# The Conformal Grid behind Escher's Division

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## Abstract

In this research, we will take a closer look at the artwork `Division' (1956) by Maurits Cornelis Escher, which is based on a grid he speci cally designed for the print. In the rst place, the aim of the research is to understand the mathematics behind the work. Thereafter, the research aims to establish (the rules for designing) a new print which matches a mathematically perfected version of the underlying grid, but still contains the characteristic properties of the original, such as a tessellation of black and white reptiles which divide in a multi-step transition. Previously, similar research has been published on `Print Gallery', another work by Escher: `Artful mathematics: The heritage of M.C. Escher' by Bart de Smit and Hendrik W. Lenstra Jr. [10]. The methods and the central theorem, by Joseph H. Silverman [14], that supplies them with a map that constructs the prints underlying grid, presented in their research, are applied in this research as well. Consequently, we will also consider how the prints relate.

First of all, it will be con rmed that the grid Escher supposedly drew for Division, lies at its base. However, comparing the grid with the print, it will become evident that the center of the print does not match the center of the grid: while the grid re nes toward the center, the print does not. This therefore raises the question if a new print, consistent with its grid, can be extended to the entire complex plane: both inward and outward. Escher had a similar issue with Print Gallery. Also similar, is the circular expansion present in both prints. Another question which will be induced in the research is whether Escher intended his grid to be induced by a conformal map.

From Escher's curved grid, we will establish a straightened grid and its symmetries on the complex plane: multiplication by $1 \quad i$ and 4. Escher's grid also contains two symmetries we can deduce: $\quad i$ and an unknown $\quad$, which is both a scaling and rotation and therefore has both a non-zero real and imaginary part. With these symmetries and a result of the mentioned theorem, we can then determine a map which will supply us with the unique mathematically generated version of Escher's grid (on $C = h \quad i; \quad i$) from the straight grid (on $C = h1 \quad i; 4i$):

$$h : C = h1 \quad i; 4i \quad ! \quad C = h \quad i; \quad i$$
$$z \; 7 \; ! \; z^{\frac{2 \quad i}{i \quad 2\log(2)}};$$

From the properties of map $h$ we can then also determine that

$$= 4^{\frac{2 \quad i}{i \quad 2\log(2)}};$$

and compare Escher's original grid with our version. As we will see, the grids are very much alike. From the process of constructing the map we can also conclude that it is very likely that Escher indeed intended his grid to be induced by a conformal mapping, which $h$ is. A similar result was found for Print Gallery.

Having established the map that sends the straight grid to the curved grid, we can start establishing the rules for designing a new print. We will consider each reptile in the print to be a isosceles right triangle, with which we can establish `matching rules' to form a tessellation. Additional rules are as follows: the symmetries are preserved, we can have a multi-step transition of division and the reptiles are black and white. Thereafter, we will determine a tting fundamental area for which we can design a tessellation. Complying with the symmetries, more speci cally symmetry , solves the issue Escher had in the center of the print, allowing us to ll the entire punctured complex plane.

Interestingly, the tessellation on the fundamental area that is least ne, constructs (almost entirely) the grid which supports one of Escher's other prints: `Smaller and Smaller' (1956). However, in this print there is no dividing of the reptiles.

Because making a new print will require a new (artistic) design, we will not be able to show a new print in this research just yet. However, we get a general idea of what a new print could look like by applying the established map $h$ to Smaller and Smaller.

The research will show that art and mathematics can be combined beyond basic mathematics and illustrating math in artworks; Escher has done this in multiple works.

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