.29) is slightly different. In both cases, the touch-down curve intersects $\ell^s$ twice, while the take-off curve intersects $\ell^u$ just once. By tuning $s$ and $u_0$, there are two solutions

\begin{align}
  u^-_0 &= \frac{1 - \sqrt{1 - 24\delta\gamma^{-2/33} - 1b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2\delta\gamma^{-2/33} - 1}, \\
  u^+_0 &= \frac{1 + \sqrt{1 - 24\delta\gamma^{-2/33} - 1b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2\delta\gamma^{-2/33} - 1}, \\
  s^+ &= c\delta\gamma^{-2/33} - 1 \left( \frac{1}{u^+_0} - 1 \right), \\
  s^- &= c\delta\gamma^{-2/33} - 1 \left( \frac{1}{u^-_0} - 1 \right).
\end{align}

When $A_2$ is satisfied, only the pair $(u^-_0, s^-)$ is an eligible solution because $u^+_0$ is not $O(1)$. To leading order in $\delta$, this results exactly in (2.29).

However, as $\beta = \frac{2}{3}(\gamma - 1)$, both intersections $u^\pm_0$ are $O(1)$, so there are two solutions under the condition that

\begin{equation}
  \frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} \leq 1.
\end{equation}

\begin{align}
  u^-_0 &= \frac{1 - \sqrt{1 - 24b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2}, \\
  u^+_0 &= \frac{1 + \sqrt{1 - 24b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2}, \\
  s^+ &= c \left( \frac{1}{u^+_0} - 1 \right), \\
  s^- &= c \left( \frac{1}{u^-_0} - 1 \right).
\end{align}

Both solutions correspond to a homoclinic 1-pulse solution of (1.3), and (2.33) defines the parameter combination at which the solutions merge in a saddle-node bifurcation, see also [18].

3. Existence of traveling multiscale periodic solutions. Given that the origin of system (1.3) is the description of vegetation patterns, the existence of periodic patterns is perhaps more relevant than the homoclinic $N$-pulse patterns. After all, homoclinic patterns represent $N$ relatively nearby stripes that are isolated in an elsewhere completely bare desert. The stripe patterns, as they are observed in the field, are naturally much better represented by periodic pulse patterns, [58]. The periodic patterns still have a slow/fast structure, and we will construct periodic patterns as depicted in figure 6. That is, the slow parts of the period patterns lie close to $M$ and connect to a fast excursion. Furthermore, we use that the fast
Figure 6: Schematic representation of a periodic slow/fast solution of which the slow part is close to the manifold $\mathcal{M}$. During the fast excursion the coordinate $u$ is constant to leading order.

parts of the constructed solutions are nearly homoclinic to the slow manifold $\mathcal{M}$; solutions stay in a neighborhood of $\mathcal{M}$ for an $O(\delta^{-1+\frac{1}{2}\beta-\gamma})$ amount of time in $\xi$. By definition, a periodic orbit cannot be homoclinic to $\mathcal{M}$, but we use in our construction that it is exponentially close to $W^u(\mathcal{M}) \cap W^s(\mathcal{M})$, see also [17]. We focus on the periodic solution that consists of one fast excursion from the slow manifold $\mathcal{M}$, and one long, slow segment near $\mathcal{M}$, per period. This periodic orbit may be distinguished from an $N$-pulse by the distance between the fast pulses. After their fast excursions, periodic solutions constructed in this section remain exponentially close to $\mathcal{M}$ for a long time in $\xi$ before making another excursion that actually marks a new period of the pattern. Therefore, the wavelength of the constructed patterns is algebraically large in $\delta^{-1}$. The $N$-pulses, however, do not return to an exponentially small neighborhood of $\mathcal{M}$ after a fast excursion, since $N$-pulses only approach $\mathcal{M}$ to an algebraically small distance in $\delta$ between consecutive fast excursions. The distance between pulses of an $N$-pulse is thus only logarithmically large in $\delta$. It is only after the $N$-th excursion that the $N$-pulse returns to $\mathcal{M}$, indefinitely.

In this section, we prove the following theorem.

Theorem 3.1. Let $a, b, c, s \in O(1)$ be given and let $A1$–$A3$ and

$$s < \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}},$$

be satisfied. Then, there exists a $\delta_0$, such that for all $\delta < \delta_0$, there exists a unique periodic
Figure 7: Simulation of (1.2) with \( A = 15, B = 1.8, C = 1 \) and \( \delta = 0.01 \), or, \( a = 3, b = 0.75, c \approx 0.447 \) with \( \gamma \approx -0.174, \beta \approx -0.190 \). The simulation was done on a spatial domain of length 10, and the patterns travel to the right over time. The left y-axes denote the unscaled \( U_0 \) (blue) and \( V \) - (red) values while the right y-axes denote the values of the scaled \( u \) and \( v \) for our choices of \( \beta, \gamma \).

solution \( \gamma_{p,s}(\xi) \) of (2.5) with a slow/fast structure. That is, for any \( s \) satisfying (3.1), there exists a traveling wave solution of (1.2), with a spatial profile which is periodic.

Figure 7 shows a simulation of (1.2) representing a solution like \( \gamma_{p,s} \).

Because \( v, q \) are exponentially small during the largest part of the period of the constructed solution, the leading order behavior of the slow parts is still governed by (2.12), and the derivation of the take-off and touch-down curves remains valid. We use the explicit formulas we have for \( T_{o} \) and \( T_{d} \) to construct periodic solutions and prove theorem 3.1. As an illustrative guide to the proof, we use Figure 8.

First, note that the slow segments of the periodic solution must lie in the area enclosed by \( \ell^{s} \) and \( \ell^{u} \), because they are separatrices in the slow manifold. Within this area, the take-off curve is monotonous as a function of \( p \), while the touch-down curve is not. This fact is key in the existence of periodic solutions.

Second, note that the geometry of \( T_{o,d} \) and \( \ell^{s,u} \) determines the eligible \( u \)-values for which the periodic pattern will make the fast excursion. For a given \( s \), the maximum of the interval of eligible \( u \)-values is always

\[
(3.2) \quad \tilde{u} = \sqrt{\frac{6b\sqrt{b}}{s}}
\]

i.e. the intersection of \( T_{o} \) wit the \( \{ p = 0 \} \)-axis. Condition (3.1) implies that the minimum is always the \( u \)-value corresponding to \( \ell^{u} \cap T_{o} \), see Figure 8.

Due to the lack of reversible symmetry, the methods developed in [17] do not provide sufficient control over the slow system to prove theorem 3.1. We therefore employ a different approach making use of a contraction argument.

We define a horizontal line segment in the \((\hat{p}, u)\)-plane, i.e. on \( \mathcal{M} \) enclosed by \( \ell^{u} \) and \( \ell^{s} \),
Figure 8: Schematic illustration of the construction of a periodic solution of (2.5). Depicted is the slow manifold $\mathcal{M}$ with the stable and unstable eigenvectors $\ell^s,u$ and take-off and touchdown curves $T_o,d$. In dark red, a schematic representation of the slow function $S$ is sketched, and the fast function $\mathcal{F}$ is dark blue. The point $P$ is at an $O(\delta^{1-\frac{\alpha}{2}})$-distance, hence the dashed lines.

and at the $u = \tilde{u}$ level.

(3.3) \[ J = \{ \tilde{p} \in \left( c - \sqrt{c^2 + 4a}, c + \sqrt{c^2 + 4a} \right) , \quad u = \tilde{u} \} . \]

and $J_R$ and $J_L$ as the parts of $J$, where $\tilde{p}$ is positive, resp. negative, see figure 8. On these intervals, we define two functions. Firstly, a map $S : J_R \to J_L$. Let $J_R$ represent a line of initial conditions of the slow flow, see (2.12). Note that $u_\xi = 0$ for $p = 0$. As the slow flow is linear and $P$ is a saddle point on $\mathcal{M}$, we know that all orbits with an initial point in $J_R$ have an intersection with $J_L$ as well. In fact, this is a one-to-one correspondence. The map $S$ assigns to every point $(\tilde{u}, p) \in J_R$, the corresponding intersection in $J_L$, and this is a bijection.

Secondly, in the parameter regime where (3.1) holds, define the map

(3.4) \[ \mathcal{F} : J_L \to J_R, \quad \mathcal{F}(\tilde{u}, p) = \left( \tilde{u}, \quad p + \frac{24sb\sqrt{b}}{p + \sqrt{p^2 + 24sb\sqrt{b}}} \right) . \]
This map represents the fast flow indicated by the take-off and touch-down curves. For initial values in $J_L$, the map is based on a concatenation of three steps. To leading order, $p$ does not change by the evolution of the slow flow. Furthermore, $T_o$ is monotonous, so by following the slow flow the initial points on $J_L$ correspond bijectively to points $T_o$. This is the first step of $F$. As the second step, dictated by the fast flow, the point on $T_o$ that results from the first step is sent to a point on $T_d$, which is also well-defined since the $u$-value remains, at leading order, constant during the fast excursion. Lastly, the slow flow takes all these touch-down points to $J_R$. Condition (3.1) implies that the range of $F$ is guaranteed to be within $J_R$. Hence, the map $F$ keeps solutions in the bounded area of $M$, shaded in Figure 8. By inverting the expressions for $T_o$ and $T_d$, it can be verified that $F$ indeed acts as given in (3.4).

Lemma 3.2. Both maps $S : J_R \to J_L$ and $F : J_L \to J_R$ are contractions.

Proof. Naturally, we use the Euclidian metric on $J_L$ and $J_R$. As explained, the map $S$ is a bijection. However, the size of the domain, $J_R$ is $c + \sqrt{c^2 + 4a}$, which is smaller than the size of codomain $J_L$, which is $c - \sqrt{c^2 + 4a}$. Since $P$ is a saddle, the flow on $M$ is linear and orbits cannot intersect, the initial points in $J_R$ remain ordered after they intersect the nullcline $p = 0$. By explicitly evaluating $S$, we indeed find that for every pair $X, Y \in J_R$, 

$$|S(X) - S(Y)| < |X - Y|,$$

and $S$ is a contraction.

For the map $F$ it suffices to check that

$$\frac{\partial}{\partial p} \left( p + \frac{24sb\sqrt{b}}{p + \sqrt{p^2 + 24sb\sqrt{b}}} \right) = \frac{p}{\sqrt{p^2 + 24sb\sqrt{b}}} < 1.$$

A derivative smaller than 1 implies contraction, so the function $F$ is a contraction, too.

Lemma 3.2 allows us to establish the first ingredient of the proof of theorem 3.1 about the existence of periodic solutions. The composition $F \circ S : J_R \to J_R$ is a contraction as well. By Banach’s fixed point theorem, this contraction has a unique fixed point in $J_R$, say $\hat{p} = \hat{p}_{p,h}^* \in J_R$. This implies that $(\hat{u}, \hat{p}_{p,h}^*)$ is an initial point for a slow segment in $M$ which intersects the take-off and touch-down curves in the same $u$-value. We label this $u$-value $u_{p,h}^*$ so that

$$(u_{p,h}^*, \hat{p}_{p,h}^*) = \left( u_{p,h}^*, \frac{1}{2} \left( su_{p,h}^* + \frac{6b\sqrt{b}}{u_{p,h}^*} \right) \right).$$

Note that $u_{p,h}^*$ is always larger than the $u_0$ of (2.29), because $u_0$ marks the left boundary of $J_L$, in which $T_o$ is monotonically increasing. In sections 4 and 5, we show that this has implications for the stability of the solutions and links directly to Ni’s conjecture [48].

A concatenation of the slow and fast parts of the constructed orbit associated with $(u_{p,h}^*, \hat{p}_{p,h}^*)$ does not immediately result in a periodic orbit, because the orbit is derived using the intersection of $W^s(M) \cap W^u(M)$ for $T_{o,d}$. The constructed orbit, we label it $\gamma_{p,h}(\xi)$, is depicted in figure 9, and is rather than being periodic, homoclinic to the manifold $M$. That
means that it makes only one excursion in the fast field, after it has returned to a neighborhood of \( \mathcal{M} \) it only gets closer and closer. Note that actually, \( \gamma_{p,h}(\xi) \) depends on the choice of \( s \). We label the speed corresponding to \( \gamma_{p,h} \) as \( s_{p,h} \).

We reset the initial point of \( \gamma_{p,h} \) so that it is a the point where \( v \) is maximal, i.e. \( q = 0 \). That is, we write

\[
\gamma_{p,h}(0) = (u_{p,h}(0), p_{p,h}(0), v_{p,h}(0), 0).
\]

We will show that a true periodic orbit exists with initial point exponentially close to \( \gamma_{p,h}(0) \). This orbit is \( \gamma_{p,s}(\xi) \) that is defined in the statement of theorem 3.1.

The proof of theorem 3.1 is inspired by that of theorem 3.1 in [17], but cannot be carried over immediately. The advection term in (1.2) breaks the reversible symmetry, which is crucial to the proofs in [17]. In the present, not symmetric case, more delicate arguments are developed to resolve that issue.

\textit{Proof.} (of theorem 3.1)

We prove the existence of periodic orbits by using arguments of intersecting manifolds, similar to the proof of theorem 2.1. The aim is to show (in four dimensions) that there is a one-dimensional periodic orbit that lies close to \( \gamma_{p,h}(\xi) \).
Define the exponentially small, two-dimensional rectangle $j$ as follows,

$$
j = \{ u = u_{p,h}(0) \} \times \left( \left( p_{p,h}(0) - k_1 e^{-k\delta - \frac{1}{2} \beta - \gamma}, p_{p,h}(0) + k_2 e^{-k\delta - \frac{1}{2} \beta - \gamma} \right) \times \left( v_{p,h}(0) - k_3 e^{-k\delta - \frac{1}{2} \beta - \gamma}, v_{p,h}(0) + k_4 e^{-k\delta - \frac{1}{2} \beta - \gamma} \right) \right) \times \{ q = 0 \},
$$

(3.6)

with $k, k_i > 0$ and $k_3 > k_4$. The value $k$ is associated with the eigenvalues of the fast limit system. Figure 10 shows the rectangle $j$ in the rescaled $e^{-k\delta - \frac{1}{2} \beta - \gamma}(v, p)$-plane. Note that $-1 + \frac{1}{2} \beta - \gamma < 0$ in the regime defined by A1–A3 (figure 2), so $-k\delta - \frac{1}{2} \beta - \gamma$ is very large and negative, and $j$ is indeed exponentially small.

We use $j$ as a set of initial conditions that lie exponentially close to the intersection $W^u(M) \cap W^s(M) \cap \{ q = 0 \}$. All orbits formed by flowing initial condition in $j$ forward, remain exponentially close to $W^s(M)$ as long as $\xi$ does not become larger than $O(\delta^{1 - \frac{1}{2} \beta - \gamma})$. Because of the exact closeness estimate of $O(e^{-k\delta - \frac{1}{2} \beta - \gamma})$, it remains close to $W^s(M)$ exactly long enough to make an $O(\delta^{3 - \beta})$ change in $p$. This is the same order of magnitude that $p$ changes during the slow part of $\gamma_{p,h}$. The three-dimensional manifold is obtained from flowing $j$ forward is labeled $J^+$. The flow depends directly on the speed of $\gamma_{p,h}$, i.e. $s_{p,h}$, so $J^+$ does so as well.

By choosing $k_i$ appropriately, $J^+$ will return to an exponentially small neighborhood of $j$, since it is exponentially close to $W^u(M)$. Thus, $J^+$ intersects $j$ in a one-dimensional manifold $j \cap J^+$; a curve in $j$. We define the map $G^+_{p,h} : j \rightarrow j$ that maps points in $j$ to their next intersection with $j$ when the forward flow induced by (2.5) is followed. Hence we find

$$
G^+_{p,h}(j) = J^+ \cap j,
$$

see Figure 10.

Similarly, we can define $J^-$ by flowing $j$ backwards in $\xi$. Again, appropriate $k_i$ will make sure that $J^-$ intersects $j$ and we label this analogous map $G^-_{p,h} : j \rightarrow j$, so that we have

$$
G^-_{p,h}(j) = J^- \cap j.
$$

See again figure 10 for an illustration. The curves $G^\pm_{p,h}(j)$ are one-dimensional and generically intersect in a point in $j$, say $R_{p,h}$. The intersection point $R_{p,h}$ is the point in $j$ that in both in forward and backward time returns to $j$. Since $R_{s_{p,h}}$ is on $G^+_{s_{p,h}}(j)$, there is a point in $j$ that is flown forward to $R_{s_{p,h}}$, say $G^+_{s_{p,h}}(R_{s_{p,h}})$, as it is a the pre-image of $R_{s_{p,h}}$. On the other hand, we do know, however, that if we consider exponentially small deviations in $s$ from $s_{p,h}$, the points $G^+_{s_{p,h}}(R_{s_{p,h}})$ and $G^-_{s_{p,h}}(R_{s_{p,h}})$ move in $j$, along curves parametrized by $s$, see figure 11. These curves, $G^\pm_{s_{p,h}}(R_{s_{p,h}})$, generically intersect in $j$ for a specific $s_{p,h}$, say in the point $X$. Exactly when this happens, both the forward flow map $G^+$ and backward flow map $G^-$ have
26 Patterns in a gKGS model

Figure 10: Schematic representation of the rectangle $\gamma$ (gray) with $G^{+}_{s_{p,h}}(\gamma)$ and $G^{-}_{s_{p,h}}(\gamma)$ as one-dimensional curves in $\gamma$ in red and blue, respectively. The point $G^{+,-1}_{s_{p,h}}(R_{s_{p,h}})$ is the original in $\gamma$ that was sent to $R_{s_{p,h}}$ following the forward flow and is therefore red, while $G^{-,-1}_{s_{p,h}}(R_{s_{p,h}})$ is the original under $G^-$ and is therefore associated with the blue curve and colored accordingly.

$X$ as a fixed point. Of course, that also implies that the intersection $G^{+}_{s_{p}}(\gamma) \cap G^{-}_{s_{p}}(\gamma)$ occurs exactly in $X$ so $R_{s_{p}} = X$.

This point $X = (v^X, p^X)$ is associated to a periodic orbit with an initial value $(u_{p,h}(0), p^X, v^X, 0)$. This periodic orbit is $\gamma_p(\xi)$ and it has a speed that is exponentially close to $s_{p,h}$. Note that, by our choice of rectangle $\gamma$, we do fix the initial condition of the periodic solution at the maximum value of $v$ (since $q = 0$) and that the $u$-value at that point is equal to the $u$-value of $\gamma_{p,h}$ where $v_{p,h}$ is maximal.

Using the direct method as developed in [16], or the exchange lemma approach with exponential errors of [34], it is possible to show that indeed this $X$ exists for a certain $s_{p}$. Moreover, this $s_{p}$ is unique, but we refrain from going into the details any further.

Remark 2. As for the existence of homoclinic singular pulse solution, see 2.1, we have assumed through $A2$ that $\beta > \frac{2}{3}(\gamma - 1)$. When $\beta = \frac{2}{3}(\gamma - 1)$ the construction of a homoclinic singular pulse solution is still possible, see Remark 1. The same holds for periodic patterns. As the assumption $A2$ attains equality, the construction of periodic solutions remains valid as the rectangle $\gamma$ remains exponentially small. The only difference to take into account is that condition (3.1) changes into

$$u_{\min} < \frac{2c}{2c + s},$$

where $u_{\min}$ is the $u$-value corresponding to the intersection of $T_o$ with $\ell^u$ in $\mathcal{M}$. 
4. Stability of singular multiscale patterns. Although the existence of many types of stripe patterns in system (1.3) is guaranteed by the analysis of sections 2 and 3, not all will be relevant in light of vegetation patterns. Of course, an unstable solution of a simplified model like (1.3) can never be observed in a natural system. In this section, we test the linear stability of the constructed patterns from the previous sections against two-dimensional perturbations. We formulate the Evans function corresponding to the linear stability problem, and make extensive use of the slow/fast structure of our solutions to evaluate it. This method is developed and described in full detail for reversible systems in [13–15]. However, like in the existence problem, the advection term breaks the symmetry, so the theory cannot be carried over completely. Nevertheless, we show that this approach can be extended and we decompose the Evans function in a fast and a slow component. As in the literature, the slow component of the Evans function is determined explicitly through a hypergeometric function approach.

Let \((U_0, V_0)\) be a traveling solution of (1.2) corresponding to a single stripe \((U_{\text{hom}}, V_{\text{hom}})\) of vegetation or a periodic pattern \((U_p, V_p)\) of vegetation stripes in \(\mathbb{R}\), trivially extended in a second spatial dimension. Since \(N\)-pulse solutions with \(N > 1\) are unstable (this is motivated in section 4.1), we do not consider the stability of these patterns in the current section. The solutions \(\gamma_{\text{hom}}\) and \(\gamma_{p,s}\) that have been established in theorems 2.1 and 3.1 may be rescaled back into \(U, V\) coordinates according to (2.4) to obtain \((U_{\text{hom}}, V_{\text{hom}})\) and \((U_p, V_p)\), respectively;
both solutions of (1.2) could be substituted for \((U_0, V_0)\). Due to the trivial structure of \((U_0, V_0)\) in \(y\)-direction we can use a Fourier ansatz and perturb the solutions as,

\[
(U(\chi, y, \tau), V(\chi, y, \tau)) = \left(U_0 + e^{i\tau + i\ell y}\hat{u}(\chi), V_0 + e^{i\tau + i\ell y}\hat{v}(\chi)\right),
\]

where \(\chi\) is the super slow traveling coordinate. Here we must make the following clarification. For periodic patterns, any perturbation in the direction transverse to the pattern (that is the \(y\)-direction), may be represented by \(\ell \in \mathbb{R}\) and a \(\gamma \in S^1\), taking into account the so-called ‘\(\gamma\)-eigenvalues’ [8]. We may then speak of eigenvalues \(\hat{\lambda}(\gamma, \ell)\) for each \(\gamma, \ell \in (\mathbb{R}, S^1)\). However, since the fundamental interval of our periodic solutions is asymptotically large and the exponential decay of \(U\) and \(V\) is fast enough so that they are both exponentially small outside \(I_f\), the entire family \(\hat{\lambda}(\gamma, \ell)\) is exponentially close to one specific value \(\hat{\lambda}(\ell)\) for each \(\ell\). We quantify this in section 4.1.3. Then, for every fixed \(\ell \in \mathbb{R}\) there exists an eigenvalue \(\hat{\lambda}(\ell)\) that determines the stability of the solution \((U_0, V_0)\), also in the case that \((U_0, V_0)\) is a spatially periodic pattern. For more details on this approach, see [8, 23, 24, 63].

If \(\hat{\lambda}\) has negative real part for every \(\ell \in \mathbb{R}\) – apart from the trivial translation eigenvalue \(\hat{\lambda}(0) = 0\) – the pattern is (spectrally) 2D-stable. Conversely, if there is one \(\ell\) for which an eigenvalue \(\hat{\lambda}(\ell)\) has positive real part, the pattern is 2D-unstable. Note that the reduction \(\ell = 0\) in (4.1) implies only perturbations in the \(\chi\)-direction. The results for \(\ell = 0\) therefore correspond to 1D-stability. In what follows, we derive the stability problem of \((U_0, V_0)\) and subsequently formulate three theorems that summarize the results. In section 4.1, the Evans function framework is established and analyzed for 1-pulse patterns. These results are expanded to also be valid for periodic patterns, so that the proofs of the theorems can be presented for all stripe patterns constructed in this article in section 4.2.

Substituting (4.1) into (1.3) and linearizing, yields the linear stability problem for \((\hat{u}, \hat{v})\),

\[
\begin{align*}
0 &= \hat{u}_{\chi\chi} + (C + S)\hat{u}_\chi - (A + \hat{\ell}^2 + \hat{\lambda} + V_0^2)\hat{u} - 2U_0V_0\hat{v}, \\
0 &= \delta^2\hat{v}_{\chi\chi} + S\hat{v}_\chi + (2U_0V_0 - \delta^2\hat{\ell}^2 - \hat{\lambda} - B)\hat{v} + V_0^2\hat{u}.
\end{align*}
\]

Rescaling according to (2.4), and introducing

\[
\begin{align*}
U_0 &= \delta^{-\gamma + \frac{3}{2}\beta}u_0, \quad \hat{u} = \delta^{-\gamma + \frac{3}{2}\beta}u, \quad \hat{\lambda} = \delta^\beta b\lambda, \\
V_0 &= \delta^{-1+\gamma - \frac{1}{2}\beta}v_0, \quad \hat{v} = \delta^{-1+\gamma - \frac{1}{2}\beta}v, \quad \hat{\ell} = \delta^\theta \sqrt{bl},
\end{align*}
\]

we derive a linear, non-autonomous, four-dimensional system of first order differential equations in the fast variable \(\xi\), much like system (2.6),

\[
\begin{align*}
u_\xi &= \delta^{\gamma - \beta}p, \\
p_\xi &= \delta^{\gamma - \beta}\left[2u_0v_0v + v_0^2u + \delta^{2+\beta}au + \left(\delta^{2+2\theta+\beta-2\gamma}\ell^2 + \delta^{2+2\beta-2\gamma}\lambda\right)bu - \delta^{1+\frac{3}{2}\beta}cp - \delta^{2+\gamma - \beta}sp\right], \\
v_\xi &= \delta^\theta q, \\
q_\xi &= [b(\lambda + 1) - 2u_0v_0]v + v_0^2u - \delta^{2+2\theta+\beta}b\ell^2v - \delta^{2(\gamma - \beta)}sq.
\end{align*}
\]
Here, the new scaling parameter $\theta$ is a dummy parameter to make the magnitude of $\ell$ explicit. In principle, 2D-stability is only guaranteed if we can make a stability statement for all $\theta \in \mathbb{R}$. In order to perform our slow/fast analysis, however, we restrict to a subclass of transverse perturbations by assuming,

\[ \theta > \gamma - \frac{1}{2} \beta - 1. \]

That is, we test stability against perturbations with a transverse wave number that may be arbitrarily small, but is not larger than $O(\delta^{\gamma - \frac{3}{2} \beta - 1})$. Since we have assumed (A3) that $\gamma - \frac{1}{2} \beta - 1 \geq 0$, this is an asymptotically large bound. Restricting to this subclass in general weakens the stability statements. However, we find that also for $\theta$ within this bound, there always exist unstable perturbations. That implies that the restriction to this subclass does not change anything for the 2D-stability of $(U_0, V_0)$.

Concerning the stability of solutions $(U_0(\xi), V_0(\xi))$ of (1.2), we formulate the following theorems.

**Theorem 4.1.** Let the assumptions A1–A3 be satisfied and let $\delta$ be small enough. Let $(U_0, V_0)$ be a slow/fast solution of (1.2), either of a 1-pulse type, or a spatially periodic pattern with asymptotically large wave length of $O(\delta^{\frac{3}{2} \beta - 1})$ as established in theorems 2.1 and 3.1. Then there is an $\ell$ such that there is a bounded solution to (4.1) for a $\lambda(\ell)$ with positive real part. That is, the 1-pulse and periodic stripe patterns constructed in this article, are 2D-unstable.

For $\ell = 0$, the stability of a solution $(U_0, V_0)$ is tested for perturbations without a component in the $y$-component. Since we have assumed the constructed solutions in this article have a trivial extension in the $y$-direction, results for $\ell = 0$ correspond to the stability of one-dimensional patterns. For this 1D-stability of a spatially one-dimensional pattern, which we will, slightly abusive, also label $(U_0, V_0)$, we formulate theorem 4.2.

For a clear presentation of the results, we introduce,

\[ C_1 = \frac{\bar{u}^2}{6b}, \quad C_2 = \frac{c^2 + 4a}{b}. \]

where $\bar{u}$ is the value of $\delta^{\gamma - \frac{3}{2} \beta - 1}U_0$ in the fast regime $I_f$, where $U_0$ is to leading order constant, see 2.6. In the case that $(U_0, V_0) = (U_{\text{hom}}, V_{\text{hom}})$, the value of $\bar{u}$ is in fact reported in (2.29).

**Theorem 4.2.** Let the assumptions A1 and A3 be satisfied and let $\beta \geq \frac{2}{3}(\gamma - 1)$. Let $(U_0, V_0)$ be a slow/fast 1-pulse solution of (1.2) in one spatial dimension. That is, let $(U_0, V_0)$ be $(U_{\text{hom}}, V_{\text{hom}})$ established in theorem 2.1 without the trivial extension in the $y$-direction. The (1D)-stability of this pattern can be summarized as follows.

(i) If $\beta < \gamma - \frac{1}{2}$, there is a bounded solution to (4.1) with $\Re(\lambda) > 0$, so $(U_0, V_0)$ is 1D-unstable.

(ii) If $\beta > \gamma - \frac{1}{2}$ and A2 is satisfied, all nontrivial eigenvalues $\lambda$ corresponding to pattern $(U_0, V_0)$ have negative real part, so the pattern is 1D-stable.

(iii) If $\beta = \gamma - \frac{1}{2}$ and A2 is satisfied, then a pair of eigenvalues $\lambda_{1,2}$ passes through the imaginary axis (i.e., a Hopf bifurcation occurs) if $C_1 = 2H^*$. Here, $H^*$ is given as the solution of an explicit expression in terms of integrals, and $H^* \approx 0.661$. Using (2.29), we equivalently formulate that for $\sqrt{\frac{b}{2c^2 + 4a}} < 2H^*$, the pattern $(U_0, V_0)$ is 1D-stable.
The transition from stable to unstable one-dimensional homoclinic and/or spatially periodic patterns (or vice versa) occurs in the parameter regime $\frac{\delta^2}{\Omega^2} = \mathcal{O}(\delta)$, i.e. $\beta = \gamma - \frac{1}{2}$ (2.4) – see (iii), (v), (vi) of theorem 4.2, (iii) of theorem 4.3 and Figure 12 (this regime is
Figure 12: Schematic representation of 1D-stability regimes as described in theorem 4.2 and 4.3. The shaded regimes are the regimes in the $(\gamma, \beta)$-plane where stripe patterns exist, defined by assumptions A1–A3 and $\beta \geq \frac{2}{3} (\gamma - 1)$. Stripe patterns with scaling parameters in the gray shaded areas are 1D-unstable. Stripe patterns with scaling parameters in the blue shaded area are 1D-stable. Stripe patterns with scaling parameters on the borders may undergo a bifurcation when parameters change as described in theorem 4.2. The regimes (i)–(vi) correspond to the subresults with the same labels in theorem 4.2, and the regimes (i)–(iii) with those of theorem 4.3.

indicated by $C_{\text{HOPF}}$ in [12]). Parameter values at which the Hopf bifurcations take place are determined in terms of explicit expressions that are obtained from numerical evaluations of integrals over (complex powers of) several polynomial functions – see section 4.1.2, [13] and especially remark 4.2 in [63]. Therefore, the proofs of the statements in theorems 4.2 and 4.3 about the case $\beta = \gamma - \frac{1}{2}$ need additional arguments. In [15], this has been worked out in detail by a combination of asymptotic and computer-assisted winding number arguments. A similar procedure can be applied in the present setting. An alternative, less parameter dependent but also less explicit, method by which the occurrence of Hopf bifurcations can be established is developed in [32, 68] – see especially the recent text [71].

Remark 4. In the case of the homoclinic pulses of theorem 4.2, the fact that the operator is sectorial immediately establishes nonlinear stability if the pattern is spectrally stable, [15, 29]. We refrain from going into details about the nonlinear stability of the periodic patterns and two-dimensional patterns: stability statements in theorem 4.3 and upcoming corollary 4.5 concern only spectral stability with respect to $O(1)$ eigenvalues. This follows the approach of [13], and more details are reported in [8].

Remark 5. The outcome of the analysis has been confirmed by numerical simulations for various parameter combinations. For example, the stable traveling patterns shown in Figures 1 and 7 indeed correspond to choices of $\gamma$ and $\beta$ that are located in the stable (blue shaded)
areas in Figure 12, where it should be noted that the values of $A, B, C$ in Figure 1 may be equivalently represented by $a \approx 0.28, b = 0.25, c \approx 0.45$ with $\gamma \approx -0.174$ and $\beta \approx -0.190$ (via rescaling (2.4)).

4.1. The Evans function and associated nonlinear eigenvalue problem. For the proof of theorems 4.1–4.3, we establish an Evans function framework for the 1-pulse patterns, $(U_{hom}, V_{hom})$ constructed in section 2. In section 4.1.3 we will show how this framework can be in essence carried over to periodic patterns with periods of sufficient asymptotic length, after which we may proceed to section 4.2 for the proofs of the theorems 4.1, 4.2. To make use of the slow/fast structure of the system in an efficient way, we will use the four-dimensional formulation of the stability problem, (4.4).

4.1.1. Homoclinic 1-pulse solutions. As an equivalent representation of (4.4), we write

$$\frac{\partial}{\partial \xi} \phi(\xi; \lambda, \ell) = A(\xi; \lambda, \ell) \phi(\xi; \lambda, \ell),$$

where $\phi(\xi; \lambda, \ell) = (u(\xi), p(\xi), v(\xi), q(\xi))$, and $A$ is a $4 \times 4$ matrix. We will lay out the stability analysis in terms of the fast variable $\xi$. For $\xi$ outside $I_f$, the solution $v_0$ is exponentially close to zero. Moreover, every term of (4.4) that involves $u_0$, which varies outside the fast regime, is multiplied with $v_0$. Hence, the matrix $A(\xi; \lambda, \ell)$ approaches a constant matrix outside that fast regime;

$$\lim_{\xi \to \pm \infty} A(\xi; \lambda, \ell) = A_\infty(\lambda, \ell),$$

with

$$A_\infty(\lambda, \ell) = \begin{pmatrix} 0 & \delta^{\gamma - \beta} & \delta^{\gamma - \beta} & 0 \\ 0 & \delta^{2\gamma - \beta} a + b(\delta^{2\theta - \beta} \ell^2 + \lambda) & -\delta^{1+\gamma - \beta} (\delta^{\beta} c + \delta^{1+\gamma - \beta} s) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b(\lambda + 1 + \delta^{2+2\theta - \beta} \ell^2) & -\delta^{2(\gamma - \beta) s} & 1 \end{pmatrix}$$

(4.10)

The matrix $A_\infty$ has four eigenvalues $\Lambda_1, 2, 3, 4$. Two of those eigenvalues are asymptotically small and two are $O(1)$. Their leading order approximations are,

$$\Lambda_1 = \sqrt{b(\lambda + 1 + \delta^{2+2\theta - \beta} \ell^2)} + \text{h.o.t.}$$

$$\Lambda_2 = \frac{1}{2} \delta^{1+\gamma - \beta} \left[ -c + \sqrt{c^2 + 4a + 4\delta^{3-2\gamma} (\delta^{2\theta - \beta} b\ell^2 + b\lambda)} \right] + \text{h.o.t.}$$

$$\Lambda_3 = \frac{1}{2} \delta^{1+\gamma - \beta} \left[ -c - \sqrt{c^2 + 4a + 4\delta^{3-2\gamma} (\delta^{2\theta - \beta} b\ell^2 + b\lambda)} \right] + \text{h.o.t.}$$

$$\Lambda_4 = -\sqrt{b(\lambda + 1 + \delta^{2+2\theta - \beta} \ell^2)} + \text{h.o.t.}$$

(4.11)
Assumptions $A_1$–$A_3$ and (4.5), imply that indeed $\Lambda_{2,3} \ll 1$, while $\Lambda_{1,4} = O(1)$. Furthermore, the eigenvalues $\Lambda_{1,2}$ have by definition positive real part and are unstable, while $\Lambda_{3,4}$ have negative real part and are stable. The corresponding eigenvectors are

\[
E_{1,4} = (0,0,1,\Lambda_{1,4})^T,
\]

\[
E_{2,3} = (1,\delta^{3-\gamma}\Lambda_{2,3},0,0)^T.
\]

The essential spectrum associated with (4.9) coincides with all $\lambda(\ell)$ for which $A_{\infty}$ has an eigenvalue $\Lambda_i \in i\mathbb{R}$, that is,

\[
\sigma_{\text{ess}} = \bigcup_{k,\ell \in \mathbb{R}} \left\{ -1 - \delta^2 + 2\theta - \beta \ell^2 - k^2, \frac{1}{4\delta^2 - 2\gamma b} \left( -4a - 4\delta^2 - 2\gamma b \ell^2 + 2ck - k^2 \right) \right\}
\]

Depending on whether $\beta \gtrsim 2\gamma$, the maximal real part of the essential spectrum is $-1$ or $-\delta^2 - \beta \frac{a}{b}$, see a schematic representation of both cases in figure 13.

The full linear stability problem (4.9), has four independent (vector-)solutions $\phi_j(\xi; \lambda, \ell)$. The theory in [14,15] explains that we may introduce eigenvectors $\phi_j$ such that,

\[
\lim_{\xi \to -\infty} \phi_j(\xi; \lambda, \ell)e^{-\Lambda_j \xi} = E_j \quad \text{for } j = 1, 2,
\]

\[
\lim_{\xi \to \infty} \phi_j(\xi; \lambda, \ell)e^{-\Lambda_j \xi} = E_j \quad \text{for } j = 3, 4.
\]

In particular, this implies that $\phi_{1,2} \to (0,0,0,0)$ as $\xi \to -\infty$ and $\phi_{3,4} \to (0,0,0,0)$ as $\xi \to \infty$. Because $\Lambda_1$ is the largest, positive eigenvalue, a general solutions $\phi$ will grow as $e^{\Lambda_1 \xi}$ as $\xi \to \infty$. We define the fast transmission function $t_f(\lambda, \ell)$, which is an analytic function over $\lambda$ as,

\[
\lim_{\xi \to \infty} \phi_1(\xi; \lambda, \ell)e^{-\Lambda_1 \xi} = t_f(\lambda, \ell)E_1(\lambda, \ell).
\]
Not all $\phi_i$ necessarily grow with the largest rate, though. In the case that $t_f(\lambda, \ell) \neq 0$, we can define $\phi_2$ uniquely by assuming it does not grow like $e^{\Lambda_1 \xi}$, i.e.
\begin{equation}
\lim_{\xi \to \infty} \phi_2 e^{-\Lambda_1 \xi} = (0, 0, 0)^T.
\end{equation}

In other words, the only vector solution that grows with the fast rate as $\xi \to \infty$ is $\phi_1(\xi)$, because (4.14) has already put a boundary condition on $\phi_2$ and $\phi_3$. A detailed justification of this procedure follows especially from Lemma 3.7 in [14]. Generically, the behavior of the second solution $\phi_2$ is dominated by the slow growth rate $\Lambda_2$ for $\xi > 0$. Hence, we define a slow transmission function, $t_s(\lambda, \ell)$.
\begin{equation}
\lim_{\xi \to \infty} \phi_2(\xi; \lambda, \ell) e^{-\Lambda_2 \xi} = t_s(\lambda, \ell) E_2(\lambda, \ell).
\end{equation}

The Evans function is the determinant of the four independent solutions of $A_\infty$,
\begin{equation}
D(\lambda, \ell) = \det [\phi_1, \phi_2, \phi_3, \phi_4] e^{-\int_0^\xi \text{Tr}(A(\eta; \lambda, \ell)) d\eta},
\end{equation}
see [14]. The Evans function is only defined for $\lambda$ outside of the essential spectrum, (4.13). Since
\[\text{Tr}(A(\eta; \lambda, \ell)) = \sum_j \Lambda_j = -\delta^{2(\gamma-\beta)} s,\]
and because the Evans function is independent of $\xi$, see [1], (4.18) is equivalent to,
\begin{equation}
D(\lambda, \ell) = \lim_{\xi \to \infty} \det \left[ \phi_1 e^{-\Lambda_1 \xi}, \phi_2 e^{-\Lambda_2 \xi}, \phi_3 e^{-\Lambda_3 \xi}, \phi_4 e^{-\Lambda_4 \xi} \right]
= \det \left[ t_f(\lambda, \ell) E_1(\lambda, \ell), t_s(\lambda, \ell) E_2(\lambda, \ell), E_3(\lambda, \ell), E_4(\lambda, \ell) \right]
= \delta^{\beta - \gamma} t_s(\lambda, \ell) t_f(\lambda, \ell)((\Lambda_4 - \Lambda_1)(\Lambda_3 - \Lambda_2)),
\end{equation}
to leading order. Zeros of the Evans function coincide with eigenvalues $\lambda(\ell)$. Since outside of the essential spectrum $\sigma_{\text{ess}}$, see (4.13), the $\Lambda_i$ never coincide, we conclude that the Evans function is zero only if the product of the transmission functions, $t_s t_f$ is zero. For the fast transmission function, the zeros can simply be found in literature. The associated fast reduced stability problem, that can be obtained from (4.4) by taking the limit $\delta \to 0$ and setting $u \equiv 0$, is
\begin{equation}
v_{\xi \xi} + 2\bar{u} v_\eta v - b(\lambda + 1)v = 0.
\end{equation}
Which equivalent to those reported in [14, 15, 58]. The fast isolated eigenvalues are
\begin{equation}
\lambda_f^0 = \frac{5}{4}, \quad \lambda_f^1 = 0, \quad \lambda_f^2 = -\frac{3}{4}.
\end{equation}
Paradoxically, the positive eigenvalue $\lambda_f^0$ does not immediately imply instability of $(u_0, v_0)$, because the slow transmission is not analytic but merely meromorphic and has a pole at $\lambda_f^0$, as we will show below. This zero-pole cancellation is explained in full detail in [14, 15]. In [14], it is also shown that $N$-pulses pick up the same fast eigenvalues but with multiplicity $N$, which are not canceled by the order 1 poles of the slow transmission function. Hence, $N$-pulses are unstable. For more details on this mechanism, see [17].
4.1.2. Zeros of the slow transmission function. We determine the zeros of the slow transmission function by matching the values of $u$ and $p$ in- and outside the fast interval $I_f$, see (2.20). The change in $u$ and $p$ is measured by the uniquely determined fundamental solution $\phi_2$. We know that it does not grow with the fast rate, see (4.16), so outside $I_f$ and with $\xi > 0$,

$$\phi_2(\xi) = t_s e^{\Lambda_2 \xi} E_2 + \tilde{t}_s e^{\Lambda_3 \xi} E_3 + \tilde{t}_f e^{\Lambda_4 \xi} E_4,$$

where $\tilde{t}_s, \tilde{t}_f$ are also transmission functions of $\lambda$ and $\ell$. The $\Lambda_4$-term does not contribute to the leading order behavior, however, because $\Lambda_4$ is large and negative. In fact, outside $I_f$, $e^{\Lambda_3 \xi}$ is already exponentially small. On the other side outside of $I_f$, where $\xi < 0$, we know by (4.14)

$$\phi_2(\xi) = e^{\Lambda_2 \xi} E_2,$$

up to exponentially small corrections. We measure the change of $\phi_2$ over $I_f$ from outside of $I_f$ (in the slow regime) first. We define the slow difference function $\Delta_s$ as follows,

$$\Delta_s \begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix} = \lim_{\xi \to -\xi_0} (t_s e^{\Lambda_2 \xi} E_2 + \tilde{t}_s e^{\Lambda_3 \xi} E_3 + \tilde{t}_f e^{\Lambda_4 \xi} E_4) - \lim_{\xi \to -\xi_0} e^{\Lambda_2 \xi} E_2.$$

Upon assuming $A1$–$A3$ and (4.5), we can always choose a $\mu > 0$ (recall (2.20)) such that

$$\Delta_s u = t_s e^{\Lambda_2 \delta^{-\mu}} E_2 + \tilde{t}_s e^{\Lambda_3 \delta^{-\mu}} E_3 + \tilde{t}_f e^{\Lambda_4 \delta^{-\mu}} E_4 - e^{\Lambda_2 \delta^{-\mu}} E_2,
= t_s + \tilde{t}_s - 1,$$

to leading order. We know that during the fast transition, the $u$-component does not change (recall (4.4)), therefore

$$\tilde{t}_s = 1 - t_s.$$

Although $u$ is constant even up to the second order corrections during the fast transition, $p$ is not constant in the first correction, see (2.6). We also measure the change in $p$ with the slow difference function,

$$\Delta_s p = t_s e^{\Lambda_2 \delta^{-\mu}} \delta^{\beta - \gamma} \Lambda_2 + \tilde{t}_s e^{\Lambda_3 \delta^{-\mu}} \delta^{\beta - \gamma} \Lambda_3 + \tilde{t}_f e^{\Lambda_4 \delta^{-\mu}} \cdot 0 - e^{\Lambda_2 \delta^{-\mu}} \delta^{\beta - \gamma} \Lambda_2,
= \delta^{\beta - \gamma} (t_s - 1)(\Lambda_2 - \Lambda_3),$$

to leading order. On the other hand, we can measure the change in $p$ over the fast regime, much like section 2, equation (2.28). We define the fast difference function $\Delta_f$ as,

$$\Delta_f p = \int_{I_f} p(\xi) d\xi,$$

$$\Delta_f p = \delta^{\beta - \gamma} \int_{I_f} \left( v_0^2 u + 2\tilde{u} v_0 v + \delta^{2+\beta} au + \delta^{2+2\gamma-2\gamma+\beta} b t^2 + \tilde{u}^2 \delta^{2+\gamma+2\beta} b \right) d\xi,$$

$$- (\delta^{1+\frac{1}{2}\beta} \tilde{c} + \delta^{2+\gamma-\beta} s u) \phi_2(\xi).$$
We use the fact that, at the boundary of $I_f$, the two difference functions must be equal; $\Delta f_p = \Delta s_p$. Moreover, as we are interested in zeros (and poles) of the slow transmission function, we set $t_s = 0$, so that

$$\delta^{\beta - \gamma}(\Lambda_3 - \Lambda_2) = \Delta f_p. \tag{4.26}$$

In the evaluation of $\Delta f_p$, we may choose $u = 1$, because it must be a constant and we use the one-parameter freedom of choice in determining an eigenfunction. If $A1$–$A3$ and $(4.5)$ are satisfied, we can always choose a $\mu$ as defined in $(2.20)$ such that $\Delta f_p$ is, at leading order, purely associated with the fast variables, that is,

$$\Delta f_p = \delta^{\gamma - \beta} \int_{I_f} 2\bar{u}v_0(\xi)v(\xi) + v_0^2(\xi)d\xi, \tag{4.27}$$

In particular, this implies we must choose

$$0 < \mu < \min\{2 + 2\theta - 2\gamma + \beta, 2 - 2\gamma + 2\beta, \gamma - \beta, 1 + \gamma - \frac{1}{2}\beta\}. \tag{4.28}$$

See figure 2 and take into account assumption $(4.5)$ to verify that in our restricted regime for $\gamma$ and $\beta$, we can always choose $\mu$ within this range.

Equation $(4.27)$ does not stand on itself but is directly linked to the stability problem $(4.9)$. Our slow eigenvalue problem thus becomes

$$\left\{ \begin{array}{l}
\delta^{2(\gamma - \beta)} \int_{I_f} 2\bar{u}v_0(\xi)v_0(\xi)d\xi = \Lambda_3 - \Lambda_2, \\
v_{0,xx} + (2\bar{u}v_0 - b(\lambda + 1))v_0 = -v_0^3
\end{array} \right. \tag{4.29}$$

We can rewrite $(4.29)$ into one equation: the so-called nonlinear eigenvalue problem (NLEP) as is used in $[13]$,

$$v_{\xi\xi} + \left( \frac{3b}{\cosh^2\left(\frac{\sqrt{b}\xi}{2}\right)} - b(\lambda + 1) \right)v = \frac{2\delta^{2(\gamma - \beta)}\bar{u}v_0^2}{\delta^{2(\gamma - \beta)}\int_{I_f} v_0^2(\xi)d\xi - \Lambda_3 + \Lambda_2} \int_{I_f} v_0(\xi)v(\xi)d\xi. \tag{4.30}$$

Using $(2.18)$, we derive

$$\int_{I_f} v_0^2(\xi)d\xi = \frac{6b\sqrt{b}}{\bar{u}^2}, \tag{4.31}$$

which simplifies equation $(4.30)$,

$$v_{0,xx} + \left( \frac{3b}{\cosh^2\left(\frac{\sqrt{b}\xi}{2}\right)} - b(\lambda + 1) \right)v = \frac{2\delta^{2(\gamma - \beta)}\bar{u}v_0^2}{\delta^{2(\gamma - \beta)}b\sqrt{b} - (\Lambda_3 - \Lambda_2)\bar{u}^2} \int_{I_f} v_0(\xi)v(\xi)d\xi. \tag{4.32}$$

Following $[13]$, we transform equation $(4.30)$ using the following substitutions,

$$\xi = \frac{2}{\sqrt{b}}t, \quad v_{0}(\xi) = y(t), \quad \lambda = \frac{1}{4}P^2 - 1. \tag{4.33}$$
We only consider \( \text{Re}(P) > 0 \), to stay away from the essential spectrum, see (4.13) and figure 13. To convert \( P \) into \( \lambda \), we use the principal square root. Substituting (4.33) into (4.32) yields

\[
y_{tt} + \left( \frac{12}{\cosh^2(t)} - P^2 \right) = \frac{K}{\cosh^4(t)} \int_I y(t) \cosh^2(t) dt,
\]

with

\[
K = \frac{9\delta^2(\gamma - \beta)}{\delta^2(\gamma - \beta) + \frac{\tilde{u}^2(\Lambda_2 - \Lambda_3)}{6b\sqrt{b}}}.
\]

Following [13], the differential equation for \( y \) can be modified to a hypergeometric differential equation for \( G(x) \) with a second substitution,

\[
y(t) = G(t)/(\cosh(t))^P, \quad t = \tanh^{-1}(2x - 1).
\]

The exact derivation of the solutions of that hypergeometric differential equation is detailed in [13], and an alternative expression for \( K \) is derived,

\[
K(P) = \frac{P(P - 1)(P - 2)(P - 3)}{16R(P)}.
\]

where

\[
R(P) = -\frac{1}{(P + 3)(P + 2)(P + 1)} \int_0^1 \int_0^\xi \xi(1 - \xi) \left( \frac{x(1 - \xi)}{\xi(1 - x)} \right)^{P/2} k(P, 1 - x)k(P, \xi) dx d\xi,
\]

where \( k(P, \xi) \) is defined as,

\[
k(P, \xi) = (P - 3)(P - 2)(P - 1) + 12(P - 3)(P - 2)\xi + 60(P - 3)\xi^2 + 120\xi^3.
\]

For consistency in \( K \), we must equate (4.35) and (4.36), and we arrive at the final formulation of the eigenvalue problem.

\[
\delta^{-2(\gamma - \beta)} (\Lambda_2 - \Lambda_3) \frac{\tilde{u}^2}{6b\sqrt{b}} = \frac{9}{K(P)} - 1,
\]

with \( K(P) \) as in the formulation (4.36). The right hand side of (4.39) has poles of \( O(1) \) for \( P = 1 \) and \( P = 3 \), see Figure 14 and [13, 14]. They link directly to two of the fast eigenvalues (4.21), \( \lambda_f^0 = \frac{\delta}{2} \) and \( \lambda_f^2 \) and hence the positive eigenvalue \( \lambda_f^0 \) does not necessarily destabilize the pattern. This phenomenon is often referred to as the ‘NLEP-paradox’ see [14].

### 4.1.3. Periodic solutions

In the previous section, we have derived equation (4.39) which determines the stability properties of a 1-pulse \((U_{hom}, V_{hom})\). In this subsection, we briefly comment on how the same equation determines stability of periodic solutions \((U_{p,s}, V_{p,s})\). For homoclinic solutions, which are localized, the eigenfunctions are localized as well. Hence, the full spectrum consists of discrete eigenvalues \( \tilde{\lambda} \) for given \( \tilde{\ell} \), united with the essential spectrum
corresponding to the background state. Perturbations of periodic solutions are in general represented by both a wave number \( \ell \) as well as a \( \gamma \in S^1 \), because the eigenfunctions are not localized. This implies that the eigenvalues \( \lambda \) outside \( \sigma_{\text{ess}} \), see (4.13), are not discrete for a given \( \ell \). Instead, there exist curves of essential spectrum parametrized by \( \gamma \), [23].

For periodic solutions with a sufficient length, these \( \gamma \)-parametrized curves of spectrum may be exponentially approximated by the discrete eigenvalue corresponding to a localized pulse, because the periodic solution is ‘nearly’ localized. The period of the solutions constructed in section 3, has a length of \( O(\ell) \) in \( \xi \). If we assume that \( q(0) = 0 \) we can construct a fundamental interval for the periodic solution as

\[
I_{\text{per}} = \left(-\frac{l}{\delta^{\gamma-\frac{1}{2}\beta+1}}, \frac{l}{\delta^{\gamma-\frac{1}{2}\beta+1}}\right),
\]

with \( l > 0 \). The eigenfunctions \((U, V)\) of (4.1) corresponding to a periodic solution \((U_{\text{per},s}, V_{\text{per},s})\) may then be called nearly localized if both \( U \) and \( V \) are exponentially small on the boundaries of \( I_{\text{per}} \). For the \( V \)-pulse, this is clear, because \( V \) decays exponentially fast to zero already within the fast interval \( I_f \), which is smaller than \( I_{\text{per}} \). What is left is to verify that the decay rate of \( U \) is also fast enough. About this decay rate and the validity of the approximation, we formulate the following lemma.

**Lemma 4.4.** For periodic solutions of (1.2) as constructed in section 3, with wave length of \( O\left(\frac{1}{\delta^{\gamma-\frac{1}{2}\beta}}\right) \), we may approximate the spectrum outside \( \sigma_{\text{ess}} \) by the discrete values that are solutions to (4.39) in any of the following regimes.

(i) \( \beta < 2\gamma \);

(ii) \( \beta \geq 2\gamma \) and \( \gamma - \frac{1}{2}\beta - 1 < \theta < \gamma \).

**Proof.** To show this, we will work with the rescaled coordinates and system (4.4). Outside the fast interval \( I_f \), the equation for \( u \) decouples because \( v \) is exponentially small. We obtain,

\[
u_{\xi\xi} = \delta^{2\gamma-2\beta} \left[ \delta^{2+\beta} au + (\delta^{2+2\theta+\beta-2\gamma} \ell^2 + \delta^{2+3\beta-2\gamma} \lambda) bu - (\delta^{1+\frac{3}{2}\beta-\gamma} c + \delta^2 s) u_\xi \right],
\]

which is a second order ODE for \( u \) with constant coefficients. The solution for \( u \) is a linear combination of exponentials in variable \( \delta^r \xi \), where

\[
(4.41) \quad r = \min\{1, 1 + \gamma - \frac{1}{2}\beta, 1 + \theta - \frac{1}{2}\beta\},
\]

is the decay rate of \( u \). If we require \( u \) to be exponentially small on the boundaries of \( I_{\text{per}} \), which is \( O\left(\frac{1}{\delta^{\gamma-\frac{1}{2}\beta}}\right) \), we must satisfy

\[
(4.42) \quad 1 + \gamma - \frac{1}{2}\beta > r = \min\{1, 1 + \gamma - \frac{1}{2}\beta, 1 + \theta - \frac{1}{2}\beta\}
\]

In other words, the length of the fundamental interval \( I_{\text{per}} \) must be asymptotically strictly larger than the decay of \( u \). If we want to exponentially approximate the spectrum corresponding to periodic solutions by discrete eigenvalues, condition (4.42) must be satisfied. Note that the strict inequality (4.42) can never be satisfied if \( r = 1 + \gamma - \frac{1}{2}\beta \), the same parameter.
combination is stated on the left side of the inequality. This is implies that if $\beta \geq 2\gamma$, a restriction must be put on $\theta$ so that $r = 1 + \theta - \frac{1}{2}\beta$, in order to satisfy (4.42). Combining this with condition (4.5), this yields the result stated in (ii). If, on the other hand, $\beta < 2\gamma$, then $r \neq 1 + \gamma - \frac{1}{2}\beta$ and this immediately implies that (4.42) is satisfied, so we arrive at the statement of (i).

Since $\theta$ is only a dummy parameter restrictions on $\theta$ change the function space of perturbations against which we will test the (in)stability of periodic patterns. By condition (4.5), we put an upper bound on the transverse wave number of the perturbation. A consequence of lemma 4.4 is that if $\beta \geq 2\gamma$, the stability statements derived from equation (4.39) for periodic solutions are only valid for a smaller subclass of perturbations. Namely, those perturbations that have a transverse wave number that is not only bounded below by (4.5), but also bounded above by requiring $\theta < \gamma$.

4.2. Proof of stability theorems. In this section, we analyze (4.39), to prove theorems 4.1 and 4.2. The right-hand side of (4.39) is plotted as a graph over $P \in \mathbb{R}$ in Figure 14. Note

![Figure 14: Black: Graph of $\frac{9}{K(P)} - 1$ with $P > 0$ and real-valued. Note that there are vertical asymptotes at $P = 1$ and $P = 3$. These are the poles of the slow transmission function, and they cancel two of the zeros of the fast transmission function, $\lambda^0_f$ and $\lambda^2_f$, see (4.21).](image)

that, as $\lambda \in \mathbb{C}$, so is $P$. The representation of figure 14 is therefore not exhaustive. Note that, indeed, the graph is singular at $P = 1$ and $P = 3$, where the function has poles of order 1. In the cases that $\lambda$ is real, however, the solutions of (4.39) may be visualized as the intersections of the graph in Figure 14 and a real-valued curve of the left hand side of (4.39) as a function over $P$. We clarify this in the proofs of the theorems 4.1–4.3. As was already obvious from their formulations, the magnitude of $\Lambda_{2,3}$ makes a crucial difference. From (4.11) which we derive

\begin{equation}
\Lambda_2 - \Lambda_3 = \delta^{1+\gamma - \frac{1}{2}\beta} \sqrt{c^2 + 4a + 4b(\delta^{2\beta-2\gamma}c^2 + \delta^{\beta-2\gamma}x)}.
\end{equation}

Note that

\begin{equation}
O(\Lambda_2 - \Lambda_3) = O(\delta^0),
\end{equation}

(4.44)
where $r$ was defined in (4.41).

Proof. (of theorem 4.1).
If the left hand side of equation (4.39) is much larger than $O(1)$, there are solutions for $P \gtrless 1$ and $P \lesssim 3$, because the right hand side has poles for $P = 1$ and $P = 3$. By choosing $\theta$ appropriately, namely

\begin{equation}
\theta < 2\gamma - \frac{3}{2}\beta - 1,
\end{equation}

we obtain $r < 2(\gamma - \beta)$, so that the left hand side of (4.39) is asymptotically large. That implies that there is always a $\ell$ such that there exists an solution $P \lesssim 3$, which corresponds to $\lambda = \frac{5}{4}$. Note that condition (4.5) is automatically satisfied in this case. Hence, perturbations with a transverse wave number that is small enough grow exponentially, so the patterns are 2D-unstable. This result holds for both the 1-pulse and periodic solutions, because (4.45) implies by assumption A2 that $\theta < \gamma$, see lemma 4.4. ■

Proof. (of theorem 4.2)
This theorem concerns only patterns in one spatial dimension. Hence, we do not consider transverse perturbations so we may set $\ell = 0$ in equation (4.39) or, more specifically, (4.43). This simplifies equation (4.39) to

\begin{equation}
\delta^{1-\gamma + \frac{3}{2}\beta} \sqrt{c^2 + 4a + 4\delta^{\gamma - 2\gamma} b \lambda} \frac{\mu^2}{6b^3} = \frac{9}{K(P)} - 1.
\end{equation}

(i) If $\beta < \gamma - \frac{1}{2}$ and $\beta \geq \frac{2}{3}(\gamma - 1)$, then the left hand side of (4.39) is much larger than $O(1)$, so the same argument as in the proof of theorem 4.1 applies. There is an eigenvalue $\lambda \lesssim \frac{5}{4}$, so the solution is 1D-unstable.

(ii) If $\beta > \gamma - \frac{1}{2}$ and $\beta > \frac{2}{3}(\gamma - 1)$, the left hand side of (4.39) is asymptotically small. Thus, we solve

\begin{equation}
0 = \frac{9}{K(P)} - 1.
\end{equation}

By a numerical evaluation of the analytic expressions (4.35)–(4.38), we find that this occurs for $P \approx 0.56 \pm 0.52i$ or $\lambda \approx -0.99 \pm 0.14i$. Hence, the pattern is 1D-stable.

(iii) If $\beta = \gamma - \frac{1}{2}$ and $\beta > \frac{2}{3}(\gamma - 1)$, recall the definition of $C_1$ in (4.6), so that the leading order of equation (4.46) becomes

\begin{equation}
\frac{1}{2} C_1 = \frac{1}{\sqrt{P^2 - 4}} \left( \frac{9}{K(P)} - 1 \right),
\end{equation}

which is equivalent to equation (5.15) of [13]. Thus, there is a Hopf bifurcation when this equation is solved for a purely imaginary set of eigenvalues $\lambda_{\pm}$. Using numerical evaluation we can derive immediately that this occurs when

\begin{equation}
\frac{1}{2} C_1 = H^* \approx 0.661,
\end{equation}

at which $\lambda \approx \pm 0.535i$. This implies that the pattern is 1D-stable for $C_1 < 2H^*$ and 1D-unstable for $C_1 > 2H^*$. Since we know that in this case $\bar{u} = u_0 = \frac{6b\sqrt{\gamma}}{\sqrt{c^2 + 4a}}$ (see 2.29), we can
simplify this to
\[ \sqrt{\frac{b}{c^2 + 4a}} < 2H^*. \]
This proves this part of the theorem.

(iv) If \( \beta > \gamma - \frac{1}{2} \) and \( \beta = \frac{2}{3}(\gamma - 1) \) equation (4.39) reduces to leading order to,

\[ (4.48) \]
\[ C_1 \sqrt{C_2} = \frac{9}{K(P)} - 1. \]

The left hand side of this equation does not depend on \( \lambda \) and is always real-valued. The stability of a pattern changes if the real part of an eigenvalue \( \lambda \) passes zero. The special case \( \lambda = 0 \) corresponds to \( P = 2 \), and a simple check with [13] yields

\[ (4.49) \]
\[ \lim_{P \to 2} \frac{9}{K(P)} - 1 = 1. \]

Which implies that (4.48) is solved for a zero eigenvalue if

\[ C_1 \sqrt{C_2} = 1. \]

Hence, this is where a bifurcation occurs. A straightforward verification yields that the pattern is 1D-stable if \( C_1 \sqrt{C_2} < 1 \), and 1D-unstable if \( C_1 \sqrt{C_2} > 1 \). Since this is the case that \( \beta = \frac{2}{3}(\gamma - 1) \), we use solutions \( u_0^\pm \) of (2.34) for \( \bar{u} \). That implies that

\[ C_1^\pm = \left(1 \pm \sqrt{1 - \frac{24b\sqrt{K}}{\sqrt{c^2 + 4a}}} \right)^2 \]

so the pattern is stable if,

\[ \left(1 \pm \sqrt{1 - \frac{24b\sqrt{K}}{\sqrt{c^2 + 4a}}} \right)^2 \frac{1}{24b\sqrt{b}} \sqrt{c^2 + 4a} < 1, \]

which is always true for \( u_0^- \), and never true for \( u_0^+ \). Hence, the pattern with \( \bar{u} = u_0^- \) is in this case always 1D-stable, and the pattern with \( \bar{u} = u_0^+ \) is always 1D-unstable. This concludes the proof of part (iv) of the theorem.

For cases (v) and (vi) we have \( \beta = -1 \) and \( \gamma = -\frac{1}{2} \), we rewrite the leading order terms of equation (4.46) to

\[ (4.50) \]
\[ C_1 = \frac{1}{\sqrt{P^2 - 4} + C_2} \left( \frac{9}{K(P)} - 1 \right). \]

Recall also the values (2.34) for \( \bar{u} \) and the existence requirement (2.33). While \( C_1 \) is real-valued, \( P \in \mathbb{C} \). We use Mathematica to solve for which values of \( P \) the right hand side of (4.50) is real-valued, and plot its values against the real part of these values of \( P \). For several values of \( C_2 \), we have illustrated this in Figure 15. The \( P \)-values of intersections of the left and right side of (4.50), correspond to eigenvalues \( \lambda \) via (4.33). Regardless of the
value of $C_2$, there is an asymptote at $P = 3 \ (\lambda = \frac{5}{4})$. Hence, a stability criterion is always upper limit for $C_1$. On the other hand, as $C_1$ approaches 0, equation (4.50) approaches case (ii) of this theorem and the pattern is stable for every $C_2$ of $O(1)$. Hence, there is always a bifurcation, although the nature of it may vary. A transition between a Hopf bifurcation or a zero eigenvalue occurs when the minimum of the right hand side of (4.50) for real-valued $P$ occurs exactly at $P = 2$. That happens for

$$C_2 = C_2^* = -\frac{9}{2K'(2)} \approx 1.333,$$

which separates the cases (v) and (vi).

(v) This case occurs if $C_2 > C_2^*$ and (2.33) is satisfied. That implies

$$\frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} < \min \left\{ \frac{24b}{\sqrt{C_2}}, 1 \right\}.$$  

If $C_2 > C_2^*$, the right hand side of (4.50) is real-valued in a neighborhood of $P = 2$. Hence, there is a bifurcation with a zero eigenvalue. We use (4.49) to derive that the pattern is 1D-stable if

$$C_1 < \frac{1}{\sqrt{C_2}},$$

and 1D-unstable if $C_1 > \frac{1}{\sqrt{C_2}}$. A straightforward computation yields that the pattern with $\bar{u} = u_0^*$ never satisfies this case but undergoes a saddle-node bifurcation when it collides with $u_0^-$, i.e. when (2.33) is equality. The pattern with $\bar{u} = u_0^-$, is always stable, apart from that same bifurcation point.

(vi) Using both conditions $C_2 \leq C_2^*$ and (2.33), we derive that case (vi) occurs for

$$\frac{24b}{\sqrt{C_2}} \leq \frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} \leq 1,$$

which is only a nonempty range if $b \leq \frac{\sqrt{c^2 + 4a}}{24} \approx 0.048$. In this case, destabilization occurs through a Hopf bifurcation. For $\text{Re}(P) > 0$, $P$ has nonzero imaginary part if and only if $\lambda$ has nonzero imaginary part. If $\lambda = i\kappa$ with $\kappa \in \mathbb{R}$, i.e. at the bifurcation point, $P = 2\sqrt{i\kappa + 1}$, see (4.33). We substitute $P = 2\sqrt{i\kappa + 1}$ in (4.50) and for every $C_2$, we solve for which $\kappa$ the right hand side of (4.50) has zero imaginary part,

$$\text{Im} \left( \frac{1}{\sqrt{4i\kappa(C_2)} + C_2} \left( \frac{9}{K \left( 2\sqrt{i\kappa(C_2)} + 1 \right)} - 1 \right) \right) = 0.$$  

This yields a curve that assigns to every $C_2 \in (0, C_2^*)$ the value of the right hand side of (4.50) at the Hopf bifurcation, see the right panel of figure 14, which is exactly the function $Z(C_2)$ defined in (4.8). A 1D-stability requirement is then,

$$C_1 < Z(C_2),$$
where equality coincides with the Hopf bifurcation. The value of the eigenvalues as function of $C_2$ is depicted in figure 16, where we see that $C_2 \downarrow 0$ returns the same result as described in case (iii) of this theorem.

Figure 15: Left: Two times the graph of the right hand side of (4.50) for two values of $C_2$, for values of $P \in \mathbb{C}$ for which the expression equals a value in $\mathbb{R}$, plotted against the real part of the corresponding $P$. When $P$ has nonzero imaginary part, the graph is gray. Dark blue corresponds to $C_2 = 4$, while lighter blue corresponds to $C_2 = 0.5$. The stars indicate where the intersections with $P = 2$ are, i.e. at the bifurcation point. Right: The function $Z(C_2)$ (see (4.8)) for those values of $C_2$ for which $Z(C_2)$ is real-valued, and with $\bar{\kappa}$ for $C_2 \in (0, C^*_2)$.

Figure 16: The function $\bar{\kappa}$ for $C_2 \in (0, C^*_2)$, which assigns to every $C_2$ the imaginary part of the eigenvalues at the Hopf bifurcation.

Proof. (of theorem 4.3). In theorem 4.3, the only change compared to theorem 4.2 is that this concerns periodic solutions with long wave lengths. Following lemma 4.4, the results for the 1-pulse may be carried over to this situation exactly if $\beta < 2\gamma$. The case (ii) of that lemma does not apply, because for one spatial dimension there is no parameter $\theta$. Selecting exactly
those parameter regimes that satisfy $\beta < 2\gamma$, yields the results stated in the theorem.

The results of the theorems 4.1 is derived from testing the stability of solutions $(U_0, V_0)$ of (1.2) against perturbations in two dimensions. Of these perturbations, the wave number of the transverse perturbation factor may be arbitrarily small, but has an upper bound defined by (4.5). The theorems 4.2 and 4.3, however, consider only patterns with one spatial dimension and hence perturbations in only one space variable. If, however, we consider a more restrictive function space for perturbations in two spatial dimensions, we may carry over the results of theorems 4.2 and 4.3 to two-dimensional stability.

Corollary 4.5. Let the assumptions $A1$ and $A3$ be satisfied and let $\beta \geq \frac{2\gamma}{3}(\gamma - 1)$. Let $(U_0, V_0)$ be a slow/fast 1-pulse solution of (1.2), trivially extended in the $y$ direction. Let $X$ be the function space spanned by perturbations

$$e^{i\ell}(\hat{u}(\chi), \hat{v}(\chi)),$$

with $\ell \ll \delta^{2\gamma - \frac{2\beta - 1}{3}}$. Then, the statements of theorem 4.2 summarize the two-dimensional stability of $(U_0, V_0)$ against functions space $X$.

Let $(U_0, V_0)$ be a slow/fast periodic solution of (1.2), trivially extended in the $y$ direction. Then, the statements of theorem 4.3 summarize the two-dimensional stability of $(U_0, V_0)$ against functions space $X$.

Proof. The proof follows immediately from the observation that if $\ell \ll \delta^{2\gamma - \frac{3\beta - 1}{2}}$, $\theta \geq 2\gamma - \frac{2\beta - 1}{3}$, which is more restrictive than (4.5). The proof of theorem 4.1 can no longer be applied. In fact, if the wave number is not too large, the leading order equation that determines the stability, (4.39), is (4.46). Hence, the results from theorems 4.2 and 4.3 follow.

The theorems 4.1–4.3 particularize the 2D-instability and 1D-stability results for a range of parameter and scaling regimes. Since the results are concluded from the Evans function that was derived in section 4.1 for 1-pulses and where it was explicitly used that the function has a slow/fast splitting, our results are not exhaustive. If wave numbers become too large (i.e. when (4.5) is violated), equation (4.39) does not describe the leading order approximation of the eigenvalues. In fact, it can be shown that the unstable eigenvalue of (the proof of) theorem 4.1 become negative as $\ell$ increases further, see also [20]. Similarly, if $\beta > 2\gamma$, the constructed periodic solutions cannot be considered ‘nearly’ localized, and therefore the associated spectrum cannot be approximated by discrete values united with $\sigma_{\text{ess}}$. In that case, the stability problem is similar to that of the periodic solutions of the Gierer-Meinhardt equations, see [63]. A full analysis of this case is, however, not part of the present article.

5. Conclusions and discussion. The need for a mathematical framework to study vegetation patterns has been acknowledged at least since [25, 35, 53]. At present, a variety of mathematical models is considered in the literature [25, 30, 35, 42, 43, 51, 53, 67, 74, e.g]. Naturally, the emphasis of the studies of these models has been on numerical simulations, although there is a growing body of literature in which aspects of the dynamics generated by these models are studied analytically [7, 26, 27, 55–58, 61, 64, e.g]. Some of this mathematical research is devoted to the onset of vegetation patterns [7, 26, 27, 64]. Although it is known from the literature that predictions based on near equilibrium patterns may be relevant far beyond the region of their mathematical validity [31, e.g], most of the mathematical analysis
of these models concern far-from-equilibrium patterns. Of course this is also natural from the ecological point of view, since the vegetation patterns of interest consist of regions of bare soil interspersed with vegetated areas (or vice versa): these clearly do not have the character of small amplitude perturbations of a homogeneously vegetated state. However, especially in the far-from-equilibrium setting there is an obvious trade-off between manageable analytics and the level of realism in a model, also in the case of singularly perturbed models (as all models in this context are, due to the strong variation of timescales in the spread of water versus biomass). For that reason, the literature in which analytical aspects of far-from-equilibrium vegetation patterns are considered is at present restricted to the most simple two-component Klausmeier-Gray-Scott type models [55–58,61,64].

There is, however, a growing mathematical literature on the dynamics of localized (far-from-equilibrium) patterns in multi-component singularly perturbed reaction-diffusion systems with more than two components which shows that many of the mathematical ideas developed in the context of two-component models can indeed be generalized, [8,65,66,72, e.g.]. Nevertheless, the analytical technicalities become more and more involved at a very fast rate – especially in the context of realistic models. Thus, although there is reason to be confident that important aspects of the mathematical far-from-equilibrium analysis of ecological vegetation models can be developed in the context of more realistic multi-component models, such analysis has so far only been applied to two-component Klausmeier-Gray-Scott type models, see [55–58,61,64, e.g]. This is also the approach taken here: although we are interested in the realistic problem of striped vegetation patterns, we only analyze the most simple model, the generalized Klausmeier-Gray-Scott model (1.2)/(2.8)/(2.9).

The ecological motivation for the analysis in the present paper comes from the dominating appearance and resilience of striped patterns on hillsides with sufficiently steep slopes that has been both observed in nature [2,9,10] and in simulations [58]. However, the emphasis of this paper lies on mathematical analysis. The simulations in [58] suggest that advection may act as an exciting new stabilization mechanism for striped patterns in reaction-diffusion equations. Even without the ecological background of this paper, this is an important topic of research in the mathematical field, as there is currently no literature in which this is studied rigorously.

In this section, we discuss potential ecological and mathematical implications of our analysis. In the ecological setting, this is embedded in the much larger issue of the ecological interpretation and relevance of insights obtained from simplified models as the generalized Klausmeier-Gray-Scott model (1.2)/(2.8)/(2.9) – which also is the subject of ongoing work in the context of a collaboration between ecologists and mathematicians as in [40,58,60,61].

5.1. Interpretation of the analysis. We have explored the existence and stability of traveling homoclinic and spatially periodic stripe patterns of the generalized Klausmeier-Gray-Scott (1.2). They arise as multiscale pulse patterns in one spatial dimension, trivially extended in the other direction. Apart from their intrinsic value, these patterns also form the foundation for complex pattern dynamics. The existence of traveling single pulse solutions is established using geometric singular perturbation theory. Such a traveling homoclinic solution corresponds to a single strip of vegetation in an elsewhere endless desert. More realistic are the periodic patterns that were constructed in section 3. The stripe patterns that correspond to
these solutions are widely observed and known as tiger bushes [73]. Both types of solutions have a positive speed, indicating that the patterns travel uphill. This phenomenon is confirmed by observations [9], and can be explained by a surplus of water on the upper side of a vegetation strip. For homoclinic traveling waves, the values (2.29) imply that as the slope $c$ decreases, the speed of the homoclinic traveling wave also decreases. This is in agreement with the Gray-Scott results, see section 1.1, where stationary pulses exist on flat terrain ($c = 0$). Note that as the rainfall, parametrized by $a$, decreases, the speed decreases. Furthermore, the water density value in the vegetation strip is smaller in the case of a traveling oasis ($u_0$) compared to traveling periodic patterns for the same set of parameters.

There are several destabilization mechanisms described by the theorems 4.1–4.3. Theorem 4.1 describes that in the validity regime of our analysis, defined by assumptions $A1$–$A3$, there is always a range transverse wave numbers of perturbations that destabilize the pattern. In other words, stripe patterns will cease to exist, as soon as the perturbations along the vegetation strip have a transverse wave number that is within this regime. Since the scaling regime bounds the slope of the terrain, $c$, this is in agreement with [58], where two-dimensional stripe patterns are numerically found to be stable only above a certain threshold for the slope – see section 2.2. Since we only consider patterns that are trivially extended, it is not yet clear if the pattern will in fact collapse to a desert state. Considering the observations of rhombic or spot patterns, one may suggest that a large wave number perturbation could generate nontrivial two-dimensional patterns, as is also confirmed by the numerical simulations in [58]. If we consider a more ‘mild’ definition of stability, with perturbations in a well-chosen function space, we can make more detailed conclusions about the resilience of stripe patterns, formulated in Corollary 4.5.

The theorems 4.2 and 4.3 describe stability of homoclinic and periodic patterns, with respect to one-dimensional (non-transversal) perturbations, respectively. In absence of transverse perturbations, the patterns may indeed be stable and several destabilization mechanisms take place as parameters vary. In the cases (iii)-(vi) in theorem 4.2 and case (iii) in 4.3, destabilization occurs as $C_1$ (defined in (4.6)) grows. Reversely, the patterns are stable as long as $C_1$ is small enough. Since $C_1$ is minimal for the homoclinic 1-pulse, because $u_0$ is smaller than $u_{p,h}^*$, the homoclinic 1-pulse is the last pattern to destabilize, i.e. ‘the most stable pattern’. This is completely in agreement with Ni’s conjecture, which was formulated for the Gierer-Meinhardt system in [48], see also [19, 20, 63]. This implies that, as either precipitation ($a$) or slope ($c$) decreases, periodic solutions with the larger water density in the vegetation strip destabilize first and the traveling oasis is the last observable pattern. Although this also confirms the numerical observations of [58, 61, 64], it should be remarked that the present analysis only holds for the validity regime of our method, which is characterized by the splitting of fast and slow behavior in the system and defined by $A1$–$A3$. Furthermore, our analysis is only valid for patterns that are ‘nearly localized’, i.e. that the pattern is a homoclinic pattern or that Lemma 3.2 holds. Outside this regime, extra measures are necessary to draw conclusions about the stability (and the existence) of stripe patterns.

The value of $\bar{u}$ – the water density within a vegetated area – for periodic patterns is not explicitly determined in section 3, which makes the interpretation of the stability results of
section 4 less straightforward. However, we do have a lower and upper bound for $\bar{u}$, namely,

$$\bar{u} \in (u_0, \bar{u}) = \left( \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}, \frac{\sqrt{6b\sqrt{b}}}{\sqrt{s}} \right),$$

see (2.29) and (3.2). The upper bound can be carried over to $C_1$ to obtain, by theorem 4.2(iii), a sufficient but not necessary condition for the stability of periodic patterns, namely

$$\frac{\sqrt{b}}{s} < 1.322,$$

see (4.47). In other words, it implies a lower bound for the migration speed of the periodic patterns. Together with the upper bound for $s$ defined in (3.1), we have a range of $s$-values for which one-dimensional periodic patterns exist and are stable,

$$s \in \left( 0.756\sqrt{b}, \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}} \right).$$

This range is nonempty if $4.539b^2 \leq c\sqrt{c^2 + 4a}$. We conclude that even if $a$ or $c$ decreases, this does not have to destabilize the pattern, as long as it adapts to a suitable speed $s$, to satisfy the stability condition.

For all patterns constructed in this manuscript, we conclude that if $a$ or $c$ decreases the parameter regime in which patterns are stable shrinks, making them less resilient. A general conclusion we may draw is that decreasing rainfall or slope may have a serious negative impact on the ecological resilience of both homoclinic and periodic patterns. This is also confirmed by [2, 10], where field observations show that stripe patterns disappear as the slope decreases.

5.2. Mathematical aspects. In mathematical terms, we have considered the existence of homoclinic and spatially periodic slow/fast patterns in a reaction-advection-diffusion model. The advection term breaks the reversibility symmetry of the original reaction-diffusion system. Since this symmetry is a key feature of the establishment of especially the existence of these types of patterns in reaction-diffusion equations – Gray-Scott, Gierer-Meinhardt – several novel ideas have been introduced to incorporate the skew effects of the advection term in the geometric singular perturbation approach. We found that the symmetry breaking effect could be incorporated in the Evans function framework for spectral stability of the patterns in a natural way. Our stability results on one-dimensional patterns provide direct generalizations of previous results in the literature on reaction-diffusion models, see [13, 32, 63, 68]. Moreover, our result on the instability of stripes to the transverse perturbations, theorem 4.1, is similar to the findings in [20, 46, 58], and generalized these to a class of reaction-advection-diffusion systems.

It should be explicitly noted that our analysis relies heavily on the scaling regimes. The clear advantages of this asymptotic approach is that we can explicitly determine the existence and stability regimes. The major drawback is, however, that we were not able to capture all types of stripe patterns. This could also be an explanation for the fact that we do not recover the stability results of [58], in which 2D-stable stripe patterns were reported for (a slightly
rescaled version of) system (1.2) with large slopes. These large slopes may be described by parameters outside the triangular regime defined by assumptions A1–A3, and therefore cannot be fully understood by our analysis. However, since this [58] is to our awareness the first mathematical result in the literature in which stable far-from-equilibrium stripes composed from homoclinic fast excursions are observed, this challenging problem is intriguing, [20, 46] (see however [36] for stable singular stripes based on ‘nearly heteroclinic’ fast excursions). Solving it requires a new mathematical approach that goes beyond the currently developed method for reaction-advection-diffusion systems. This is the topic of ongoing research.

5.3. Limitations and observations. In any article where mathematical models are used to describe a natural phenomenon, many simplifications need to be done to keep the work manageable. This, in turn, implies the need for a brief discussion of the conclusions drawn according to the simplified model. As mentioned before, our analysis does not extend beyond stripe patterns, and hence does not describe the widely observed gap or labyrinth patterns described in, for example, [26, 27]. Many of the assumptions made in this article cannot be made in the case where the pattern is not trivially extended in one spatial dimension. Perhaps another type of symmetry could resolve this. Moreover, apart from restricting our analysis to a strongly simplified two-component model, we did not incorporate nonlinear diffusion, nor did we examine a spatially dependent slope $C(x, y)$ or a nonconstant speed $S(t)$.

A more general, but prudent remark that should be made, is that the comparison of mathematical results with observation from the field remains to be a delicate task. We did not attempt to estimate the magnitude of parameters, let alone the values of the parameters themselves, from field data. In fact, it is not even clear whether it is possible to make realistic estimates of parameters in a highly simplified model as (1.2) – see also the discussion in [59]. Of course, this makes it difficult to distinguish which of our conclusions are relevant to the ecological system it applies to, and which are predominantly relevant from the mathematical point of view. Especially in the case of stability, it is for instance not easily identified whether stability should indeed be tested against perturbations with all possible transverse wave numbers. It is the subject of ongoing research to test and interpret the findings of the present paper – and those of earlier mathematical model studies [58, 61, 64] – against observations.

Acknowledgments. We gratefully acknowledge support from NWO through the NDNS$^+$ cluster, and Robbin Bastiaansen for his help with the simulations of Figures 1 and 7.

REFERENCES


Patterns in a gKGS model


